An Introduction to Differential Topology, de Rham Theory and Morse Theory

(Subject to permanent revision)

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June 2, 2005

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Chapter I

Why Differential Topology?

General topology arose by abstracting from the "usual spaces" of euclidean or noneuclidean geometry and defining more general notions of 'spaces'. One such generalization is that of a metric space. Abstracting further one is led to the very general concept of a topological space, which is just specific enough to talk about notions like neighborhoods, convergence and continuity. However, in order to prove non-trivial results one is immediately forced to define and impose additional properties that a topological space may or may not possess: the Hausdorff property, regularity, normality, first and second countability, compactness, local compactness, σ-compactness, paracompactness, metrizability, etc. (The book [29] considers 61 such attributes without being at all exhaustive.) This is not to say that there is anything wrong with general topology, but it is clear that one needs to consider more restrictive classes of spaces than those listed above in order for the intuition provided by more traditional notions of geometry to be of any use.

For this reason, general topology also introduces spaces that are made up in a specific way of components of a regular and well understood shape, like simplicial complexes and, more generally, CW-complexes. In particular the latter occupy a central position in homotopy theory and by implication in all of algebraic topology.

Another important notion considered in general topology is that of the dimension of a space as studied in dimension theory, one of the oldest branches of topology. In the case of a space X that is composed of simpler components X_i, one typically has $\dim X = \sup_i \dim X_i$. It is natural to ask for spaces which have a homogeneous notion of dimension, i.e. which all points have neighborhoods of the same dimension. This desirable property is captured in a precise way by the notion of a topological manifold, which will be given in our first definition.

However, for many purposes like those of analysis, topological manifolds are still not nice or regular enough. There is a special class of manifolds, the smooth ones, which with all justification can be called the nicest spaces considered in topology. (For example, real or complex algebraic varieties without singularities are smooth manifolds.) Smooth manifolds form the subject of differential topology, a branch of topology with a very distinct, at times very geometric and intuitive, flavor.

The importance of smooth manifolds is (at least) fourfold. To begin with, smooth manifolds are an extremely important (and beautiful) subject in themselves. Secondly, many interesting and important structures arise by equipping a smooth manifold with some additional structure, leading to Lie groups, riemannian, symplectic, Kähler or Poisson manifolds, etc.) Differential topology is as basic and fundamental for these fields as general topology is, e.g., for functional analysis and algebraic topology. Thirdly, even though many spaces encountered in practice are not smooth manifolds, the theory of the latter is a very natural point of departure towards generalizations. E. g., there is a topological approach to real and complex algebraic varieties with singularities, and there are the theories of manifolds with corners and of orbifolds (quotient spaces of smooth manifolds by non-free group actions), etc. A thorough understanding of the theory of smooth manifolds is necessary prerequisite for the study of these subjects. Finally, a solid study of the algebraic topology of manifolds is very useful to obtain an intuition for the more abstract and difficult algebraic topology of general spaces. (This is the philosophy behind the masterly book [4] on which we lean in Chapter 3 of these notes.)

We conclude with a very brief overview over the organization of these notes. In Chapter II we give an introduction to some of the basic concepts and results of differential topology. For the time being, suffice it to say that the most important *concept* of differential topology is that of transversality (or general position), which will pervade Sections IV.1-V.4. The three most important technical *tools* are the rank theorem, partitions of unity and Sard's theorem. In Chapter VII we define and study the cohomology theory of de Rham, which is the easiest way to approach the algebraic topology of manifolds. We will try to emphasize the connections with Chapter II as much as possible, based on notions like the degree, the Euler characteristic and vector bundles. Chapter IX is an introduction to a more advanced branch of differential topology: Morse theory. Its main idea is to study the (differential) topology of a manifold using the smooth functions living on it and their critical points. On the one hand, Morse theory is extremely important in the classification programme of manifolds. On the other hand, the flow associated with any Morse function can be used to define homology theory of manifolds in a very beautiful and natural way. We will also show that the dual Morse co-homology with $\mathbb{R}$-coefficients is naturally isomorphic to de Rham cohomology. In the final chapter we will briefly highlight the perspective on our subject matter afforded by the combinatorial approach of singular (co)homology theory and by analysis on manifolds, to wit Hodge theory.

Chapter II

Basics of Differentiable Manifolds

II.1 Topological and smooth manifolds

II.1.1 Topological manifolds

In these notes we will prove no results that belong to general (=set theoretic topology). The facts that we need (and many more) are contained in the first chapter (62 pages) of [6]. (This book also contains a good its introduction to differential topology.) For an equally beautiful and even more concise (40 pages) summary of general topology see Chapter 1 of [24].

We recall some definitions. 'Space' will always mean topological space. We recall some definitions.

II.1.1 Definition *A space M is locally euclidean if every $p \in M$ has an open neighborhood U for which there exists a homeomorphism $\phi : U \to V$ to some open $V \subset \mathbb{R}^n$, where V has the subspace topology.*

II.1.2 1. Note that the open subsets $U \subset M, V \subset \mathbb{R}^n$ and the homeomorphisms ϕ are not part of the structure. The requirement is only that for every $p \in M$ one can find U, V, ϕ as stated.

2. Let $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ be non-empty open sets admitting a homeomorphism $\phi : U \to V$. Then the 'invariance of domain' theorem from algebraic topology, cf. [6, Section IV.19], implies $m = n$. Thus the dimension n of a neighborhood of some point is well defined, and is easily seen to be locally constant. Thus every connected component of M has a well defined dimension. We will soon restrict ourselves to spaces where the dimension is the same for all connected components.

3. It is immediate that a locally euclidean space X inherits all local properties from $\mathbb{R}^n$. Thus (a) X is locally path connected, and therefore connected components and path components coincide. (b) X is locally simply connected, implying that every connected component of X has a universal covering space. (c) X is locally compact, i.e. every $p \in X$ has a compact neighborhood K. We quickly prove this. Let $\tilde{U} \ni p$ be open and small enough so that there exists a homeomorphism $\phi : \tilde{U} \to V$ with $V \subset \mathbb{R}^n$ open. Clearly V contains some open sphere $B(\phi(p), \varepsilon), \varepsilon > 0$. Now $K = \phi^{-1}(\overline{B(\phi(p), \varepsilon/2)})$ and $U = \phi^{-1}(B(\phi(p), \varepsilon/3))$ do the job. (d) M is first countable, i.e. every $p \in M$ has a countable neighborhood base.

II.1.3 Recall that a space X is Hausdorff if for every $p, q \in X$, $x \neq y$ there are open sets $U \ni p, V \ni q$ such that $U \cap V = \emptyset$. One might think that a locally euclidean space is automatically Hausdorff. That this is not true is exemplified by the space X that is constructed as follows. Let Y be the disjoint union of two copies of $\mathbb{R}$, realized as $Y = \mathbb{R} \times 0 \cup \mathbb{R} \times 1$. Now define an equivalence relation $\sim$ on Y by declaring $(x, 0) \sim (x, 1) \Leftrightarrow x \neq 0$. (Of course we also have $(x, i) \sim (x, i)$.) Let $X = Y/\sim$ with the quotient topology ($V \subset Y$ is open iff $f^{-1}(V)$ is open) and let $\pi : Y \to X$ be the quotient map. Write $p = \pi(0, 0), q = \pi(0, 1)$, and let $U \ni p, V \ni q$ be open neighborhoods. Then there exists (exercise!) $\varepsilon > 0$ such that $0 < |x| < \varepsilon$ implies $\pi(x, 0) = \pi(x, 1) \in U \cap V$. Thus X is non-Hausdorff.

II.1.4 Recall that a space X with topology τ is second countable if there exists a countable family $F \subset \tau$ of open sets such that every $U \in \tau$ is a union of sets in F. One can construct spaces that are Hausdorff and locally $\mathbb{R}^n$ but not second countable, e.g., the 'long line' which is locally 1-dimensional. The assumption of second countability mainly serves to deduce paracompactness, cf. Section II.10, which is needed for the construction of 'partitions of unity'. As we will see many times, the latter in turn is crucial for the passage from certain local to global constructions. For this reason we will not consider spaces that are more general than in the following definition.

II.1.5 Definition *A topological manifold of dimension $n \in \mathbb{N}$ (or n-manifold) is a second countable Hausdorff space M such that every $p \in M$ has an open neighborhood U such that there is a homeomorphism $\phi : U \to V$, where V is an open subset of $\mathbb{R}^n$.*

For later use we recall the following fact from general topology:

II.1.6 Proposition *Let X be a second countable space. Then every open cover $(U_i)_{i\in I}$ (i.e. the $U_i \subset X$ are open and $\cup_{i\in I} U_i = X$) admits a countable subcover, i.e. there is a countable subset $I_0 \subset I$ such that $\cup_{i\in I_0} U_i = X$.*

Proof. Let $(V_j, j \in J)$ be a countable basis for the topology, and recall that every open $U \subset X$ is the union of all the V_j contained in U. Define $J_i = \{j \in J \mid V_j \subset U_i\}$ and $J_0 = \cup_{i\in I} J_i$, and for every $j \in J_0$ pick a $s(j) \in I$ such that $V_j \subset U_{s(j)}$. Clearly $J_0 \subset J$ and $I_0 := s(J_0)$ are countable and we have

$$X = \bigcup_{i\in I} U_i = \bigcup_{i\in I} \bigcup_{j\in J_i} V_j = \bigcup_{j\in \cup_{i\in I} J_i} V_j = \bigcup_{j\in J_0} V_j \subset \bigcup_{j\in J_0} U_{s(j)} = \bigcup_{i\in I_0} U_i.$$

The converse inclusion being obvious, this proves that I_0 does the job. ∎

II.1.2 Differentiable manifolds and their maps

There is a highly developed theory of topological manifolds with many non-trivial results. For most of the applications in other areas of mathematics, however, one needs more structure, in particular in order to do analysis on M. This leads to the following notion.

II.1.7 Definition *Let M be a n-dimensional topological manifold. A chart (U, ϕ) consists of an open set $U \subset M$ and a continuous map $\phi : U \to \mathbb{R}^n$ such that $\phi(U)$ is open and $\phi : U \to \phi(U)$ is a homeomorphism. For $0 \le r \le \infty$, a C^r-atlas on a n-dimensional topological manifold consists of a family of charts (U_i, ϕ_i) such that the U_i cover M and such that the map*

$$\phi_j \circ \phi_i^{-1} : \quad \mathbb{R}^n \supset \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j) \subset \mathbb{R}^n$$

is r times continuously differentiable whenever $U_i \cap U_j \neq \emptyset$. A chart (U, ϕ) is compatible with a C^r-atlas $\mathcal{A} = \{(U_i, \phi_i)\}$ iff $\mathcal{A} \cup (U, \phi)$ is a C^r-atlas. Two C^r-atlasses $\mathcal{A}, \mathcal{A}'$ are compatible if the union $\mathcal{A} \cup \mathcal{A}'$ is a C^r-atlas. A maximal C^r-atlas is a C^r-atlas that cannot be enlarged by adding compatible charts.

II.1.8 Lemma *Every C^r-atlas $\mathcal{A}$ on a topological manifold M is contained in a unique maximal atlas, consisting of all charts that are compatible with $\mathcal{A}$. Two C^r-atlasses $\mathcal{A}$ and $\mathcal{A}'$ are equivalent iff they are contained in the same maximal atlas.*

Proof. Obvious. ∎

II.1.9 DEFINITION *A C^r-differential structure on a topological manifold M is given by specifying a maximal C^r-atlas on M or, equivalently, by giving an equivalence class of (not necessarily) maximal atlasses $[\mathcal{A}]$. A C^r-manifold is a pair $(M, [\mathcal{A}])$ consisting of a topological manifold and a C^r-differential structure on it.*

II.1.10 REMARK The notion of a differential manifold is not nearly as abstract as it may seem. In practice one does not work with maximal atlasses but rather with a single representant $\mathcal{A}$ of an equivalence class $[\mathcal{A}]$. One adds or removes compatible charts as is convenient. Whenever we speak of charts on a differential manifolds we mean charts that are compatible with a given differential structure! □

A morphism in the category of topological manifolds just is a continuous map. (Thus the topological manifolds form a full subcategory of the category of topological spaces and continuous maps.) For differentiable manifolds we need restrictions on the admissible maps:

II.1.11 DEFINITION *Let M, N be C^r-manifolds, $0 \leq r \leq \infty$, with atlasses (U_i, ϕ_i) and (V_j, ψ_j). Let $n \leq r$. A map $f : M \to N$ is C^n if the composite*

$$\psi_j \circ f \circ \phi_i^{-1} : \quad \mathbb{R}^m \supset \phi_i(U_i \cap f^{-1}(V_j)) \to \psi_j(V_j) \subset \mathbb{R}^n$$

is C^n whenever $f(U_i) \cap V_j$ is non-empty. The set of smooth maps from M to N is denoted by $C^\infty(M, N)$. For $C^\infty(M, \mathbb{R})$ we just write $C^\infty(M)$.

II.1.12 REMARK 1. Note that this is well defined since the transition maps $\phi_{i'}^{-1} \circ \phi_i$ and $\psi_{j'}^{-1} \circ \psi_j$ are C^r and $r \geq n$. Thus composing with them does not lead out of the class of C^n-functions.

2. Manifolds and maps that are C^∞ are called smooth.

3. Now that we are able to say what a C^r (smooth) map from M to $\mathbb{R}^n$ is we see that a chart (U, ϕ) of a C^r (smooth) manifold is just a C^r (smooth) diffeomorphism from U to an open subset of $\mathbb{R}^n$.

