On TQFTs

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1 Introduction

Note that many of the subjects treated by us have never appeared in textbook/monograph form! (Triangulation tqft in $d = 1+1$, the improved approach of [8, 21] to triangulation TQFTs in $d = 2+1$, the Freed-Quinn TQFTs, the open-closed TQFTs in $d = 1+1$ of Moore-Segal, etc.)

Conventions: $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2 Atiyah-style definition of TQFTs

2.1 Some reminders of differential topology

2.2 Atiyah's axioms

notion of equivalence of TQFTs? Will be defined later in terms of equivalence of tensor functors.

2.1 Definition *An (Atiyah-style) TQFT (V, Z) is complete if*

$$V(\Sigma) = \mathrm{span}_k\{Z(M) \mid \partial M = \Sigma\}$$

for every closed oriented s-manifold Σ.

Note this definition is useful only when every closed oriented s-manifold Σ arises as the boundary ∂M of a compact oriented d-manifold M. This is obviously false for $d = 1$, obviously true for $d = 2$ and also true (by a non-trivial theorem of Rokhlin) for $d = 3$.

2.3 TQFTs in $0 + 1$ dimensions

3 Tensor categories and all that

3.1 Basic notions of categories

3.2 Tensor categories, coherence, tensor functors, monoidal natural transformations, braidings

$$\begin{array}{ccc} (X \otimes \mathbf{1}) \otimes Y & \xrightarrow{\alpha_{X,\mathbf{1},Y}} & X \otimes (\mathbf{1} \otimes Y) \\ \Big\downarrow{\scriptstyle r_X \otimes \mathrm{id}_Y} & & \Big\downarrow{\scriptstyle \mathrm{id}_X \otimes l_Y} \\ X \otimes Y & = = = = & X \otimes Y \end{array} \tag{3.1}$$

3.3 The group categories $\mathcal{C}(G, A)$

In this subsection we will consider a class of moderately interesting tensor categories associated with finite groups. These categories will play a certain rôle in our discussion of triangulation TQFTs in $d = 2+1$ dimensions, but for the time being they serve to illustrate the cohomological meaning of the pentagon identity.

Let G be a group and A an abelian group. We define a category $\mathcal{C}(G, A)$ by $\mathrm{Obj}\,\mathcal{C} = G$,

$$\mathrm{Hom}_{\mathcal{C}(G,A)}(g, h) = \left\{ \begin{array}{ll} A & \text{if } g = h \\ \emptyset & \text{if } g \neq h \end{array} \right.$$

and $\mathrm{id}_g = e$ for all $g \in G$. For $s, t =\in \mathrm{End}\, g = A$, we define $t \circ s = ts$. It is easy to see that $\mathcal{C}(G, A)$ is a category. Our aim is to study monoidal structures on $\mathcal{C}(G, A)$, always defining $g \otimes h = gh$. The unit object will be $\mathbf{1} = e$, and since this is a strict unit, the unit constraints can be taken to be identity morphisms. The tensor product of morphisms is defined by

$$s \in \mathrm{End}\, g,\ \ t \in \mathrm{End}\, h \quad \Rightarrow \quad s \otimes t = st \in A = \mathrm{End}\, gh,$$

and one verifies the interchange law $a \otimes b \circ c \otimes d = (a \circ c) \otimes (b \circ d)$. It remains to consider the associativity constraint $\alpha_{g,h,k} : (g \otimes h) \otimes k \to g \otimes (h \otimes k)$. In view of $(g \otimes h) \otimes k = g \otimes (h \otimes k) = ghk$, we have $\alpha_{g,h,k} \in \mathrm{End}\, ghk = A$. Thus α is a map $G \times G \times G \to A$, and the pentagon axiom (**??**) takes the form

$$\alpha(g, h, k)\alpha(g, hk, l)\alpha(h, k, l) = \alpha(gh, k, l)\alpha(g, h, kl) \quad \forall g, h, k, l \in G. \tag{3.2}$$

(Since A is abelian, the orders of the factors don't matter.) We denote by $Z^3(G, A)$ the set of maps $G \times G \times G \to A$ satisfying (**??**). Since the unit constraints were taken to be identities, (3.1) implies $\alpha(g, e, h) = 1$ for all $g, h \in G$.

3.4 A glimpse of group cohomology

4 The functorial definition of TQFTs

4.1 Cobordisms

4.2 The tensor category Cob_{s+1}

The symmetry of Cob_{s+1} is defined as follows. Let Σ_1, Σ_2 be closed oriented s-manifolds. Let $M = (\Sigma_1 \coprod \Sigma_2) \times I$ be the cylinder cobordism and $f_i : \Sigma_1 \coprod \Sigma_2 \to \partial M$ be the obvious diffeomorphisms onto the components of ∂M at $z = 0$ and $z = 1$, respectively. Recall that (M, f_1, f_2) is the identity morphism $\mathrm{id}_{\Sigma_1 \coprod \Sigma_2}$. Now let $\alpha : \Sigma_2 \coprod \Sigma_1 \to \Sigma_1 \coprod \Sigma_2$ be the canonical flip diffeomorphism. Now the symmetry $c(\Sigma_1, \Sigma_2)$ is the cobordism $(M, f_1, f_2 \circ \alpha)$ from $\Sigma_1 \coprod \Sigma_2$ to $\Sigma_2 \coprod \Sigma_1$.

4.3 TQFTs as tensor functors. Relation to Atiyah's definition

comment on symmetry requirement on $F : \mathrm{Cob}_{1+1} \to \mathrm{Vect}_k$. Note that the flip cobordism $M : \Sigma_1 \otimes \Sigma_2 \to \Sigma_2 \otimes \Sigma_1$ is topologically just the cylinder $(\Sigma_1 \coprod \Sigma_2) \times I$....

4.1 Remark Let $F : \mathrm{Cob}_{s+1} \to \mathrm{Vect}_k$ be the functorial TQFT corresponding to the Atiyah-style TQFT (V, Z). Then $F(\Sigma) = V(\Sigma)$ for each closed s-manifold and $F(M) = V(M)$, where M is interpreted as a cobordism $\emptyset \to \partial M$. Thus (Z, V) is complete in the sense of Definition 2.1 iff

$$F(\Sigma) = \mathrm{span}_k\{F(M) \mid M : \emptyset \to \Sigma\}$$

for every closed oriented s-manifold Σ.

4.4 Unitary TQFTs

4.2 Definition *Define* $* : \mathrm{Cob}_{s+1}(\Sigma_1, \Sigma_2) \to \mathrm{Cob}_{s+1}(\Sigma_2, \Sigma_1)$.

$*$ is a covariant and involutive ($** = \mathrm{id}$) endofunctor of Cob_{s+1}.
Let $\mathcal{H}$ denote the symmetric tensor $*$-category of finite dimensional Hilbert spaces.

4.3 Definition *A unitary TQFT in* $s+1$ *dimensions is a symmetric tensor functor* $F : \mathrm{Cob}_{s+1} \to \mathcal{H}$ *such that* $F(M^*) = F(M)^*$ *for all morphisms* M.

5 TQFTs in $d = 1 + 1$ vs. commutative Frobenius algebras

Here comes an account of the standard theory, cf. e.g. [29]. history: [12, 48, 1]
NOTE: For us, all Frobenius algebras are unital!!

5.1 Theorem *There is an equivalence of categories between TQFTs in* $1 + 1$ *dimensions (with isomorphisms of TQFTs) and commutative Frobenius algebras (with isomorphisms).*

Note: If the cFA $((A, m, e), \varepsilon)$ satisfies $\Delta \circ m = \mathrm{id}$ then the associated TQFT is trivial in the sense that $Z(M) = \dim A$ for every closed oriented 2-manifold.

5.1 The semisimple and unitary cases

*** TQFTs corresponding to SEMISIMPLE commutative FAs.

5.2 Definition *A TQFT in* $1 + 1$ *dimensions is called semisimple if the associated commutative Frobenius algebra is semisimple.*

5.3 Remark How should one define semisimplicity of TQFTs in $> 1+1$ dimensions??

5.4 Proposition *A Frobenius algebra V is semisimple if and only if the element $m \circ b \in V$ is invertible.*

If A is semisimple then

$$Z_A(M) = \sum_{i \in I} c_i^{1-g(M)}$$

for every closed oriented 2-manifold M.

If, in addition, A is special (i.e. $m \circ \Delta = \mathrm{id}$) then $c_i = 1$ for all i, implying $Z_A(M) = |I|$ for every closed connected M. Multiplicativity then implies $Z_A(M) = |I|^n$ for closed manifolds with n connected components. I.e., the TQFT can detect only the number of connected components (and nothing at all if $|I| = 1$).

*** classification of unitary TQFTs in $1+1$ dimensions, cf. [15]: unitary TQFTs are semisimple and $|c_i| = 1$ for all $i \in I$.

6 TQFTs from triangulations: $d = 1 + 1$

In Section 5 we have obtained an essentially complete (modulo the classification of commutative Frobenius algebras) theory of TQFTs in $1+1$ dimensions. So why should we bother to study a different approach to constructing TQFTs in two dimensions? There are at least two reasons. On the one hand, our constructions of TQFTs by triangulation will proceed in two steps: First we construct an invariant Z of closed oriented 2-manifolds, and in a second step we lift Z to a TQFT. While the second step will be essentially independent of the dimension, the first step is considerably more involved for $d = 2+1$, the case we are really interested in, than for $d = 1+1$. The lower dimensional case may thus serve as a playground on which to get familiar with the principles. On the other hand, some interesting things *do* happen in triangulation TQFTs in $d = 1+1$, allowing us to completely clarify the relationship between triangulation TQFTs and the general construction in Section 5. Some of these results have intriguing analogues in $2+1$ dimensions where, however, many open questions remain!

The presentation in this section is inspired mainly by [34] and [20] (see also [5]), but we will work harder trying to answer all naturally arising questions. In particular, we will begin with a minimal set of assumptions and show that one can make them more restictive withoul sacrificing generality, cf. Exercise 6.23 and Proposition **??**. In Subsections 6.1 and 6.2 we will only work with triangulations in terms of simplicial complexes!

6.1 Invariant of closed oriented triangulated 2-manifolds

The aim of this subsection is to use triangulations to define an invariant of closed oriented 2-manifolds. In analogy to the general theory of $d = 1+1$ TQFTs, which turned out to be classified by commutative Frobenius algebras, we base our construction on a certain algebraic structure. (Further conditions will be introduced below.)

6.1 Definition *A good triple is of the form (V, a, μ), where*

- *V is a finite dimensional vector space over some field k.*
- *$a \in V \otimes V \otimes V$ is $\mathbb{Z}_3$-invariant, i.e. invariant under the automorphism $\tau_3 : u \otimes v \otimes w \mapsto v \otimes w \otimes u$ of $V^{\otimes 3}$.*
- *$\mu \in (V \otimes V)^*$ is symmetric.*

Let M be a closed oriented 2-manifold and T a triangulation. (Note that there are no compatibility conditions between orientation and triangulation.) For any good triple (V, a, μ), we define an element $Z_{(V,a,\mu)}(M,T) \in k$ as follows: We use the orientation of M to assign a direction to every edge in the triangulation as follows: Every 2-simplex $S \subset M$ inherits an orientation from M, and we give the three edges which constitute ∂S the usual boundary orientation. (Then the three edges constitute an oriented loop.) For each of the n_2 2-simplices in the triangulation we take one copy of $a \in V \otimes V \otimes V$ and associate each edge of that 2-simplex with one of the three 'legs' of a in an orientation preserving way. Now we tensor the a's together, obtaining and element of $V^{\otimes 3n_2}$. Since every 1-simplex of the triangulation belongs to precisely two 2-simplices, we can use $\mu \in (V \otimes V)^*$ to contract all the pairs of legs of $a^{\otimes n_2}$ corresponding to the same edge. In view of the cyclic invariance of a and the symmetry of μ, this procedure gives a well-defined element $Z_{(V,a,\mu)}(M,T) \in k$. Since the triple (V, a, μ) usually is fixed, we may simply write $Z(M,T)$. (So far, we didn't use the non-degeneracy of μ.)

Our next aim is to find conditions on a good triple that make $Z(M,T)$ independent of the triangulation T. In view of Theorem A.68, this amounts to showing that $Z(M,T)$ is invariant under the Pachner moves. In two dimensions, there are the two moves shown in Figures 1 and 2.

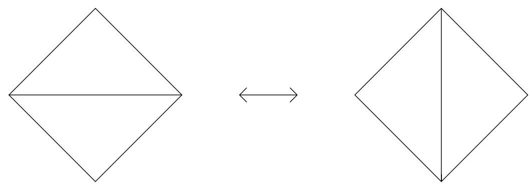


Figure 1: $d = 2$: Pachner move of order 1

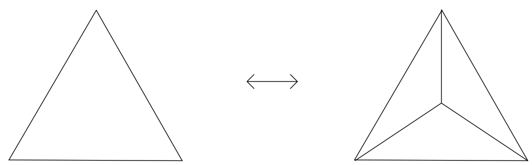


Figure 2: $d = 2$: Pachner moves of orders 0 and 2

6.2 PROPOSITION *(a) $Z(M,T)$ is invariant under the 2-2 Pachner move (and its inverse) if and only if*

$$(\mathrm{id} \otimes \mathrm{id} \otimes \mu \otimes \mathrm{id} \otimes \mathrm{id})(a \otimes a) = \tau_4[(\mathrm{id} \otimes \mathrm{id} \otimes \mu \otimes \mathrm{id} \otimes \mathrm{id})(a \otimes a)], \tag{6.1}$$

where $\tau_4 : s \otimes t \otimes u \otimes v \mapsto t \otimes u \otimes v \otimes s$. (I.e., $(\mathrm{id} \otimes \mathrm{id} \otimes \mu \otimes \mathrm{id} \otimes \mathrm{id})(a \otimes a) \in V^{\otimes 4}$ is $\mathbb{Z}_4$-invariant.)

(b) $Z(M,T)$ is invariant under the 2-2 Pachner move (and its inverse) if and only if

$$a = (\mu_{19}\mu_{34}\mu_{67})(a \otimes a \otimes a), \tag{6.2}$$

where μ_{ij} denotes contraction of the i-th and j-th component using μ.

Proof. (a) Consider the situation at the l.h.s. of Figure 3 with a numbering of the edges compatible with the orientation: The two 2-simplices correspond to a tensor $a \otimes a \in V^{\otimes 6}$. Using μ_{34} to contract the two components corresponding to the edge $3 = 4$, we obtain $(\mathrm{id} \otimes \mathrm{id} \otimes \mu \otimes \mathrm{id} \otimes \mathrm{id})(a \otimes a) \in V^{\otimes 4}$, where the four components of the tensor correspond to the edges $1, 2, 5, 6$. (In the computation of $Z(M,T)$, this tensor is part of a bigger expression, and the four exterior edges are contracted with those of the adjacent 2-simplices.) The r.h.s. of Figure 1, which is identical to the preceding one up to a rotation by $\pi/2$, gives rise to the same expression. Now the components of the tensor correspond to the edges $2, 5, 6, 1$, which differs from $1, 2, 5, 6$ by a cyclic permutation. Thus $Z(M,T)$ is invariant under the 2-2 Pachner move if and only if (6.1) holds.

Figure 3: Invariance under the Pachner moves

(b) Consider the right diagram in Figure 3. It gives rise to the tensor $(\mu_{19}\mu_{34}\mu_{67})(a \otimes a \otimes a) \in V \otimes V \otimes V$, whose components correspond to the edges $2, 5, 8$ of the boundary. Obviously, $Z(M, T)$ is invariant under the 1-3 Pachner move if and only if this expression equals a. ■

6.3 Definition *A very good triple is a good triple (V, a, μ) satisfying (6.1) and (6.2).*

6.4 Theorem *A very good triple (V, a, μ) gives rise to an invariant $M \mapsto Z_{(V,a,\mu)}(M) \in k$ of closed oriented 2-manifolds.*

Proof. Define $Z(M) = Z(M, T)$, where T is *any* triangulation of M. ■

6.5 Remark 1. In terms of the Sweedler notation, which writes $a \in V^{\otimes 3}$ as $a_1 \otimes a_2 \otimes a_3$ (ignoring the fact that a need not be a simple tensor), the conditions (6.1) and (6.2) are equivalent to

$$\begin{aligned} \mu(a_3, a_1')\, a_1 \otimes a_2 \otimes a_2' \otimes a_3' &= \mu(a_1, a_2') a_1' \otimes a_2 \otimes a_3 \otimes a_3', \\ a_1 \otimes a_2 \otimes a_3 &= \mu(a_1, a_3'')\mu(a_3, a_1')\mu(a_3', a_1'') a_2 \otimes a_2' \otimes a_2'', \end{aligned}$$

respectively, where a, a', a'' are copies of a.

2. Changing the orientation of M reverses the direction of the edges, which results in replacing a by $a^t = \sigma_{13}(a)$, where $\sigma_{13}(a) : x \otimes y \otimes z \mapsto z \otimes y \otimes x$. Thus $Z_{(V,a,\mu)}(-M) = Z_{(V,a^t,\mu)}(M)$, and Z lifts to an invariant of unoriented (or even unorientable) manifolds if and only if $a = a^t$. In conjunction with the cyclic invariance of a, this is equivalent to full S_3-invariance of a. However, we will see that this condition leads to a trivial theory.

3. A more transparent interpretation for the conditions (6.1) and (6.2) will be provided in Subsection 6.5. But first we will lift the invariant $Z(M)$ to a TQFT.

6.2 The triangulation TQFT in $1+1$ dimensions

Throughout this subsection, fix a very good triple (V, a, μ). Our aim is to construct a TQFT in $1+1$ dimensions, i.e. a symmetric tensor functor $F : \mathrm{Cob}_{1+1} \to \mathrm{Vect}_k$. We will follow the strategy of [21, Section 7], with the obvious simplifications due to the lower dimension.

6.6 Definition *Let Σ be a closed oriented 1-manifold with triangulation T. Then $V(\Sigma, T)$ is the vector space $V^{\otimes n_1(T)}$, where $n_1(T)$ is the number of edges appearing in the triangulation. (It is always understood that we choose (and remember) a bijection between the edges of T and the copies of V.)*

(Note in particular that $F(\emptyset) = k$.) We now need a slight generalization of Theorem 6.4. If M is a closed oriented 2-manifold and T_∂ a triangulation of ∂M, we may appeal to Theorem A.59 to choose a triangulation T of M that restricts to T_∂ on ∂M. Using μ to contract $a^{\otimes n_2(T)}$ over all interior edges in T as in the preceding subsection, we obtain an element $Z(M, T) \in V(\Sigma, T_\partial)$, and the same argument as in the proof of Proposition 6.2 shows that $Z(M, T)$ is independent of the chosen triangulation T (for fixed T_∂), allowing to denote it by $Z(M, T_\partial)$.

In order to obtain a functor-style TQFT, we need a suitable category of cobordisms between triangulated manifolds:

6.7 Definition *The category* $TCob_{s+1}$ *is defined like* Cob_{s+1}*, except that the objects (closed oriented s-manifolds) come with a triangulation. (The morphisms (=cobordisms) are not supposed to come with triangulations!)*

We now define $F_0(\Sigma, T) := V(\Sigma, T)$ and would like to complete this to a (tensor) functor $F_0 : \mathrm{TCob}_{s+1} \to \mathrm{Vect}_k$. Let $(\Sigma_1, T_1), (\Sigma_2, T_2)$ be objects in TCob_{s+1} and $M \in \mathrm{TCob}_{1+1}((\Sigma_1, T_1), (\Sigma_2, T_2)) = \mathrm{Cob}_{1+1}(\Sigma_1, \Sigma_2)$. Recall that the latter means that M is a compact oriented manifold equipped with smooth maps $\Sigma_1 \stackrel{f_1}{\to} \partial M \stackrel{f_2}{\leftarrow} \Sigma_2$ such that ∂M is the disjoint union of $f_1(\Sigma_1)$ and $f_2(\Sigma_2)$ and f_1 and f_2 are orientation-reversing and orientation-preserving, respectively, diffeomorphisms on their images. The diffeomorphisms f_1, f_2 induce a triangulation $T_\partial =$"$f_1(T_1) \cup f_2(T_2)$" of ∂M. Now the construction above gives rise to a vector $Z(M, T_\partial) \in V(\partial M, T_\partial)$. By construction it is clear that $V(\partial M, T_\partial) \cong V(\Sigma_1, T_1) \otimes V(\Sigma_2, T_2)$. Applying $n_1(T_1)$ copies of the linear map $\gamma : V \to V^*, x \mapsto \mu(x, \cdot)$ to the tensor factor of $Z(M, T_\partial)$ corresponding to Σ_1, we obtain an element of $V(\Sigma_1, T_1)^* \otimes V(\Sigma_2, T_2)$. We interpret the latter as a linear map $V(\Sigma_1, T_1) \to V(\Sigma_2, T_2)$ and denote it $F_0(M)$. The question now is whether F_0 is a tensor functor.

