TQFT'S ASSOCIATED TO FINITE GROUPS

1. MOTIVATION FOR TQFT

Almost all examples of TQFT's involve "counting stuff" over a manifold over which the theory is defined. This is inherent to the physical origin of the subject, and we therefore briefly discuss the physics motivation for TQFT. As the name suggests, a TQFT is a very special kind of *Quantum Field Theory*, which in turn is a Quantum mechanical counterpart of a *Classical Field Theory*. The latter is defined as follows: For a $(d+1)$-dimensional manifold X, in physics terminology "space-time", one considers a space of fields $\mathscr{C}(X)$. Fields are assumed to be *local* objects: roughly speaking, this means that we can define them locally and there is a way to "glue" locally defined fields to a global field defined over all of X. A typical example is given by

$$\mathscr{C}(X) := C^\infty(X, \mathbb{R}),$$

but one should allow for more exotic examples, such as

$$\mathscr{C}(X) := \{\text{isomorphism classes of principal } G\text{-bundles over } X\},$$

where G is a Lie group. (For the definition of a principal bundle, see below.) This is important for so-called gauge theories.

The second ingredient of the theory is an action

$$S_X : \mathscr{C}(X) \to \mathbb{R},$$

which is "locally defined" in the sense that

$$S_X = \int_X \mathscr{L}_X,$$

where the Lagangian $\mathscr{L}_X : \mathscr{C}(X) \to \Omega^{top}(X)$ is a local morphism. In the Quantum theory, a fundamental object is the so-called partition function

$$Z(X) = \int_{\mathscr{C}(X)} e^{2\pi i S_X} d\mu,$$

where μ is a measure on $\mathscr{C}(X)$. Two remarks are in order about this:

- $\mathscr{C}(X)$ is usually an infinite dimensional space, so that the existence of a measure is a nontrivial issue. Usually in Physics, the existence, together with some naive properties is simply assumed.
- In some theories, such as the one we will discuss, there exist *gauge symmetries* that act on the space of fields $\mathscr{C}(X)$. These leave the action invariant and one is actually more interested in

$$Z(X) = \int_{\overline{\mathscr{C}}(X)} e^{2\pi i S_X} d\overline{\mu}, \tag{1.1}$$

where $\overline{\mathscr{C}}$ is the quotient space with respect to the gauge symmetries. Usually the gauge symmetries form a group, in our case they form a groupoid.

The fundamental observation due to Witten was that if the action functional S_X and the metric does not depend on the choice of a metric, the partition function $Z(X)$ is a quantum invariant, and should extend to a full TQFT when considering manifolds with boundary, i.e., cobordisms.

2. Principal bundles

Definition 2.1. A *fiber bundle* over X with fiber F is a continuous surjection $\pi : E \to X$ which is locally trivial in the following sense: each $x \in X$ has an open neighbourhood U for which there is a homeomorphism $\phi : \pi^{-1}(U) \to U \times F$, which makes the following diagram commute

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\ \phi\ } & U \times F \\ {\scriptstyle \pi}\downarrow & \swarrow{\scriptstyle \mathrm{proj}_1} & \\ U & & \end{array}$$

where proj_1 is the projection onto the first factor.

Notice that it follows that $\pi^{-1}(x) \cong F$ for each $x \in X$. When X is manifold, it is natural to require a manifold structure on E and smoothness of the structure maps. We will do that from now on. Let G be a Lie group.

Definition 2.2. A principal G-bundle is a fiber bundle $p : P \to X$ with a free fiberwise right action of G on P such that $P/G \cong X$.

It follows that G can be taken to be the fiber of the bundle. A morphism of principal bundles over X is a smooth map $f : P_1 \to P_2$ which commutes with the right G-action, and induces the identity mapping on X. A section s of a principal G-bunde is a smooth map $s : X \to P$ satisfying $p \circ s =$identity.

Exercise 2.3. Show that with this notion of morphism, principal G-bundles form a category, in fact a groupoid.