4. It is clear that a differential C^0-manifold is essentially the same as a topological manifold. It suffices to observe that given two charts $(U, \phi), (U', \phi')$, the map $\phi' \circ \phi^{-1} : \phi(U \cap U') \to \phi'(U \cap U')$ is automatically C^0. Thus any two C^0-atlasses on M are compatible and there is exactly one C^0-structure on M. Similarly, any continuous map between C^0-manifolds is C^0 in the sense of Definition II.1.11 □

II.1.13 DEFINITION *A C^n-diffeomorphism is a C^n-map $f : M \to N$ that has a C^n inverse. We write $\mathrm{Diff}(M)$ for the set of C^∞ diffeomorphisms $M \to M$. Clearly, this is a group with id_M as unit.*

II.1.14 LEMMA *Let M, N be C^r (smooth) manifolds. Then the product space $M \times N$ has a canonical C^r (smooth) structure such that the projections $\pi_1 : M \times N \to M$, $\pi_2 : M \times N \to N$ are C^r (smooth). $M \times N$ is called the product manifold.*

Proof. Let $\{(U_i, \phi_i)_{i \in I}\}, \{(V_j, \psi_j)_{j \in J}\}$ be atlasses for M, N, respectively. Then $\{(U_i \times V_j)_{(i,j) \in I \times J}\}$ is an open cover of $M \times N$ and the coordinate maps $\phi_i \times \psi_j : U_i \times V_j \to \mathbb{R}^{m+n}$ define an atlas. The easy verifications of compatibility and of smoothness of π_1, π_2 are omitted. ■

From the next section on we will exclusively consider smooth, i.e. C^∞-manifolds. Yet we think it would be inexcusable not to comment briefly on the extremely interesting relations between the categories of $C^0, C^r (r \in \mathbb{N})$ and C^∞-manifolds. If desired, the rest of this section can be ignored.

II.1.3 Remarks

The following result, proven e.g. in [13, Chapter 2], shows that there is no real reason to consider C^r-manifolds with $1 \leq r < \infty$:

II.1.15 Theorem *Let M be a C^r-manifold, where $r \geq 1$. There exists a C^∞-manifold $\tilde{M}$ and a C^r-diffeomorphism $\phi : M \to \tilde{M}$. If $\tilde{M}'$ is another C^∞-manifold that is C^r-diffeomorphic to M then there is a C^∞-diffeomorphism $\tilde{M} \to \tilde{M}'$.*

Thus every C^r-manifold ($r \geq 1$) can be smoothed in an essentially unique way. (An equivalent of way of putting this is: Every maximal C^r atlas contains a C^∞ atlas.) Yet for some applications it may still be necessary to consider C^r-maps ($r < \infty$) between C^∞-manifolds.

II.1.16 Remark It is very important to note that the above theorem is false for $r = 0$. We list some results that are relevant in this context. (Each of them is deeper than anything studied in these notes.)

1. There are topological (i.e. C^0-)manifolds that do not admit any smooth structure, cf. [42, 38].

2. There are topological manifolds that admit more than one differential structure. For example, Milnor discovered that the (topological) sphere S^7 admits inequivalent differential structures, and together with Kervaire he showed that there are 28, cf. [43]. Brieskorn [37] has given a relatively concrete representation of these manifolds: Consider the subset $X_k \subset \mathbb{C}^5$ defined by the equations
$$\begin{aligned} |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 &= 1, \\ z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6k-1} &= 0. \end{aligned}$$
The first equation is real and its solution set clearly is S^9, whereas the second equation has two real components. Brieskorn has shown that $X_k,\ k = 1, \ldots, 28$ is a (topological) 7-manifold homeomorphic to S^7 and that the differential structures induced from $\mathbb{C}^5 \cong \mathbb{R}^{10}$ correspond to the 28 possibilities classified in [43]. The proof requires non-trivial techniques both from algebraic topology and algebraic geometry.

3. In four dimensions, Donaldson [38] has shown that $\mathbb{R}^4$ (as topological manifold) admits smooth structures that are inequivalent to the usual ones. Smooth 4-spheres are still not completely understood.

4. None of the above can happen in dimensions 1,2,3: In these dimensions every topological manifold admits a unique smooth structure (and a unique piecewise linear structure or triangulation). Thus the homeomorphism classes of topological 1-manifolds are in bijective correspondence with the diffeomorphism classes of smooth 1-manifolds which we will classify in Theorem II.12.1. 2-manifolds have been classified, see e.g. [31, Chapter 2] in the topological and [13, Chapter 9] in the smooth category. The classification of 3-manifolds is still incomplete, but very recently (2002) there has been spectacular progress due to Perelman.

5. There is an interesting connection between differential topology and real algebraic geometry: Every compact connected C^∞-manifold is diffeomorphic to a connected component of a non-singular real algebraic variety. Cf. [45] and later work.

6. Complex manifolds are defined as real ones with two changes: 1. The charts take values in $\mathbb{C}^n$. 2. The transition functions $\phi_j^{-1} \circ \phi_i$ are assumed *complex differentiable* (i.e. real differentiable with $\mathbb{C}$-linear derivative maps), equivalently holomorphic. Thus a complex manifold is in particular a smooth real manifold of even (real) dimension, but only very few even dimensional manifolds admit a complex structure. For the basics of complex manifolds see [33] and for the close connections with complex algebraic geometry see [30, 11].

7. Finally, we mention that smooth manifolds can be considered over fields other than $\mathbb{R}$ or $\mathbb{C}$. For any complete normed field K (like $\mathbb{Q}_p$) we can consider complete normed K-vector spaces and differentiable maps between such. Then a manifold over K is defined as before, with charts taking values in K^n.

□

II.1.4 Quotient manifolds

So far our only examples of smooth manifolds are obvious examples $\mathbb{R}^n, S^n$ and direct products of manifolds. Quotient spaces of manifolds by equivalence relations often fail to be manifolds. (They needn't even be Hausdorff.) But there is a very useful special case:

II.1.17 Definition *An action of a (discrete) group G on a topological space X is a group homomorphism $\gamma : G \to \mathrm{Homeo}(X)$, $g \mapsto \gamma_g$, where $\mathrm{Homeo}(X)$ is the group of homeomorphisms of X. (When there is no risk of confusion between different actions we often write gx instead of $\gamma_g(x)$.) The action is called totally discontinuous if every $p \in X$ has an open neighborhood U such that $U \cap \gamma_g(U) = \emptyset$ for all $g \neq e$. If X is a smooth manifold M we replace $\mathrm{Homeo}(X)$ by the group $\mathrm{Diff}(M)$ of smooth diffeomorphisms.*

II.1.18 Definition *A map $f : X \to Y$ of topological spaces is a covering map if every $y \in Y$ has an open neighborhood such that*

$$f^{-1}(V) = \bigcup_{i \in I} U_i,$$

where the U_i, $i \in I$ are pairwise disjoint open sets and the restrictions $f_i : U_i \to V$, $i \in I$ are homeomorphisms. (Equivalently, $f^{-1}(V)$ is homeomorphic to $V \times I$, where I has the discrete topology.)

II.1.19 Exercise The cardinality of the index set I may depend on y. Show that it is constant on the connected components of Y. □

II.1.20 Definition *If a group G acts on a space X we write X/G for the quotient space $X/\sim$, where $x \sim y$ iff there is $g \in G$ such that $y = gx$. (Thus X/G is the orbit space of the action.)*

II.1.21 Lemma *Let X be a topological space and G a (discrete) group acting on X by homeomorphisms. Then the quotient map $f : X \to X/G$ is continuous and open. If the action of G is totally discontinuous then f is a covering map. If in addition G is finite and X is Hausdorff then X/G is Hausdorff.*

Proof. f is continuous by definition of the quotient topology. For $U \subset X$ define $\widehat{U} = \cup_{g \in G} gU$. Now, for any $U \subset X$ we have $f^{-1}(f(U)) = \widehat{U}$. If U is open then $\widehat{U}$ is open, implying that f is an open map.

Let $y \in X/G$ and pick $x \in X$ such that $f(x) = y$. Assuming the action of G to be totally discontinuous, there exists an open neighborhood $U \subset X$ of x such that $gU \cap hU = \emptyset$ whenever $g \neq h$. By the above, $V = f(U)$ is an open neighborhood of y and

$$f^{-1}(V) = \widehat{U} = \bigcup_{g \in G} gU.$$

Here the right hand side is a disjoint union of open sets. Since no two elements of U are in the same G-orbit, the restricted map $f_g : gU \to V$ is injective. Surjectivity and continuity are trivial and openness has been proven above. Thus $f_g : gU \to V$ is a homeomorphism for every $g \in G$, and f is a covering map.

Assume now that X is Hausdorff, G is finite and acts discontinuously. Consider $x, y \in X/G,\ x \neq y$. Let $a, b \in X$ such that $f(a) = x, f(b) = y$. Since $x \neq y$, we have $ga \neq b$ for all $g \in G$. Since X is Hausdorff we thus have open neighborhoods $U_g \ni ga$ and $V_g \ni b$ such that $U_g \cap V_g = \emptyset$ for all $g \in G$. Then $U = \cap_g g^{-1} U_g$ and $V = \cap_g V_g$ are open (here we need finiteness of G) neighborhoods of a, b, respectively, and satisfy $hU \cap V \subset U_h \cap V_h = \emptyset$ for all $h \in G$. Thus $f(U)$ and $f(V)$ are disjoint open neighborhoods of x and y, respectively. This proves that X/G is Hausdorff. ■

II.1.22 Remark The conclusion that X/G is Hausdorff also holds for infinite G if one assumes that X is locally compact Hausdorff and that the action of G is totally discontinuous and proper, i.e. for compact $K, L \subset X$ the set $\{g \in G \mid gK \cap L \neq \emptyset\}$ is finite. □

II.1.23 Proposition *Let γ be a totally discontinuous action of a finite group G on a manifold M. Then the quotient map $f : M \to M/G$ is a covering map and M/G has a natural smooth structure w.r.t. which f is smooth.*

Proof. By the lemma, M is Hausdorff and f is open. Second countability clearly is inherited from M. Let $\mathcal{A}$ be a maximal atlas of M. We define an atlas $\mathcal{A}_G$ as given by

$$\mathcal{A}_G = \{(f(U), \phi \circ f^{-1})\},$$

where we consider those charts (U, ϕ) of $\mathcal{A}$ such that $f(U) \subset M/G$ satisfies the condition in Definition II.1.18 and U is one of the components of $f^{-1}(f(U))$. Then f^{-1} denotes the inverse of the homeomorphism $f : U \to f(U)$. Since $f : M \to M/G$ is a covering map, these sets $f(U)$ clearly cover M/G. Overlapping charts of M/G are now of the form $(f(U), \Phi = \phi \circ f^{-1}), (f(V), \Psi = \psi \circ f^{-1}) \in \mathcal{A}_G$ with $(U, \phi), (V, \psi) \in \mathcal{A}$ and $f(U) \cap f(V) \neq \emptyset$. Then $W = f(U) \cap f(V)$ is open and $f^{-1}(W) = \cup_{i \in I} W_i$, and there are unique $j, k \in I$ such that $W_j \subset U$ and $W_k \subset V$. On the domain $\Psi(f(U) \cap f(V))$ we have

$$\Psi \circ \Phi^{-1} = \phi \circ f_k^{-1} \circ f \circ \psi^{-1},$$

where f_k^{-1} is the inverse of $f : W_k \to W$. By the above, the action of G permutes the $W_i's$ transitively, thus there exists $g \in G$ such that $f_k^{-1} \circ f = \gamma_g$. Since γ_g is a (smooth) diffeomorphism, $\phi \circ \gamma_g \circ \psi^{-1}$ is a smooth map between open sets in $\mathbb{R}^n$. Thus the charts $(f(U), \phi \circ f^{-1}), (f(V), \psi \circ f^{-1})$ are compatible and $\mathcal{A}_G$ is an atlas. That f is smooth w.r.t. $\mathcal{A}, \mathcal{A}_G$ is obvious. ■

II.1.24 Definition *For $n \in \mathbb{N}$, the real projective space $\mathbb{R}P^n$ is the quotient space $(\mathbb{R}^{n+1} - \{0\})/\sim$, where $x, y \in \mathbb{R}^{n+1} - \{0\}$ are equivalent iff there is $\lambda \in \mathbb{R}^*$ such that $y = \lambda x$.*

II.1.25 Exercise Show that S^n can be considered as the quotient space $(\mathbb{R}^{n+1} - \{0\})/\sim$, where $x, y \in \mathbb{R}^{n+1} - \{0\}$ are equivalent iff there is $\lambda > 0$ such that $y = \lambda x$. Conclude that $\mathbb{R}P^n \cong S^n/\pm$. Equivalently, $\mathbb{R}P^n \cong S^n/\mathbb{Z}_2$, where the non-trivial element of $\mathbb{Z}_2 = \{e, g\}$ acts by $gx = -x$. Show that the proposition can be applied to conclude that $\mathbb{R}P^n$ is a manifold. □

II.2 The tangent space

II.2.1 The tangent space according to the geometer and the physicist

If a smooth n-manifold M is given as a submanifold of some euclidean space $\mathbb{R}^N$ (the precise meaning of submanifolds will be defined later) one can imagine, at every point $p \in M$, a plane tangent to M. This tangent plane can be considered as n-dimensional vector space. (Translating it such that p arrives at $0 \in \mathbb{R}^N$ we obtain a sub-vector space of $\mathbb{R}^N$.) The aim of this section is to give an intrinsic definition of the tangent space at a point p, independent of any embedding of M into euclidean space.

In fact, we will consider three different but equivalent definitions, following [7]. All three definitions, which we denote $T_p^G M, T_p^P M, T_p^A M$ until we have proven their equivalence, appear very frequently in the literature and their comparison is quite instructive.

II.2.1 Definition *Let $p \in M$. A chart (U, ϕ) such that $p \in U$ and $\phi(p) = 0$ will be called a chart around p.*

II.2.2 Definition (of the geometer) *A germ of a function at p is a pair (V, h) where $V \subset M$ is an open set containing p and $h : V \to \mathbb{R}$ is a smooth map. A germ of a curve through p is a pair (U, c) where $U \subset \mathbb{R}$ is an open set containing 0 and $c : U \to M$ is a smooth map satisfying $c(0) = p$. We define a pairing between germs of curves and germs of functions by*

$$\langle (U, c), (V, h) \rangle = \frac{d}{dt} h(c(t))_{|t=0}.$$

(This is well defined since $c(U) \cap V$ contains some neighborhood of p.) We define an equivalence relation on the germs of curves through p by

$$(U, c) \simeq (U', c') \ \Leftrightarrow \ \langle (U, c), (V, h) \rangle = \langle (U', c'), (V, h) \rangle \quad \text{for all germs } (V, h) \text{ of functions at } p.$$

Now we define $T_p^G M = \{\text{germs of curves through } p\}/\sim$. Such equivalence classes will be denoted $[c]$, dropping the inessential neighborhood U.