6.8 Proposition *Modulo the requirement that cylinder cobordisms* $\mathrm{id}_\Sigma = \Sigma \times I$ *be mapped to* $\mathrm{id}_{V(\Sigma)}$, F_0 *is a symmetric tensor functor* $\mathrm{TCob}_{s+1} \to \mathrm{Vect}_k$ *that satisfies all axioms of a TQFT.*

Proof. Let $(\Sigma_1, T_1), (\Sigma_2, T_2), (\Sigma_3, T_3)$ be closed oriented triangulated 1-manifolds and $M_1 : \Sigma_1 \to \Sigma_2, M_2 : \Sigma_2 \to \Sigma_3$ cobordisms. Let $M_3 : \Sigma_1 \to \Sigma_3$ be the cobordism obtained by gluing. One can find a triangulation $\widehat{T}_3$ of M_3 that restricts to T_i on $\Sigma_i, 1 \le i \le 3$ and to triangulations $\widehat{T}_1, \widehat{T}_2$ of the submanifolds M_1 and M_2. Now taking the tensor product of copies of a and contracting over the interior edges in $\widehat{T}_1$ and $\widehat{T}_2$ gives elements of $p \in V(\Sigma_1, T_1) \otimes V(\Sigma_2, T_2)$ and $q \in V(\Sigma_2, T_2) \otimes V(\Sigma_3, T_3)$. By definition, $F_0(M_1) = (\gamma^{\otimes n_1(T_1)} \otimes \mathrm{id}_{V(\Sigma_2, T_2)})(p)$ and $F_0(M_2) = (\gamma^{\otimes n_1(T_2)} \otimes \mathrm{id}_{V(\Sigma_3, T_3)})(q)$. Thus, somewhat symbolically,

$$\begin{aligned} F_0(M_2) \circ F_0(M_1) &= (\gamma^{\otimes n_1(T_2)} \otimes \mathrm{id}_{V(\Sigma_3,T_3)})(q) \circ (\gamma^{\otimes n_1(T_1)} \otimes \mathrm{id}_{V(\Sigma_2,T_2)})(p) \\ &= (\gamma^{\otimes n_1(T_1)} \otimes \mu^{\otimes n_1(T_2)} \otimes \mathrm{id}_{V(\Sigma_3,T_3)})(p \otimes q). \end{aligned}$$

The second line means that we use μ to contract over the two copies of $V(\Sigma_2, T_2)$ and apply $(\gamma^{\otimes n_1(T_1)} \otimes \mathrm{id}_{V(\Sigma_3,T_3)})$ to the resulting element of $V(\Sigma_1, T_1) \otimes V(\Sigma_3, T_3)$. The result of this clearly being equal to $F_0(M_3)$, we have proven $F_0(M_3) = F_0(M_2) \circ F_0(M_1)$. Thus modulo the requirement $F_0(\Sigma \times I) = \mathrm{id}_{F_0(\Sigma)}$, F_0 is a functor.

In view of Definition 6.7 it is clear that there are canonical isomorphisms $F_0(\Sigma_1 \coprod \Sigma_2, T_1 \coprod T_2) \cong F_0(\Sigma_1, T_1) \otimes F_0(\Sigma_2, T_2)$ satisfying the coherence axioms. (Just recall that the entries in the tensor products defining $F_0(\Sigma, T)$ correspond bijectively in the edges of T.) We also have $F_0(\emptyset, \emptyset) = \mathbf{1}_{\mathrm{Vect}_k} = k$. In view of the definition of the symmetry of Cob_{1+1} in Subsection 4.2, it is clear that F_0 maps the latter to the symmetry of Vect_k, thus F_0 is a symmetric tensor functor. ■

6.9 Remark The main defect of F_0 is not its failure of being unit-preserving but the fact that $F_0(\Sigma, T)$ depends strongly on T. However, solving the former problem will also do away with the latter, cf. Lemma 6.12.

6.10 Definition/Proposition *Let* $(\Sigma, T) \in \mathrm{Obj}\,\mathrm{TCob}_{1+1}$ *and consider the identity cobordism* $M = \Sigma \times I \in \mathrm{TCob}_{1+1}((\Sigma, T), (\Sigma, T))$*. Then*

$$p(\Sigma, T) = F_0(M) \in \mathrm{End}\, F_0(\Sigma, T)$$

is an idempotent.

Proof. Clearly $M \circ M$ is diffeomorphic to M. Now the claim follows from the fact that F_0 respects composition and is diffeomorphism invariant. ■

6.11 Theorem *Defining*

$$F_1(\Sigma, T) = p(\Sigma, T)F_0(\Sigma, T) \;\subset\; F_0(\Sigma, T),$$

we have

$$F_0(M)F_1(\Sigma_1, T_1) \subset F_1(\Sigma_2, T_2) \qquad \forall\, M \in \mathrm{TCob}_{1+1}((\Sigma_1, T_1), (\Sigma_2, T_2)), \tag{6.3}$$

thus we can define $F_1(M) := F_0(M) \restriction F_1(\Sigma, T)$. *Now,*

$$F_1(\Sigma \times I) = \mathrm{id}_{F_1(\Sigma,T)}, \tag{6.4}$$

and F_1 *is a functor. The latter indeed satisfies all axioms of a TQFT (on* TCob_{1+1}*).*

Proof. Any $M \in \mathrm{TCob}_{1+1}((\Sigma_1, T_1), (\Sigma_2, T_2))$ is diffeomorphic to its composite $(\Sigma_2 \times I) \circ M$ with the identity cobordism of Σ_2. Thus $F_0(M) = F_0(\Sigma_2 \times I) \circ F_0(M) = p(\Sigma_2, T_2) \circ F_0(M)$, implying (6.3). Eq. (6.4) follows from the fact that $p(\Sigma, T)$ is idempotent. That F_1 is a tensor functor follows from the fact that F_0 respects tensor products and $p(\Sigma_1 \coprod \Sigma_2, T_1 \coprod T_2) = p(\Sigma_1, T_1) \otimes p(\Sigma_2, T_2)$. ■

We now have obtained a tensor functor $F_1 : \mathrm{TCob}_{1+1} \to \mathrm{Vect}_k$, but this is not yet a TQFT in the usual sense. In order to obtain one, we need to get rid of the triangulations (of the objects), i.e. replace TCob_{1+1} by Cob_{1+1}. An obvious question is how the vector space $F(\Sigma, T)$ depends on the triangulation T:

6.12 Lemma *Let* $F_1 : \mathrm{TCob}_{1+1} \to \mathrm{Vect}_k$ *as constructed above. Let* Σ *be a closed oriented 1-manifold and* T_1, T_2 *triangulations of* Σ*. Then the linear maps* $\alpha_{T_1,T_2} = F_1(\Sigma \times I) : F_1(\Sigma, T_1) \to F_1(\Sigma, T_2)$ *are isomorphisms such that*

$$\alpha_{T,T} = \mathrm{id}, \qquad \alpha_{T_2,T_3} \circ \alpha_{T_1,T_2} = \alpha_{T_1,T_3}, \tag{6.5}$$

and

$$\begin{array}{ccc} F_1(\Sigma_1, T_{11}) & \xrightarrow{F_1(M)} & F_1(\Sigma_2, T_{21}) \\ \Big\downarrow{\scriptstyle \alpha_{T_{11},T_{12}}} & & \Big\downarrow{\scriptstyle \alpha_{T_{21},T_{22}}} \\ F_1(\Sigma_1, T_{12}) & \xrightarrow[F_1(M)]{} & F_1(\Sigma_2, T_{22}) \end{array} \tag{6.6}$$

commutes for any $M \in \mathrm{Cob}_{1+1}(\Sigma_1, \Sigma_2)$ *and triangulations* T_{ij} *of* Σ_i*.*

Proof. We claim that $F_1(\Sigma \times I) : F_1(\Sigma, T_2) \to F_1(\Sigma, T_1)$ is an inverse of the map under consideration. Indeed, composing the identity cobordism $\Sigma \times I$ with itself reproduces it. Thus

$$F_1(\Sigma, T_1) \xrightarrow{F_1(\Sigma \times I)} F_1(\Sigma, T_2) \xrightarrow{F_1(\Sigma \times I)} F_1(\Sigma, T_1)$$

equals $F_1(\Sigma \times I) = \mathrm{id}_{F_1(\Sigma,T_1)}$, and analogously for the opposite order of composition. The rest is essentially obvious. ■

It remains to get rid of the triangulations, i.e. construct a functor $F : \mathrm{Cob}_{1+1} \to \mathrm{Vect}_k$. This is achieved by an inverse limit construction:

6.13 Definition *For a closed oriented 1-manifold* Σ, *let* $\mathcal{T}_\Sigma$ *be the set of triangulations of* Σ. *Furthermore, we define*

$$\begin{aligned} F(\Sigma) &= \lim_{\substack{\longleftarrow \\ T\in\mathcal{T}_\Sigma}} F_1(\Sigma,T) \\ &= \left\{ v_\bullet \in \prod_{T\in\mathcal{T}_\Sigma} F_1(\Sigma,T) \mid \alpha_{T,T'}(v_T) = v_{T'} \ \forall T,T' \right\}. \end{aligned}$$

The vector space structure on $F(\Sigma)$ *is given by componentwise addition and action of the scalars.*

6.14 Remark How big is $F(\Sigma)$? In view of (6.5), the map $p_{T_0} : F(\Sigma) \to F_1(\Sigma, T_0)$, $v_\bullet \mapsto v_{T_0}$, is an isomorphism for every $T_0 \in \mathcal{T}_\Sigma$: Its inverse is $p_{T_0}^{-1} : F_1(\Sigma,T_0) \to F(\Sigma)$, $w \mapsto v_\bullet$, where $v_T = \alpha_{T_0,T}(w)$. Thus the construction of F really "does nothing" but formally removing the T-dependence by considering all triangulations at the same time.

Now we can define F on the morphisms:

6.15 Definition *Let* $M \in \mathrm{Cob}_{1+1}(\Sigma_1,\Sigma_2)$ *and* $v_\bullet \in F(\Sigma_1) \subset \prod_{T_1\in\mathcal{T}_{\Sigma_1}} F_1(\Sigma,T_1)$. *Then we define* $w_\bullet = F(M)(v_\bullet) \in F(\Sigma_2) \subset \prod_{T_2\in\mathcal{T}_{\Sigma_2}} F_1(\Sigma_2,T_2)$ *by* $w_{T_2} = F_1(M)(v_{T_1}) \in F_1(\Sigma_2,T_2)$, *where* $T_1 \in \mathcal{T}_{\Sigma_1}$ *is chosen arbitrarily and* M *is considered as an element of* $\mathrm{TCob}_{1+1}((\Sigma_1,T_1),(\Sigma_2,T_2))$. *(The choice of* T_1 *does not matter by the compatibility* $\alpha_{T_{11},T_{12}}(v_{T_{11}}) = v_{T_{12}}$ *between the components of* $v_\bullet$ *and the commutativity of (6.6).)*

Finally:

6.16 Theorem $F : \mathrm{Cob}_{1+1} \to \mathrm{Vect}_k$ *is a TQFT.*

Proof. One first has to verify that $F(M)$ as given in Definition 6.15 is in $F(\Sigma_1)$. This follows from the commutativity of (6.6). It is clear that F respects composition of morphisms and maps identity cobordisms to identity linear maps. The tensor structure is inherited from F_1 in an obvious way. ■

6.17 Exercise *We know that a TQFT in* $d = s+1$ *dimensions gives rise to an invariant* Z *of closed oriented d-manifolds. For the TQFT obtained from a very good triple, show that the induced invariant of closed 2-manifolds is the one constructed in Subsection 6.1.*

6.18 Remark Inverse (=projective) limit constructions are not to be confused with direct (=inductive) limits! Both constructions are very important throughout mathematics. E.g., profinite groups, i.e. inverse limits of finite groups, play a central rôle in Galois theory. It is important to realize that inverse limit constructions can have quite non-trivial effects, making it difficult to analyze the structure of the object they produce. In our case, in view of Remark 6.14, the construction of F via an inverse limit of F_1 over $T \in \mathcal{T}_\Sigma$ amounted to pure bookkeeping. But this was only so since we first normalized F_0. Cf. the next exercise.

6.19 Exercise *Prove that the functor* F *can be obtained directly by*

$$F(\Sigma) = \lim_{\substack{\longleftarrow \\ T\in\mathcal{T}_\Sigma}} F_0(\Sigma,T).$$

(Thus here it is clearly not true that $F(\Sigma)$ *is isomorphic to* $F_0(\Sigma,T)$ *for any* T*!)*

6.20 Remark That the inverse limit over F_1 is essentially trivial whereas the one over F_0 does have a non-trivial effect is due to the fact that the 1-cocycle α from Lemma 6.12 has 'trivial holonomy', whereas the analogous $\beta_{T_1,T_2} = F_0(\Sigma\times I) : F_0(\Sigma,T_1) \to F_0(\Sigma,T_2)$ does not.

6.21 Remark Since our construction of the TQFT $F_{(V,a,\mu)}$ was independent of any non-degeneracy assumptions on (V,a,μ), we cannot exclude the possibility that $F(\Sigma)=0$ for all Σ. In Subsection 6.5 we will show that the TQFT is non-zero whenever (V,a,μ) is perfect in the sense of Definition 6.26.

6.3 Two reductions

6.22 Definition *Let $(V,a,\mu),(V',a',\mu')$ be very good triples. We call them isomorphic if there is an isomorphism $\alpha: V\to V'$ such that $(\alpha\otimes\alpha\otimes\alpha)(a)=a'$ and $\mu'\circ\alpha\otimes\alpha=\mu$.*

The triples are called equivalent if they give rise to isomorphic TQFTs in $d=1+1$ (thus in particular the same invariant $Z(M)$).

(It should be obvious that isomorphism implies equivalence.) So far, we made no assumptions of non-degeneracy on either μ or a. As we will see below, one consequence of this is that non-isomorphic very good triples can be equivalent. Imposing non-degeneracy of μ and a will reduce this phenomenon (without eliminating it completely), but, more importantly, it will considerably simplify the determination of the commutative Frobenius algebra corresponding to a triangulation TQFT in Subsection 6.5.

6.23 Exercise *Let (V,a,μ) be a very good triple and define*

$$V_0=\{x\in V\mid \mu(x,y)=0\ \forall y\in V\}$$

and $\widetilde{V}=V/V_0$. Let $\phi: V\to\widetilde{V}$ be the quotient homomorphism.

(a) Show that there is a unique $\widetilde{\mu}\in(\widetilde{V}\otimes\widetilde{V})^*$ such that $\mu(x,y)=\widetilde{\mu}(\phi(x),\phi(y))$. Show that $\widetilde{\mu}$ is non-degenerate.

(b) Define $\widetilde{a}=(\phi\otimes\phi\otimes\phi)(a)\in\widetilde{V}\otimes\widetilde{V}\otimes\widetilde{V}$ and show that $(\widetilde{V},\widetilde{a},\widetilde{\mu})$ is a very good triple.

(c) Show that the invariants $M\mapsto Z(M)$ of closed oriented 2-manifolds, constructed as in Subsection 6.1, from $(\widetilde{V},\widetilde{a},\widetilde{\mu})$ and (V,a,μ) coincide.

(d) Show that the TQFT arising from $(\widetilde{V},\widetilde{a},\widetilde{\mu})$ is isomorphic to the one arising from (V,a,μ).

Now we will address the potential degeneracy of a:

6.24 Proposition *Let (V,a,μ) be a very good triple. Defining*

$$\widetilde{V}=\{(\mathrm{id}\otimes\phi)(a)\mid \phi\in(V\otimes V)^*\}\ \subset V,$$

we have:

(i) $a\in\widetilde{V}\otimes\widetilde{V}\otimes\widetilde{V}$. Furthermore, $\widetilde{V}$ is the smallest subspace of V for which this is true.

(ii) Define $\widetilde{\mu}=\mu\restriction\widetilde{V}\otimes\widetilde{V}$ and let $\widetilde{a}=a$ (considered as an element of $\widetilde{V}\otimes\widetilde{V}\otimes\widetilde{V}$). Then $(\widetilde{V},\widetilde{a},\widetilde{\mu})$ is a very good triple.

(iv) We have $Z_{(\widetilde{V},\widetilde{a},\widetilde{\mu})}(M)=Z_{(V,a,\mu)}(M)$ for all closed oriented 2-manifolds M.

(v) There is an isomorphism $F_{(\widetilde{V},\widetilde{a},\widetilde{\mu})}\cong F_{(V,a,\mu)}$ of TQFTs

(vi) If μ is non-degenerate then $\widetilde{\mu}$ is non-degenerate.

6.25 Exercise *Prove (i)-(v).*

Claim (vi) is more difficult and will be proven later.

6.26 Definition *A perfect triple is a very good triple (V, a, μ) such that*

1. *μ is non-degenerate.*
2. *$V = \{(\mathrm{id} \otimes \phi)(a) \mid \phi \in (V \otimes V)^*\}$.*

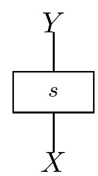
6.27 Corollary *Every very good triple is equivalent to a perfect triple.*

Proof. Apply both constructions, in the above order. ■

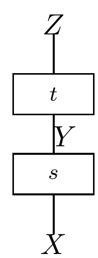
6.4 Notational interlude

Before we continue our analysis of triangulation TQFTs, we will briefly introduce graphical notation for morphisms in tensor categories. While this formalism will prove its full usefulness only in $2+1$ dimensions, it is helpful even in $1+1$ dimensions.

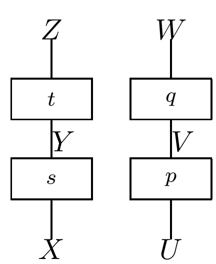
We denote objects in a category by lines and morphisms boxes with two lines entering. The morphism $s : X \to Y$ is represented by



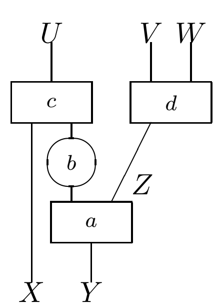
I.e., our convention is that that morphisms are drawn upwards. Composition is denoted by vertical stacking of boxes: If $s : X \to Y$ and $t : Y \to Z$, then $t \circ s : X \to Z$ is represented by



In a tensor category, two parallel lines labeled X and Y, respectively, represent the tensor product $X \otimes Y$, and similarly, the tensor product of morphisms is represented by placing the corresponding boxes next to each other horizontally. This implies that the diagram

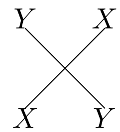


can be read as $(t \circ s) \otimes (q \circ p)$ or as $(t \otimes q) \circ (s \otimes p)$. This does not cause problems since these expressions coincide by the interchange law holding in all tensor categories! Here is a non-trivial example: The diagram

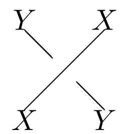
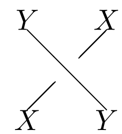


represents the formula $(c \otimes d) \circ (\mathrm{id}_X \otimes b \otimes \mathrm{id}_Z) \circ (\mathrm{id}_X \otimes a)$, which is quite unintelligible in view of the different numbers of objects entering and leaving the morphisms. Notice that the unit object **1** is not drawn! (Notice that the graphical formalism works best for strict tensor categories. But it can be used also in the non-strict case, when one implicitly inserts the associativity and unit constraints wherever they are needed.)

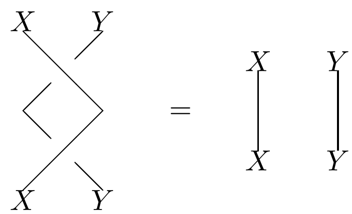
Finally, when $\mathcal{C}$ is a symmetric tensor category, we represent the symmetry $c(X,Y) : X \otimes Y \to Y \otimes X$ by



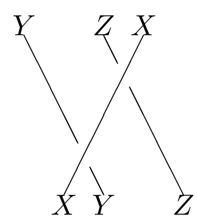
In a braided category, where $c(Y,X)$ need not equal $c(X,Y)^{-1}$, we draw these morphisms as follows:

$c(X,Y) =$  and $c(Y,X)^{-1} =$ 

The identity $c(X,Y)^{-1} \circ c(X,Y) = \mathrm{id}_{X \otimes Y}$ then looks like

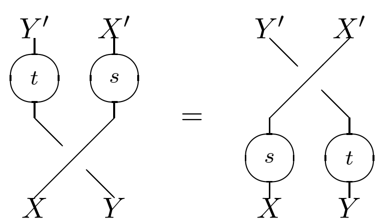


thus lines can be pulled over each other and back as long as they are not cut. Notice also that



can be interpreted both as $c(X, Y \otimes Z)$ or as $(\mathrm{id}_Y \otimes c(X,Z)) \circ (c(X,Y) \otimes \mathrm{id}_Z)$, which is perfectly consistent, since this relation holds in every braided tensor category.

Also the naturality of the symmetry/braiding has a nice graphical interpretation: If $s : X \to X', t : Y \to Y'$ then



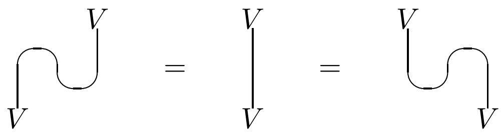
Thus, the boxes representing morphisms can be 'dragged' over the crossing representing the braiding.

6.28 Remark It should be emphasized that the above rules are not just a heuristic computational tool of dubious rigor but can be made perfectly rigorous – at the expense of considerable work. There is a (rigid, see Subsection 7.3) tensor category **Tan** of 'tangle diagrams' (lines, boxes, crossings as above), and for every tensor category $\mathcal{C}$ one has the category $\mathcal{C}$-**Tan** of '$\mathcal{C}$-labeled tangles', where each line is labeled by an object of $\mathcal{C}$ and each box by a morphism in $\mathcal{C}$ with the right in- and outputs. One can then prove the existence of a unique tensor functor $\mathcal{C}$-**Tan**$\to \mathcal{C}$ with the obvious properties. For all this, see e.g. [55, 26].

6.5 Relation to the classification of TQFTs in $d = 1+1$

In the preceding subsection, we have constructed a TQFTs in $1+1$ dimensions starting from a very good triple (V, a, μ). The aim of this subsection is to determine the commutative Frobenius algebra (A, m, ε) corresponding to the TQFT via the classification theory of Section 5. In view of Exercise 6.23 we may and will limit ourselves to the case of non-degenerate μ. We first need some complements to our graphical formalism.