There is a "cocycle view" on principal bundles over X as follows: by definition, we can find an open covering $\{U_\alpha\}_{\alpha \in I}$ of X such that P has local trivialisations $\phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times G$. Two local trivializations (U_α, ϕ_α), and (U_β, ϕ_β) define a smooth map

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G,$$

by

$$(\phi_\alpha \circ \phi_\beta^{-1})(x, g) = (x, g g_{\alpha\beta}(x)).$$

These functions are called *transition functions*. One easily verifies the following conditions satisfied by the transition functions of a principal bundle:

i) for three local trivializations (U_α, h_α), (U_β, h_β) and (U_γ, h_γ),

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1,$$

on $U_\alpha \cap U_\beta \cap U_\gamma$,

ii) for each local trivialization (U_α, h_α),

$$g_{\alpha\alpha} = 1.$$

Exercise 2.4. Suppose we are given the following data: a open covering $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ of X, and functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ satisfying *i* and *ii*) above. Construct a principal G-bundle out of these data. Fixing the covering $\mathcal{U}$, when do two collections of transition functions $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ define isomorphic vector bundles? What happens for different coverings?

Of course, we always have the trivial G-bundle over X, namely $P := X \times G$. In general we say that a G-bundle is trivial if it is isomorphic to this bundle.

Proposition 2.5. *A G-bundle P is trivial if and only if it has a global section.*

Exercise 2.6. Let $f : X \to Y$ be a continuous map and $\pi : P \to Y$ a principal G-bundle. Show that

$$f^*P := \{(p,x),\ \pi(p) = f(x)\}$$

has the structure of a principal G-bundle. This defines the so-called *pull-back* along f.

3. The case of a finite group: coverings

From now on we focus on the case that $G = \Gamma$, a finite group. Then things get considerably easier.

3.1. **Covering spaces and π_1.** The contents of this section should be known, but we give a brief refresher for the benefit of the reader. Let $f, g : X \to Y$ be two continuous maps between topological spaces X and Y. They are said to be homotopic if there exists a continuous map $H : X \times I \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$. The fundamental group of X (relative to x_0) is defined as

$$\pi_1(X, x_0) := \{\gamma : I \to X,\ \gamma(0) = x_0 = \gamma(1)\}/\sim$$

where $\sim$ means homotopy relative to the endpoints. In other words homotopies $H : I \times I \to X$ as above are required to satisfy $H(0,t) = x_0 = H(1,t)$. The group structure is induced by concatenation.

Definition 3.1. A covering space is given by a continuous map $\pi : \tilde{X} \to X$ with the property that any $x \in X$ has an open neighborhood U_x such that $\pi^{-1}(U_x)$ is a disjoint union of open sets, to each of which π restricts to a homeomorphism.

Let $f : Z \to X$ be a continuous map. A *lifting* of f is a continuous map $\tilde{f} : Z \to \tilde{X}$ such that $\pi \circ \tilde{f} = f$.

Proposition 3.2 (Path lifting property)**.** *Let $\pi : \tilde{X} \to X$ be a covering and $\gamma : I \to X$ a path in X. Given a point $\tilde{x} \in \pi^{-1}(\gamma^{-1}(0))$, there exists a unique lift of γ starting in $\tilde{x}$.*

Definition 3.3. Let $\pi : \tilde{X} \to X$ be a covering. The group of automorphisms $Aut(\tilde{X}/X)$ is called the group of *covering transformations*.

Theorem 3.4. *Let $p : Y \to X$ be a covering. Then $p_*\pi_1(Y, y_0) \subseteq \pi_1(X, x_0)$ is a normal subgroup and there exists an isomorphism*

$$\pi_1(X, x_0)/p_*\pi_1(Y, y_0) \cong Aut(Y/X).$$

Corollary 3.5. *When Y is simply-connected $\pi_1(X, x_0) \cong Aut(Y/X)$.*

Such a simply connected covering is called the *universal covering*; it is unique up to isomorphism. It can be shown that any reasonable space (such as a manifold) has a universal covering.