In order to elucidate the structure of $T_p^G M$, consider a chart $\Phi = (U, \phi)$ around p. In view of $h \circ c = (h \circ \phi^{-1}) \circ (\phi \circ c)$ (valid in a neighborhood of 0) we have

$$\langle c, h \rangle = \frac{d}{dt} h(c(t))_{|t=0} = \sum_{i=1}^{n} \frac{\partial (h \circ \phi^{-1}(x_1, \ldots, x_n))}{\partial x_i}_{x_1 = \cdots = x_n = 0} \frac{d(\phi_i(c(t)))}{dt}_{|t=0}. \tag{II.1}$$

Two germs c, c' of curves through p therefore define the same element of $T_p^G M$ iff $d(\phi_i(c(t)))/dt_{|t=0} = d(\phi_i(c'(t)))/dt_{|t=0}$ for $i = 1, \ldots, n$. Thus the map $T_p^G M \to \mathbb{R}^n$ given by $[c] \mapsto (d(\phi_i(c(t)))/dt_{|t=0})$ is injective. On the other hand, for every $v \in \mathbb{R}^n$ there is a germ of a curve through p defined by $c(t) = \phi^{-1}(tv)$ on some neighborhood of $0 \in \mathbb{R}$. Obviously, $d(\phi_i(c(t)))/dt_{|t=0} = v_i$, and therefore the map $T_p^G M \to \mathbb{R}^n$ is surjective, thus a bijection. This bijection can be used to transfer the linear structure of $\mathbb{R}^n$ to $T_p^G M$, and in particular it shows that $\dim_{\mathbb{R}} T_p^G M = n$. It remains to show that the linear structure is independent of the chart Φ we used. Let $\Phi' = (U', \phi')$ be another chart around p. Then we have $\phi'_i \circ c = \phi'_i \circ \phi^{-1} \circ \phi \circ c$ and thus

$$\frac{d(\phi'_j(c(t)))}{dt}_{|t=0} = \sum_{i=1}^{n} \frac{\partial (\phi'_j \circ \phi^{-1}(x_1, \ldots, x_n))}{\partial x_i}_{x_1 = \cdots = x_n = 0} \frac{d(\phi_i(c(t)))}{dt}_{|t=0}$$

This computation motivates the definition:

II.2.3 Definition (of the physicist) *Let M be a manifold of dimension n and let $p \in M$. Consider pairs (Φ, v), where $\Phi = (U, \phi)$ is a chart around p and $v \in \mathbb{R}^n$. Two such pairs $(\Phi, v), (\Phi', v')$ are declared equivalent if*

$$v'_j = \sum_{i=1}^{n} v_i \left. \frac{\partial (\phi'_j \circ \phi^{-1}(x_1, \ldots, x_n))}{\partial x_i} \right|_{x_1 = \ldots = x_n = 0}, \qquad j = 1, \ldots, n.$$

The set of equivalence classes $[\Phi, v]$ is called the tangent space $T_p^P M$ of M at p. We define a vector space structure on $T_p^P M$ by $a[\Phi, v] + b[\Phi, v'] = [\Phi, av + bv']$ for $a, b \in \mathbb{R}$. (This definition makes sense since by definition of $\sim$ given two charts Φ, Φ' around p and $v \in \mathbb{R}^n$ there exists a unique $v' \in \mathbb{R}^n$ such that $[\Phi, v] \sim [\Phi', v']$.)

The isomorphism $\alpha_p^M : T_p^G M \to T_p^P M$ is now given by $[c] \mapsto [\Phi, v]$, where $\Phi = (U, \phi)$ is any chart around p and $v = (d(\phi_i(c(t)))/dt_{|t=0})$. From now on we identify $T_p^G M$ and $T_p^P M$ and omit the superscript.

II.2.4 Remark Given a chart (U, ϕ) around p, a basis of T_pM is given by the symbols $\partial/\partial x_1, \dots, \partial/\partial x_n$. This is to be interpreted in the sense of

$$\left\langle \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}, h \right\rangle = \sum_{i=1}^{n} \alpha_i \, \frac{\partial(h \circ \phi^{-1}(x_1, \dots, x_n))}{\partial x_i}, \qquad \alpha_1, \dots, \alpha_n \in \mathbb{R}.$$

□

II.2.2 The tangent space according to the algebraist

Being manifestly independent of coordinate charts, the 'geometer's' definition is conceptually more satisfactory than the 'physicist's', but we needed the latter to identify the vector space structure on T_pM. We now show how this can actually be done in an intrinsic albeit less intuitive way. The considerations of this subsection will not be used later.

II.2.5 Definition (of the algebraist) *Let $(V_1, h_1), (V_2, h_2)$ be germs of functions at $p \in M$. Defining*

$$\begin{aligned} a(V_1, h_1) + b(V_2, h_2) &= (V_1 \cap V_2, ah_1 + bh_2), \\ (V_1, h_1) \cdot (V_2, h_2) &= (V_1 \cap V_2, h_1 h_2), \end{aligned}$$

we turn the set of germs of functions at p into an $\mathbb{R}$-algebra A_pM. A derivation of A_pM is a map $\partial : A_pM \to \mathbb{R}$ that is $\mathbb{R}$-linear (i.e. $\partial(ax + by) = a\partial x + b\partial y$ for $x, y \in A_pM$ and $a, b \in \mathbb{R}$) such that $\partial(xy) = y(p)\partial x + x(p)\partial y$ for all $x, y \in A_pM$. We denote the set of derivations of A_pM by $D(A_pM)$ and turn it into an $\mathbb{R}$-vector space by $(a\partial + b\partial')(x) = a\partial x + b\partial' x$ for $a, b \in \mathbb{R}$, $\partial, \partial' \in D(A_pM)$ and $x \in A_pM$. (Clearly, $D(A_pM)$ is a subspace of $(A_pM)^*$.)

II.2.6 Remark Note that the notion of derivation used in differential topology differs from the usual one in algebra and functional analysis. (By a derivation of a, not necessarily commutative, k-algebra A one usually means a k-linear map $\partial : A \to A$ such that $\partial(xy) = x\partial(y) + \partial(x)y$.) □

In our definition of the geometer's tangent space $T_p^G M$ in terms of equivalence classes of germs of curves through p we have considered a pairing $\langle c, h \rangle$ between germs of curves and germs of functions. In view of (II.1), and writing h_1 instead of (V_1, h_1) etc., it is clear that

$$\langle c, ah_1 + bh_2 \rangle = a\langle c, h_1 \rangle + b\langle c, h_2 \rangle,$$

$$\langle c, h_1 h_2 \rangle = h_1(p)\langle c, h_2 \rangle + h_2(p)\langle c, h_1 \rangle,$$

thus $\langle c, \cdot \rangle : h \mapsto \langle c, h \rangle$ is a derivation on A_pM. In this we way get an injective map $T_pM \to D(A_pM)$. The following lemmas will show that this map is an isomorphism, allowing to consider $D(A_pM)$ as an alternative definition of the tangent space. The latter might be denoted $T_p^A M$, the algebraist's version of the tangent space.

II.2.7 Lemma *Let $A_p^0 M \subset A_pM$ be the ideal of (germs of) functions vanishing at p. Then $(A_p^0)^2 \equiv \{\sum_i a_i b_i,\ a_i, b_j \in A_p^0 M\}$ (finite sums of products of two elements of $A_p^0 M$) is an ideal in $A_p^0 M$. Then*

(a) $\partial \in D(A_pM) \subset (A_p^0 M)^*$ vanishes on $(A_p^0 M)^2$, thus defines an element of $(A_p^0 M/(A_p^0 M)^2)^*$.

(b) *Let $\varphi \in (A_p^0M/(A_p^0M)^2)^*$. Then the map $h \mapsto \varphi([h-h(p)1])$ is in $D(A_pM)$, and the maps between $D(A_pM)$ and $(A_p^0M/(A_p^0M)^2)^*$ thus obtained are mutually inverse.*

Proof. Ad (a) If $h, h' \in A_p^0M$ and $\partial \in D(A_pM)$ then $\partial(hh') = h(p)\partial h' + h'(p)\partial h = 0$. Thus $(A_p^0M)^2 \subset \ker \partial$ and therefore $\partial \in (A_p^0M/(A_p^0M)^2)^*$. Ad (b), given $\varphi \in (A_p^0M/(A_p^0M)^2)^*$ we define $\partial h = \varphi([h-h(p)1])$ for any germ $h \in A_pM$. Here 1 is the constant function and $[\cdots]$ means the coset in $A_p^0M/(A_p^0M)^2$. Clearly ∂ is $\mathbb{R}$-linear and it remains to show the derivation property. We have

$$hh' - h(p)h'(p)1 = h(p)(h' - h'(p)1) + h'(p)(h - h(p)1) + (h - h(p)1)(h' - h'(p)1)$$

in A_p^0M. Since $(h-h(p)1)(h'-h'(p)1) \in (A_p^0M)^2$ we have

$$[hh' - h(p)h'(p)1] = [h(p)(h' - h'(p)1) + h'(p)(h - h(p)1)]$$

in $A_p^0M/(A_p^0M)^2$. Applying φ and using $\mathbb{R}$-linearity we have

$$\partial(hh') = \varphi([hh' - h(p)h'(p)1]) = h(p)\varphi([h' - h'(p)1]) + h'(p)\varphi([h - h(p)1]) = h(p)\partial h' + h'(p)\partial h,$$

thus $\partial : h \mapsto \varphi([h-h(p)1])$ is a derivation on A_pM. ■

II.2.8 Exercise Complete the proof by showing that the above maps between $(A_p^0M/(A_p^0M)^2)^*$ and $D(A_pM)$ are mutually inverse. □

We cite the following lemma from analysis without proof.

II.2.9 Lemma *Let $h : U \to \mathbb{R}$ be a C^2-function on a convex open set $U \subset \mathbb{R}^n$. Then*

$$h(q) = h(p) + \sum_i (q_i - p_i)\frac{\partial h}{\partial x_i}_{|x=p} + \sum_{i,j}(q_i - p_i)(q_j - p_j)\int_0^1 (1-t)\frac{\partial^2 h}{\partial x_i \partial x_j}_{|x=p+t(q-p)} dt \qquad \text{(II.2)}$$

for all $p, q \in U$. If h is smooth then the term with the integral is smooth as a function of p.

II.2.10 Lemma *With the above notation, $T_pM \cong (A_p^0M/(A_p^0M)^2)^*$.*

Proof. Let (U, ϕ) be a chart around $p \in M$ and $(V, h) \in A_p^0M$ be a germ vanishing at p. Applying Lemma II.2.10 to $h \circ \phi^{-1}$ and observing that $\phi(p) = 0$ we obtain

$$h(q) = \sum_i \phi_i(q)\frac{\partial(h \circ \phi^{-1}(x))}{\partial x_i}_{|x=0} + \sum_{i,j}\phi_i(q)\phi_j(q)\int_0^1 (1-t)\frac{\partial^2(h \circ \phi^{-1}(x))}{\partial x_i \partial x_j}_{|x=t\phi(q)} dt$$

in some neighborhood of p. Since $\phi_i(\cdot) \in A_p^0M$ for all i and the integral is smooth, the last summand is in $(A_p^0)^2$, thus

$$h(q) \equiv \sum_i \phi_i(q)\frac{\partial(h \circ \phi^{-1}(x))}{\partial x_i}_{|x=0} \quad (\bmod (A_p^0)^2).$$

This means that the algebra $A_p^0M/(A_p^0M)^2$ is spanned by the (classes of the) coordinate functions $[\phi_i(\cdot)]$, $i = 1, \ldots, n$. (Thus $\dim_{\mathbb{R}}(A_p^0M/(A_p^0M)^2) \leq \dim M$.) Let $\partial_1, \ldots, \partial_n$ be the basis of T_pM associated with the chart (U, ϕ) as in Remark II.2.4. Then

$$\langle \partial_i, \phi_j \rangle = \frac{\partial \phi_j(\phi^{-1}(x_1, \ldots, x_n))}{\partial x_i} = \frac{\partial x_j}{\partial x_i} = \delta_{i,j}. \qquad \text{(II.3)}$$

This proves that the $[\phi_i(\cdot)] \in A_p^0M/(A_p^0M)^2$ are linearly independent. (Let $\sum_i c_i[\phi_i(\cdot)] = 0$. Then $c_i = \langle \partial_i, \sum_j c_j[\phi_j(\cdot)]\rangle = 0$ for all i.) Furthermore, in view of (II.3), T_pM is the dual space of $A_p^0M/(A_p^0M)^2$, concluding the proof. ■

Putting the two lemmas together we obtain

II.2.11 Proposition *The map $T_pM \to D(A_pM)$ given by $[c] \mapsto \langle c, \cdot \rangle$ is an isomorphism of vector spaces.*

II.2.12 Remark 1. One can show that for non-smooth C^r manifolds, the quotients $A_p^0M/(A_p^0M)^2$ typically are infinite dimensional, thus the isomorphism with T_pM breaks down.

2. In algebraic geometry, one defines A_pM (A_p^0M) as the algebra of germs of 'regular functions' defined near p (and vanishing at p). The analogue of Lemma II.2.7 holds, thus one can *define* the tangent space T_pM to be either $D(A_pM)$ or $(A_p^0M/(A_p^0M)^2)^*$. The above considerations show that in the case of a smooth algebraic variety over $\mathbb{R}$ this definition is consistent with our earlier (geometrical and 'physical') ones.

3. The dual vector space $T_p^*M := (T_pM)^*$, called the cotangent space, will play an important rôle in the theory of differential forms, cf. Chapter VII. □

We summarize: The geometer's definition is probably the most intuitive one, but it does not give the linear structure. Furthermore, it is the least suited for manifolds with boundary (cf. Section II.6). The algebraist's approach is somewhat unintuitive but conceptually the nicest, and it is the way the tangent space is defined in algebraic geometry. It has the disadvantage of breaking down for non-smooth C^r-manifolds. The 'physicist' approach is the least elegant but it works in all situations, including non-smooth manifolds and manifolds with boundary.