Let $\{e_i\}$ be a basis of V^*. Then there are unique $f_i \in V^*$ such that $\mu = \sum_i e_i \otimes f_i$. Now non-degeneracy of μ is equivalent to $\{f_i\}$ being a basis of V^*. Pick dual bases $\{E_i\}$ and $\{F_i\}$ (satisfying $e_i(E_j) = \delta_{i,j}$ and $f_i(F_j) = \delta_{i,j}$), we define $b = \sum_i F_i \otimes E_i \in V \otimes V$. In view of the fact that k is the tensor unit of Vect_k, we may represent $\mu : V \otimes V \to k$ by $\frown$ and $b \in V \otimes V \cong \mathrm{Hom}(k, V \otimes V)$ by $\smile$. Now one easily verifies:



(6.7)

(E.g., the l.h.s. amounts to $\sum_{ij} f_i(F_j) E_i e_j = \sum_i E_i e_i = \mathrm{id}_V$.) The converse is also true:

6.29 Lemma *Let $\mu : V \otimes V \to k$. Then μ is non-degenerate iff there exists $b \in V \otimes V$ such that (6.7) holds. In that case, b is symmetric iff μ is.*

6.30 Exercise *Prove this.*

We will now show that the definition of the invariant Z of closed 2-manifolds discussed in Subsection 6.1 has a convenient interpretation in terms of the graphical formalism of Subsection 6.4, augmented by the symbols $\frown$ and $\smile$ for μ and b. Let $T = (X_\bullet, \alpha)$ be a triangulation of M. The vertices and edges (0- and 1-simplices) of the s.s.s. $X_\bullet$ constitute a graph $\mathcal{G}$. (This graph has the special property that the minimal loops consist of three edges, but it has no reason to be planar.) Let $\mathcal{G}^d$ be the dual graph. Then $\mathcal{G}^d$ has one vertex for each 2-simplex of $X_\bullet$ and an edge connecting two vertices when the corresponding 2-simplices are adjacent. In terms of the dual graph $\mathcal{G}^d$, the Pachner moves look as in Figures 4 and 5.

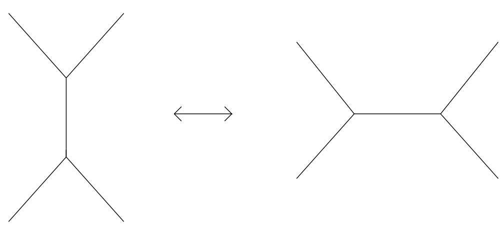


Figure 4: Dual of Pachner move of order 1

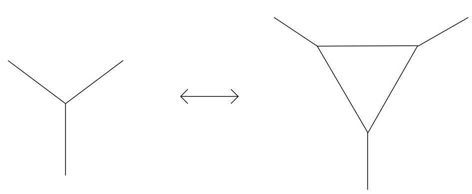
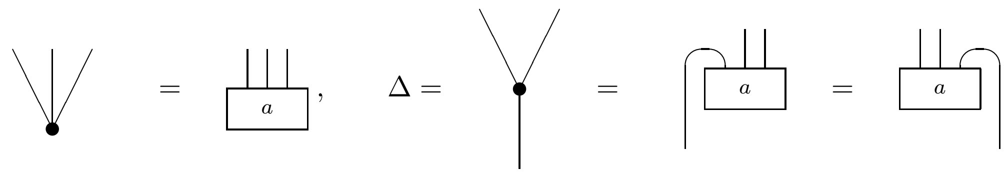


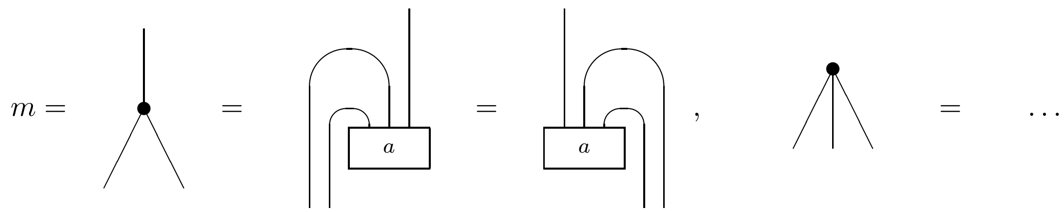
Figure 5: Duals of Pachner moves of orders 0 and 2

To each 2-simplex in $X_\bullet$ we associated one copy of $a \in V \otimes V \otimes V$ and we used μ to contract the two copies of V corresponding to the same edge (but appearing in two different 2-simplices). In terms of $\mathcal{G}^d$, this amounts to tensoring copies of $a \in V \otimes V \otimes V$ for each vertex, where each 'leg' of a

corresponds to one edge connecting to the corresponding vertex. Now we contract over those pairs of V connected by an edge of $\mathcal{G}^d$. This can be interpreted in terms of our graphical formalism as follows. (We only give a rough description and leave it to the reader to make this rigorous.) Draw $\mathcal{G}^d$ in a bounded area of the plane in a regular way, i.e. all edges intersect transversally, no two vertices of $\mathcal{G}^d$ are mapped to the same point of $\mathbb{R}^2$ etc. Furthermore, we forbid horizontal line segments, which allows us to interpret the lines as representing morphisms id_V. Each vertex is trivalent, but the lines may enter it from above or below. We interpret the various constellations as morphisms in Vect_k as follows, taking the opportunity to introduce the notations $m : V \otimes V \to V$ and $\Delta : V \to V \otimes V$:



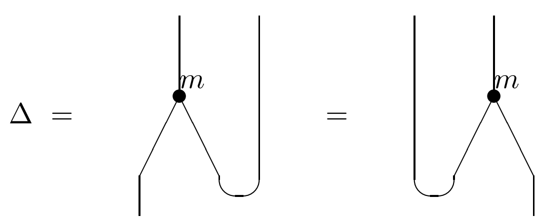
(6.8)



(6.9)

6.31 Remark 1. Notice that all these definitions maintain the cyclic order of the lines incident to each vertex!

2. In view of the duality relations (6.7), it is clear that the morphisms $m : V \otimes V \to V$ and $\Delta : V \to V \otimes V$ defined above are related by many identities like



(6.10)

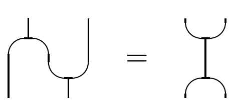
The following two exercises reformulate the conditions (6.1) and (6.2) in terms of m and Δ:

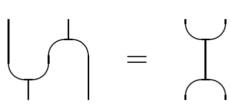
6.32 Exercise *Let $a \in V \otimes V \otimes V$ be $\mathbb{Z}_3$-invariant, $\mu : V \otimes V \to k$ non-degenerate, and $b \in V \otimes V$, $m : V \otimes V \to V$, $\Delta : V \to V \otimes V$ as defined above. Then the following are equivalent:*

(i) Condition (6.1) (as expressed graphically in Figure 4).

(ii) Associativity of m.

(iii) Coassociativity of Δ.

(iv) 

(v) 

6.33 Remark The equivalences shown above imply that every morphism $V^{\otimes p} \to V^{\otimes q}$ that is built out of m and Δ (or a, b, μ) and which can be represented by a tree graph (i.e. no loops) is equal to $\Delta^{(q-1)} \circ m^{(p-1)}$, which is a coherence result like those discussed in Section 3.

6.34 EXERCISE *Under the assumptions of the preceding exercise and the equivalent conditions given there, the following are equivalent:*

(i) *Condition (6.2) (as expressed graphically in Figure 5).*

(ii) $m \circ \Delta \circ m = m$.

(iii) $m \circ \mathrm{id} \otimes (m \circ \Delta) = m$.

(iv) $m \circ (m \circ \Delta) \otimes \mathrm{id} = m$.

6.35 REMARK 1. Using the identity in Figure 5 or the equivalent conditions above, one can show by induction that every morphism whose graphical representation is planar and contains one loop to which ≥ 3 lines connect can be contracted to a vertex, i.e. a morphism of the type in Remark 6.33. However, the analogous statement for loops with one or two legs does not follow from the axioms (6.1) and (6.2). (The loop with zero legs represents the morphism $\mu(b)$, which equals $\dim V \cdot \mathrm{id}_{\mathbf{1}}$.) The following proposition shows that this is indeed a condition on the triple (V, a, μ), namely precisely condition 2. in Definition 6.26.

2. Defining $p_0 = m \circ \Delta$, Exercise 6.34 (ii) implies that p_0 is idempotent and $m = p_0 \circ m = m \circ p_0 \otimes \mathrm{id} = m \circ \mathrm{id} \otimes p_0$. If $\widetilde{V} = p_0 V \subset V$, the identities (ii)-(iv) in Exercise 6.34 mean that $m(x, y) \in \widetilde{V}$ for all $x, y \in V$ and $m(x, y) = 0$ if x or y is in the subspace $(1 - p_0)V$. This implies that m cannot have a unit if $p_0 \neq 1$. More precisely:

6.36 PROPOSITION *Let (V, a, μ) be a very good triple with μ non-degenerate, and let b, m, Δ as defined above. With p_0 as above, the following are equivalent:*

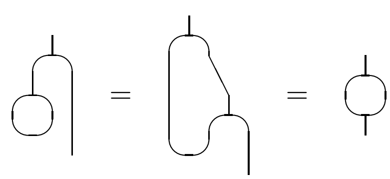
(i) *The element $e = m(b) \in V$ is a unit for the associative algebra (V, m).*

(ii) *The associative algebra (V, m) admits a unit e.*

(iii) $p_0 = \mathrm{id}_V$.

(iv) $V = \{(\mathrm{id} \otimes \phi)(a) \mid \phi \in (V \otimes V)^*\}$.

Proof. The implication (i)⇒(ii) is trivial. (ii)⇒(iii) follows, e.g., by multiplying equation (ii) in Exercise 6.34 by $e \otimes \mathrm{id}$ on the right. The equivalence (iii)⇔(i) follows from



(6.11)

where we used associativity of m and Remark 6.31.2.

The definition of m implies

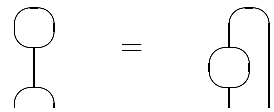
$$\mathrm{Im}\, m = m(V \otimes V) = \{(\mathrm{id} \otimes \phi)(a) \mid \phi \in (V \otimes V)^*\}.$$

Thus the equivalence (iii)⇔(iv) follows once we show that $\mathrm{Im}\, m = \mathrm{Im}\, p_0$. Now, the fact $m = p_0 \circ m$ ((ii) of Proposition 6.36) implies $\mathrm{Im}\, m \subset \mathrm{Im}\, p_0$. On the other hand, $x \in \mathrm{Im}\, p_0$ implies $x = p_0(x) = m \circ \Delta(x)$, thus $x \in \mathrm{Im}\, m$. ■

In Proposition 6.24 we showed that restriction to $\widetilde{V}$ enforces the above condition (iv) without changing the TQFT, but we need to complete its proof:

Proof of Proposition 6.24 (vi). Define $\varepsilon = \mu \circ \Delta$. Computations that are standard by now give

$$\varepsilon \circ m =$$



(6.12)

When $x \in \widetilde{V} = \mathrm{Im}\, p_0,\ y \in V$, this implies $\mu(x, y) = \varepsilon(m(x, y))$ and by (iii) of Exercise 6.34 this equals $\varepsilon(m(x, p_0(y))) = \mu(x, p_0(y))$. If $0 \neq x \in \widetilde{V}$, by non-degeneracy of μ there is $y \in V$ such that $\mu(x, y) \neq 0$. Since this equals $\mu(x, p_0(y))$ and we have $p_0(y) \in \widetilde{V}$, we conclude that $\widetilde{\mu}$ is non-degenerate. ■

From now on we may and will restrict ourselves to perfect triples, but see Remark 6.49 below. It turns out that they are equivalent to a more familiar structure:

6.37 Definition *A Frobenius algebra $((V, m, e), \varepsilon)$ is called symmetric if $\varepsilon(xy) = \varepsilon(yx)$ for all $x, y \in A$. (Obviously, every commutative Frobenius algebra is symmetric, but not every non-commutative one.) A Frobenius algebra is called special if $m \circ \Delta = \mathrm{id}_V$.*

6.38 Remark The first use of 'special' seems to have occurred in [48, A.4.7]. Note however that we use 'special' for the 'superspecial' of [48]. (Note that what [48] calls ambialgebras are precisely the symmetric Frobenius algebras.)

6.39 Proposition *Let (V, a, μ) be a perfect triple and b, m, Δ as defined above. Defining $e = m \circ b$ and $\varepsilon = \mu \circ \Delta$, we have $\mu = \varepsilon \circ m$, and $((V, m, e), \varepsilon)$ is a (unital) special symmetric Frobenius algebra. This defines a bijection between perfect triples (V, a, μ) and special symmetric Frobenius algebras.*

Proof. For a perfect triple, Proposition 6.36 gives that e is a unit and $p = m \circ \Delta = \mathrm{id}_V$. The latter fact together with (6.12) implies $\mu = \varepsilon \circ m$.

Proving the last claim amounts to showing that every special symmetric Frobenius algebra arises from a perfect triple. But this just amounts to reversing the above arguments: Given a special symmetric Frobenius algebra $((V, m, e), \varepsilon)$, define $\mu = \varepsilon \circ m$, obtain its dual morphism b and use that to define $a \in V \otimes V \otimes V$ in terms of m or Δ. $\mathbb{Z}_3$-invariance of a is an easy consequence of (co)associativity. Finally, the perfectness of (V, a, μ) follows from the specialness of the Frobenius algebra by Proposition 6.36. ■

Returning to the computation of $Z(M, T)$ from $\mathcal{G}^d$, crossing lines in the representation of $\mathcal{G}^d$ in the plane are interpreted in terms of the symmetry of Vect_k, cf. Subsection 6.4. With these conventions, the graph $\mathcal{G}^d$ represents a morphism in $\mathrm{End}\,\mathbf{1} = k$ and thus an element of k. Using (6.7) to remove possible 'wiggles' in the graph, we see that the graph $\mathcal{G}^d$ precisely leads to $Z(M, T)$, as defined earlier.

In order to illustrate this, we consider the torus with the triangulation from Figure 15. The dual graph $\mathcal{G}^d$, drawn in a plane, looks like Figure 6, and 'normalizing' into a tangle diagram we obtain

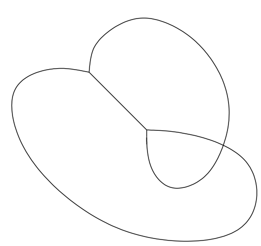
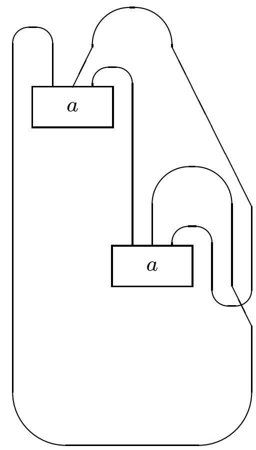


Figure 6: $\mathcal{G}^d$ for the standard triangulation of the torus



which finally represents the formula

$$(\mu_{15}\mu_{26}\mu_{34})(a \otimes a).$$

One easily verifies that this coincides with the result of the original prescription as applied to the graph $\mathcal{G}$ in Figure 15.

Turning to the the discussion of the TQFT, let $(\Sigma_1, T_1), (\Sigma_2, T_2)$ be triangulated 1-manifolds, $M : \Sigma_1 \to \Sigma_2$ a cobordism and T a triangulation of M restricting to $\partial T = T_1 \cup T_2$ on ∂M. The dual graph $\mathcal{G}^d$ corresponding to the triangulation T now may have various 'loose lines' representing the edges of ∂T. Bending them suitably using μ, b, we obtain a diagram that represents a morphism (i.e. linear map) $V^{\otimes m_1} \to V^{\otimes m_1}$, where $m_i = n_1(T_i)$. We will encounter several examples of this below.

We now start determining the commutative Frobenius algebra (A, m, ε) from the triangulation TQFT $F = F_{(V,a,\mu)}$. We recall from Section 5 that $A = F(S^1)$. If T is the "triangulation" of S^1 having just one edge, we have $F_0(S^1, T) = V$. The idempotent $p(S^1, T)$ arises from the triangulated cylinder cobordism $S^1 \times I$. With our graphical formalism we can easily interpret the dual graph $\mathcal{G}^d$

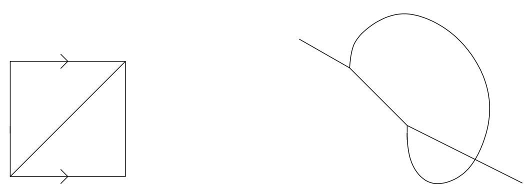


Figure 7: $\mathcal{G}$ and $\mathcal{G}^d$ for the standard triangulation of the cylinder cobordism

as a linear map $p(S^1, T) : V \to V$:

$$p(S^1, T)(x) = (\mu_{26}\mu_{34}\mu_{57})(a \otimes a \otimes x) \quad \forall x \in V.$$

From now on we will abbreviate $p = p(S^1, T)$. (This should not be confused with the $p_0 \in \operatorname{End} V$ used above. After all $p_0 = \mathrm{id}_V$ for perfect triples.) We already know that p is idempotent, but this can also be verified differently:

6.40 Exercise *Use the identities in Figures 4 and 5 to show that p is an idempotent in $\operatorname{End} V$.*

By definition, $F_1(S^1, T) = \operatorname{Im}(p(S^1, T)) \subset F_0(S^1, T)$, and in view of Lemma 6.12 we may define

$$A = \operatorname{Im}(p(S^1, T)) \subset V.$$

Before we continue elaborating the TQFT $F_{(V,a,\mu)}$, we give a purely algebraic discussion of A:

6.41 Theorem *Let (V, a, μ) be a perfect triple and μ, b, m as defined above. Then*

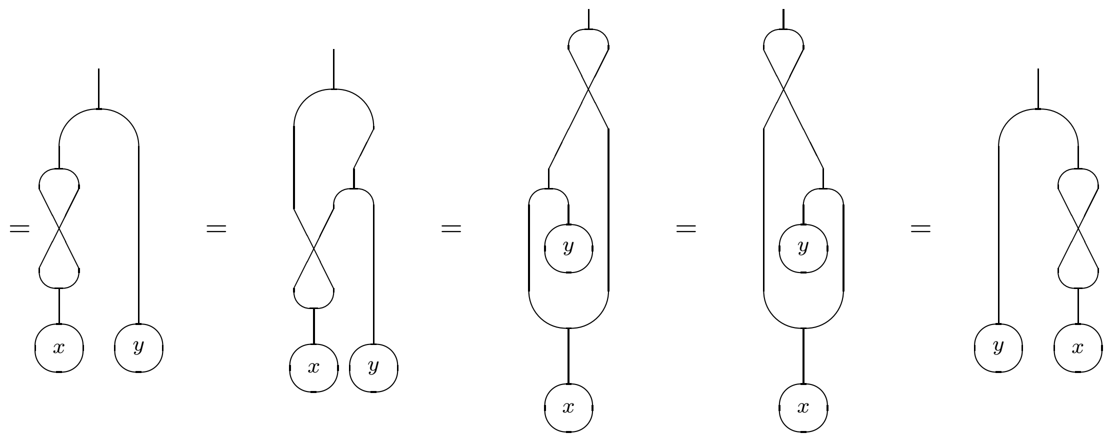
(a) $A = Z(V, m)$.

(b) $\mu_A = \mu \restriction A \otimes A$ *is non-degenerate.*

Thus $((A, m, e), \varepsilon)$ *is a commutative Frobenius algebra.*

Proof. (a) Let $x \in A,\ y \in V$. Then $p(x) = x$ and thus

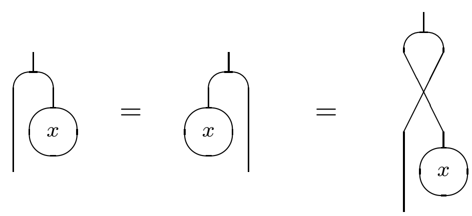
$m(x, y) = m(p(x), y) =$



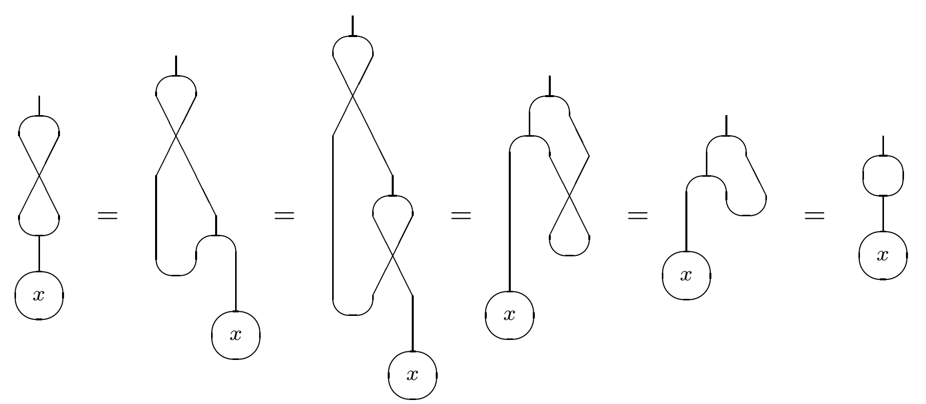
$$= m(y, p(x)) = m(y, x).$$

Thus $x \in Z(V, m)$, implying $A \subset Z(V, m)$.

As to the inclusion $Z(V, m) \subset A$, let $x \in Z(V, m)$. This means



which implies

$p \circ x =$  $= x,$

thus $x \in A$, and we have $A = Z(V, m)$.