3.2. **Γ-coverings; relation to $B\Gamma$.** As before, Γ will be a finite group, and we will consider Γ-bundles over a given space X.

Proposition 3.6. *A principal Γ-bundle over X is the same as a covering space $\pi : \tilde{X} \to X$ together with an isomorphism $\Gamma \cong Aut(\tilde{X}/X)$.*

Proof. Omitted. □

It makes sense to call such bundles "Γ-coverings".

Proposition 3.7. *Γ-coverings over X are classified up to isomorphism by homotopy classes of maps from X to $B\Gamma$.*

Proof. (Sketch) We have already seen the "universal Γ-bundle" $E\Gamma \to B\Gamma$ given by the free action of Γ on the contractible space $E\Gamma$. Therefore, given a map $F : X \to B\Gamma$, define the principal Γ-bundle $P_F := F^* E\Gamma$ with projection map $\pi_F : P_F \to X$. Now suppose that F and G are homotopic via a map $H : X \times I \to B\Gamma$. Using the path lifting property, $H(x,t) : \pi_F^{-1}(x) \to \pi_G^{-1}(x)$ defines a isomorphism of Γ-coverings.

Conversely, given a Γ-covering $P \to X$, consider the bundle

$$(E\Gamma \times P)/\Gamma \to X.$$

Its fiber over $x \in X$ is given by $(E\Gamma \times P_x)/\Gamma \cong E\Gamma$ which is contractible. Therefore all sections are homotopic and this fixes a Γ-homotopy class of maps $P \to E\Gamma$. On the quotient this induces the desired homotopy class $X \to B\Gamma$. These two constructions are inverses of each other. □

Anticipating the application to TQFT, we can rewrite this proposition as follows: Let $\mathscr{C}(X)$ be the groupoid of Γ-coverings of X. The arrows in this groupoid define an equivalence relation $\sim$ on $\mathscr{C}(X)$ and we write $\overline{\mathscr{C}}(X)$ for the associated quotient. With this we have

$$\overline{\mathscr{C}}(X) \cong [X, B\Gamma], \tag{3.1}$$

where the square brackets stands for the set of homotopy classes of maps between the spaces.

Exercise 3.8. There is an another proof of this isomorphism using local trivializations, cf. the "cocycle view" on principal bundles. Given a covering of X by opens over which the bundle can be trivialized, use the transition functions and the cocycle identity to construct an explicit map $X \to B\Gamma$, and show that this indeed is a classifying map for the given bundle. (Hint: show that the transition functions are locally constant and recall that we constructed $B\Gamma$ as the geometric realization of a simplicial set!)

Proposition 3.9. *There is a bijection of sets:*

$$\overline{\mathscr{C}}(X) \cong \mathrm{Hom}(\pi_1(X, x_0), \Gamma)/\Gamma$$

Proof. By the isomorphism (3.1), the left hand side is isomorphic to $[X, B\Gamma]$. A homotopy class of maps $X \to B\Gamma$ induces a homomorphism $\pi_1(X, x_0) \to \pi_1(B\Gamma, b) = \Gamma$, where $b \in BG$ is a basepoint. Changing this base point means conjugating with elements in Γ and this induces the map $\overline{\mathscr{C}}(X) \to \mathrm{Hom}(\pi_1(X, x_0), \Gamma)/\Gamma$.