II.3 The differential of a smooth map

The 'first derivative' or 'differential' of a smooth map $f : M \to N$ should be a collection of linear maps $T_pf : T_pM \to T_{f(p)}N$ of the tangent spaces for all $p \in M$. (Instead of T_pf one often writes f_*, but we will try to stick to T_pf.) According to the chosen definition of the tangent spaces there are different but equivalent definitions of T_pf.

II.3.1 Definition (Geometer) *Let $f : M \to N$ be smooth manifolds. Define $T_p^Gf : T_p^GM \to T_{f(p)}^GN$ by*

$$T_p^Gf : [c] \mapsto [f \circ c].$$

II.3.2 Exercise Show that this is well defined. □

II.3.3 Definition (Physicist) *Let $f : M \to N$ be smooth manifolds of dimensions m, n. Let $\Phi = (U, \phi)$ and $\Psi = (V, \psi)$ be charts around p and $f(p)$, respectively. For $[\Phi, v] \in T_p^PM$ we define $T_p^Pf([\Phi, v]) = [\Psi, v']$ where $v' \in \mathbb{R}^n$ is given by*

$$v'_j = \sum_{i=1}^m v_i \, \frac{\partial(\psi_j \circ f \circ \phi^{-1}(x_1, \ldots, x_m))}{\partial x_i}_{|x=0}, \quad j = 1, \ldots, n.$$

In Section II.2 we have found isomorphisms $\alpha_p^M : T_p^GM \to T_p^PM$ between the two different definitions of the tangent space of M at p. Now that we also have induced maps $T_p^Gf : T_p^GM \to T_{f(p)}^GN$ and $T_p^Pf : T_p^PM \to T_{f(p)}^PN$, their compatibility becomes an issue. The precise answer is given by the following

II.3.4 Exercise Consider the map $\alpha_p^M : T_p^GM \to T_p^PM$ given by $[c] \mapsto [\Phi, v]$, where $\Phi = (U, \phi)$ is a chart around p and $v = (d(\phi_i(c(t)))/dt_{t=0})$, as discussed in the previous section. Show that α_p is a natural transformation, i.e. the diagram

$$\begin{tikzcd}
T_p^G M \arrow[r, "T_f^G M"] \arrow[d, "\alpha_p^M"'] & T^G_{f(p)} N \arrow[d, "\alpha^N_{f(p)}"] \\
T_p^P M \arrow[r, "T_f^P M"'] & T^G_{f(p)} N
\end{tikzcd}$$

commutes for every smooth map $f : M \to N$ and every $p \in M$. □

II.3.5 Lemma *Let $f : M \to N,\ g : N \to P$ be smooth maps. Then the differentials $T_p f, T_{f(p)} g, T_p(g \circ f)$ satisfy the 'chain rule' $T_p(g \circ f) = T_{f(p)} g \circ T_p f$ as linear maps $T_p M \to T_{g \circ f(p)} P$.*

Proof. Obvious, e.g., in the geometer's definition of the differential. ■

II.3.6 Exercise Let $f : M \to N$ be a diffeomorphism. Then $T_p f : T_p M \to T_{f(p)} N$ is a linear isomorphism for every $p \in M$. □

A very important rôle in differential topology is played by the inverse function theorem:

II.3.7 Theorem (Inverse function theorem) *Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^n$ a C^r-function where $r \in \{1, 2, \ldots, \infty\}$. If $p \in U$ and $T_p f : \mathbb{R}^n \to \mathbb{R}^n$ is invertible (equivalently, the matrix $(\partial f_i / \partial x_j)_{x=p}$ is invertible) then there is an open $V \subset U$ such that $f : V \to f(V)$ is a bijection with C^r inverse function.*

For a proof see, e.g., [25] (where it is proven only for $r = 1$) or [6, Section II.1] (all $r \in \mathbb{N} \cup \{\infty\}$). One can give 'elementary' proofs using only classical differential and integral calculus and induction on the dimension n, but it has become standard to apply the Banach fixpoint theorem. The latter proof has the advantage of working also in infinite dimensions.

II.3.8 Corollary *Let $f : M \to N$ be smooth and $T_p f : T_p M \to T_{f(p)} N$ invertible for some $p \in M$. Then there exists an open neighborhood $U \ni p$ such that $f(U)$ is open and $f : U \to f(U)$ is a diffeomorphism.*

Proof. Let $(U', \phi), (V, \psi)$ be charts around p and $f(p)$. Apply the inverse function theorem to $\psi \circ f \circ \phi^{-1}$ and conclude the claim for some $U \subset U'$. ■

We recall that a differentiable homeomorphism $f : M \to N$ need not have a differentiable inverse, e.g. $x \mapsto x^3$. The preceding corollary allows to exclude this nuissance at least locally (and Exercise II.3.6 shows that invertibility of $T_p f$ is also necessary). Note that a map need not be globally invertible even iff $T_p f$ is invertible everywhere: Consider $f : \mathbb{C} \to \mathbb{C},\ x \mapsto e^x$. We will later return to the problem of proving that a map f is globally a diffeomorphism.

II.4 The Tangent Bundle and Vector Fields

In this section we introduce a formal construction whose importance will become clear later. Let M be a manifold of dimension m and consider the disjoint union

$$TM = \coprod_{p \in M} T_p M,$$

which is called the *tangent bundle* of M. Its elements are denoted (p, v), where $p \in M$ and $v \in T_p M$. There is a canonical surjection $\pi : TM \to M,\ (p, v) \mapsto p$

II.4.1 Proposition *The tangent bundle TM admits the structure of a manifold of dimension $2m$ such that the following holds: For every $p \in M$ there is a neighborhood U and a diffeomorphism $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^m$ such that the diagram*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\psi"] \arrow[dr, "\pi"'] & U \times \mathbb{R}^m \arrow[d, "p_1"] \\ & U \end{tikzcd}$$

commutes and such that for each $p \in M$ the map $\pi^{-1}(p) = T_pM \to \{x\} \times \mathbb{R}^m$ is an isomorphism of vector spaces.

Proof. Let $(U_i, \phi_i)_{i \in I}$ be an atlas of M. We define an atlas $(V_i, \psi_i)_{i\in I}$ of TM by

$$V_i = \pi^{-1}(U_i) = \coprod_{p \in U_i} T_pM,$$

the coordinate maps $\psi_i : V_i \to \mathbb{R}^{2m}$ being given by

$$\psi_i(p, v) = (\phi_i(p), T_p\phi_i(v)).$$

(We use the canonical isomorphism $T_x\mathbb{R}^n \cong \mathbb{R}^n$.) For overlapping charts and $(x, u) \in \psi_i(V_i)$ we have

$$\psi_j \circ \psi_i^{-1}(x, u) = (\phi_j \circ \phi_i^{-1}(x), T_p\phi_j \circ (T_p\phi_i)^{-1}(u)) = (\phi_j \circ \phi_i^{-1}(x), D_xu),$$

where $D_x = T_x(\phi_j \circ \phi_i^{-1}) = (\partial(\phi_j \circ \phi^{-1}(x_1, \dots, x_n))/\partial x_i)$. Since the matrix D_x depends smoothly on x, $\psi_j \circ \psi_i^{-1}$ is a smooth map, and thus $(V_i, \psi_i)_{i\in I}$ defines a manifold structure on TM. The rest is now obvious: For $p \in M$, let U be the domain U_i of a chart containing p. Then the coordinate map $\psi_i : V_i = \pi^{-1}(U_i) \to U \times \mathbb{R}^m$ is the diffeomorphism whose existence is claimed in the proposition. ■

The differentials T_pf for $p \in M$ combine to a map between the tangent bundles:

II.4.2 Proposition *Let $f : M \to N$ be a smooth map. Define a map $Tf : TM \to TN$ by $TM((p, v)) = (f(p), T_pf(v))$. Then Tf is smooth and the diagram*

$$\begin{tikzcd} T_pM \arrow[r, hook, "\iota"] \arrow[d, "T_pf"'] & TM \arrow[r, "\pi"] \arrow[d, "Tf"] & M \arrow[d, "f"] \\ T_{f(p)}N \arrow[r, hook, "\iota"'] & TN \arrow[r, "\pi"'] & N \end{tikzcd}$$

commutes, where $\iota : T_pM \to TM$ is given by $v \mapsto (p, v)$.

Proof. That Tf is a smooth map is immediate by definition of the tangent bundle. Commutativity of the diagram is trivial. ■

II.4.3 Definition *A vector field on M is a section of the tangent bundle TM, to wit a smooth map $v : M \to TM$ such that $\pi \circ v = \mathrm{id}_M$. Thus, to every $p \in M$ we assign a tangent vector $v(p) \in T_pM$, and this is done in a smooth way. The set of all vector fields on M is denoted by $\Gamma(TM)$.*

II.4.4 Remark A vector field on an open set $\Omega \subset \mathbb{R}^n$ (considered as a manifold with one chart (Ω, id)) is just a smooth map $v : \Omega \to \mathbb{R}^n$. But this shouldn't lead one to forget that $v(p)$ is an element of T_pM and behaves accordingly under smooth maps and changes of charts. □

II.4.5 Exercise As noted before, a chart (U, ϕ) around p gives rise to bases $\{\partial/\partial x_i, i = 1, \ldots, n\}$ of T_pM for all $p \in U$, thus the (restriction to U of the) vector field can be written as

$$v(p) = \sum_{i=1}^n v_i(p)\, \partial/\partial x_i.$$

Then smoothness of $v : M \to TM$ is just smoothness of the $\mathbb{R}$-valued functions $v_1, \ldots, v_n$ w.r.t. any chart (U, ϕ). □

II.4.6 Exercise (a) In the definition of the tangent space T_pM, the pairing $\langle c, h\rangle$ between (germs of) curves through p and functions at p was crucial. Given a map (not assumed continuous) $v : M \to TM$ satisfying $\pi \circ v = \mathrm{id}_M$ and a function $f \in C^\infty(M)$ we obtain a function $\langle v, f\rangle : M \to \mathbb{R}$, $p \mapsto \langle v(p), f\rangle$. Show that $v : M \to TM$ is smooth iff $\langle v, f\rangle \in C^\infty(M)$ for every $f \in C^\infty(M)$.

(b) If this is the case then $f \mapsto \langle v, f\rangle$ is a derivation (in the usual sense) of the algebra $C^\infty(M)$, i.e. a linear map $\partial C^\infty(M) \to C^\infty(M)$ satisfying $\partial(fg) = f\partial(g) + g\partial(f)$. □

II.4.7 Remark The tangent bundle of a manifold plays a fundamental rôle in the Lagrangian formulation of classical mechanics, see [2]. In the latter, the set of possible positions of the N particles under consideration (the configuration space) constitutes a smooth manifold M. Then the state space of the system is just TM, consisting of the positions of the N particles and their velocities. Now the dynamics of the system is determined by a smooth function $L : TM \to \mathbb{R}$, the Lagrangian function. Provided that L satisfies a certain technical condition, the Lagrangian equations define a vector field $v : TM \to TTM$, and the flow obtained by integrating the latter (see the next section) describes the time development $\mathbb{R} \times TM \to TM$ of the system. □

II.5 Vector fields, Flows and Diffeomorphism Groups

In this section we consider the relation between vector fields on a manifold and flows, to be defined soon. To begin with, let $I \subset \mathbb{R}$ be connected and consider a smooth curve $c : I \to M$. For every $t \in I$, the latter has a velocity $T_tc(1) \in T_{c(t)}M$, where we have used the canonical identification $T_t\mathbb{R} \cong \mathbb{R}$. If c is injective, this defines a unique element of T_pM for every $p \in c(I)$. However, $c(I)$ only is a one-dimensional subset of M (at this point we don't make this precise). One way to obtain a vector field defined on all of M is the following:

II.5.1 Lemma *Let M be a (smooth) manifold and $\Lambda : \mathbb{R} \times M \to M$ a smooth map such that $\Lambda(0, p) = p$ for all $p \in M$. Then $\mathbb{R} \to M, t \mapsto \Lambda(t, p)$ is a curve through p and the map $v_\Lambda : M \to TM$ defined by*

$$v_\Lambda : p \mapsto [\Lambda(t, p)] \in T_p^{(G)}M$$

is a smooth vector field.

Proof. Consider the case $M = \mathbb{R}^n$. Then $v(p) \in T_p\mathbb{R}^n \cong \mathbb{R}^n$ is given by

$$v(p) = \frac{\partial \Lambda}{\partial t}(0, p).$$

Since Λ is smooth, $v(p)$ clearly is smooth, too. In the general case, let (U, ϕ) be a chart around p. Then $\widetilde{\Lambda} := \phi \circ \Lambda \circ (\mathrm{id} \times \phi^{-1})$ maps a neighborhood of $0 \times \phi(p) \in \mathbb{R} \times \mathbb{R}^n$ to $\phi(U) \subset \mathbb{R}^n$. Then the preceding argument implies that $\widetilde{v} = \partial\widetilde{\Lambda}/\partial t_{t=0}$ is smooth. Since ϕ and $\mathrm{id} \times \phi$ are diffeomorphisms, Λ is smooth. ■

Note that there are many maps $\Lambda : \mathbb{R} \times M \to M$ giving rise to the same vector field v since only the behavior in a neighborhood of $\{0\} \times M$ matters. We will now show that every vector field arises from a map Λ as above. We cite the following results from the theory of first order ordinary differential equations:

II.5.2 THEOREM *Let $\Omega \subset \mathbb{R}^n$ be open and $v : \Omega \to \mathbb{R}^n$ smooth. For every $x \in \Omega$ there exists an open interval $I_x \subset \mathbb{R}$ containing* 0 *and a smooth map $\Lambda_x : I_x \to \Omega$ such that*

1. *$\Lambda_x(0) = x$.*
2. *$\frac{d\Lambda_x(t)}{dt} = v(\Lambda_x(t))$.*
3. *I_x cannot be enlarged without losing 1-2, and every solution of 1-2 is obtained from Λ_x by restriction to a subinterval of I_x.*
4. *The set $A = \cup_{x\in\Omega} I_x \times \{x\} \subset \mathbb{R} \times \Omega$ is open and the map $\Lambda : A \to \Omega$ given by $(t, x) \mapsto \Lambda_x(t)$ is smooth.*
5. *We have $\Lambda(t, \Lambda(s, x)) = \Lambda(t + s, x)$ whenever both sides are defined.*

II.5.3 REMARK Statements 1-2 mean that there is a smooth solution $t \mapsto x(t)$ to the initial value problem $x'(t) = v(x(t)), x(0) = x_0$. Statement 4 means that the solution depends smoothly on the initial value x_0. Statement 5 is a straightforward consequence of the fact the the differential equation in statement 2 is autonomous, to wit the right hand side does not depend on t other than through $x(t)$.