(b) This is somewhat analogous to the proof (involving p_0) of (vi) in Proposition 6.24. An easy diagrammatic argument gives

$$\mu \circ p \otimes \mathrm{id}_V = \mu \circ \mathrm{id}_V \otimes p.$$

Since p is idempotent, we have $\mu \circ p \otimes \mathrm{id}_V = \mu \circ p \otimes p$. If now $0 \neq x \in A$, by non-degeneracy of μ there is a $y \in V$ such that $\mu(x, y) \neq 0$. But then $0 \neq \mu(x, y) = \mu(p(x), y) = \mu(p(x), p(y)) = \mu(x, p(y))$. Since $p(y) \in A$, we see that $\mu_A = \mu \restriction A \otimes A$ is non-degenerate. ■

Since the preceding (purely algebraic) results have provided us with a commutative Frobenius algebra, it is natural to conjecture that this is precisely the commutative Frobenius algebra corresponding to the TQFT $F_{(V,a,\mu)}$. Indeed:

6.42 LEMMA *Let $u : \emptyset \to S^1$ be the unit cobordism, cf. **??**, and let S^1 be triangulated by the T from above. Then $F_1(u) = m(b) = e$.*

Proof. Consider the triangulation T_1 of the unit cobordism $u : \emptyset \to S^1$ in Figure 8. The boundary of

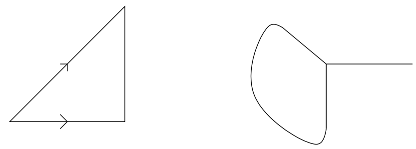


Figure 8: $\mathcal{G}$ and $\mathcal{G}^d$ for the standard triangulation of the unit cobordism

(u, T_1) is S^1 with the 1-edge triangulation T. The morphism $F_0(u, T_1) \in \mathrm{Hom}(\mathbf{1}, V) \cong V$ corresponding to the dual graph is just $m(b) \in V$. (As we have seen in Theorem 6.41, $e := m(b)$ actually is in $A \subset V$, as it must in view of (6.3)). ■

Entirely analogously one proves:

6.43 LEMMA *Let $v : S^1 \to \emptyset$ be the counit cobordism, cf. **??**, where S^1 is triangulated by T. Then $F_1(v) = \mu \circ \Delta = \varepsilon$.*

The following is only a bit more involved:

6.44 PROPOSITION *Let $w : S^1 \times S^1 \to S^1$ be the pants cobordism, cf. [], interpreted as element of* $\mathrm{Hom}_{\mathrm{Cob}_{1+1}}((S^1, T) \coprod (S^1, T), (S^1, T))$. *Then $F_1(w) = m : A \times A \to A$.*

Proof. A triangulation T_w of w is given by Figure 9. Notice that the pairwise identifications of the four

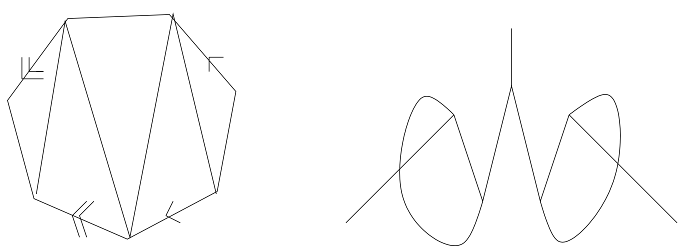


Figure 9: $\mathcal{G}$ and $\mathcal{G}^d$ for the triangulation T_w of the pants cobordism w

marked edges give rise to three boundary components S^1, each of which is triangulated in terms of one edge. Thus $F_0(w)$ is a linear map $V \otimes V \to V$ as needed. The morphism $F_0(w, T_w)$ corresponding to the dual graph $\mathcal{G}^d$ visibly just is

$$m \circ p \otimes p : V \otimes V \to V,$$

where $p = p(s^1, T)$. Restricting this to $A = pV$ and using that m restricts to a multiplication on A (part (a) of Theorem 6.41), we see that $F_1(w) = m : A \otimes A \to A$. ■

Thus:

6.45 THEOREM *Let (V, a, μ) be a perfect triple. Then the commutative Frobenius algebra corresponding to the TQFT $F_{(V,a,\mu)}$ is (isomorphic to) the $((A, m, e), \varepsilon)$ defined in Theorem 6.41.*

6.46 Remark 1. In the preceding arguments we didn't bother to work with the unwieldy functor F defined in terms of an inverse limit over triangulations. Instead we chose the simplest possible triangulations and worked with F_1, as is justified by Lemma 6.12.

2. Theorem 6.45 implies all of Theorem 6.41 with one exception: It only shows that m restricts to A and that (A, m) is commutative, whereas Theorem 6.41 actually gave $A = Z(V, m)$. Anyway, the direct proof of Theorem 6.41 is simpler and quite instructive.

6.47 Corollary *Two perfect triples give rise to isomorphic TQFTs if and only if the special symmetric Frobenius algebras associated to them by Proposition 6.39 have isomorphic centers.*

Proof. This follows from Theorems 6.45 and 5.1. ■

6.48 Remark It should be obvious that isomorphism of the centers is a weaker condition than having an isomorphism between the Frobenius algebras themselves! We will see that a strikingly similar result holds in $2+1$ dimensions, cf. Remark 7.30. Also there, we will have two different approaches to constructing TQFTs, which are (conjecturally) related by passing to a 'center', albeit a considerably more complicated one.

6.49 Remark The above results from Proposition 6.39 through Corollary 6.47 were proven under the restriction to perfect triples, as was justified by Corollary 6.27. If we only assume μ to be non-degenerate, but not the second condition in Definition 6.26, then the algebra (V, m) is not unital, cf. Proposition 6.36. One can nevertheless show that

- $A \subset Z(V, m)$, but the inclusion is proper.
- The multiplication m restricts to A.
- $e = m \circ b$ satisfies $e = p_0 \circ e$, thus $e \in A$, and it is a unit of A (despite the fact that (V, m) does not have a unit).
- $((A, m, e), \varepsilon)$ is a commutative Frobenius algebra, and Theorem 6.45 remains true.

However, the fact that (V, m) is non-unital makes the theory uglier and somewhat more involved. (E.g., in the perfect case the result that m restricts to A does not require an independent proof since it follows from $A = Z(V, m)$. A similar comment applies to the unit property of e.)

Restricting our attention to perfect triples was sufficiently justified by Corollary 6.27. However it may be instructive to give a geometric interpretation of the specialness axiom $m \circ \Delta = \mathrm{id}$. We know that it is a strengthening of the condition in Figure 5, formulated in terms of the dual graph $\mathcal{G}^d$ of the triangulation. If one translates this back to the triangulation T one finds that it expresses invariance of $Z(M, T)$ under the move in Figure 10, where one 'squeezes' two new triangles between two adjacent ones. Notice that the new triangles have two edges in common, which cannot happen in a simplicial complex. This is the reason why the stronger condition $m \circ \Delta = \mathrm{id}$ does not follow from Pachner's theorem, since the latter was formulated (and proven!) for triangulations in terms of simplicial complexes!

In Subsections 6.1 and 6.2 we used only triangulations by simplicial complexes. But in the present section, we used manifestly non-simplicial Δ-complexes to triangulate the unit, counit, pants and cylinder cobordisms. It is therefore not surprising that we need to impose further conditions on the triple (V, m, μ). (The authors do not know whether a version of Pachner's theorem for Δ-complexes has been worked out, i.e. whether the 2-2 Pachner move and the 'hairsplitting move' are sufficient to transform all triangulations in terms of Δ-complexes into each other.)

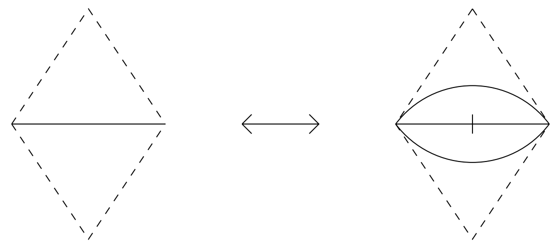


Figure 10: Hairsplitting move

6.50 THEOREM *A perfect triple (V, a, μ) where a is S_3-invariant (not just $\mathbb{Z}_3$-invariant) gives rise to a TQFT that is trivial in the sense that for every closed oriented 2-manifolds we have $Z(M) = (\dim V)^n$, where n is the number of connected components of M. This generalizes to very good triples provided* $\dim V$ *is replaced by the dimension of $\widetilde{V}$ in the perfect triple $(\widetilde{V}, \widetilde{a}, \widetilde{\mu})$ equivalent to (V, a, μ).*

Proof. We give two proofs: (A) Since (V, m) is commutative, Theorem 6.45 implies that the commutative Frobenius algebra corresponding to the TQFT is just the $((V, m, e), \varepsilon)$ from Proposition 6.39, which is special. But in Section 5 we have seen that the TQFT corresponding to a special commutative Frobenius algebra A satisfies $Z(M) = (\dim A)^n$ and we are done since $V = A$.

(B) An alternative, more pedestrian argument goes as follows: Since we work with perfect triples, every morphism in $\mathrm{End}\,\mathbf{1}$ built from m, Δ, μ, b that can be represented by a planar connected graph is equal to $\mu(b) = \dim V \,\mathrm{id}_{\mathbf{1}}$. If a is S_3-invariant, then m is commutative and the crossings in $\mathcal{G}^d$ can be eliminated successively by interchanging edges. Thus *every* connected morphism constructed from m, Δ, μ, b equals $\dim V \,\mathrm{id}_{\mathbf{1}}$. If M is connected, then the dual graph $\mathcal{G}^d$ corresponding to a triangulation T is connected, and the claim follows. The general result follows by multiplicativity.

A glance at the constructions in Subsection 6.3 shows that replacing (V, a, μ) by an equivalent perfect triple does not affect the commutativity of a, implying the last claim. ■

6.51 REMARK Above, we have considered symmetric Frobenius algebras $((V, m, e), \varepsilon)$ that are special, i.e. satisfy $m \circ \Delta = \mathrm{id}_V$. It is important to understand that the latter condition does not imply that the center $A = Z(V)$ is special! Since we have seen in Section 5 that commutative special Frobenius algebras give rise to trivial TQFTs, this would imply that all triangulation-TQFTs are trivial. This is not the case, as we will see in the next subsection.

The reason why $m_A \circ \Delta_A = \mathrm{id}_A$ usually fails to hold is that, while m_A is the restriction of m to A, Δ_A is not the restriction of Δ. Rather, Δ_A is obtained by 'bending up' one input of m_A using $b_A \in A \otimes A$, cf. Remark 6.31. The latter is dual to $\mu_A \in (A \otimes A)^*$ which is the (non-degenerate) restriction of $\mu \in (V \otimes V)^*$. This implies that $\Delta_A = p \otimes p \circ \Delta \restriction A$ and thus

$$m_A \circ \Delta_A = m \circ p \otimes p \circ \Delta \restriction A.$$

That this should be equal to id_A does not follow from $m \circ \Delta = \mathrm{id}_V$.

As we have seen, there is a bijection between perfect triples and special symmetric Frobenius algebras. We close the general discussion of triangulation TQFTs in 1+1 dimensions by reducing this further.

6.52 LEMMA *Let $((V, m, 1), \varepsilon)$ be a Frobenius algebra. Then $m \circ \Delta(1) = \sum_i x_i y_i$, where $\{x_i\}$ is a basis of V and $\{y_i\}$ is the dual basis satisfying $\varepsilon(y_i x_j) = \delta_{i,j}$.*

Proof.

■

6.53 PROPOSITION *Every special symmetric Frobenius algebra is semisimple. Conversely, every semisimple algebra admits a unique linear form ε turning it into a special symmetric Frobenius algebra, where the restriction of ε to a direct summand $M_n(k)$ is given by $n\mathrm{Tr}$ (i.e. $\varepsilon(1_{M_n(k)}) = n^2$).*

Proof. The specialness assumption implies that $\sum_i x_i y_i = m\Delta(1) = 1$ is invertible. As shown in [48], a symmetric Frobenius algebra with invertible $\sum_i x_i y_i$ is semisimple. As to the converse, let $(V, m, 1)$ be semisimple. If $\overline{k} \supset k$ is an algebraic closure, we have

$$V \otimes_k \overline{k} \;\cong\; \bigoplus_{j \in J} M_{n_j}(\overline{k}).$$

If Tr_j is the standard trace on $M_{n_j}(\overline{k})$ (normalized such that $\mathrm{Tr}(1) = n_j$) and $c \in \overline{k}^J$ is a vector with no zero component, then $\varepsilon = \sum_{j \in J} c_j \mathrm{Tr}_j$ is a non-degenerate trace on $V \otimes_k \overline{k}$. (Furthermore, every non-degenerate trace on $V \otimes_k \overline{k}$ is of this form.) Its restriction ε to V is non-degenerate, thus $((v, m, 1), \varepsilon)$ is a symmetric Frobenius algebra. It remains to prove that there is a unique choice ε implying specialness. It is sufficient to do this for a simple matrix algebra $V = M_n(k)$. Every trace on $M_n(k)$ is a multiple of the standard trace: $\varepsilon = c\mathrm{Tr}$. The matrix units $\{x_{ij} = e_{ij}\}$ form a basis, and the dual basis w.r.t. $\varepsilon = c\mathrm{Tr}$ is $\{y_{ij} = c^{-1} e_{ji}\}$. Now,

$$\sum_{ij} x_{ij} y_{ij} = c^{-1} \sum_{ij} e_{ij} e_{ji} = c^{-1} \sum_{ij} e_{ii} = c^{-1} n \,\mathrm{id}_V,$$

which is id_V precisely if $c = n$. ■

6.54 COROLLARY *For every very good triple, the associated triangulation TQFT in $1+1$ dimensions is semisimple (in the sense of Definition 5.2).*

Proof. Let (V, a, μ) be a very good triple and $F_{(V,a,\mu)}$ the TQFT obtained from it. By Corollary 6.27, (V, a, μ) is equivalent to a perfect triple, which corresponds to a special symmetric Frobenius algebra V by Proposition 6.39. By Proposition 6.53, the latter is semisimple, thus also its center is semisimple. Now the claim follows from Theorem 6.45, according to which the commutative Frobenius algebra associated with $F_{(V,a,\mu)}$ is just that center. ■

Since non-semisimple commutative Frobenius algebras exist, cf. Example **??**, we see that the triangulation approach does not produce all TQFTs in $d = 1+1$. In fact, it does not even give the TQFTs corresponding to all commutative semisimple Frobenius algebras A, since the numbers $\varepsilon(p)$, where $p \in A$ is a minimal idempotent, can assume only values in $\{n^2 \mid n \in \mathbb{N}\}$.

6.6 Example: (G, c)

Let k be a field and G a finite group, whose order $|G|$ is not a multiple of char k, and consider the group algebra kG. (This is the vector space V spanned by $\{x_g,\ g \in G\}$, with product given by $x_g x_h = x_{gh}$ and unit $1 = x_e$.) Define $\varepsilon \in (kG)^*$ by $\varepsilon : x_g \mapsto |G|\,\delta_{g,e}$.

6.55 EXERCISE *(a) Prove that $((kG, m, 1), \varepsilon)$ is a symmetric Frobenius algebra.*

(b) Determine the element $b \in kG \otimes kG$ dual to $\mu = \varepsilon \circ m$ and use it to compute Δ. (Warning: Δ is **not** *the usual coproduct of kG as a Hopf algebra!)*

(c) Show that the Frobenius algebra is special.

(d) *Show that the perfect triple (V,a,μ) corresponding to the (special symmetric) Frobenius algebra kG is given by $V = kG$ (as a vector space) and*

$$a = |G|^{-2} \sum_{\substack{g,h,k\in G \\ ghk=e}} x_g \otimes x_h \otimes x_k.$$

$$\mu(x_g, x_h) = |G|\,\delta_{gh,e}.$$

6.56 Remark In combination with Proposition 6.53, the above (c) implies that kG is semisimple when $|G|$ is not divided by char k (which includes the case char $k = 0$). This – and the converse – can be shown without referring to Frobenius algebras. But notice that the first and motivating examples of Frobenius algebras were precisely the group algebras kG!

What can be said about the TQFT $F_{(V,a,\mu)}$ arising from the triple (V,a,μ)? Since the TQFT will be considered in greater generality in Section 8, we here limit ourselves to determining the corresponding invariant $Z_{kG}(M)$ of closed oriented 2-manifolds. (For more on this TQFT in $1+1$ dimensions cf. [34, Section 3.2].) There are actually two ways of computing $Z_{kG}(M)$.

On the one hand, we can use the definition of $Z(M)$ in terms of triangulations. Thus let M be a closed oriented 2-manifold and T a triangulation of M. Consider the set $E(T,G)$ of pairs (t,e), where t is a triangle of T and e is an edge involved in t. (Notice that for every edge e of T, there are precisely two triangles t such that $(t,e) \in E(T,G)$.) Now let $L(T,G)$ be the set of maps $f : E(T,G) \to G$ satisfying the following conditions:

(i) If t_1, t_2 are the two triangles containing the edge e, we have $f(t_1,e) = f(t_2,e)^{-1}$.

(ii) If t is a triangle consisting of the edges e_1, e_2, e_3 (in an order consistent with the orientation of t as induced by the orientation of M), then $f(t,e_1)f(t,e_2)f(t,e_3) = 1$.

6.57 Proposition *Let G be a finite group and M a closed oriented 2-manifold. Let $M \mapsto Z(M)$ be the invariant of closed 2-manifolds arising from the (special symmetric) Frobenius algebra $\mathbb{C}G$. Then, for any triangulation T of M,*

$$Z(M) = |G|^{-2n_2(T)+n_1(T)}\,|L(T,G)|.$$

Proof. $Z_{(V,a,\mu)}$ is defined as in Subsection 6.1. Thus we apply $n_1(T)$ contractions μ to $a^{\otimes n_2(T)}$. In view of the normalizations of μ and a found in part (d) of the Exercise, this accounts for the numerical factor. Apart from this normalization, it is clear from the definition of a that we are just counting labellings of the edges by elements of G, where every edge is considered twice: once for each triangle in which it occurs. This is the reason why we defined $E(T,G)$. The condition $ghk = e$ in the formula for a leads to condition (ii) in the definition of $L(T,G)$, whereas (i) is implied by the definition of μ. ■

6.58 Theorem *Let M be as above and connected and $* \in M$. Then*

$$Z(M) = |G|^{\chi(M)-1}\,|\mathrm{Hom}(\pi_1(M,*),G)|, \tag{6.13}$$

where $\chi(M) = 2-2g$ is the Euler characteristic.

Proof.
****

■

The other way of computing $Z_{kG}(M)$ is to apply the methods of Section 5 to the center $A(G) = Z(kG)$ of the Frobenius algebra $V = kG$. Now, the center of kG is spanned by the elements $x_c =$

$\sum_{g \in c} x_g$, where $c \subset G$ is a conjugacy class. Denoting the set of conjugacy classes of G by $C(G)$, we have $A(G) = \bigoplus_{c \in C(G)} k$, and the Frobenius form ε acts by multiplication with $|c|$ on the copy of k corresponding to $c \in C(G)$. Together with the results of Subsection 5.1, this implies

$$Z(M) = \sum_{c \in C(G)} |c|^{1-g(M)}. \tag{6.14}$$

Since there are (non-canonical) bijections $c \leftrightarrow \pi$ between $C(G)$ and the set $\widehat{G}$ of equivalence classes of irreducible representations of G under which $|c| = \dim \pi$, we also have

$$Z(M) = \sum_{\pi \in \widehat{G}} (\dim \pi)^{1-g(M)}. \tag{6.15}$$

Taking a field of characteristic zero, e.g. $k = \mathbb{Q}$, and comparing this with (6.13), we obtain the remarkable identity

$$|G|^{\chi(M)-1} \, |\mathrm{Hom}(\pi_1(M,*), G)| = \sum_{c \in C(G)} |c|^{1-g(M)},$$

valid for every closed oriented 2-manifold and every finite group G.

6.59 Remark 1. The above identity was discovered by Mednykh in 1978, cf. also [34]. In [51] it was shown that the identity also holds for non-orientable surfaces.

2. That the invariant $Z_{kG}(M)$ can be obtained from either $g(M)$ or $\pi_1(M)$ is not surprising since closed connected oriented 2-manifolds are classified by their genus and the genus can be recovered from $\pi_1(M)$ via $H_1(M) = \pi_1(M,*)$ and $H_1(M) \cong \mathbb{Z}^{2g}$. We conclude that, by varying G we can distinguish all closed 2-manifolds.

3. In Section 8, we will construct analogous TQFTs in all dimensions, and the result $Z(M) = |\mathrm{Hom}(\pi_1(M,*), G)/G|$ for connected M will generalize to them. Thus for $d > 2$ the TQFTs (for varying G) do not give us too much information about M. But notice that $\pi_1(M)$ is typically known only in terms of generators and relations, which makes if very difficult (or impossible) to understand its structure.

We now briefly mention a generalization of the TQFT discussed above. Let $c \in H^2(G, k^*)$. (Return to Appendix A if necessary.) Let $\alpha \in Z^2(G, k^*)$ such that $[\alpha] = c$. Thus by definition, α satisfies

$$\alpha(g,h)\alpha(gh,k) = \alpha(h,k)\alpha(g,hk) \quad \forall g,h,k \in G. \tag{6.16}$$

Considering the vector space $k^{|G|}$ as before, but now with multiplication m defined by $x_g x_h := \alpha(g,h)x_{gh}$, the 2-cocycle identity implies that m is associative. We denote the resulting algebra by $k^\alpha G$. Taking $g = e$ in (6.16) one finds that $\alpha(e,h)$ is independent of h. Similarly, $\alpha(g,e)$ is independent of g. Thus there is a $z \in k^*$ such that

$$\alpha(g,e) = \alpha(e,g) = z \quad \forall g \in G. \tag{6.17}$$

This implies that $z^{-1}x_e$ is a unit for (kG, m). (Notice that one can also find an α' with $[\alpha'] = c$ such that $z = 1$.) If $[\alpha'] = [\alpha]$ then there is an isomorphism $k^{\alpha'}G \cong k^\alpha G$ of algebras, thus the isomorphism class of $k^\alpha G$ depends only on $c = [\alpha]$. One can show that $k^\alpha G$ is semisimple if and only if $\mathrm{char}\, k$ does not divide $|G|$. Under this condition, Proposition 6.53 shows that $k^\alpha G$ has a unique Frobenius structure. Therefore, we have a TQFT in $d = 1+1$ for each pair (G, c), where G is a finite group and $c \in H^2(G, k^*)$. (6.15) remains true if $\widehat{G}$ is replaced by the set $\widehat{G}_\alpha$ of irreducible projective representations w.r.t. the 2-cocycle α. For more on this TQFT we refer to [58].