Conversely, for a homomorphism $\phi : \pi_1(X, x_0) \to \Gamma$, let P_ϕ be the Γ-bundle given by

$$P_\phi := (\tilde{X} \times \Gamma)/\pi_1(X, x_0),$$

where $\pi_1(X, x_))$ acts on Γ via ϕ. The isomorphism class of this bundle does not change if we conjugate the homomorphism ϕ in Γ and therefore this defines a map $\mathrm{Hom}(\pi_1(X, x_0), \Gamma)/\Gamma \to \overline{\mathscr{C}}(C)$. We leave it to the reader to check that this is the inverse to the map defined above. □

4. The quantum invariant

Here we will define the so-called Dijkgraaf–Witten invariant of a d-dimensional manifold **[?]**. It is a quantum invariant defined by actually making sense of the partition function (1.1) in a specific case. The input data for this are:

- a finite group Γ
- a class $\hat{\alpha} \in H^d(B\Gamma, \mathbb{R}/\mathbb{Z})$ of degree d in group cohomology.

As explained in the first section, we need three definitions to make sense of the partition function:

1) The space of fields $\mathscr{C}(X)$, or rather its quotient $\overline{\mathscr{C}}(X)$.
2) A gauge-invariant action $S_X : \mathscr{C}(X) \to \mathbb{R}/\mathbb{Z}$
3) The measure μ on $\mathscr{C}(X)$.

To start with the first item of our list, $\mathscr{C}(X)$ will be the category of principal Γ-bundles over X. One might object that this is not a *space* of fields, i.e., they do even form a set, but $\mathscr{C}(X)$ is in a natural way a groupoid, and if we mod out the morphisms (which are thereby coined to be the gauge symmetries) we do end up with a set,

$$\overline{\mathscr{C}}(X) = \mathrm{Hom}(\pi_1(X, x_0), \Gamma)/\Gamma,$$

which is even a *finite* set.

Next, we will define the action S_X. Assume X to be closed and define

$$S_{X,\alpha}(P) := \langle [X], F_P^* \alpha \rangle \in \mathbb{T}. \tag{4.1}$$

Here $[X] \in H_d(X, \mathbb{T})$ is the fundamental class of X, constructed in **??**, and $F_P : X \to B\Gamma$ is a classifying map for the Γ-covering P.

Lemma 4.1. *S_X is well defined and invariant under gauge transformations.*

Proof. As we have seen F_P is uniquely defined up to homotopy. However, homotopic maps induce the same morphisms on cohomology, so $F_P^* \alpha = G_P^* \alpha$ in $H^{s+1}(X)$, for any two classifying maps F_P and G_P of P. Since the pairing with the fundamental class depends only on the cohomology class (so not the choice of cocycle), this shows that the action is well-defined. The same argument also shows that S_X is invariant under gauge symmetries, i.e., the morphisms in the groupoid $\mathscr{C}(X)$. □

The final ingredient is the measure on the space of fields $\mathscr{C}(X)$. Since in our case the quotient space $\overline{\mathscr{C}}(X)$ is a discrete space, in fact finite by Proposition 3.9, our measure will be essentially a multiple of the counting measure. Given a Γ-bundle $P \to X$, set

$$\mu(P) = \frac{1}{\#\mathrm{Aut}(P/X)}.$$

If P and P' are isomorphic in $\mathscr{C}(X)$, $\mathrm{Aut}(P) \cong \mathrm{Aut}(P')$ and therefore the measure passes to the quotient $\overline{\mathscr{C}}(X)$.

As remarked above, $\overline{\mathscr{C}}(X)$ is a finite set, therefore the partition function

$$Z(X, \alpha) = \int_{\overline{\mathscr{C}}(X)} e^{2\pi i S_X([P])} d\mu([P]),$$

where $S_{X,\alpha}([P])$ is the action (4.1), is well-defined.

Before extending this invariant to a full TQFT, let us see what exactly it measures. Suppose that $\alpha = 0$, then clearly $S_X = 0$ and we have

$$Z(X, 0) = \sum_{[P]} \frac{1}{\#\mathrm{Aut}(P/X)},$$

i.e., $Z(X, 0)$ gives the weighted number of Γ-bundles over X.

For nonzero α it is not so easy to give an interpretation of this kind to the invariant $Z(X, \alpha)$.