For a proof see, e.g., [34]. It is interesting to note that the standard proof of Theorem II.5.2 uses the Banach fixpoint theorem. The latter thus is essential for both of the analytical results that are central in differential topology (Theorems II.3.7 and II.5.2). □

For $M = \mathbb{R}$ this solves our problem: We have

$$\frac{\partial\Lambda}{\partial t}(0, x) = v(\Lambda_x(0)) = v(x),$$

thus the vector field v arises from Λ as in Lemma II.5.1. For general manifold the claim reduces to $M = \mathbb{R}^n$ by using charts. First a definition:

II.5.4 DEFINITION *A local flow on M consists of an open neighborhood A of $\{0\} \times M \subset \mathbb{R} \times M$ and a smooth map $\Lambda : A \to M$ satisfying*

1. *$\Lambda(0, p) = p \quad \forall p \in M$.*
2. *$\Lambda(t, \Lambda(s, p)) = \Lambda(t + s, p)$ whenever both sides are defined.*

II.5.5 THEOREM *Let M be a manifold and $v \in \Gamma(TM)$. Then there exists a local flow Λ on M such that $v_\Lambda = v$.*

Proof. Let $p \in M$ and (U, ϕ) a chart around p. Then $\Omega = \phi(U) \subset \mathbb{R}^n$ is open and

$$\tilde{v}(x) = T_{\phi^{-1}(x)}\phi(v(\phi^{-1}(x))) : \Omega \to T_x\mathbb{R}^n \cong \mathbb{R}^n$$

is a smooth vector field on Ω. By Theorem II.5.2 there exist $\widetilde{A} \subset \mathbb{R} \times \Omega$ and a local flow $\widetilde{\Lambda} : \widetilde{A} \to \mathbb{R}^n$ such that $(\partial\Lambda/\partial t)(0, x) = \tilde{v}(x)$. Denoting $A = (\mathrm{id} \times \phi^{-1})(\widetilde{A})$ we defining $\Lambda : A \to M$ by

$$\Lambda(t, p) = \phi^{-1}(\widetilde{\Lambda}(t, \phi(p))).$$

Since our charts cover M we can define $\Lambda : A \to M$ for some open neighborhood A of $\{0\} \times M$. (In the intersection of the domains of two charts we obtain consistent results. Why?) Comparing our construction of Λ with the definition of v_Λ in Lemma II.5.1 it is clear that $v_\Lambda = v$. ■

II.5.6 Definition *A (global) flow on a manifold M is a local flow Λ defined on all of $\mathbb{R} \times M$.*

II.5.7 Remark 1. Let $\Lambda : \mathbb{R} \times M \to M$ be a global flow. Then, for every $t \in \mathbb{R}$, $\Lambda_t : p \mapsto \Lambda(t, p)$ has the smooth inverse Λ_{-t}, thus is a diffeomorphisms of M. The map $\mathbb{R} \to \mathrm{Diff}\, M,\ t \mapsto \Lambda_t$ satisfies $\Lambda_0 = \mathrm{id}_M$ and $\Lambda_{s+t} = \Lambda_s \circ \Lambda_t$, and therefore is called a one-parameter group of diffeomorphisms of M. Note that $\mathbb{R} \to \mathrm{Diff}\, M$ is an action of $\mathbb{R}$ on M in the sense of Definition II.1.17, but it is totally discontinuous only if it is trivial ($\Lambda(t, \cdot) = \mathrm{id}_M\ \forall t$)!

2. The above considerations have a natural interpretation in terms of dynamical systems. Considering the points of M as states of a physical system and assuming that there are no time dependent external forces, there should be a map $\Lambda_t : M \to M$ describing the time development of the system. (In terms of the above, $\Lambda_t = \Lambda(t, \cdot)$.) Property 2 in Definition II.5.4 then just means that waiting s seconds and then t seconds has the same effect as waiting $s + t$ seconds.

2. Unfortunately, not every vector field integrates to a global flow. Consider e.g. the vector fields $v(x) = 1$ on $M = (0, 1) \subset \mathbb{R}$ or $v(x) = x^2$ on $M = \mathbb{R}$. They integrate to $\Lambda(t, x) = x + t$ and $\Lambda(t, x) = x/(1 - tx)$, respectively, both of which run out of M in finite time. However, we have the following result. □

II.5.8 Proposition *If the support $S = \overline{\{p \in M \mid v(p) \neq 0\}}$ of $v \in \Gamma(TM)$ is compact (thus in particular if M is compact) then there exists a global flow $\Lambda : \mathbb{R} \times M \to M$ such that $v_\Lambda = v$.*

Proof. Suppose first that M is compact. By the above, the local flow Λ is defined on an open neighborhood $A \subset \mathbb{R} \times M$ of $\{0\} \times M$. Thus for every $p \in M$ there exists an open $U_p \subset M$ and an $\varepsilon_p > 0$ such that $(-\varepsilon_p, \varepsilon_p) \times U_p \subset A$. By compactness there exists a finite subset $F \subset M$ such that $\cup_{p \in F} U_p = M$. Let $\varepsilon = \min_{p \in F} \varepsilon_p$. Then $\varepsilon > 0$ and $(-\varepsilon, \varepsilon) \times M \subset A$. But now we can define $\Lambda_t(\cdot)$ for $t \in (-2\varepsilon, 2\varepsilon)$ by $\Lambda_t = \Lambda_{t/2} \circ \Lambda_{t/2}$. It is easy to see that this gives rise to a local flow on $(-2\varepsilon, 2\varepsilon) \times M$ with velocity v. Since A was maximal, we conclude $(-2\varepsilon, \varepsilon) \times M \subset A$. Iterating this argument we see that $A = \mathbb{R} \times M$.

Now consider non-compact M with compactly supported v. We can find an open $N \subset M$ with compact closure such that $S \subset N$. N is a manifold of the same dimension as N and it is compact in the relative topology since M is locally compact, cf. e.g. [6]. Thus the above considerations give rise to a global flow $\widetilde{\Lambda}$ on N such that $v_{\widetilde{\Lambda}} = v \upharpoonright N$. We extend Λ to all of M by setting $\Lambda(t, p) = p\ \forall p \in M - N, t \in \mathbb{R}$. This is a smooth map $\mathbb{R} \times M \to M$ since $\Lambda(t, p) = p$ also holds on the open set $N - S$, and it is a flow since Λ leaves S stable. ■

II.6 Manifolds with boundary

For many purposes, like the formulation of Stokes' theorem, manifolds as defined above are not sufficiently general, but a very harmless generalization turns out to be sufficient for most applications. We write $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0\}$ and $\partial\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 = 0\}$.

II.6.1 Definition *A (smooth) manifold of dimension $n \in \mathbb{N}$ with boundary is a second countable Hausdorff space M such that every $p \in M$ has an open neighborhood U such that there is a homeomorphism $\phi : U \to V$, where V is an open subset of $\mathbb{R}^n$ or $\mathbb{R}^n_+$, and such that the transition maps $\phi' \circ \phi^{-1} : \phi(U \cap U') \to \phi'(U \cap U')$ between any two charts $(U, \phi), (U', \phi')$ are smooth.*

II.6.2 Lemma *Let $p \in M$. If there is an $\mathbb{R}^n_+$-valued chart (U, ϕ) around p such that $\phi(p) \in \partial\mathbb{R}^n_+$ then $\phi(p) \in \partial\mathbb{R}^n_+$ holds in any chart around p.*

Proof. Let (U', ϕ') be another chart around p and assume that $\phi(p)$ is an interior point of $\mathbb{R}^n_+$. Applying the inverse function theorem to the smooth and invertible map $\phi' \circ \phi^{-1}$ we see that also $\phi'(p)$ is an interior point of $\mathbb{R}^n_+$. ■

Thus a point $p \in M$ is mapped to $\partial\mathbb{R}^n_+$ by all charts or by no chart. (The same result holds for topological manifolds, but to prove this one needs to invoke the 'invariance of the domain' already alluded to.)

II.6.3 Definition *The boundary of M is*

$$\partial M = \{p \in M \mid \phi(p) \in \partial\mathbb{R}^n_+ \textit{ for some } \mathbb{R}^n_+\textit{-valued chart } (U, \phi) \textit{ around } p\}.$$

If $f : M \to N$, we write $\partial f = f \restriction \partial M : \partial M \to N$.

II.6.4 Remark Since $\mathbb{R}^n$ is diffeomorphic to an open ball in $\mathbb{R}^n$, one could also require all charts to take values in $\mathbb{R}^n_+$. However, the flexibility gained by allowing charts taking values in $\mathbb{R}^n$ is convenient since now a manifold in the sense of Definition II.1.9 manifestly also is a manifold with boundary with $\partial M = \emptyset$. □

II.6.5 Lemma *The boundary ∂M of an n-manifold M is a $(n-1)$-manifold without boundary, thus $\partial\partial M = \emptyset$.*

Proof. For every $p \in \partial M$ there is a chart (U, ϕ) of M around p such that $\phi(U \cap \partial M)$ is an open neighborhood of 0 in $0 \times \mathbb{R}^{n-1}$. Forgetting the first coordinate, $(U \cap \partial M, \phi \restriction U \cap \partial M)$ is a chart of ∂M mapping an open neighborhood of p to an open subset of $\mathbb{R}^{n-1}$. One verifies that the atlas of M gives rise to an atlas of ∂M. Clearly ∂M has no boundary, since we have $\mathbb{R}^{n-1}$ in the preceding sentence, not $\mathbb{R}^{n-1}_+$. ■

II.6.6 Exercise If M is a manifold with boundary then $M - \partial M$ is a manifold without boundary (or empty boundary), called the interior of M. □

II.6.7 Exercise Show that $M = \partial M$ implies $M = \emptyset$. □

II.6.8 We now must reconsider the notions of tangent space and differential for manifolds with boundary. The important point is that we want T_pM to be a vector space, not a half space, even if $p \in \partial M$. The 'physicist's' and the 'algebraist's' definition of the tangent space are clearly applicable also in the presence of a boundary and give a vector space T_pM for all $p \in M$. The 'geometer's' definition is more problematic since a (germ of a) curve may run into the boundary. (One may try to make this definition work restricting oneself to germs of the form $[0, \varepsilon) \to M$, but this becomes somewhat tedious.) Now it is clear that also the differentials $T_pf : T_pM \to T_{f(p)}N$ and $TF : TM \to TN$ can be defined as for boundaryless manifolds

The following is obvious but quite important:

II.6.9 Proposition *Let M be a manifold with boundary $\partial M \neq \emptyset$. For every $p \in \partial M$ there is a canonical linear inclusion map $T_p\partial M \hookrightarrow T_pM$ given by $[c] \mapsto [c]$. (Here the left $[c]$ is an equivalence class of curves in ∂M through p and the right $[c]$ is the same curve considered as curve in M.) Let $\Phi = (U, \phi)$ be a chart around $p \in \partial M$. Then the 'physicist pictures' of the tangent spaces are related via*

$$T_p^P\partial M \;\cong\; \{[(\Phi, v)] \in T_p^P M \mid v_1 = 0\}.$$

II.6.10 Exercise Let M, N be n-manifolds, possibly with boundary, with atlasses $(U_i, \phi_i), (V_j, \psi_j)$, respectively. Then the disjoint union $M + N = M \coprod N$ with the atlas $\{(U_i, \phi_i)\} \coprod \{(V_j, \psi_j)\}$ is an n-manifold and $\partial(M + N) = \partial M + \partial N$. □

II.6.11 Exercise Let M, N are manifolds, where $\partial N = \emptyset$, with atlasses $(U_i, \phi_i), (V_j, \psi_j)$, respectively. Then $M \times N$ with the atlas $(U_i \times V_j, \phi_i \times \psi_j)$ is a manifold of dimension $\dim M + \dim N$, and $\partial(M \times N) = \partial M \times N$. □

II.6.12 Remark If $\partial M \neq \emptyset \neq \partial N$ then $M \times N$ is not a manifold! If $p \in \partial M, q \in \partial N$ then $p \times q$ has a neighborhood in $M \times N$ that is homeomorphic to an open neighborhood of $0 \in \mathbb{R}^{m+n-2} \times \mathbb{R}_+ \times \mathbb{R}_+$ but not to any open subset of $\mathbb{R}^{m+n-1} \times \mathbb{R}_+$. However, $M \times N$ is a *manifold with corners*, i.e. a second countable Hausdorff space where every point p has a neighborhood that is homeomorphic to an open subset of $(\mathbb{R}_+)^n$.) The latter are a straightforward generalization of manifolds with boundary, but we will not consider them any further in this course. □

From now on 'manifold' will mean 'manifold with boundary'. Of course, the boundary may be empty. If this is required to be the case we will say 'manifold without boundary'. Note that in the literature very often compact manifolds without boundary are called *closed*. Less frequently, an *open* manifold is meant to be a manifold without boundary such that all connected components are non-compact. We don't use either of these terms.

II.7 Locally compact spaces

We begin by recalling a few facts concerning compact spaces that should be known from general topology.

II.7.1 Definition *A normal space is a Hausdorff space such that for any two disjoint closed sets C_1, C_2 there are disjoint open sets $U_i \supset C_1$, $i = 1, 2$.*

The importance of the normality property derives from the following

II.7.2 Lemma *(Urysohn) A Hausdorff space X is normal iff for any two disjoint closed sets C_1, C_2 there is a continuous function $f : X \to [0, 1]$ such that $f \restriction C_1 \equiv 0$ and $f \restriction C_2 \equiv 1$.*

Proof. As to $\Leftarrow$, let C_1, C_2, f as stated. Then $U_1 = f^{-1}([0, 1/3))$ and $U_2 = f^{-1}((2/3, 1])$ are disjoint open sets containing C_1, C_2, respectively. For the $\Rightarrow$ direction see any book on general topology. ■

II.7.3 Proposition *Every compact Hausdorff space is normal.*

Proof. See any book on general topology. ■

II.7.4 Definition *A space X is locally compact if for every $x \in X$ there are an open U and a compact K such that $x \in U \subset K$.*

Unfortunately, not every locally compact Hausdorff space is normal, cf. [29]. Before we return to this question we consider some basic properties of locally compact spaces.