6.60 REMARK 1. In the next section, we will use the categories $\mathcal{C}(G,c)$ from Subsection 3.3 to define an analogous class of TQFTs in $2+1$ dimensions for any pair (G,c), where $c \in H^3(G,k^*)$. In Section 8, we will construct a class of TQFTs in all dimensions reducing to the above for $d = 1+1$ and $d = 2+1$. This flavor of this construction will be much more geometric than the rather combinatorial ones considered so far.

2. The Frobenius structure of $k^\alpha G$ can be described explicitly: Taking $k = g = h^{-1}$ in (6.16) and using (6.17), we get $\alpha(g, g^{-1}) = \alpha(g^{-1}, g)\ \forall g \in G$. Defining now $\varepsilon : x_g \mapsto \delta_{g,e}$ as above, ε is non-degenerate and symmetric:

$$\varepsilon(x_g x_h) = \alpha(g,h)\varepsilon(x_{gh}) = \alpha(g,g^{-1})\delta_{g,h^{-1}} = \alpha(g^{-1},g)\delta_{g,h^{-1}} = \alpha(h,g)\varepsilon(x_{hg}) = \varepsilon(x_h x_g).$$

Thus $k^\alpha G$ is a symmetric Frobenius algebra, and generalizing the computation in Exercise 6.55 shows that it is special.

7 TQFTs from triangulations: $d = 2+1$

7.1 Introduction

In our discussion of TQFTs in $1+1$ dimensions, we followed two different approaches (which we then related to each other): On the one hand, the fact that 2-manifolds are very well understood allows to give a complete classification of TQFTs in terms of commutative Frobenius algebras. While this in principle means that everything is known about $d = 1+1$ TQFTs, on the other hand, one can construct TQFTs in terms of triangulations. In $2+1$ the situation is different since we have no complete classification of 3-manifolds to rely on. This is the reason why (as yet) there is no classification of TQFTs but only various ways of constructing them. In this section we will consider the triangulation approach.

In Section 6, we constructed a TQFT in $1+1$ dimensions from a good triple (V, a, μ), which we interpreted as a (non-unital, non-commutative) Frobenius algebra. Roughly, a corresponded to the 2-simplices of a triangulation and μ to the 1-simplices. (V does not correspond to 0-simplices, but just sets the 'scene' on which a, μ act.) In $2+1$ dimensions, the triangulation of a manifold consists of 3-simplices (=tetrahedra), and it is clear that we will have to start from a more complicated algebraic structure. It will turn out that a good point of departure is provided by a certain class of tensor categories. (Now the 1-simplices of the triangulation will be related to the objects $X, Y, Z, \ldots$ of the category, the 2-simplices to certain morphisms in $\mathrm{Hom}(\mathbf{1}, X \otimes Y \otimes Z)$, where X, Y, Z are the objects associated with the edges of the 2-simplex, and the 3-simplices to a way of assigning a number to the four morphisms associated with the (codimension one) faces of a 3-simplex.) The first construction of a 3-manifold invariant using triangulations was given by Turaev and Viro, cf. [59]. The authors actually started from the representation category of the quantum group $U_q(sl_2)$. This was generalized to 'modular categories', which are special braided tensor categories, in [55]. Only somewhat later it was realized that a braiding on the category is not needed, cf. the independent papers [8, 21]. This fact was actually observed earlier by A. Ocneanu (1991, unpublished, but see the account in [16]), who constructed triangulation invariants of 3-manifolds starting from a 'type II_1 subfactor of finite index and depth'. Now, such a subfactor contains much more information than needed, for example regarding the isomorphism classes of the factors involved. Even if one disregards this complicated analytic information, a subfactor gives rise to more algebraic data than needed for the construction of the invariant. This in fact leads to a non-trivial result, mentioned in Remark 7.30.

7.2 Semisimple categories

7.1 DEFINITION *Let k be a field (or a commutative unital ring). A k-linear category is a category where each hom-set is a k-vector space and the composition $(s,t) \mapsto t \circ s$ is k-bilinear. A k-linear tensor* *category is a tensor category that is k-linear and where the tensor product operation $(s,t) \mapsto s \otimes t$ on morphisms is k-bilinear.*

7.2 Definition *Let $\mathcal{C}$ be a k-linear (or, more generally, an Ab-category) and X, Y, Z objects in $\mathcal{C}$. We say that Z is a direct sum of X and Y ($Z \cong X \oplus Y$) if there are morphisms*

$$X \underset{s'}{\overset{s}{\rightleftarrows}} Z \underset{t'}{\overset{t}{\leftrightarrows}} Y$$

satisfying

$$s' \circ s = \mathrm{id}_X, \qquad t' \circ t = \mathrm{id}_Y, \qquad s \circ s' + t \circ t' = \mathrm{id}_Z.$$

7.3 Exercise *Show that*

(a) *If $Z \cong X \oplus Y$ and $X' \cong X$, $Y' \cong Y$ then $Z \cong X' \oplus Y'$. (This is why we don't write $Z = X \oplus Y$.)*

(b) *$s' \circ t = t' \circ s = 0$.*

If $Z \cong X \oplus Y$ with s, s', t, t' as in Definition 7.2, then $s \circ s'$ and $t \circ t'$ are idempotents in $\mathrm{End}\, Z$ corresponding to the direct summands X and Y respectively. This suggests the following

7.4 Definition *A k-linear (or just Ab-) category $\mathcal{C}$ is said to have splitting idempotents (or be 'Karoubian') if for each $p = p^2 \in \mathrm{End}\, X$ there exists an object Y together with morphisms $s : Y \to X$, $s' : X \to Y$ such that $s' \circ s = \mathrm{id}_Y$ and $s \circ s' = p$. (Since also $1 - p$ is idempotent, it then follows that Y is a direct summand of X.)*

7.5 Definition *An object $X \in \mathcal{C}$ is called a zero object if $\mathrm{Hom}(X, Y)$ and $\mathrm{Hom}(Y, X)$ contain precisely one element for each $Y \in \mathcal{C}$.*

A k-linear category is called additive if it has a zero object and direct sum exists for all pairs of objects.

7.6 Exercise *If Z is a zero object then $X \cong X \oplus Z$ for all X.*

7.7 Definition *An object in a k-linear category is called simple if $\mathrm{End}\, X = k\, \mathrm{id}_X$.*

7.8 Remark The above definition is convenient for our purposes, but somewhat non-standard. In the context of abelian categories one says that X is simple if every injection $Y \hookrightarrow X$ is an isomorphism. Our 'simple' objects are then called 'absolutely simple', since it is stronger than simplicity.

7.9 Definition *A k-linear category $\mathcal{C}$ is called semisimple if there is a family $\{X_i,\ i \in I\}$ of simple objects such that the canonical map*

$$\bigoplus_{i \in I} \mathrm{Hom}(X, X_i) \otimes \mathrm{Hom}(X_i, Y) \to \mathrm{Hom}(X, Y)$$

is an isomorphism. If I is finite, $\mathcal{C}$ is called finitely semisimple.

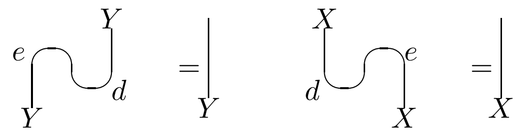
7.3 Dual objects and spherical categories

Throughout this subsection we consider a strict tensor category $(\mathcal{C}, \otimes, \mathbf{1})$.

7.10 Definition *An object Y is called a left dual of $X \in \mathcal{C}$ if there are morphisms $e : Y \otimes X \to \mathbf{1},\ d : \mathbf{1} \to X \otimes Y$ satisfying the 'duality equations'*

$$e \otimes \mathrm{id}_Y \circ \mathrm{id}_Y \otimes d = \mathrm{id}_Y, \qquad \mathrm{id}_X \otimes e \circ d \otimes \mathrm{id}_X = \mathrm{id}_X,$$

or



Right duals are defined analogously with $e : X \otimes Y \to \mathbf{1},\ d : \mathbf{1} \to Y \otimes X$.

7.11 Remark 1. Beware: Some authors (< 50%) interchange the above definitions of left and right duals.

2. If Y is a left dual of X and $Y' \cong Y$, then Y' is a left dual of X.

3. Let $Y,\ Y'$ be left duals of X w.r.t. the morphisms d, e and d', e'. Then the duality equations imply that the morphisms

$$a = e \otimes \mathrm{id}_{Y'} \circ \mathrm{id}_Y \otimes d' :\ Y \to Y', \qquad a' = e' \otimes \mathrm{id}_Y \circ \mathrm{id}_{Y'} \otimes d :\ Y' \to Y$$

are inverses of each other. Thus the left dual object of X, if it exists, is unique up to isomorphism and we denote it by ${}^\vee V$.

4. Right duals $V^\vee$ are defined similarly by exchanging X and Y in Definition 7.10.

5. Even if an object X has left and right duals ${}^\vee V,\ V^\vee$, they don't need to be isomorphic. If they are, we speak of a two-sided dual and write $\overline{X}$ instead of ${}^\vee V,\ V^\vee$.

6. The considerations at the beginning of Subsection 6.5 show that a finite dimensional vector space V equipped with a non-degenerate bilinear form μ is (two-sided) self-dual in the sense $\overline{V} \cong V$.

The following will not be used, but is instructive nevertheless:

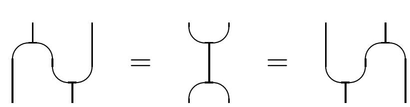
7.12 Lemma *Let $\mathcal{C}$ be a strict tensor category with braiding c. Then if $X \in \mathcal{C}$ has a left dual (Y, e, d), then Y is also a right dual of X. (Thus Y is a two-sided dual $\overline{X}$.)*

Proof. We define $e_r : X \otimes Y \to \mathbf{1}$ and $d_r : \mathbf{1} \to Y \otimes X$ by

$$e_r = e \circ c(X, Y), \qquad d_r = c(Y, X)^{-1} \circ d.$$

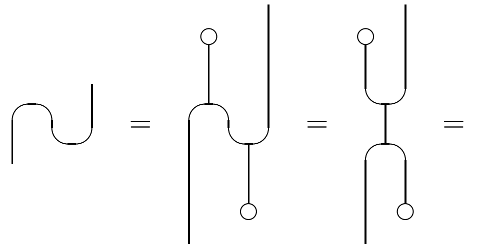
Now it is easy to verify that (Y, e_r, d_r) is a right dual of Y. ■

7.13 Remark Let $((V, m, 1), \varepsilon)$ be a Frobenius algebra over a field k. Then $\mu := \varepsilon \circ m$ is a non-degenerate bilinear form $V \otimes V \to k$. Define the dual morphism $b \in V \otimes V$ as in Subsection 6.4. Now we can define $\Delta : V \to V \otimes V$ as in (6.10) and one verifies easily that (V, Δ, ε) is a coassociative coalgebra. Furthermore, m and Δ satisfy the 'Frobenius identity'



(7.1)

Conversely, let $(V, m, 1, \Delta, \varepsilon)$ be such that $(V, m, 1)$ is an associative algebra, (V, Δ, ε) is a coassociative coalgebra and (7.1) is satisfied. In this situation, define $\mu = \varepsilon \circ m : V \otimes V \to k$ and $b = \Delta(1) \in V \otimes V$. Then



(The first equality holds by definition, the second by (7.1) and the third one since η (ε) is the unit (counit) for m (Δ).) The second duality equation is verified analogously. Now Lemma 6.29 implies that $\mu : V \otimes V \to k$ is non-degenerate. Thus $((V, m, 1), \varepsilon)$ is a Frobenius algebra. It is clear that the two constructions are inverses of each other. This shows that Frobenius algebras, as defined usually, are the same as Frobenius algebras in the tensor category Vect_k, as defined below. The new definition has the advantage of being more symmetric and making sense in any tensor category. (In the non-strict case, insert the associativity and unit constraints in the obvious places.)

7.14 Definition *Let $\mathcal{C}$ be a strict tensor category. Then a Frobenius algebra in $\mathcal{C}$ is a quintuple $(V, m, \eta, \Delta, \varepsilon)$ such that (V, m, η) is an algebra (we drop the '(co)associative'), (V, Δ, ε) is a coalgebra and (7.1) is satisfied.*

7.15 Remark Let $\mathcal{C}$ be a strict tensor category and X an object having a two-sided dual $\overline{X}$. I.e. there are morphisms $e : \overline{X} \otimes X \to \mathbf{1}$, $d : \mathbf{1} \to X \otimes \overline{X}$, $e' : X \otimes \overline{X} \to \mathbf{1}$, $d' : \mathbf{1} \to \overline{X} \otimes X$ satisfying the usual identities. Defining $\Gamma = X \otimes \overline{X}$ and the morphisms

$$\eta = d : \mathbf{1} \to \Gamma, \quad m = \mathrm{id}_X \otimes e \otimes \mathrm{id}_{\overline{X}} : \Gamma \otimes \Gamma \to \mathbf{1},$$

$$\varepsilon = e' : \Gamma \to \mathbf{1}, \quad \Delta = \mathrm{id}_X \otimes d' \otimes \mathrm{id}_{\overline{X}} : \Gamma \to \Gamma \otimes \Gamma,$$

it is straightforward to verify that (Γ, m, η) is a monoid in $\mathcal{C}$, $(\Gamma, \Delta, \varepsilon)$ is a comonoid in $\mathcal{C}$ and the identity

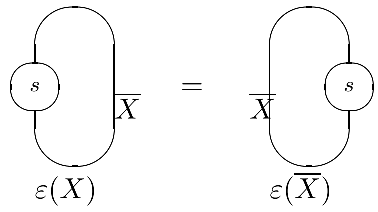
$$m \otimes \mathrm{id}_\Gamma \circ \mathrm{id}_\Gamma \otimes \Delta = \Delta \circ m = \mathrm{id}_\Gamma \otimes m \circ \Delta \otimes \mathrm{id}_\Gamma$$

is satisfied. This means that $(\Gamma, m, \eta, \Delta, \varepsilon)$ is a Frobenius algebra in $\mathcal{C}$ in the sense of the new Definition 7.14! For a converse of this cf. [40]. Here we limit ourselves to the remark that Frobenius algebras in tensor categories other than Vect_k are very important in subjects as diverse as subfactor theory and conformal field theory.

7.16 Remark It is hard to find a field of mathematics exhibiting worse a terminological mess than that of duality in tensor categories: rigid, (compact) closed (Kelly/Laplaza), autonomous (FY), ribbon (Turaev ?)=tortile (Shum, Joyal/Street ?), sovereign (FY) Maltsiniotis)=pivotal (FY), balanced (?, Deligne/Milne) =(?) spherical (Barrett/Westbury), category with conjugates (Doplicher/Roberts/Longo)

7.17 Definition *A spherical category is a pivotal category where $\mathrm{Tr}_L(s) = \mathrm{Tr}_R(s)$ for every $s \in \mathrm{End}\, X$.*

7.18 Remark In terms of the graphical notation, sphericity means that



(7.2)

This can be interpreted by saying that the diagrams live on a sphere instead of the plane. Invariance under isotopies then introduces the new relation (7.2) since one can 'pull a line around the back side of the sphere'.

7.19 DEFINITION *If $\mathcal{C}$ is a spherical category and $X \in \mathcal{C}$ then the dimension $d(X) \in \mathrm{End}\,\mathbf{1}$ is defined by*

$$d(X) = \mathrm{Tr}(\mathrm{id}_X).$$

(If $\mathcal{C}$ is k-linear with $\mathrm{End}\,\mathbf{1} = k\,\mathrm{id}_{\mathbf{1}}$ then $d(X)$ is usually considered as an element of k.)

7.20 DEFINITION *A fusion category is a finitely semisimple k-linear strict spherical tensor category with $\mathrm{End}\,\mathbf{1} = k\,\mathrm{id}_{\mathbf{1}}$ and $\dim \mathcal{C} \neq 0$.*

7.21 LEMMA *Let $\mathcal{C}$ be a fusion category. Then the pairing between $\mathrm{Hom}(X,Y)$ and $\mathrm{Hom}(Y,X)$ defined by $\langle s,t\rangle = \mathrm{Tr}_X(t \circ s) = \mathrm{Tr}_Y(s \circ t)$ is non-degenerate. It therefore induces canonical isomorphisms $\mathrm{Hom}(X,Y)^* \cong \mathrm{Hom}(Y,X)$.*

Proof.

■

7.22 DEFINITION *If $\mathcal{C}$ is a fusion category, we define*

$$\dim \mathcal{C} = \sum_{i \in I} d(X_i)^2 \in k.$$

7.23 REMARK If H is a finite dimensional semisimple and cosemisimple (i.e. the dual Hopf algebra $\widehat{H}$ is semisimple) Hopf algebra, then $H-\mathrm{Mod}$ is a spherical category. (Though not a strict one, but we can obtain a strict one by strictification.) $H-\mathrm{Mod}$ comes with a forgetful functor to Vect_k, and the spherical structure on $H-\mathrm{Mod}$ is the obvious one which makes this functor spherical. This implies that the categorical dimension of a representation of H is just the dimension of the representation space. Now the semisimplicity of H implies $\dim H - \mathrm{Mod} = \dim_k H$. In particular, $\dim \mathrm{Rep_k G} = \dim kG - \mathrm{Mod} = |G|$ (provided $\mathrm{char}\,k$ does not divide $|G|$).

7.4 Definition of $Z(M,T)$

Our strategy will be the following: Starting from a closed 3-manifold M, a triangulation T and a nice category,

(a) Define $Z(M,T) \in k$.

(b) Show that $Z(M,T)$ is invariant under Pachner moves, allowing to define $Z(M) = Z(M,T)$.

(c) Define a TQFT in $2+1$ dimensions and study (some of) its properties.

In this subsection we will concentrate on (a), while the other points will be addressed in subsequent subsections.

Throughout, we fix a closed oriented 3-manifold M, a *simplicial* triangulation T of M and a nice category $\mathcal{C}$ and let I denote its set of (isomorphism classes of) simple objects of $\mathcal{C}$. For the time being, we fix a map $l : E \to I$, where E is the set of edges (1-simplices) in T.

The standard 3-simplex $\Delta_3 = \{0123\}$ comes with a canonical embedding in $\mathbb{R}^4$, and this defines an orientation on Δ_3. Let A be a tetrahedron (3-simplex) in T. We consider two cases:
(Case A) The orientation of A is compatible (via the homeomorphism $M \to |X_\bullet|$) with that of M.

Let e_{ij} be the edge of A that connects the vertices $i,j \in \{0,1,2,3\}$ of Δ_3. (Note that the practice of [8], where e_{ij} denotes the 1-simplex $\partial_i\partial_j A$ is inconsistent with the usage of e_{ij} in the rest of that paper!)

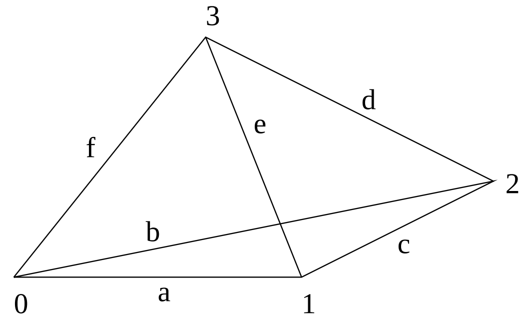


Figure 11: Tetrahedron A

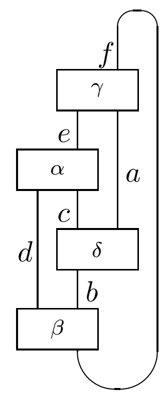
Let a, b, c, d, e, f be the objects associated (by the map $l : E \to I$) to the edges $01, 02, 12, 23, 13, 03$, respectively, of A as in Figure 11.

We define a linear functional

$$\phi_A : \mathrm{Hom}(d \otimes c, e) \otimes_k \mathrm{Hom}(f, d \otimes b) \otimes_k \mathrm{Hom}(e \otimes a, f) \otimes_k \mathrm{Hom}(b, c \otimes a) \to k, \tag{7.3}$$

by

$$\alpha \otimes_k \beta \otimes_k \gamma \otimes_k \delta \mapsto Tr(\gamma \circ \alpha \otimes \mathrm{id}_a \circ \mathrm{id}_d \otimes \delta \circ \beta) =$$



$\in k$

7.24 Remark 1. Note that it does not matter whether we take the left or right trace since $\mathcal{C}$ is spherical.

2. We will often denote the tensor product in Vect_k by $\otimes_k$ in order to avoid confusion with that of $\mathcal{C}$, as could arise here.)

3. Notice that the morphisms $\alpha, \beta, \gamma, \delta$ correspond to the faces (2-simplices) of A, and they involve the objects by which the edges of the respective face are labelled.

Note that drawing this as in Figure 12, we see that the categorical diagram representing the number $\phi_A(\alpha \otimes_k \beta \otimes_k \gamma \otimes_k \delta) \in k$ is just the dual A' of the tetrahedron of A! The vertices (edges, faces) of A' correspond to the faces (edges, vertices) of A. This is analogous to the discussion in Subsection 6.5, where $Z(M, T) \in k$ was obtained by interpreting the dual graph $\mathcal{G}^d$ of the triangulation T as a morphism in the category Vect_k. (In that case, each edge was interpreted as the object V and each vertex as either m or Δ.)

As defined above, we have

$$\begin{aligned} \phi_A &\in (\mathrm{Hom}(d \otimes c, e) \otimes_k \mathrm{Hom}(f, d \otimes b) \otimes_k \mathrm{Hom}(e \otimes a, f) \otimes_k \mathrm{Hom}(b, c \otimes a))^* \\ &\cong \mathrm{Hom}(d \otimes c, e)^* \otimes_k \mathrm{Hom}(f, d \otimes b)^* \otimes_k \mathrm{Hom}(e \otimes a, f)^* \otimes_k \mathrm{Hom}(b, c \otimes a)^*. \end{aligned}$$

Using the canonical isomorphisms provided by Lemma 7.21, we obtain an element

$$\psi_A \in \mathrm{Hom}(e, d \otimes c) \otimes_k \mathrm{Hom}(f, d \otimes b)^* \otimes_k \mathrm{Hom}(f, e \otimes a) \otimes_k \mathrm{Hom}(b, c \otimes a)^*. \tag{7.4}$$

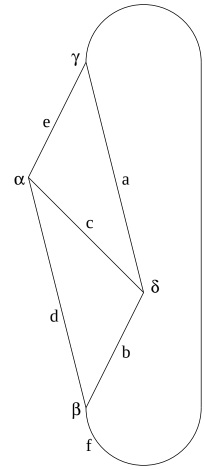


Figure 12: Poincaré dual A' of the tetrahedron A

(Case B) The canonical orientations of A and the one inherited from M are opposite.

In this case we use the linear functional

$$\phi_{-A} : \mathrm{Hom}(e, d \otimes c) \otimes_k \mathrm{Hom}(d \otimes b, f) \otimes_k \mathrm{Hom}(f, e \otimes a) \otimes_k \mathrm{Hom}(c \otimes a, b) \;\rightarrow\; k,$$

$$a \otimes_k b \otimes_k c \otimes_k d \mapsto \mathrm{Tr}(\beta \circ \mathrm{id}_d \otimes \delta \circ \alpha \otimes \mathrm{id}_a \circ \gamma),$$

thus

$$\begin{aligned}\phi_{-A} \in (\mathrm{Hom}(e, d \otimes c) \otimes_k \mathrm{Hom}(d \otimes b, f) \otimes_k \mathrm{Hom}(f, e \otimes a) \otimes_k \mathrm{Hom}(c \otimes a, b))^* \\ \cong \mathrm{Hom}(e, d \otimes c)^* \otimes_k \mathrm{Hom}(d \otimes b, f)^* \otimes_k \mathrm{Hom}(f, e \otimes a)^* \otimes_k \mathrm{Hom}(c \otimes a, b)^* \\ \cong \mathrm{Hom}(e, d \otimes c)^* \otimes_k \mathrm{Hom}(f, d \otimes b) \otimes_k \mathrm{Hom}(f, e \otimes a)^* \otimes_k \mathrm{Hom}(b, c \otimes a),\end{aligned}$$

and we define

$$\psi_{-A} \in \mathrm{Hom}(e, d \otimes c)^* \otimes_k \mathrm{Hom}(f, d \otimes b) \otimes_k \mathrm{Hom}(f, e \otimes a)^* \otimes_k \mathrm{Hom}(b, c \otimes a) \tag{7.5}$$

by these isomorphisms.

Observing that the oriented boundary of $A = \{0123\}$ is given by $\partial A = \{123\} - \{023\} + \{013\} - \{012\}$, thus $\partial(-A) = -\{123\} + \{023\} - \{013\} + \{012\}$, and comparing this with (7.4) and (7.5) we see that the state space corresponding to a positively oriented face of A appear without dualization (i.e. $*$) and that corresponding to a negatively oriented face of A with dualization. Notice that every triangle (2-simplex) in T belongs to the boundary of precisely two tetrahedra, and these two copies appear with opposite orientations. Tensoring the ψ's of all tetrahedra together and contracting the mutually dual spaces, we get an element $Z(M, T, f) \in \mathrm{End}\,\mathbf{1} \cong k$. Now we define

$$Z(M, T) = (\dim \mathcal{C})^{-n_1(T)} \sum_{l:E \to I} Z(M, T, f) \prod_{e \in E} d(f(e)). \tag{7.6}$$

(I.e., we multiply $Z(M, T, f)$ by the dimensions of all objects labelling the edges and sum over all labellings. As before, $n_1(T)$ is the number of vertices of T.)

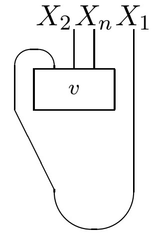
Before we can turn to proving that $Z(M, T)$ is invariant under Pachner moves, we must show that it is actually well-defined.

7.25 Definition *Let $\mathcal{C}$ be a tensor category and $X_1, \ldots, X_n \in \mathcal{C}$. Then the state space $V(X_1, \ldots, X_n)$ is defined as*

$$V(X_1, \ldots, X_n) = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, X_1 \otimes \cdots \otimes X_n).$$

7.26 Proposition *Let $\mathcal{C}$ be a strict spherical category. There is an isomorphism $\sigma : V(X_1, \ldots, X_n) \to V(X_2, \ldots, X_n, X_1)$ such that $\sigma^n = \mathrm{id}_{V(X_1,\ldots,X_n)}$.*

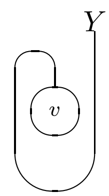
Proof. For $v \in V(X_1, \ldots, X_n)$ we define $\sigma(v) \in V(X_2, \ldots, X_n, X_1)$ by

$\sigma(v) =$ 

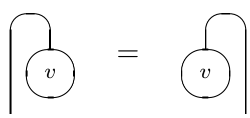
In view of duality, this map clearly is an isomorphism. Now σ^n amounts to wrapping all lines corresponding to $X_1, \ldots, X_n$ around v. Writing $Y = X_1 \otimes \cdots X_n$, we have $v \in \mathrm{Hom}(\mathbf{1}, Y)$, and using

$$\varepsilon(Y) = \mathrm{id}_{X_1 \otimes \cdots X_{n-1}} \otimes \varepsilon(X_n) \otimes \mathrm{id}_{\overline{X_{n-1}} \otimes \cdots \otimes \overline{X_1}} \circ \mathrm{id}_{X_1 \otimes \cdots X_{n-2}} \otimes \varepsilon(X_{n-1}) \otimes \mathrm{id}_{\overline{X_{n-2}} \otimes \cdots \otimes \overline{X_1}} \circ \cdots \circ \varepsilon(X_n),$$

we have

$\sigma^n(v) =$ 

That this equals v follows from

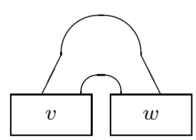


which in turn follows from
****

7.27 Proposition *Let $X_1, \ldots, X_n \in \mathcal{C}$. Then there is a canonical isomorphism*

$$V(X_1, \ldots, X_n)^* \cong V(\overline{X_n}, \ldots, \overline{X_1}).$$

Proof. This follows from non-degeneracy of the pairing $V(X_1, \ldots, X_n) \times V(\overline{X_n}, \ldots, \overline{X_1}) \to k$ defined by

$\langle v, w \rangle =$ 

■

7.28 Corollary *Let $s \in S_3$. Then*

$$V(X_{s(1)}, X_{s(2)}, X_{s(3)}) \cong \begin{cases} V(X_1, X_2, X_3) & \text{if } S \text{ is even} \\ V(\overline{X_1}, \overline{X_2}, \overline{X_3})^* & \text{if } S \text{ is odd} \end{cases}$$

Proof. The even permutations in S_3 are precisely the cyclic ones, to which Proposition **??** applies. The odd ones are obtained by combining a cyclic permutation with $(123) \to (321)$, to which Proposition **??** applies. ■

7.5 Invariance of $Z(M,T)$ under Pachner moves

In Exercise A.66 we have already encountered the Pachner moves for $n = 3$, cf. Figures 13 and 14.

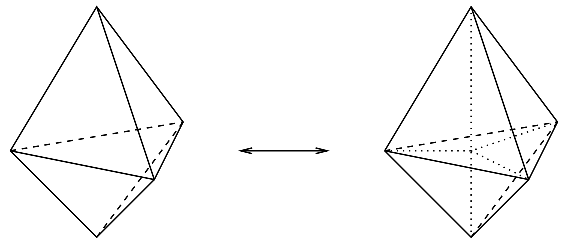


Figure 13: $d = 3$: Pachner move of orders 1 and 2

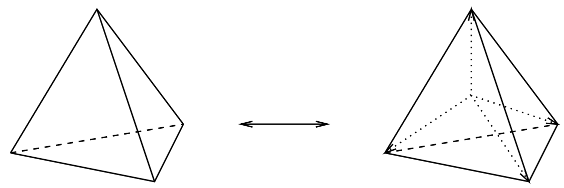


Figure 14: $d = 3$: Pachner move of orders 0 and 3

7.6 The triangulation TQFT in $2+1$ dimensions

In this subsection we will lift the invariant $Z_{\mathcal{C}}$ of oriented 3-manifolds constructed in the preceding subsection to a TQFT. The construction is very similar to the one given in Subsection 6.2, and we will limit ourselves to indicating the minor changes.

7.29 Remark In view of Corollary 6.54, it is natural to ask whether the triangulation TQFTs $F_{\mathcal{C}}$ in $d = 2+1$ are semisimple.
****

(If every TQFT comes from a modular category $\mathcal{M}$ via RT (more generally Kerler/Lyubashenko), then this is true since we will have $\mathcal{M} = Z(\mathcal{C})$, which is semisimple, cf. [41].)

7.30 Remark When are the TQFTs $F_{\mathcal{C}}, F_{\mathcal{C}'}$ arising from two spherical categories $\mathcal{C}, \mathcal{C}'$ isomorphic? This is not yet completely understood, but in [40] it has been shown that $F_{\mathcal{C}} \cong F_{\mathcal{C}'}$ when there is a 'weak monoidal Morita equivalence' $\mathcal{C} \approx \mathcal{C}'$. (The equivalence relation $\approx$ is much weaker than ordinary equivalence of tensor categories. E.g., if H is a finite dimensional semisimple Hopf algebra such that also the dual Hopf algebra $\widehat{H}$ is semisimple, then one has $H-\mathrm{Mod} \approx \widehat{H}-\mathrm{Mod}$.) Furthermore, one has that $\mathcal{C} \approx \mathcal{C}'$ if and only if $\mathcal{C}$ and $\mathcal{C}'$ have equivalent centers, cf. [28]. Here 'center' refers to a somewhat involved construction that associates a braided tensor category $Z(\mathcal{C})$ to any tensor category $\mathcal{C}$, cf. e.g. [26]. Thus there are considerable similarities with the situation in $d = 1+1$, as described in Remark 6.46. The analogy becomes complete if one assumes the implication $F_{\mathcal{C}} \cong F_{\mathcal{C}'} \Rightarrow \mathcal{C} \approx \mathcal{C}'$, which is almost certainly true. Under this assumption we have $F_{\mathcal{C}} \cong F_{\mathcal{C}'} \Leftrightarrow Z(\mathcal{C}) \simeq Z(\mathcal{C}')$.

This analogy (and there are more!) between TQFTs in 1+1 and 2+1 dimensions suggest that there should be a general theory for all dimensions to which results like $F_{(V,a,\mu)} \cong F_{(V',a',\mu')} \Leftrightarrow Z(V,a,\mu) \cong Z(V',a',\mu')$ and $F_{\mathcal{C}} \cong F_{\mathcal{C}'} \Leftrightarrow Z(\mathcal{C}) \simeq Z(\mathcal{C}')$ generalize! So far, this is science fiction, and even the known results about TQFTs in $3+1$ dimensions, whether via triangulations or otherwise, are quite scarce. It is therefore quite gratifying that there is at least one family of TQFTs that can be defined in all dimensions. They will be the subject of Section 8.

7.7 Example: The Dijkgraaf-Witten TQFT: $\mathcal{C} = \mathcal{C}(G, \alpha)$

In Subsection 3.3 we defined a k-linear tensor category $\mathcal{C}(G, A, \alpha)$. Here we slightly modify the definition in order to obtain a semisimple category. Let k be a field and $\alpha \in Z^3(G, k^*)$. We define $\mathcal{C}_0(G, k, \alpha)$ as in Subsection 3.3, except that

$$\mathrm{Hom}_{\mathcal{C}(G,k,\alpha)}(g, h) = \begin{cases} k & \text{if } g = h \\ \{0\} & \text{if } g \neq h \end{cases}$$

every finite group G and $\alpha \in H^3(G, k^*)$. This category is semisimple and spherical etc., allowing us to consider the associated TQFT $F_{\mathcal{C}(G,\alpha)}$. This TQFT in $d = 2 + 1$ was first discussed in heuristic (path integral terms) in [14] and more rigorously in [2] and [60].

7.8 Ocneanu's tube algebra

The aim of this subsection is to begin the analysis of the triangulation TQFT $F_{\mathcal{C}}$ associated to the fusion category $\mathcal{C}$. The ultimate aim is to show that $F_{\mathcal{C}} \cong F^{RT}_{Z(\mathcal{C})}$, where F^{RT} is the Reshetikhin-Turaev TQFT and $Z(\mathcal{C})$ is the center, cf. [26], of $\mathcal{C}$, which is modular by [41], but we pose ourselves a more modest goal.

7.31 Definition *For every $s \in \mathbb{N}_0$, the functor $\mathcal{S} : \mathrm{Cob}_{s+1} \to \mathrm{Cob}_{(s+1)+1}$ is defined as $\times S^1$. I.e., the closed oriented s-dimensional surfaces Σ and the cobordisms M between them are multiplied by S^1. Clearly, this is a symmetric tensor functor.*

If $F : \mathrm{Cob}_{(s+1)+1} \to \mathrm{Vect}_k$ is a TQFT in $s+1$ dimensions, then $F \circ \mathcal{S}$ is a TQFT in $s+1$ dimensions. In this way, every TQFT in $2 + 1$ dimensions gives rise to a TQFT in $1 + 1$ dimensions. Since we know that the latter corresponds to a commutative Frobenius algebra (cFA), the question arisis of computing the cFA $A(\mathcal{C})$ corresponding to the 2d TQFT $F_{\mathcal{C}} \circ \mathcal{S}$, where $\mathcal{C}$ is a fusion category and $F_{\mathcal{C}}$ is the corresponding 3d triangulation TQFT. Determining the latter will be remarkably similar to the computation of the cFA corresponding to a triangulation TQFT in $1+1$ dimensions in Subsection 6.5.

8 TQFTs in all dimensions: The Freed-Quinn TQFTs

8.1 The classical field theories

cf. [17, Section 1].

8.2 Integration theory

cf. [17, Appendix B].

8.3 The quantum field theories

cf. [17, Section 2].

9 Further topics

9.1 Generalization of $d = 1 + 1$-TQFT: $\otimes$-functors $\mathrm{Cob}_{1+1} \to \mathcal{C}$

9.2 Moore-Segal: Open-closed TQFTs in $d = 1 + 1$

9.3 Homotopy TQFT (à la Turaev)

9.4 More on the triangulation invariant in $d = 2 + 1$

- Invariance under weak Morita equivalence.

- Definition of $Z(\mathcal{C})$.
- Tube algebra, proof of GK conjecture.
- more general conjecture: $RT(M, Z(\mathcal{C})) = BW(M, \mathcal{C})$ for every spherical semisimple $\mathcal{C}$.
- Kitaev/MM: $\mathcal{C}_1 \approx \mathcal{C}_2 \Leftrightarrow Z(\mathcal{C}_1) \simeq Z(\mathcal{C}_2)$.
- ...

9.5 More on extended TQFTs

9.6 Reshetikhin-Turaev invariant and TQFT (incl. lagrangian subspaces etc.)

A Simplicial stuff

A.1 Semisimplicial sets and their morphisms

A.1 Definition *A semisimplicial set (s.s.s.)*

$$X_0 \;\underset{\partial_1}{\overset{\partial_0}{\leftleftarrows}}\; X_1 \;\overset{\partial_0,\,\partial_1,\,\partial_2}{\longleftarrow\!\!\longleftarrow\!\!\longleftarrow}\; X_2 \;\overset{\partial_0,\,\partial_1,\,\partial_2,\,\partial_3}{\longleftarrow\!\!\longleftarrow\!\!\longleftarrow\!\!\longleftarrow}\; X_3 \qquad \cdots$$

consists of a family $X_\bullet = (X_n)$ of sets $X_0, X_1, \dots$ and maps $\partial_i : X_n \to X_{n-1}$, where $n \geq 1$ and $i \in \{0, 1, \dots, n\}$, satisfying

$$\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i \quad \textit{whenever} \quad i < j. \tag{A.1}$$

(Both sides being understood as maps $X_n \to X_{n-2}$, where $n \geq 2$.)

A.2 Remark 1. It might be said that we should write $\partial_i^{(n)} : X_n \to X_{n-1}$, but this would be as uncustomary as unnecessary, since in general there is no danger of confusion.

2. A semisimplicial set will also be called an 'abstract Δ-complex' in view of the relation to the (concrete) Δ-complexes appearing later, cf. also [22].

3. Eq. (A.1) can be completed to

$$\partial_i \circ \partial_j = \begin{cases} \partial_{j-1} \circ \partial_i & \text{when} \quad i < j \\ \partial_i \circ \partial_i & \text{when} \quad i = j \\ \partial_j \circ \partial_{i+1} & \text{when} \quad i > j \end{cases},$$

but this contains no new information.

4. A simplicial set is a semisimplicial set together with maps $s_i : X_{n-1} \to X_n$, with $n \geq 1$, $i \in \{0, 1, \dots, n\}$, where the s_i satisfy relations similar to the ∂_i and there are relations between the s_i and the ∂_j. (E.g., $\partial_i \circ s_i = \mathrm{id}$, i.e. s_i is a right-inverse of ∂_i.) There is a category $\mathcal{S}$, the simplicial category, such that a simplicial set is the same as a functor $\mathcal{S}^{\mathrm{op}} \to \mathrm{Sets}$. (Similarly for s.s.s. and a suitable subcategory of $\mathcal{S}$.) But we will not need these notions. For more on this see [61, Chapter 8], and [37].

A.3 Definition *A map (or morphism) $s : X_\bullet \to Y_\bullet$ of semisimplicial sets $X_\bullet, Y_\bullet$ is a family of maps $s_n : X_n \to Y_n$, for $n \in \mathbb{N}_0$, such that*

$$\begin{array}{ccc} X_n & \xrightarrow{\partial_i} & X_{n-1} \\ {\scriptstyle s_n}\downarrow & & \downarrow{\scriptstyle s_{n-1}} \\ Y_n & \xrightarrow[\partial_i]{} & Y_{n-1} \end{array}$$

commutes for all $n \geq 1, i \in \{0, \ldots, n\}$.

It should be clear that semisimplicial sets and morphisms between them form a category, which we denote **SSS**. (In the interpretation of s.s.s. as functors alluded to in Remark A.2.4, morphisms of s.s.s. are natural transformations.)

The remainder of this subsection is not essential for the sequel. We will briefly comment on the relation between s.s.s. (=abstract Δ-complexes) and the notion of an '(abstract) simplicial complex' that one encounters occasionally, cf. e.g. [9, 22, 61].

A.4 Definition *An (abstract) simplicial complex* (V, F) *consists of a set* V *(the 'vertices') and a set* F *of of finite subsets of* V *(the 'faces') satisfying*

1. $v \in V \Rightarrow \{v\} \in F$.
2. $f' \subset f \in F \;\Rightarrow\; f' \in F$.

Every simplicial complex gives rise to a s.s.s., though not uniquely:

A.5 Proposition *Let* (V, F) *be a simplicial complex and* $\leq$ *a total order on* V*. For* $n \in \mathbb{N}_0$ *we define* $X_n = \{f \in F \mid |f| = n+1\}$*. If* $f \in X_n$ *(in particular,* f *contains* $n+1$ *elements of* V*) and* $0 \leq i \leq n$*, we define* $\partial_i(f) \in X_{n-1}$ *to be the subset of* f *obtained by removing the* $(i+1)$*-th element w.r.t. the total order* $\leq$*. Then* $X_\bullet$ *is a s.s.s.*

Proof. Easy exercise. ■

A.6 Remark Note that the s.s.s. $X_\bullet$ obtained from a simplicial complex (V, F) (and an order $\leq$ on V) has a property not shared by every s.s.s.: Every face $f \in X_n$ is uniquely characterized by its $n+1$ 'corners' (faces of codimension n) $x_0 = 1, \ldots, x_n = 1$. There is no such requirement for s.s.s. Occasionally, one sees '(geometric) simplicial complex' used to denote the geometric realization, cf. Subsection A.5, of an abstract simplicial complex, cf. e.g. [9, Section IV.21]. We have two reasons for preferring the more general notion of s.s.s.: (i) It is more useful also for an integrated treatment of (co)homology, and (ii): The geometric realizations of s.s.s. (=geometric Δ-complexes) are much more convenient to work with than the geometric simplicial complexes. Cf. also Remark A.60 concerning triangulations.