II.7.5 Lemma *If X is such that every neighborhood of a point contains a compact neighborhood of that point then X is locally compact. If X is Hausdorff the converse is also true.*

Proof. The implication $\Rightarrow$ is trivial. As to the $\Leftarrow$ direction, let $p \in V$ be given with V open. By local compactness there are $p \in U \subset K$ with U open and K compact (thus closed since X is Hausdorff). Then $U \cap V \ni p$ is open and $\overline{U \cap V} \subset K$ is compact, thus normal by Proposition II.7.3. Thus there exist

****

■

II.7.6 Remark Occasionally one sees locally compact spaces defined as spaces where every neighborhood of a point contains a compact neighborhood. This is objectionable since in the non-Hausdorff case it is a stronger assumption. More importantly, it clashes with the terminology according to which a space X is 'locally P', where P is any property that a topological may or may not have, if every point has a neighborhood V (not necessarily open) that is P. Besides P='compact', other examples for P are 'path connected', 'euclidean' (homeomorphic to $\mathbb{R}^n$), etc. □

II.7.7 Lemma *Let X be locally compact and $K \subset X$ compact. Then there is an open $U \supset K$ such that $\overline{U}$ is compact.*

Proof. By local compactness there exists a cover (V_j) of X by open sets with compact closures. Since $K \subset X$ is compact it is covered by finitely many of the V_j. The union of the latter is an open set with compact closure. ■

II.7.8 Exercise Every open and every closed subset of a locally compact Hausdorff space is locally compact Hausdorff in the relative topology. □

II.7.9 Definition *Let X be a topological space with topology $\mathcal{T}$. We write $X^+ = X \coprod \{\infty\}$ and identify X with the obvious subset of X^+. We define a topology $\mathcal{T}^+$ on X^+ by declaring $U \subset X^+$ to be open if it doesn't contain ∞ and is open in X or it does contain ∞ and $X^+ - U$ is closed and compact in X.*

II.7.10 Exercise Prove the following claims:

1. $(X^+, \mathcal{T}^+)$ is a topological space.
2. X is an open subset of X^+ and the subspace topology on X obtained from X^+ coincides with $\mathcal{T}$. Thus the inclusion map $X \hookrightarrow X^+$ is a homeomorphism of X onto its image.
3. X^+ is compact.
4. If X is compact then X is closed in X^+, thus $X^+ = X \coprod \{\infty\}$ as topological spaces.
5. If X is non-compact then $\overline{X} = X^+$, thus X is dense in X^+.
6. X^+ is Hausdorff iff X is locally compact Hausdorff.

□

II.7.11 Remark Given a space X, a space $Y \supset X$ is called a compactification of X if (1) the subspace topology on $X \subset Y$ coincides with the given one on X, (2) Y is compact and (3) X is dense in Y. Thus X^+ is a compactification of X provided X is non-compact. It is called the one-point (or Alexandroff) compactification. There are many other compactifications, working under different assumptions on X. □

II.7.12 Remark The last part of Exercise II.7.10 already shows the usefulness of locally compact (Hausdorff) spaces. We mention four further reasons:

1. Abstract measure theory on locally compact spaces: On locally compact spaces there exists a nice generalization of Lebesgue's measure and integration theory for $\mathbb{R}^n$. Cf. [26].

2. Gelfand duality: For a loc. cp. space X define

$$C_0(X) = \{f \in C(X, \mathbb{C}) \mid \forall \varepsilon > 0\ \exists K \subset X \text{ compact such that } |f(x)| < \varepsilon\ \forall x \in X - K\}.$$

   (Equivalently, $C_0(X) = \{f^+ \restriction X \mid f^+ \in C(X^+),\ f^+(\infty) = 0\}$.) Then $C_0(X)$ is a complex algebra, complete w.r.t. the sup-norm and therefore a commutative C^*-algebra. Now, every commutative C^*-algebra is isomorphic to $C_0(X)$ for a locally compact Hausdorff space that is unique up to homeomorphism. Cf. e.g. [24, Section 4.3].

3. The Haar measure on locally compact groups: On every locally compact group there exists a unique (up to normalization) left invariant measure in the setting of 1. [23, Section 1.2].

4. Pontrjagin duality for locally compact abelian groups: Let G be a loc. cp. abelian group and write $\widehat{G}$ for the group of continuous homomorphisms $G \to \{z \in \mathbb{C} \mid |z| = 1\}$, with a suitable topology. It is not difficult to show that $\widehat{G}$ is a loc. cp. abelian group. Using all of 1-3 above one proves Pontrjagin duality: $\widehat{\widehat{G}} \cong G$. [23, Chapter 3].

□

We now examine the separation properties of locally compact Hausdorff spaces.

II.7.13 Proposition *Let X be locally compact Hausdorff. Let $K \subset U \subset X$ with K compact and U open. Then there exists a continuous function $f : X \to [0,1]$ such that $f \restriction K \equiv 1$ and* $\mathrm{supp} f \equiv \overline{\{x \in X \mid f(x) \neq 0\}}$ *is compact and contained in U.*

Proof. By Exercise II.7.10, the one-point compactification X^+ is compact Hausdorff and thus normal by Proposition II.7.3. By Lemma II.7.7 there exists an open $V \supset K$ with $\overline{V}$ closed. Applying Urysohn's lemma to the closed sets K (since X is Hausdorff) and $X^+ - (U \cap V)$ we obtain a continuous function $f : X^+ \to [0,1]$ such that $f \restriction K \equiv 1$ and $\{x \in X \mid f(x) > 0\} \subset U \cap V$. Since $\overline{U \cap V}$ is compact, we are done. ■

II.7.14 Remark Let X be locally compact Hausdorff and let $K \subset X$ be compact and $C \subset X$ closed such that $K \cap C = \emptyset$. Then Proposition II.7.13 applies to $K \subset U = X - C$, thus there is a continuous function $f : X \to [0,1]$ such that $f \restriction C \equiv 1$ and $f \restriction C \equiv 0$. Thus a continuous function separating disjoint closed sets exists (as is the case in normal spaces) if at least one of the sets is compact.

Given any closed set $C \subset X$ and an $x \in X - C$, $K = \{x\}$ is compact and we obtain a continuous function separating x and C. Such a space is called *completely regular*. In turn, it follows that X is *regular* in the sense that a closed set $C \subset X$ and $x \notin C$ are contained in disjoint open sets. □

If we insist on normality, we need stronger assumptions on X. The following property is sufficient:

II.7.15 Definition *A space X is σ-compact if it is the union of a countable family C_n of compact subsets.*

II.7.16 Proposition *A locally compact σ-compact Hausdorff space is normal.*

Proof. See [24, Proposition 1.7.8]. For the class of second countable locally compact Hausdorff spaces this will follow from the following lemma and the results in the next subsection. ■

II.7.17 Lemma *A second countable locally compact space X is σ-compact.*

Proof. By local compactness, there exists a cover $(U_i,\, i \in I)$ of X by open sets with compact closures. By Proposition II.1.6, there is a countable subcover $I_0 \subset I$. Now $(\overline{U_i},\, i \in I_0)$ is a countable family of compact sets covering X. ■

II.7.18 Remark Actually, every second countable locally compact Hausdorff space is metrizable and thus has all properties of metric spaces, like normality and paracompactness (see below). □

II.8 Proper maps

It is natural to ask whether a continuous map between locally compact spaces can be extended continuously to the one-point compactifications. For this we need the following

II.8.1 Definition *A function $f : X \to Y$ (not necessarily continuous) between topological spaces is proper iff $f^{-1}(K) \subset X$ is compact for every compact $K \subset Y$.*

II.8.2 Exercise Let $f : X \to Y$ be a continuous map between locally compact Hausdorff spaces. Extend f to $f^+ : X^+ \to Y^+$ by setting $f^+(\infty) = \infty$. Show that f^+ is continuous iff f is proper. □

II.8.3 Exercise Prove the following claims.

1. If $f : A \to B$ and $g : B \to C$ are proper then $g \circ f : A \to C$ is proper.
2. If A is Hausdorff, B is compact and $f : A \to B$ is proper then f is continuous.

□

II.8.4 Proposition *Let B, C be Hausdorff spaces. If $f : A \to B$ and $g : A \to C$ are continuous and f is proper then $h = (f, g) : A \to B \times C$ is proper.*

Proof. Let $X \subset B \times C$ be compact. If $p_1 : B \times C \to B$ is the projection onto B, then $p_1(X) \subset B$ is compact, thus $f^{-1}(p_1(X))$ is compact. Since B, C are Hausdorff, $B \times C$ is Hausdorff, thus the compact subset X is closed. Since f, g, thus h, are continuous, $h^{-1}(X)$ is a closed and thus compact since $h^{-1}(X) \subset f^{-1}(p_1(X))$, which is clear by $X \subset p_1(X) \times C$. ■

II.8.5 Exercise Let G be a toplogical group and X a topological space. An action of G on X is continuous if the map $G \times X \to X,\ (g, x) \mapsto gx$ is continuous. An action is called proper if the map $G \times X \to X \times X, (g, x) \mapsto (gx, x)$ is proper. Show that if G is discrete, this condition is equivalent to the one given in Remark II.1.22. □

The following result will play an important rôle in the proof of the embedding theorem.

II.8.6 Lemma *Let $f : X \to Y$ be an injective map of locally compact Hausdorff spaces. Then the following are equivalent:*

(i) *$f(X)$ is closed and $f : X \to f(X)$ is a homeomorphism w.r.t. the subset topology.*

(ii) *f is closed, i.e. $f(C)$ is closed for every closed $C \subset X$.*

(iii) *f is proper.*

Proof. (ii)⇒(i): Since f is injective, (i) holds iff $f^{-1}: f(X) \to X$ is continuous, which is the case if $f(Z)$ is open in $f(X)$ for every open $Z \subset X$. Let $Z \subset X$ be open. By (ii), $f(X - Z)$ is closed in Y, thus closed in $f(X)$. Since f is injective, we have $f(Z) = f(X) - f(X - Z)$, thus $f(Z)$ is open in $f(X)$.

(i)⇒(iii): Let $K \subset Y$ be compact. Then $K \cap f(X)$ is compact in $f(X)$, thus $f^{-1}(K) = f^{-1}(K \cap f(X))$ is compact in X by (i).

(iii)⇒(ii): Since f is proper, it extends to a continuous map $\hat{f}: \hat{X} \to \hat{Y}$ of the 1-point compactifications such that $\hat{f}(\infty) = \infty$. Let $C \subset X$ be closed. Then $C \cup \{\infty\} \subset \hat{X}$ is closed and thus compact. Thus $f(C \cup \{\infty\}) \subset \hat{Y}$ is compact, thus closed (since $\hat{Y}$ is Hausdorff). Thus $\widehat{Y} - f(C \cup \{\infty\}) = Y - f(C)$ is open in $\widehat{Y}$ and thus in Y, thus $f(C) \subset Y$ is closed. ■

II.9 Paracompact spaces

II.9.1 Definition *A cover $(U_i)_{i \in I}$ of a space X is locally finite if every $p \in X$ has a neighborhood U such that the set $\{i \in I \mid U \cap U_i \neq \emptyset\}$ is finite. A refinement of a cover $(U_i)_{i \in I}$ is a cover $(V_j)_{j \in J}$ such that every V_j is contained in some U_i. A space X is called paracompact if every cover admits a locally finite refinement.*

II.9.2 Remark Some authors include Hausdorffness in the definition. Anyway, we'll only consider paracompact spaces that are also Hausdorff. □

Paracompact spaces are nicely behaved:

II.9.3 Proposition *A paracompact Hausdorff space is normal.*

Proof. We beginn by showing that a paracompact space X is regular, cf. Remark II.7.14. Thus suppose such x, C are given. For each point $y \in C$ there are disjoint open sets $U_y \ni x$ and $V_y \ni y$. Now $\{X - C\} \cup \{V_y,\ y \in C\}$ is an open cover of X. By paracompactness there is a locally finite refinement $(W_i, i \in I)$. Let $U = \cup\{W_i \mid W_i \subset V_y \text{ for some } y \in C\}$ and note that this contains C. By local finiteness of the cover (W_i) we have $\overline{U} = \cup\{W_i \mid \overline{W_i} \subset V_y \text{ for some } y \in C\}$. Now, x is not contained in any of the $\overline{W_i}$, thus $x \notin \overline{U}$. Therefore we have disjoint open sets $U \supset C$ and $X - \overline{U} \ni x$, as required.

To prove normality, let C, D be disjoint closed sets and repeat the argument, replacing x by D. ■

II.9.4 Definition *A shrinking of a cover $(U_i)_{i \in I}$ is a cover $(V_i)_{i \in I}$ (same index set!!) such that $\overline{V_i} \subset U_i$ for all $i \in I$.*

II.9.5 Lemma *Every locally finite cover $(U_i)_{i \in I}$ of a paracompact Hausdorff space admits a shrinking.*

Proof. Let $(U_i)_{i \in I}$ be a locally finite open cover of X. For every $x \in X$ chose a U_i containing x and call it U_x. By regularity, cf. Proposition II.9.3, there are disjoint open sets $Y_x \ni x$ and $Z_x \supset X - U_x$. Thus $Y_x \subset X - Z_x$ and therefore $\overline{Y_x} \subset \overline{X - Z_x} = X - Z_x \subset U_x$. Clearly $(Y_x)_{x \in X}$ is an open cover, and by paracompactness we can choose a locally finite refinement $(W_j)_{j \in J}$. Let $V_i = \cup\{W_j \mid \overline{W_j} \subset U_i\}$. Clearly, $(V_i)_{i \in I}$ is again an open cover. As above, local finiteness of (W_j) implies $\overline{V_i} = \cup\{\overline{W_j} \mid \overline{W_j} \subset U_i\} \subset U_i$, thus (V_i) is a shrinking of (U_i). ■

II.9.6 Definition *Let X be a topological space. A family $(\lambda_i : X \to [0,1])_{i \in I}$ of continuous functions is a partition of unity if (a) it is locally finite, i.e. every $x \in X$ has a neighborhood on which all but finitely many of the f_i are identically zero and (b) for every $x \in X$ we have*

$$\sum_{i \in I} \lambda_i(x) = 1.$$

(The sum makes sense by (a).) Clearly $(V_i = \{x \in X \mid f_i(x) > 0\})_{i\in I}$ *is an open cover, and we say that* $(f_i)_{i\in I}$ *is subordinate to a given cover* (U_j) *if* (V_i) *is a shrinking of* (U_i).