A.2 Example: Groups

An important class of examples of s.s.s. is provided by the following theorem:

A.7 Theorem *(i) Let* G *be a (discrete) group. Define* $X_0(G) = \{e\}$ *(a one-point set) and* $X_n(G) = G^{\times n}$ *for* $n \geq 1$*. With* $\partial_0 = \partial_1 : X_1(G) \to X_0(G),\ g \mapsto e$ *and, for* $n \geq 2$*,*

$$\partial_i : X_n(G) \to X_{n-1}(G),\ (g_1, \ldots, g_n) \mapsto \begin{cases} (g_2, g_3, \ldots, g_n) & (i = 0) \\ (g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_n) & (0 < i < n) \\ (g_1, g_2, \ldots, g_{n-1}) & (i = n) \end{cases}$$

Then $X_\bullet(G)$ *is a semisimplicial set.*

(ii) A homomorphism $a : G \to H$ *of groups gives rise to a morphism* $s : X_\bullet(G) \to X_\bullet(H)$ *of s.s.s. by* $s_n : X_n(G) \to X_n(H),\ (g_1, \ldots, g_n) \mapsto (a(g_1), \ldots, a(g_n))$.

(iii) $X_\bullet$ *is a functor* **Grp**$\to$**SSS** *(where* **Grp** *is the category of groups and homomorphisms).*

A.8 Exercise *Prove this!*

A.9 Remark In Subsection A.6 we will see that $X_\bullet(G)$ is a quotient of an even simpler s.s.s.

A.3 Example: The singular s.s.s. of a topological space

A.10 Definition *The standard n-simplex Δ_n is defined as*

$$\Delta_n = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0 \geq 0, \ldots, x_n \geq 0, \sum_i x_i = 1\}.$$

It is understood as a topological space with the subspace topology inherited from $\mathbb{R}^{n+1}$.

A.11 Definition *Let $S \subset \{0, 1, \ldots, n\}$ with $|S| \leq n$. The subset*

$$\Delta_n^S = \{x \in \Delta_n \mid x_i = 0 \; \forall \; i \in S\}$$

is called a face of Δ_n of codimension $|S|$.

Clearly, $\Delta_n^S \cong \Delta_{n-|S|}$. In fact, using the unique increasing bijection $\iota_S : \{0, \ldots, n - |S|\} \to \{0, \ldots, n\} - S$, there is a unique homeomorphism $\alpha_n^S : \Delta_{n-|S|} \to \Delta_n^S$ such that $\alpha_n^S(x)_{\iota_S(i)} = x_i$ for all $0 \leq i \leq n - |S|$. It is customary to introduce some notation for the codimension-one case:

A.12 Definition *For $n \geq 1$, define $d_i : \Delta_{n-1} \to \Delta_n$ by $d_i = \alpha_n^{\{i\}}$. Concretely: $d_i : (x_0, \ldots, x_{n-1}) \mapsto (x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1})$.*

A.13 Lemma *For $n \geq 1$, we have*

$$d_j \circ d_i = d_i \circ d_{j-1} \quad \text{whenever} \quad i < j.$$

A.14 Exercise *Prove this!*

A.15 Definition *Let Z be a topological space. For $n \in \mathbb{N}_0$, we define $X_n(Z) = C(\Delta_n, Z)$ (continuous maps) and maps $\partial_i : X_n(Z) \to X_{n-1}(Z)$, $f \mapsto f \circ d_i$.*

A.16 Theorem *(i) $X_\bullet(Z)$ is a s.s.s, called the singular s.s.s. of Z.*

(ii) A continuous map $g : Y \to Z$ gives rise to a morphism $s : X_\bullet(Y) \to X_\bullet(Z)$ of s.s.s. by $s_n : X_n(Y) \to X_n(Z)$, $f \mapsto g \circ f$.

(iii) $X_\bullet$ is a functor **Top→SSS** *(where* **Top** *is the category of topological spaces and continuous maps).*

Proof. (i) Immediate by Lemma A.13.
(ii) That s is a morphism of s.s.s. is just the observation that $g \circ (f \circ d_i) = (g \circ f) \circ d_i$.
(iii) Should be clear. ■

A.4 Homology and Cohomology of a s.s.s.

A.4.1 Homology

A.17 Definition *Let X be a set. Then*

$$\mathbb{Z}^{\oplus X} \equiv \bigoplus_{x \in X} \mathbb{Z} := \{a : X \to \mathbb{Z} \mid |\{x \in X \mid a(x) \neq 0\}| < \infty\}.$$

(The set of functions $X \to \mathbb{Z}$ of finite support.) $\mathbb{Z}^{\oplus X}$ is an abelian group w.r.t. pointwise addition, inverse and unit. Identifying $x \in X$ with the δ-function $\delta_x \in \mathbb{Z}^{\oplus X}$, the element of $\mathbb{Z}^{\oplus X}$ represented by the function $a : X \to \mathbb{Z}$ can also written as $\sum_{x \in X} a(x)x$. (It is clear that the $x \in X_n$ are linearly independent and span $\mathbb{Z}^{\oplus X}$, thus $\mathbb{Z}^{\oplus X}$ is just a free abelian group of rank $|X|$.)

A.18 Definition *Let $X_\bullet$ be a s.s.s. We define $C_n(X) = \mathbb{Z}^{\oplus X_n}$ $(n \geq 0)$. For $n \geq 1$, we define group homomorphisms*

$$\partial : C_n(X) \to C_{n-1}(X), \quad \sum_{x \in X_n} a(x)x \mapsto \sum_{x \in X_n} a(x) \sum_{i=0}^{n} (-1)^i \partial_i(x).$$

(Thus the generator $x \in X_n$ of $C_n(X)$ is sent to $\sum_{i=0}^{n}(-1)^i \partial_i(x)$, which is not in X_{n-1} but in $C_{n-1}(X)$.)

A.19 Proposition *For every $n \geq 2$, the composite $C_n(X) \xrightarrow{\partial} C_{n-1}(X) \xrightarrow{\partial} C_{n-2}$ is the zero map.*

A.20 Exercise *Prove this!*

This motivates the following

A.21 Definition *A diagram $C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} C_2 \xleftarrow{\partial} \cdots$ of abelian groups and homomorphisms satisfying $\partial \circ \partial = 0$ everywhere is called a chain complex.*

A.22 Definition *Let $C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} C_2 \xleftarrow{\partial} \cdots$ be a chain complex. Define*

$$\begin{aligned} Z_0 &= C_0, \\ Z_n &= \ker(\partial : C_n \to C_{n-1}), \quad (n \geq 1), \\ B_n &= \operatorname{im}(\partial : C_{n+1} \to C_n). \end{aligned}$$

The elements of C_n, Z_n, B_n are called chains, cycles, boundaries, respectively. By definition, $Z_n \subset C_n$, and $\partial^2 = 0$ implies $B_n \subset Z_n$ for all $n \geq 0$. ($\subset$ meaning 'subgroup'.) We define $H_n = Z_n/B_n$ and call it the 'n-th homology group $H_n(C)$ of the complex $C_\bullet$'.

Often it is convenient to introduce a dependence on an abelian group A of 'coefficients':

A.23 Definition *Let $C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} C_2 \xleftarrow{\partial} \cdots$ be a chain complex and A an abelian group. Then*

$$C_0 \otimes_{\mathbb{Z}} A \xleftarrow{\partial \otimes \mathrm{id}} C_1 \otimes_{\mathbb{Z}} A \xleftarrow{\partial \otimes \mathrm{id}} C_2 \otimes_{\mathbb{Z}} A \ \cdots$$

is a chain complex, and its homology groups are denoted $H_n(C, A)$, the 'homology groups of $C_\bullet$ with coefficients in A'.

A.24 Remark On might guess that $H_n(X, A) \cong H_n(X) \otimes_{\mathbb{Z}} A$ for all n. (If this was true, introducing coefficients would be relatively pointless.) But the situation is more complicated, i.e. interesting: For each n, one has an injective homomorphism $i : H_n(X) \otimes_{\mathbb{Z}} A \to H_n(X, A)$, which need not be surjective. In fact, the quotient $H_n(X, A)/i(H_n(X) \otimes_{\mathbb{Z}} A)$ is given by $\mathrm{Tor}(H_{n-1}(X), A)$, where $\mathrm{Tor}(A, B)$ is a certain abelian group. Concerning its properties we limit us to saying that $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(B, A)$ and that $\mathrm{Tor}(A, B)$ is trivial for all B when A is torsionfree, e.g. when A is a field of characteristic zero. Thus $H_n(X, k) \cong H_n(X) \otimes_{\mathbb{Z}} k$ when $k = \mathbb{Q}$ or $k = \mathbb{R}$. For much more on this see, e.g., [22, Section 3.A].

Now we study the behavior of the homologies of s.s.s. under morphisms of s.s.s.

A.25 Definition *Let $(C_\bullet, \partial), (C'_\bullet, \partial')$ be chain complexes. A chain map $f : C_\bullet \to C'_\bullet$ is a family of homomorphisms $f_n : C_n \to C'_n$ such that*

$$\begin{array}{ccc} C_n & \xrightarrow{\partial} & C_{n-1} \\ \downarrow f_n & & \downarrow f_{n-1} \\ C'_n & \xrightarrow[\partial']{} & C'_{n-1} \end{array}$$

commutes for all $n \geq 1$. Chain complexes and chain maps between them form a category, denoted **CC**.

A.26 PROPOSITION *Let $(C_\bullet, \partial), (C'_\bullet, \partial')$ be chain complexes and $f : C_\bullet \to C'_\bullet$ a chain map. Then there are unique homomorphisms $\widehat{f_n} : H_n(C) \to H_n(C')$ satisfying $\widehat{f_n}([c]) = [f_n(c)]$ for all $c \in Z_n$. (Here $[c]$ is the class in $H_n = Z_n/B_n$ of $c \in Z_n$.) With these, we have functors $H_n : \mathbf{CC} \to \mathbf{Ab}$.*

Proof. Let $c \in Z_n$, i.e. $c \in C_n$ and $\partial c = 0$. Since f is a chain map, we have $\partial'(f_n(c)) = f_{n-1}(\partial c) = f_{n-1}(0) = 0$. Thus $f_n(c) \in Z'_n$. Similarly, one shows that $c \in B_n$ implies $f_n(c) \in B'_n$. Thus the map $f_n : Z_n \to Z'_n$ descends to the equivalence classes, giving rise to a map $\widehat{f_n}$ with the claimed properties. ■

A.27 REMARK The same holds for any coefficient group A. The proofs are essentially the same.

A.28 PROPOSITION *Let $X_\bullet, X'_\bullet$ be s.s.s. and let $C_\bullet, C'_\bullet$ the associated chain complexes arising from them as in Proposition A.19. Let $s : X_\bullet \to X'_\bullet$ be a morphism of s.s.s. Then*

$$f_i : C_n \to C'_n, \ \sum_{x \in X_n} a(x)x \mapsto \sum_{x \in X_n} a(x)s_n(x)$$

defines a chain map $f : C_\bullet \to C'_\bullet$. In fact, one has a functor $f : \mathbf{SSS} \to \mathbf{CC}$.

A.29 EXERCISE *Prove this!*

Combining Propositions A.26 and A.28 we have:

A.30 COROLLARY *For any abelian group A and $n \in \mathbb{N}_0$, we have a functor $H_n(\cdot, A) : \mathbf{SSS} \to \mathbf{Ab}$.*

Applying the constructions of this section to the examples of s.s.s. considered earlier, we obtain:

A.31 DEFINITION *(a) If G is a group, the homology groups of the complex $C_\bullet(G)$ with coefficients A are denoted $H_n(G, A)$. For each $n \in \mathbb{N}_0$ and abelian A, $G \mapsto H_n(G, A)$ is a functor* **Grp→Ab** *(the category of abelian groups).*

(b) If Z is a topological space, the homology groups of the complex $C_\bullet(Z)$ with coefficients A are denoted $H_n(Z, A)$. Again $Z \mapsto H_n(Z, A)$ is a functor **Top→Ab**.

A.32 EXERCISE *Prove that $H_1(G) \cong G_{ab} := G/[G, G]$. ($G_{ab}$ is the abelianization of G, i.e. the 'biggest' abelian quotient of G. Every homomorphism $G \to A$ with A abelian factors through G_{ab}.)*

A.33 REMARK 1. The functors $H_n(\cdot, A) : \mathbf{Top} \to \mathbf{SSS}$ are called 'singular homology' since they are defined in terms of maps $\Delta_n \to Z$ that are only required to be continuous, but whose image may intersect itself.

2. It should go without saying that the functors $H_n(\cdot, A) : \mathbf{Grp} \to \mathbf{Ab}$ and $H_n(\cdot, A) : \mathbf{Top} \to \mathbf{Ab}$ have many properties not discussed here: long exact sequences, homotopy invariance, Künneth formula, excision etc. Cf. [9, 22] for topological homology and [61] for group homology.

A.4.2 Cohomology

We will need the dual notion for chain complexes:

A.34 Definition *A diagram* $C_0 \xrightarrow{d} C_1 \xrightarrow{d} C_2 \xrightarrow{d} \cdots$ *of abelian groups and homomorphisms satisfying* $d \circ d = 0$ *everywhere is called a cochain complex. We define* $Z^n = \ker(d : C^n \to C^{n+1}),\ B^n = \mathrm{im}(d : C^{n-1} \to C^n)$ *(with* $B^0 = 0$*), observe* $B^n \subset Z^n$ *and define* $H^n = Z^n/B^n$*. The elements of* C^n, Z^n, B^n *are called cochains, cocycles, coboundaries, respectively, and* H^n *is the 'n-th cohomology group of* $C_\bullet$ *with coefficients in A'.*

A.35 Remark The notions of chain complex and cochain complex become equivalent if one considers sequences of abelian groups indexed by $\mathbb{Z}$ instead of $\mathbb{N}_0$. (Define $C^n = C_{-n}$ etc.) But over $\mathbb{N}_0$ these notions are distinct.

A.36 Definition/Proposition *Let* $C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} C_2 \xleftarrow{\partial} \cdots$ *be a chain complex. Let A be an abelian group.*

We define abelian groups $C^n = \mathrm{Hom}(C_n, A),\ n \in \mathbb{N}_0$ *and homomorphisms* $d : C^n(A) \to C^{n+1}(A)$ *by* $dc(x) = c(\partial x)$ *for* $c \in C^n(A)$ *and* $x \in C_{n+1}$*. Then* $C_0 \xrightarrow{d} C_1 \xrightarrow{d} C_2 \xrightarrow{d} \cdots$ *is a cochain complex.*

Proof. $ddc(x) = c(\partial\partial x) = c(0) = 0$. ■

A.37 Definition *Let* $X_\bullet$ *be a s.s.s. and A an abelian group. Let* $(C_\bullet(X), \partial)$ *be the chain complex of Proposition A.19. Then* $C^n(X, A)$ *denotes the cochain complex obtained via Definition/Proposition A.36 from* $(C_\bullet(X), \partial)$*.*

A.38 Remark Where it not for the warning provided by Remark A.24, on might conjecture that $H^n(C_\bullet, A) \cong \mathrm{Hom}(H_n(C_\bullet), A)$ for all n. In fact this is not true in general. However, for each n there is a surjective homomorphism $p : H^n(C, A) \to \mathrm{Hom}(H_n(C), A)$, whose kernel is given by the abelian group $\mathrm{Ext}(H_{n-1}(C), A)$. Concerning $\mathrm{Ext}(A, B)$ we only note that it is trivial for all B when the abelian group A is free. For much more on this see, e.g., [22, Section 3.1].

By definition, we have a pairing $\langle\cdot,\cdot\rangle : C^n \times C_n \to A,\ (f, c) \mapsto f(c)$. We now show that the latter induces a pairing $\langle\cdot,\cdot\rangle : H^n \times H_n \to A$. (Note that both pairings are additive w.r.t. both arguments.)

A.39 Proposition *Let* $C_\bullet$ *be a chain complex. There is a unique map* $\langle\cdot,\cdot\rangle : H^n \times H_n \to A$ *such that* $\langle [f], [c] \rangle = f(c)$ *for all* $c \in Z_n, f \in Z^n$*.*

Proof. We begin by observing that $f(c) = 0$ when $c \in Z_n, f \in B^n$ or $c \in B_n, f \in Z^n$: In the first case we have $f = dg$ with $g \in Z^{n+1}$. Thus $f(c) = (dg)(c) = g(\partial c) = g(0) = 0$, where we have used $c \in Z_n$, i.e. $dc = 0$, and the definition of d. The second claim is shown analogously.

Given classes $\overline{c} \in H_n,\ \overline{f} \in H^n$, choose representers $c \in Z_n, f \in Z^n$ and observe that $f(c)$ is independent of the chosen representers by the observation just made. Thus we can define $\langle \overline{f}, \overline{c} \rangle = f(c)$. ■

We now briefly discuss the functoriality properties of cohomology. First one defines the notion of a map of cochain complexes. This is done exactly as in Definition A.25, simply reversing the horizontal morphisms. Now let $C_\bullet, C'_\bullet$ be chain complexes and A an abelian group. It is evident that a chain map $C_\bullet \to C'_\bullet$ gives rise to a map of cochain complexes in the opposite direction: $C'^\bullet(A) \to C^\bullet(A)$. In complete analogy to Proposition A.26, the latter induces homomorphisms $H^\bullet(C') \to H^\bullet(C)$. Thus:

A.40 Proposition *For each abelian group A and* $n \in \mathbb{N}_0$*, there is a contravariant functor* $H^n(\cdot, A)$: **CC**→**Ab**.

Returning to our examples of group and topological (co)homology, and observing that we have covariant functors from **Grp** to **CC** (category of chain complexes) and from **Top** to **CC**, we have in analogy to Definition A.31:

A.41 Definition *(a) If G is a group, the cohomology groups of the complex $C_\bullet(G)$ with coefficients A are denoted $H^n(G,A)$. For fixed A,n, this is a contravariant functor* **Grp**→**Ab**.

(b) If Z is a topological space, the homology groups of the complex $C_\bullet(Z)$ with coefficients A are denoted $H^n(Z,A)$. For fixed A,n, this is a contravariant functor **Top**→**Ab**.

A.42 Exercise *Prove that $H^1(G,A)$ is isomorphic to the group of homomorphisms $G \to A$ (with pointwise addition etc.)*

A.43 Exercise *Prove that $H^n(G,A)$ is isomorphic to $H^n(G,A)$ as defined earlier in terms of $C^n = \mathrm{Fun}(G^{\times n},A)$ and, for $f \in C^n(G,A)$,*

$$df(g_1,\ldots,g_{n+1}) = f(g_2,\ldots,g_{n+1}) + \sum_{i=1}^{n}(-1)^i f(g_1,\ldots,g_ig_{i+1},\ldots,g_{n+1}) + (-1)^{n+1}f(g_1,\ldots,g_n).$$

Hint: If C_n is the free abelian group over a set X, then $\mathrm{Hom}(C_n,A)$ is isomorphic to the product $\prod_{x\in X} A$, i.e. the set of **all** *functions $f : X \to A$ (as opposed to the ones with finite support) with pointwise group operations. Prove and use this!*

A.44 Remark We have seen earlier that the cohomology groups $H^n(G,A)$ of a group arise in certain algebraic classification problems (at least for $n = 2,3$). The reason for looking at topological cohomology $H^n(Z,A)$ has to do with the fact that cohomology has 'more structure' than homology: The contravariance of H^n makes it possible to define very useful product operations $H^n(Z,A) \times H^m(Z,A) \to H^{n+m}(Z,A)$, whereas this can be done in homology only for very special spaces Z ('H-spaces'). Furthermore, if M is a smooth manifold then $H^n(M,\mathbb{R})$ is isomorphic to the geometrically defined n-th de Rham-cohomology group $H^n_{dR}(M)$.

A.5 Geometric realization

A.45 Definition *Let $X_\bullet$ be a s.s.s. Then the geometric realization $|X_\bullet|$ of $X_\bullet$ is the topological space*

$$|X_\bullet| = \left(\coprod_{n\in\mathbb{N}_0} X_n \times \Delta_n\right) / \sim,$$

where the X_n are given the discrete topology and $\sim$ is the equivalence relation generated by

$$(x, d_i(y)) \sim (\partial_i(x), y), \quad \text{where} \quad x \in X_n,\ y \in \Delta_{n-1}\ (n \geq 1,\ 0 \leq i \leq n).$$

(I.e., we begin with disjoint copies of Δ_n for each $n \in \mathbb{N}_0$ and each $x \in X_n$. Then, for each $x \in X_n$, we use the maps $d_i : \Delta_{n-1} \to \Delta_n$ to identify the standard $(n-1)$-simplex corresponding to $\partial_i(x)$ with a codimension-one face of the standard n-simplex corresponding to x.)

A topological space of the form $|X_\bullet|$ is called a (geometric) Δ-complex, cf. [22].

A.46 Remark For many purposes, one is more interested in the geometric realization $|X_\bullet|$ of a s.s.s. than in $X_\bullet$ itself. However, the distinction between abstract and geometric Δ-complexes is warranted by the rôle of the former in the discussion of (co)homology. For much more on (semi)simplicial sets and their geometric realizations see [37].

By now, it should not come as a surprise that geometric realization is a functor:

A.47 Proposition *A morphism $s : X_\bullet \to Y_\bullet$ of s.s.s. gives rise to a continuous map $|s| : |X_\bullet| \to |Y_\bullet|$ between the geometric realizations. With these maps we have a functor $|\cdot| :$ **SSS**$\to$**Top**.*

Proof. We define

$$|s|_0 : \coprod_{n\in\mathbb{N}_0} X_n \times \Delta_n \to \coprod_{n\in\mathbb{N}_0} Y_n \times \Delta_n,\ (x,y) \mapsto (s_n(x), y) \quad \text{for} \quad x \in X_n,\ y \in \Delta_n,$$

which clearly is a continuous map. Thus, if $n \geq 1$, $0 \leq i \leq n$, $x \in X_n$, $y \in \Delta_{n-1}$, we have, $|s|(x, d_i(y)) = (s_n(x), d_i(y))$ and $|s|(\partial_i(x), y) = (s_n(\partial_i(x)), y)$. Since s is a morphism of s.s.s., this equals $(\partial_i(s_n(x)), y)$, which is equivalent (w.r.t. $\sim_Y$) to $(s_n, d_i(y))$. Thus the map $|s|_0$ is compatible with the equivalence relations and gives rise to a map $|s| : |X_\bullet| \to |Y_\bullet|$. It remains to show its continuity. Now, the composite of maps $\coprod_{n\in\mathbb{N}_0} X_n \times \Delta_n \stackrel{|s|_0}{\longrightarrow} \coprod_{n\in\mathbb{N}_0} Y_n \times \Delta_n \stackrel{p_Y}{\longrightarrow} |Y_\bullet|$ clearly is continuous. Since $|s|^{-1}(U) = p_X(|s|_0^{-1}(U))$ for any open $U \subset |Y_\bullet|$ and the quotient map $p_X : \coprod_{n\in\mathbb{N}_0} X_n \times \Delta_n \to |X_\bullet|$ is open (why?), $|s|^{-1}(U) \subset |X_\bullet|$ is open, thus $|s|$ is continuous. ■

It is important to note that the two functors $X_\bullet :$ **Top**$\to$**SSS** and $|\cdot| :$ **SSS**$\to$**Top** are **not inverses** of each other! E.g., the singular s.s.s. associated to the geometric realization $|X_\bullet|$ of a s.s.s. $X_\bullet$ is much bigger than the s.s.s. $X_\bullet$. Nevertheless, there is a weaker connection between these functors, namely they are adjoints of each other. We refrain from going into this and limit ourselves to state an important result on the level of homology:

A.48 Theorem *Let $X_\bullet$ be a s.s.s. and A an abelian group. Then there are natural isomorphisms $H_n(X, A) \cong H_n(|X_\bullet|, A)$ and similarly for cohomology.*

Proof. Since the proof is somewhat involved, we'll limit ourselves to giving some indications: Given a s.s.s. $X_\bullet$, there is a canonical morphism $f : X_\bullet \to X_\bullet(|X_\bullet|)$ of s.s.s., whose n-th component f_n sends $x \in X_n$ to the singular simplex

$$\Delta_n \to X_n \times \Delta_n \hookrightarrow \coprod_{n\in\mathbb{N}_0} X_n \times \Delta_n \to |X_\bullet|, \tag{A.2}$$

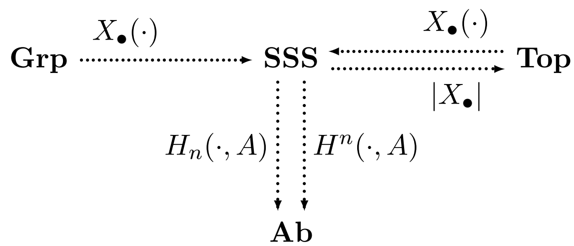
where the first arrow is given by $\Delta_n \to X_n \times \Delta_n, y \mapsto (x, y)$ and the other maps are obvious. The bulk of the proof now consists in showing that the induced maps (by Proposition A.26) of homology groups are isomorphisms. Cf. e.g. [22, p. 128-131]. ■

A.49 Remark This may be a suitable moment to address a natural question: Why simplices?? To begin with: The standard simplex Δ_n is a simple (no pun intended) model space of dimension n (in the sense of dimension theory) . Furthermore, its boundary $\partial\Delta_n$ is a union of finitely many (namely $n+1$) copies of the simplex Δ_{n-1}. This does not quite distinguish the simplices since we could use the cubes $C_n = [0,1]^n$, which satisfy $\partial C_n \sim 2n\, C_{n-1}$. One slight advantage of simplices is that the faces of Δ_n are indexed by $\{0, 1, \ldots, n\}$, whereas those of C_n are indexed by $\{1, \ldots, n\} \times \{+, -\}$. This is one reason why 'triangulations', to be discussed soon, are preferred over 'cubations'.

But apart from that, singular homology $H_n(Z)$ can also be defined in terms of continuous maps $C_n \to Z$, and one can show that the functors H_n (called cubical singular homology, cf. e.g. [23, Section 8.3]) thus obtained are naturally isomorphic to the ones of simplicial homology. The proof [23, Section 8.4] is non-trivial, and it would seem that one has to provide another such proof one for any choice of model spaces. Luckily, this is not necessary since (a) (co)homology can be defined without any use of model spaces like Δ_n, C_n, namely in terms of homotopy classes (e.g. $H^1(X, A)$ is the set of homotopy classes of continuous maps $X \to BA$, cf. e.g. [22, 38]) and (b) *any* (co)homology theory satisfying certain axioms is isomorphic to the one defined in terms of homotopy.

Also in the discussion of group (co)homology one can do away with simplicial methods by using the more fundamental approach of derived functors, cf. e.g. [61]. The latter also permits the generalization where the coefficient group A carries a non-trivial G-module structure, i.e. an action of G by automorphisms. (This generalization can be discussed in the simplicial formalism, but this is a bit tricky and the derived functor approach works better.)

We now summarize the various functors considered so far in a diagram:



Composing one of the functors $G \to X_\bullet(G)$ or $Z \to X_\bullet(Z)$ with one of the vertical ones yields the (co)homology theories for groups and topological spaces, respectively. But there is one more thing we can do, cf. the next subsection.

A.6 The classifying spaces BG and EG

We can look at the geometric realization of the s.s.s. $X_\bullet(G)$ arising from a group G, and this will turn out to be very useful:

A.50 Definition *Let G be a group. Then we write $BG = |X_\bullet(G)|$ (the geometric realization of the s.s.s. $X_\bullet$) and call it the classifying space of G. As a composite of the functors* $\mathbf{Grp} \xrightarrow{X_\bullet} \mathbf{SSS} \xrightarrow{|\cdot|} \mathbf{Top}$, *$BG$ also is a functor.*

An immediate consequence of Theorem A.48 is:

A.51 Corollary *We have natural isomorphisms $H_n(BG, A) \cong H_n(G, A)$, and similarly for cohomology. (On the left we have the topological (co)homology of the classifying space BG and on the right the purely algebraic (co)homology of G.)*

The usefulness of BG does not primarily arise from the preceding fact, but from the further properties of BG, beginning with:

A.52 Theorem *For any group G, the classifying space BG has the following properties:*

(i) BG is path-connected.

(ii) $\pi_1(BG, p) \cong G$ for any $p \in BG$.

(iii) $\pi_n(BG, p) \cong 0$ for any $p \in BG$ and $n \geq 2$.

Proof. (i) The s.s.s. $X_\bullet(G)$ has only one 0-simplex $e \in X_0(G)$. By construction of the geometric realization, every corner (=face of maximal codimension) of the standard n-simplices in $|X_\bullet(G)|$ is identified with the standard 0-simplex (=point) corresponding to $e \in X_0(G)$. Thus $|X_\bullet(G)|$ is connected.

(ii) and (iii) follow from the fact, proven below, that there is a geometric Δ-complex EG which is connected, contractible and on which G acts freely such that $EG/G \cong BG$. Namely, EG is a simply connected principal G-bundle over BG, implying $\pi_1(BG) \cong G$. Furthermore, the quotient map $p : EG \to BG$ induces isomorphisms $\pi_n(EG, *) \to \pi_n(BG, p(*))$ for all $n \geq 2$ (follows from the long exact homotopy sequence for fibrations), implying triviality of $\pi_n(BG)$ for $n \geq 2$. ∎

The space EG used in the preceding proof is defined as follows:

A.53 Definition/Proposition *Let G be a group. For all $n \in \mathbb{N}_0$, define $Y_n(G) = G^{\times(n+1)}$. Furthermore, for $n \geq 1,\ 0 \leq i \leq n$ we define*

$$\partial_i : Y_n(G) \to Y_{n-1}(G),\ (g_0, \ldots, g_n) \mapsto (g_0, \ldots, \widehat{g_i}, \ldots, g_n),$$

where the hat means that the i-th element g_i is omitted. Then $Y_\bullet$ is a s.s.s. Now we define $EG := |Y_\bullet(G)|$.

Finally, we define an action of G on each set $Y_n(G)$ by $g(g_0, \ldots, g_n) = (gg_0, \ldots, gg_n)$. Clearly, this action is free, i.e. if $g \in G$ acts trivially on any $(g_0, \ldots, g_n)$ then $g = e$. The G-action commutes with the maps ∂_i, thus every $g \in G$ gives rise to a morphism $s(g) : Y_\bullet(G) \to Y_\bullet(G)$ of s.s.s. By Proposition A.47, we have a G-action $g \mapsto |s(g)|$ on EG.

A.54 Theorem *(i) The G-action on EG is free.*

(ii) There is a canonical homeomorphism $EG/G \to BG$.

(iii) EG is contractible. (Thus in particular connected.)

Proof. (i) Essentially obvious in view of freeness of the G-action on $Y_\bullet(G)$.

(ii) Since the G-action on EG is simplicial, i.e. maps $(x, y) \in \coprod_{n\in\mathbb{N}_0} X_n \times \Delta_n$ to (gx, y), we have $EG/G = |Y_\bullet(G)|/G \cong |Y_\bullet(G)/G|$, where $Y_\bullet/G$ is the s.s.s. of G-orbits in $X_\bullet$. With this preparation, our claim will follow from $Y_\bullet(G)/G \cong X_\bullet(G)$, which we now set out to prove. The action of $g \in G$ on $Y_n(G)$ carries $(g_0, \ldots, g_n)$ to $(gg_0, \ldots, gg_n)$. Thus the G-orbit of $(g_0, \ldots, g_n)$ contains precisely one element of the form $(e, g_1' \ldots, g_n')$, and the latter has a unique representation of the form $(e, g_1'', g_1''g_2'', \ldots, g_1'' \cdots g_n'')$. Thus we can identify $[(g_0, \ldots, g_n)] \in Y_n(G)/G$ with $(g_1'', \ldots, g_n'') \in G^{\times n} = X_n(G)$. Now $\partial_i : Y_n(G) \to Y_{n-1}(G)$ acts by omitting g_i. In terms of the representation $(g_1'', \ldots, g_n'') \in G^{\times n}$ this amounts to omitting g_i'' if $i = 0$ or $i = n$ and to $(g_1'', \ldots, g_n'') \mapsto (g_1'', \ldots, g_i''g_{i+1}'', \ldots, g_n'')$ if $0 < i < n$. Since this was exactly the definition of the ∂_i in the s.s.s. $X_\bullet(G)$, we are done.

(iii) Each point of EG is represented by a $\sim$-equivalence class $[((g_0, \ldots, g_n), y)]$, where $(g_0, \ldots, g_n) \in Y_n(G)$ and $y = (y_0, \ldots, y_n) \in \Delta_n$. Define $h_t : EG \to EG,\ t \in [0, 1]$, by

$$h_t([((g_0, \ldots, g_n), y)]) = [((e, g_0, \ldots, g_n), (1 - t, ty_0, \ldots, ty_n))]. \tag{A.3}$$

Observe that (a): $(1 - t) + ty_0 + \cdots + ty_n = 1$, thus $(1 - t, ty_0, \ldots, ty_n) \in \Delta_{n+1}$. (b): h_t is compatible with the identification of standard simplices in EG, thus (A.3) defines a map of EG to itself. The latter clearly depends continuously on t. (c): For $t = 1$, the r.h.s. of (A.3) reduces to $[((e, g_0, \ldots, g_n), (0, y_0, \ldots, y_n))]$, and this point is identified with $[((g_0, \ldots, g_n), y)]$ by the definition of EG. Thus $h_1 = \mathrm{id}_{EG}$. (d): For $t = 0$, the r.h.s. of (A.3) reduces to $[((e, g_0, \ldots, g_n), (1, 0, \ldots, 0))] = [(e), (1)]$. Thus h_0 maps EG to the point $[(e), (1)] \in EG$. (iv): Thus h_t is a contraction. ■

A.55 Remark Since we constructed the classifying space BG in a very specific way, it may be worth noticing that it is essentially uniquely characterized by its properties: One can show that every CW-complex (a notion generalizing Δ-complexes) X having the properties (i)-(iii) in Theorem A.52 is homotopy equivalent to BG. Since homotopy equivalent spaces have the same homology and cohomology groups, this result can be used to *define* $H_n(G, A) := H_n(BG, A)$ and similarly for cohomology. This is what happened historically and what led to the discovery that $H_n(G, A)$ and $H^n(G, A)$ can be computed in a purely algebraic way!

A.7 Principal bundles, relation to BG and π_1

To be written later. A good reference for most of this is [53].

A.8 Triangulations and Pachner moves

A.56 Definition *A triangulation of a topological space Z is a pair $(X_\bullet, \alpha)$, where $X_\bullet$ is a s.s.s. and $\alpha : |X_\bullet| \to Z$ is a homeomorphism. A space is called triangulizable if it admits a triangulation.*

A.57 Theorem *(Whitehead, 1940) A smooth manifold of any dimension is triangulizable.*

Proof. Omitted. See [62] or, for more detail, the second half of [43]. ■

A.58 Remark In fact also topological d-manifolds with $d \in \{1, 2, 3\}$ are triangulizable. For $d = 1$ this is more or less obvious, whereas the case $d = 2$ was proven by ?? in ?? and $d = 3$ by Bing and Moise in 1952-1954. For $d \geq 4$, one needs to distinguish between topological manifolds (Top), triangulizable topological manifolds (PL for piecewise linear) and smooth manifolds (Diff).

We will also need the following intuitively plausible result, which we cite without proof.

A.59 Theorem *Let M be a smooth manifold and T_∂ a triangulation of the boundary ∂M. Then there is a triangulation T of M whose restriction to ∂M coincides with T_∂. (I.e. there is an injection $\iota : Y_\bullet \to X_\bullet$ of s.s.s. such that $\alpha(i_*(|Y_\bullet|)) = \partial M$.)*

A.60 Remark 1. The torus $T^2 = S^1 \times S^1$ can be obtained from the square $[0,1]^2$ by identifying its boundary edges pairwise in the familiar fashion. Since the square can be triangulated by cutting it along a diagonal, we obtain a triangulation of T^2 in terms of Δ-complex $X_\bullet$ with just two 2-simplices, cf. Figure 15. Note however, that all four corners of the square get identified, thus both 2-simplices have

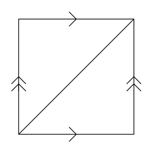


Figure 15: Economic triangulation of the 2-torus

only one (in particular the same) corner, and our geometric Δ-complex is not a geometric simplicial complex. (Recall that no two simplices in a simplicial complex can have the same set of corners.) One can show that a simplicial complex triangulating T^2 has at least 14 2-simplices! Since the torus will play an important rôle in some of our constructions, the corresponding proofs would be forbiddingly involved in terms of simplicial complexes.

2. Despite the fact that Δ-complexes are more general than simplicial complexes, every Δ-complexes can be subdivided into a simplicial complex. Cf. the instructions to Exercise 23 on p. 133 of [22].

A.61 Definition *The dimension of a (non-empty) semisimplicial set $X_\bullet$ (and its geometric realization) is defined as $d(X_\bullet) = \max\{i \,|\, X_i \neq \emptyset\} \in \mathbb{N}_0 \cup \{\infty\}$. A semisimplicial set $X_\bullet$ (and its geometric realization) is called finite if $\cup_i X_i$ is finite. (Obviously, a finite s.s.s. is finite dimensional.)*

A.62 Theorem *Let M be a (finite dimensional) manifold admitting a triangulation $(X_\bullet, \alpha)$. Then $X_\bullet$ is finite dimensional and $d(X_\bullet) = d(M)$. $|X_\bullet|$ is the union of its $d(M)$-simplices. If M is compact then $X_\bullet$ is finite.*

A.63 Exercise *Try to prove this!*

It is clear that a triangulizable space will admit many different triangulations. Thus the questions arises how the different triangulations of a space are related. In order to formulate Pachner's theorem, the topological notion of the 'join' of two spaces is useful:

A.64 Definition *Let X, Y be topological spaces, and let $I = [0,1]$. When $X \neq \emptyset \neq Y$, the join $X * Y$ is defined as the quotient space of $X \times Y \times I$ under the identifications $(x, y, 0) \sim (x, y', 0)$ and $(x, y, 1) \sim (x', y, 1)$. If $X \neq \emptyset$, we define $X * \emptyset = \emptyset * X := X$, and finally $\emptyset * \emptyset := \emptyset$.*

If X, Y are both non-empty, we see that the map $x \mapsto [(x, ?, 0)]$ is a bijection between X and $\{[(x, y, 0)]\} \subset X * Y$ and $y \mapsto [(?, y, 1)]$ is a bijection between Y and $\{[(x, y, 1)]\} \subset X * Y$. In fact, these maps are homeomorphisms onto their images, the latter equipped with the subspace topology. (The special prescriptions in the cases when one of the spaces X, Y is empty are made to maintain this desirable property of the join. Otherwise $Y = \emptyset$ would imply $X * Y = \emptyset$ since $X \times Y \times I = \emptyset$.) Thus $X * Y$ can be understood (quite vaguely) as a space that interpolates between X and Y, and more accurately as the "space of all line segments joining a point of X with a point of Y" (whence the name). Cf. the picture at [22, p. 9]. The relevance of $*$ for us that it can be used to generate all standard simplices Δ_n:

A.65 Exercise *Prove the following claims:*

1. *There are homeomorphisms $X * Y \cong Y * X$ and $(X * Y) * Z \cong X * (Y * Z)$.*
2. *If $X \neq \emptyset$ then $X * \{p\} \cong (X \times I)/\sim'$, where $(x, z) \sim' (x', z')$ iff $z = z' = 1$.*
3. *$\Delta_n * \{p\} \cong \Delta_{n+1}$.*
4. *With $P = \{p\}$ and $n \in \mathbb{N}$ we have $P^{*n} \cong \Delta_{n-1}$.*
5. *$\Delta_n * \Delta_m \cong \Delta_{n+m+1}$.*

A.66 Exercise *Let $p, q \in \mathbb{N}_0$. Show that*

(a) The geometric Δ-complexes given by the joins $\partial\Delta_p * \Delta_q$ and $\Delta_p * \partial\Delta_q$ are homeomorphic to the solid ball B^{p+q} (and thus to each other).

(b) $\partial\Delta_p * \Delta_q$ and $\Delta_p * \partial\Delta_q$ both have the boundary $\partial\Delta_p * \partial\Delta_q$. (Note that $\partial\Delta_0 = \emptyset$.)

(c) Explain why Figures 1, 2, 13 and 14 represent the cases $(p, q) \in \{(1,1), (2,0), (2,1), (3,0)\}$, in this order.

Hint: It may be useful to look at (c) first to get some intuition!

The point of Pachner's theorem is that any two triangulations of the same manifold M are related by a finite number of local modifications, where we replace one triangulation of a closed ball in M by another one:

A.67 Definition *If $0 \le k \le d$ and M is an d-manifold with an identification of a component of ∂M with $\partial\Delta_k * \partial\Delta_{d-k}$, then the bistellar move (or Pachner move) of order k refers to the replacement*

$$M \cup (\Delta_k * \partial\Delta_{d-k}) \quad \longrightarrow \quad M \cup (\partial\Delta_k * \Delta_{d-k}).$$

A.68 Theorem (Pachner) *Any two triangulations of a smooth closed manifold can be transformed into each other by a finite number of bistellar moves.*

Proof. Omitted. See the papers [46] and the background references cited there. ■

A.69 Remark The converse of the statement is obvious by Definition A.67 and Exercise A.66: Pachner moves don't change the homeomorphism class of the manifold.

A.9 Fundamental class of a closed manifold

In our discussion of the Freed-Quinn TQFTs in Section 8 we will need the fact that an orientation for a (smooth) closed n-manifold gives rise to a canonical element $[M] \in H_n(M)$, the fundamental class, defined as follows. Appealing to Theorem A.57 we choose a triangulation $T = (X_\bullet, \alpha)$ of M. By Theorem A.62, all the X_i are finite, $X_i = \emptyset$ for $i > n$ and $|X_\bullet|$ is the union of its n-simplices. Now for every $x \in X_n$, there is a map $i_x : \Delta_n \to |X_\bullet| \stackrel{\alpha}{\to} M$, where the first arrow is the map given in (A.2), defining a singular n-simplex in M, and thus an element of $[x] \in H_n(M)$. Since X_n is finite, we can define $[M] \in H_n(M)$ by

$$[M] = \sum_{x \in X_n} s(x, \alpha)\, [x],$$

where $s(x, \alpha) = 1$ if i_x is orientation preserving and $s(x, \alpha) = -1$ if i_x is orientation reversing. In this, we give Δ_n the standard positive orientation $(e_0, \ldots, e_n)$ that it inherits from the ambient $\mathbb{R}^{n+1}$.

There are two problems with this definition: One must show independence of the choice of the triangulation, which can be done e.g. using Pachner moves. More seriously, one must deal that the fact that i_x is continuous, but in general not differentiable. Thus we cannot appeal to the definition of 'orientation preserving' in terms of the derivative $Ti_x : T\Delta_n \to TM$. One way out would be to work only with triangulations such that the restriction of i_x to the interior of the standard simplex is smooth, but then one needs to show that such triangulations exist. The better way is to define 'orientation preserving' in terms of algebraic topology, cf. the literature cited below.

A.70 Remark The approach we adopted to defining the fundamental class is decidedly old-fashioned (not to say outdated), cf. [50]. We used it since it is straightforward to state once one knows about triangulations. For more modern approaches see [22, 9, 38].

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