II.9.7 PROPOSITION *A Hausdorff space is paracompact iff every open cover admits a subordinate partition of unity.*

Proof. As to $\Leftarrow$, let (U_i) be an open cover and (λ_i) a subordinate partition of unity. Then $(V_i = \{x \in X \mid \lambda_i(x) > 0\})_{i\in I}$ is a locally finite open cover subordinate to (U_i).

As to $\Rightarrow$, let X be paracompact and (U_i) an open cover. Let $(V_j)_{j\in J}$ be a locally finite refinement of (U_i) and $(W_j)_{j\in J}$ a shrinking of (V_j). Since X is normal, Urysohn's lemma provides us with functions $f_j : X \to [0,1]$, $j \in J$ such that $f_j \restriction \overline{W_j} \equiv 1$ and $f_j \restriction X - U_j \equiv 0$. Thus the family (f_j) is locally finite and $f = \sum_j f_j$ vanishes nowhere. Now $\lambda_j = f_j/f$ is a partition of unity subordinate to (V_j) and thus to (U_i). ■

In view of these results we need criteria telling providing us with paracompact spaces. The first one shows that paracompactness is independent of local compactness since, e.g., infinite dimensional Banach spaces are not locally compact.

II.9.8 THEOREM *Every metric space is paracompact.*

Proof. The original proof by Stone was quite complicated, but now there are simple one-page proofs. See M. E. Rudin, Proc. AMS **20** (1969) and D. Ornstein, Proc. AMS **21** (1969). ■

Together with the fact that every second countable locally compact Hausdorff space is metrizable this implies paracompactness of these spaces. However, we give a more direct proof.

II.9.9 THEOREM *A locally compact Hausdorff space* X *is paracompact iff every connected component of* X *is* σ*-compact.*

Proof. For the $\Rightarrow$-direction, which we don't need here, see [6]. As to $\Leftarrow$, it is clear that a topological direct sum of paracompact spaces is paracompact. It is thus sufficient to prove that a locally compact σ-compact Hausdorff space is paracompact.

By σ-compactness we have compact sets $(C_n)_{n\in\mathbb{N}}$ covering X. Using Lemma II.7.7 we can choose, for every $n \in \mathbb{N}$, an open set E_n having compact closure and containing $\overline{E_{n-1}} \cup C_n$ (with $E_0 = \emptyset$). Clearly $\cup_n E_n = X$. Now $A_n = \overline{E_n} - E_{n-1}$ defines a sequence of compact sets such that $\bigcup_n A_n = X$ and $A_i \cap A_j = \emptyset$ whenever $|i-j| > 1$. Now we choose another sequence (B_n) of compact sets such that $A_n \subset B_n^{int}$ and $B_i \cap B_j = \emptyset$ whenever $|i-j| > 1$. For this purpose it is enough that each olE_n is compact and thus normal.

After these preparations, let $(U_i)_{i\in I}$ be an arbitrary open cover of X. By compactness, each A_n is covered by finitely many $U_i's$, thus

$$A_n \subset \bigcup_{r=1}^{s_n} U_{i(n,r)}.$$

Then the family

$$\{\ V_{n,r} = U_{i(n,r)} \cap B_n^{int} \mid n \in \mathbb{N},\ r = 1,\dots,s_n\ \}$$

is an open cover refining $(U_i)_{i\in I}$. Every $x \in X$ has a neighborhood U contained in some B_m^{int} and therefore meets at most in the finite family $\{V_{n,r} \mid |n-m| \le 1, r = 1,\dots,s_n\}$. Thus the new cover is locally finite. ■

For our purposes in the sequel, the following is the upshot of this section:

II.9.10 COROLLARY *Every second countable locally compact Hausdorff space, in particular every topological manifold, is paracompact and normal.*

Proof. Lemma II.7.17 gives σ-compactness. Now apply Theorem II.9.9 and Proposition II.9.3. ■

II.10 Smooth partitions of unity

By definition, manifolds are spaces obtained by gluing together open neighborhoods in $\mathbb{R}^n$. Many proofs require in a certain sense to undo this operation. This is done using partitions of unity. We have already seen that manifolds are paracompact and therefore admit continuous partitions of unity. For the purposes of differential topology this is not good enough since we need the functions λ_j to be smooth. We therefore give a proof.

II.10.1 Lemma *Let $K \subset U \subset M$ with K compact and U open. Then there exists a smooth function $g : M \to [0, \infty)$ such that $g(x) > 0$ for all $x \in K$ and $\operatorname{supp} g \subset U$.*

Proof. Let $F : \mathbb{R} \to \mathbb{R}$ be given by $F(x) = e^{-1/(x-1)^2} e^{-1/(x+1)^2}$ for $|x| < 1$ and by $F(x) = 0$ otherwise. Then F is a smooth and satisfies $F(x) > 0$ iff $x \in (-1, 1)$. Now let $p \in U \subset M$ with U open. Take a chart $(\tilde{U}, \phi)$ around p and $\varepsilon > 0$ such that $\phi(U \cap \tilde{U}) \subset \mathbb{R}^n$ contains the cube

$$\{(x_1, \ldots, x_n) \,|\, |x_i| \leq \varepsilon \text{ for all } i\}.$$

Then the function $q \mapsto F(\phi_1(q)/\varepsilon) \cdot \ldots \cdot F(\phi_n(q)/\varepsilon)$ extends to a smooth function $g_p : M \to [0, 1]$ such that $g(p) > 0$ and $\operatorname{supp} g_p \subset U$.

To prove the lemma, take such a function g_p for every $p \in K$. The sets $\{x \in M \mid g_p(x) > 0\}$ are open and cover K. Thus a finite number of them covers K. The sum g of the corresponding functions g_p has the desired properties. ∎

II.10.2 Theorem *Let M be a manifold and (U_i) an open cover. Then there exists a smooth partition of unity $(\lambda_j)_{j \in J}$ subordinate to (U_i).*

Proof. By paracompactness there exist a locally finite refinement $(V_j)_{j \in J}$ of (U_i) and, by Lemma II.9.5, a shrinking $(W_j)_{j \in J}$ of $(V_j)_{j \in J}$. We may also assume that each V_j is contained in the domain of a coordinate chart and that each $\overline{W_j}$ is compact. Using Lemma II.10.1 we construct, for every $j \in J$, a smooth function $g_j : M \to [0, \infty)$ such that $g_j(x) > 0$ if $x \in \overline{W_j}$ and $\operatorname{supp} g_j \subset V_j$. By local finiteness, $g(p) = \sum_i g_i(x)$ exists as a smooth function that vanishes nowhere (since $\cup_j W_j = M$). Now $\lambda_i = g_i/g$ has all desired properties. ∎

As a first application we consider the extension problem of smooth functions defined on an open neighborhood of a compact subset of a manifold. (We need the open neighborhood U of K since there is no way to define smoothness of a function defined on an arbitrary closed subset $K \subset M$.)

II.10.3 Proposition *Let M be a manifold and consider $C \subset U \subset M$, where C is closed and U is open. Then for any smooth function $f : U \to \mathbb{R}$ there exists a smooth function $\overline{f} : M \to \mathbb{R}$ that coincides with f on C.*

Proof. $\{U, M - C\}$ is an open cover of M which is trivially locally finite. Thus there is a subordinate open cover, to wit smooth functions $\lambda_U, \lambda_{M-C} : M \to \mathbb{R}$ such that $\lambda_U + \lambda_{M-C} \equiv 1$ and $\operatorname{supp}\lambda_U \subset U$, $\operatorname{supp}\lambda_{M-C} \subset M - C$. Defining $\overline{f}(p)$ to be $\lambda_U(p) f(p)$ for $p \in U$ and zero otherwise, $\overline{f}$ is smooth. If $p \in C$ then $p \notin M - C$, thus $\lambda_{M-C}(p) = 0$, therefore $\lambda_U(p) = 1$ and $\overline{f}(p) = f(p)$. ∎

II.11 Basics of Riemannian Manifolds

II.11.1 Definition *A riemannian metric on a manifold M is a family $\{\langle \cdot, \cdot \rangle_p,\ p \in M\}$ of symmetric positive definite bilinear forms on the tangent spaces T_pM such that the map $p \mapsto \langle v(p), v(p) \rangle_p$ is smooth for any (smooth) vector field v. A manifold together with a (chosen) riemannian metric is a riemannian manifold.*

II.11.2 Remark One application of riemannian metrics is the assignment of a length to a smooth curve segment in M. If $f : [a,b] \to M$ is smooth we use the identification $T_t\mathbb{R} \equiv \mathbb{R}$ to define $v(t) \in T_{f(t)}M$ by $v(t) = (T_tf)(1)$. Then

$$L(f) = \int_a^b dt\,\sqrt{\langle v(t), v(t)\rangle_{f(t)}}$$

defines the length of f. For $M = \mathbb{R}^n$ this is easily seen to reduce to the usual definition. As mentioned before, every manifold is metrizable. In fact, a connected riemannian manifold has a canonical metric inducing the given topology:

$$d(p,q) = \inf\{L(f) \mid f \text{ a smooth curve connecting } p \text{ and } q\}.$$

(Showing that this is a metric requires some work, the most difficult part being $d(p,q) > 0$ if $p \neq q$.) Then the interesting question arises whether the metric space (M,d) is complete. More on this later. □

II.11.3 Remark Let $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle^\sim$ be riemannian metrics on M. Then

$$\langle\cdot,\cdot\rangle^t = t\langle\cdot,\cdot\rangle + (1-t)\langle\cdot,\cdot\rangle^\sim, \qquad t \in [0,1]$$

is a continuous family metrices such that $\langle\cdot,\cdot\rangle^1 = \langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle^0 = \langle\cdot,\cdot\rangle^\sim$. Thus the space of all metrices (with a suitable topology) is connected, and in a sense all metrices are equivalent. Indeed, for many purposes it is sufficient to know that M admits some riemannian metric. Proving this will be our first application of partitions of unity. □

II.11.4 Lemma *Every manifold admits a riemannian metric.*

Proof. Let $\{U_i, i \in I\}$ be a locally finite open cover and $\{\lambda_i, i \in I\}$ a subordinate partition of unity. For each $i \in I$, let $\langle\cdot,\cdot\rangle_i$ be a positive definite symmetric quadratic form on $\mathbb{R}^n$ and for $X, Y \in T_pM$ we define

$$\langle X, Y\rangle_p = \sum_{i\in I} \lambda_i(p)\, \langle T_p\phi_i(X), T_p\phi_i(Y)\rangle_i.$$

Here the i-th summand is understood to be zero if $p \notin U_i$. This is well defined by local finiteness of the partition. Symmetry and positivity are obvious. If $X \neq 0$ then $\langle T_p\phi_i(X), T_p\phi_i(X)\rangle_i > 0$ whenever $p \in U_i$, implying positive definiteness. Given vector fields X, Y, smoothness of $p \mapsto \langle X(p), Y(p)\rangle_p$ is easily seen using local coordinates. ■

II.12 Classification of smooth 1-manifolds

We first remark that manifolds are locally path connected, thus the path components are closed and open connected components. Thus a manifold is topologically a direct sum of its path components and it is sufficient to classify (path) connected manifolds.

In this section we will give a complete classification of smooth connected 1-manifolds. While the statement of the latter may seem obvious, giving a proper proof is not entirely trivial. Corollary II.12.3 will turn out to be useful later on. Our approach is inspired by [19, Appendix]. (There, however, only manifolds embedded in some $\mathbb{R}^m$ are considered.)

II.12.1 Theorem *Let M be a connected smooth 1-manifold. Then M is diffeomorphic to one of the following:* $[0,1], [0,\infty), \mathbb{R}, S^1$.

II.12.2 Remark Of the 1-manifolds in the theorem, the compact ones are S^1 and $[0,1]$, the boundaryless ones are S^1 and $\mathbb{R}$. □

II.12.3 Corollary *Let M be a compact smooth 1-manifold. Then ∂M consists of an even (finite) number of points.*

Proof. Since M is compact, the number of connected components is finite. The claim follows since a connected compact 1-manifold is either S^1 (no boundary) or $[0,1]$ (two boundary points). ■

In the sequel we will call a connected non empty subset of $\mathbb{R}$ an interval. It should be clear that every interval is diffeomorphic to one of the first three alternatives in Theorem II.12.1.

II.12.4 Definition *Let M be a smooth Riemannian 1-manifold and I an interval. A map $f: I \to M$ is a parametrization if f maps I diffeomorphically onto an open subset of M. It is called a parametrization by arc length if the 'velocity' $v(x) = T_x f(1) \in T_{f(p)}M$ has length one for all $x \in I$, i.e. $\langle v(x), v(x)\rangle_{f(x)} = 1\ \forall x \in I$.*

II.12.5 Exercise M must have boundary points whenever I has. *Hint:* $f(I) \subset M$ is open. □

II.12.6 Lemma *Any given local parametrization $f: I \to M$ can be transformed into a parametrization by arc length by a transformation of variables.*

Proof. As usual, $v(x) = T_x f(1)$. Pick a point $x_0 \in I$. Then the map

$$\beta: I \to \mathbb{R}, \quad x \mapsto \int_{x_0}^{x} dt\, \sqrt{\langle v(t), v(t)\rangle_{f(t)}}$$

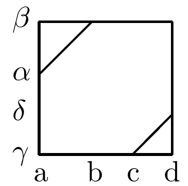
is smooth and satisfies $\beta'(x) > 0$ for all $x \in I$. Thus it has a smooth inverse $\beta^{-1}: \beta(I) \to I$. Then $\tilde{f} = f \circ \beta^{-1},\ \beta(I) \to M$ is a parametrization by arc length. (Verify!) ■

II.12.7 Proposition *If $f: I \to M$ and $g: J \to M$ are parametrizations by arc length then $f(I) \cap g(J)$ has at most two components. If it has one component then f can be extended to a parametrization of $f(I) \cup g(J)$ by arc length. If it has two components then M is diffeomorphic to S^1.*

Proof. Clearly $g^{-1} \circ f$ maps the open subset $f^{-1}(g(J)) \subset I$ diffeomorphically onto the open subset $g^{-1}(f(I)) \subset J$, and the derivative of this map is ± 1 everywhere. The subset $\Gamma = \{(s,t) \mid f(s) = g(t)\}$ of $I \times J$ consists of line segments of slope ± 1. Since Γ is closed and $g^{-1} \circ f$ is locally a diffeomorphism, these line segments cannot end in the interior of $I \times J$, but must extend to the boundary. Since $g \circ f^{-1}$ is injective and single valued, there can be at most one of these segments ending on each of the four edges of the rectangle $I \times J$. It follows that Γ has at most two components. If there are two, they must have the same slope ± 1.

If Γ is connected then $g^{-1} \circ f$ extends to a linear map $\ell: \mathbb{R} \to \mathbb{R}$. Now f and $g \circ \ell$ fit together and define a map $I \cap \ell^{-1}(J) \to f(I) \cup g(J)$.

If Γ has two components, both with slope $+1$ say, we have $\Gamma = \overline{(a,\alpha)(b,\beta)} \cup \overline{(c,\gamma)(d,\delta)}$:



Translating the interval $J = (\gamma, \beta)$ if necessary, we may assume that $\gamma = c$ and $\delta = d$ so that

$$a < b \le c < d \le \alpha < \beta:$$

Identifying S^1 with the unit circle in $\mathbb{C}$ and setting $\theta = 2\pi t/(\alpha - a)$ we define

$$h : S^1 \to M, \quad e^{i\theta} \mapsto \begin{cases} f(t) & \text{if } \ a < t < d, \\ g(t) & \text{if } \ c < t < \beta. \end{cases}$$

By definition, h is injective. The image $h(S^1)$ is open and compact, thus closed in M. Since M is connected h is surjective. ■

Proof of the Theorem. We use Lemma II.11.4 to equip M with a riemannian metric. Let $p \in M$ and (U, ϕ) a chart around p. Using Lemma II.12.6 we obtain an arc-length parametrization of U by an open interval (a, b). If $U = M$ we are done. If not, at least one of the points a, b is not a boundary point of M. Assume this point is a and choose a chart (V, ψ) around a. Clearly $V \not\subset U$. Again, we can replace ψ by an arc-length parametrization of V. We have $U \cap V \neq \emptyset$, thus by Proposition II.12.7 $U \cap V$ has one or two components. If it has one, Proposition II.12.7 provides us with an arc-length parametrization of $U \cup V$, and continuing like this we obtain an arc-length parametrization of a certain maximal subset N of M. If $N = M$ we are done. Assume $N \neq M$. Then for every $p \in M - N$ and every open neighborhood V of p, the intersection $V \cap N$ has two components. In that case, chosing an arc-length parametrization of such a neighborhood V, Proposition II.12.7 provides a diffeomorphism between M and S^1. ■

II.12.8 REMARK 1. Theorem II.12.1 implies that all connected smooth 1-manifolds of the same type ($[0, 1]$, $[0, \infty)$, $\mathbb{R}$, S^1) are diffeomorphic. Thus each of the latter carries a unique smooth structure.

2. By similar methods as used in the proof of Lemma II.12.6 one can show that every connected topological 1-manifold M is *homeomorphic* to one of the four above types. This homomorphism can be used to define a smooth structure on M, which is unique by 1.

3. The classification of smooth 1-manifolds can also be obtained using some elementary Morse theory see [12, Appendix 2]. For a Morse theory approach to the classification of smooth compact 2-manifolds see [13, Chapter 9]. Topological surfaces (i.e. 2-dimensional C^0-manifolds) can also be classified using elementary topological considerations, cf. [31]. However, proving that topological manifolds of dimensions 2 and 3 admit a unique smooth structure is more involved. □

Chapter III

Local structure of smooth maps

III.1 The rank theorem

In Exercise II.3.6 we have seen that $T_p f$ is an isomorphism for all p when f is a diffeomorphism. For general maps this will not be the case, certainly not if M, N have different dimensions. Thus it is natural to consider the rank of the linear map $T_p f$ as $p \in M$ varies.

III.1.1 Proposition *Consider $f : M \to N$ and let $p \in M$. If $\operatorname{rank}(T_p f : T_p M \to T_{f(p)} N) = r$, there exists an open $U \ni p$ such that $\operatorname{rank}(T_q f : T_q M \to T_{f(q)} N) \geq r$ for all $q \in U$.*

Proof. Let $(U, \phi), (V, \psi)$ be charts around $p, f(p)$, respectively. W.r.t. these charts the differential $T_p f$ is described by the $n \times m$-matrix $A = (\partial(\psi_j \circ f \circ \phi^{-1}(x_1, \ldots, x_m))/\partial x_i)$. That the rank of A is r means that A has an invertible $r \times r$ submatrix but no invertible submatrix of size $(r+1) \times (r+1)$. Invertibility is equivalent of non-vanishing of the determinant. Now, the determinant of the submatrix under question is a continuous function of p and therefore does not vanish in a sufficiently small neighborhood of p. In such a neighborhood the rank of $T_p f$ cannot be smaller than r, as claimed. ■

III.1.2 Remark 1. This result can be restated by saying that the map $p \mapsto \operatorname{rank} T_p f$ is lower semicontinuous. (A function f is lower semicontinuous if $\lim_{q \to p} f(q) \geq f(p)$.)

2. Note that the rank of $T_q f$ may be bigger than that of $T_p f$ arbitrarily close to p: Consider $f : \mathbb{R} \to \mathbb{R} : \ p \mapsto p^2$ at $p = 0$. □

The following proof is taken from [7].

III.1.3 Theorem *Let M, N be manifolds without boundary. Consider $f : M \to N$ and assume that $\operatorname{rank} T_p f$ is constant on some neighborhood U of $p \in M$. Then there are charts (V, ϕ) and (W, ψ) around p and $f(p)$, respectively, such that $f(V) \subset W$ and $\widetilde{f} = \psi \circ f \circ \phi^{-1} : \phi(V) \to \psi(W)$ has the form $\widetilde{f} : (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_r, 0, \ldots, 0)$.*

Proof. We may right away restrict to maps $f : U \to \mathbb{R}^n$ where $U \subset \mathbb{R}^m$ is a neighborhood of zero and $f(0) = 0$. Now there exists a $(r \times r)$-submatrix of $T_p f$ that is invertible at $p = 0$. Suitably renaming the coordinates of $\mathbb{R}^m$ and $\mathbb{R}^n$ we may assume that the matrix $(\partial f_i / \partial x_j)_{1 \leq i,j \leq r}$ is invertible at $x = 0$. Let $h : U \to \mathbb{R}^m$ be given by $(x_1, \ldots, x_m) \mapsto (f_1(x), \ldots, f_r(x), x_{r+1}, \ldots, x_m)$. The Jacobi matrix of h

has the form

$$
T_0h = \begin{array}{cc}
\begin{array}{cc} r & n-r \end{array} & \\
\left[\begin{array}{c|c} \multicolumn{2}{c}{\frac{\partial f_i}{\partial x_j}} \\ \hline \mathbf{0} & I \end{array}\right] & \begin{array}{c} r \\ m-r \end{array}
\end{array}
$$

Now we have $\det T_0h = \det(\partial f_i/\partial x_j)_{1\le i,j\le r} \neq 0$, thus by the inverse function theorem there is a local inverse $h^{-1} : V' \to U'$ bijectively mapping some open neighborhood V' of 0 to some $U' \subset U$, and the diagrams

$$
\begin{tikzcd}
\mathbb{R}^m \supset U' \arrow[r, "h"] \arrow[dr, "f"'] & V' \subset \mathbb{R}^m \arrow[d, "f \circ h^{-1}"] \\
& \mathbb{R}^n
\end{tikzcd}
\qquad
\begin{tikzcd}
(x_1, \ldots, x_m) \arrow[r, mapsto] \arrow[dr, "f"'] & (f_1(x), \ldots, f_r(x), x_{r+1}, \ldots, x_m) \arrow[d, mapsto] \\
& (f_1(x), \ldots, f_n(x))
\end{tikzcd}
$$

commute. Thus the map $g = f\circ h^{-1} : V' \to \mathbb{R}^n$ has the form $(z_1, \ldots, z_m) \mapsto (z_1, \ldots, z_r, g_{r+1}(z), \ldots, g_n(z))$ and therefore the Jacobi determinant

$$
T_0g = \begin{array}{cc}
\begin{array}{cc} r & n-r \end{array} & \\
\left[\begin{array}{c|c} I & 0 \\ \hline ? & A(z) \end{array}\right] & \begin{array}{c} r \\ m-r \end{array}
\end{array}
$$

where $A(z) = (\partial g_i/\partial z_j)$. Since $\operatorname{rank} f = \operatorname{rank} g = \operatorname{rank} T_0g = r$ in a neighborhood of zero, we must have $A(z) = 0$ in this neighborhood. Thus

$$
\frac{\partial g_i}{\partial z_j} = 0, \qquad r+1 \le i \le n,\ r+1 \le j \le m. \tag{III.1}
$$

Let now

$$
k : (y_1, \ldots, y_n) \mapsto (y_1, \ldots, y_r, y_{r+1} - g_{r+1}(y_1, \ldots, y_r, 0, \ldots, 0), \ldots, y_n - g_n(y_1, \ldots, y_r, 0, \ldots, 0)).
$$

The Jacobi matrix $(\partial k_i/\partial y_j)$ of k is

$$
T_0k = \begin{array}{cc}
\begin{array}{cc} r & n-r \end{array} & \\
\left[\begin{array}{c|c} I & 0 \\ \hline ? & I \end{array}\right] & \begin{array}{c} r \\ n-r \end{array}
\end{array}
$$

thus k is invertible in some neighborhood of zero, and $k \circ f \circ h^{-1} = k \circ g$ is represented by the composition

$$\begin{aligned}(z_1, \ldots, z_m) &\overset{g}{\mapsto} (z_1, \ldots, z_r, g_{r+1}(z), \ldots, g_n(z)) \\ &\overset{k}{\mapsto} (z_1, \ldots, z_r, g_{r+1}(z) - g_{r+1}(z_1, \ldots, z_r, 0, \ldots, 0), \ldots, g_n(z) - g_n(z_1, \ldots, z_r, 0, \ldots, 0)).\end{aligned}$$

For $(z_1, \ldots, z_m)$ in a sufficiently small neighborhood of 0 and $r + 1 \le i \le n$ we have $g_i(z_1, \ldots, z_n) - g_i(z_1, \ldots, z_r, 0, \ldots, 0) = 0$ because of (III.1), thus $k \circ g = k \circ f \circ h^{-1}$ is represented by

$$(z_1, \ldots, z_m) \mapsto (z_1, \ldots, z_r, 0, \ldots, 0),$$

as claimed. In the above argument we didn't care precisely about the open neighborhoods of zero on which our maps were defined, but is clear that we can replace V by a smaller open neighborhood V' of zero such that $f(V') \subset W$. Then $\widetilde{f} = \psi \circ f \circ \phi^{-1}$ is defined on all of $\phi(V)$. ■

In order to apply the theorem one must show that the rank of $T_p f$ is constant in a neighborhood of p, to wit one must exclude the possibility mentioned in Remark III.1.2.2. Without further information, this is difficult (but see Theorem III.2.9 for a situation where it can be done). If, however, the rank of $T_p f$ at p is maximal, i.e. $\operatorname{rank} T_p f = \min(\dim M, \dim N)$, it cannot increase, thus Theorem III.1.3 applies. This motivates a detailed study of the two cases $\operatorname{rank} T_p f = \dim N \le \dim M$ and $\operatorname{rank} T_p f = \dim M \le \dim N$. (In fact, most books prove the rank theorem only for these special cases, giving two different arguments. This seems somewhat unsatisfactory.)

III.1.4 Definition *A map $f : M \to N$ is an immersion (or immersive) at p if the linear map $T_p f : T_p M \to T_{f(p)} N$ is injective. It is called a submersion (or submersive) at p if $T_p f : T_p M \to T_{f(p)} N$ is surjective. A map is an immersion (submersion) if it is immersive (submersive) for all $p \in M$.*

III.1.5 Remark Clearly, $f : M \to N$ is an immersion iff $\operatorname{rank} T_p f = \dim M \le \dim N$ for all $p \in M$. Similarly, f is a submersion iff $\operatorname{rank} T_p f = \dim N \le \dim M$ for all $p \in M$. □

The special significance of immersions and submersions will become clear in Sections III.3 and IV.1, respectively. Here we only note that an immersion need not be injective: consider the map from S^1 to the 'figure 8' in $\mathbb{R}^2$.

III.2 Submanifolds

We begin by considering manifolds without boundary.

III.2.1 Definition *Let M be a manifold of dimension m without boundary and let $n \le m$. A subset $N \subset M$ is a submanifold of dimension n if for every $p \in N$ there is a chart (U, ϕ) of M around p such that $\phi(U \cap N) = \phi(U) \cap \mathbb{R}^n$, where $\mathbb{R}^n$ is identified with the subset $\{(x_1, \ldots, x_n, 0, \ldots, 0) \mid (x_1, \ldots, x_n) \in \mathbb{R}^n\}$ of $\mathbb{R}^m$.*

III.2.2 Proposition *If $N \subset M$ is a submanifold of a manifold M then N is a manifold in a canonical way. With respect to this differential structure, the inclusion map $N \hookrightarrow M$ is a smooth injective immersion.*

Proof. Let $p \in N$ and (U, ϕ) as in Definition III.2.1. Then $(U \cap N, \widetilde{\phi})$, where $\widetilde{\phi} : U \cap N \to \mathbb{R}^n$ is given by $p \mapsto (\phi_1(p), \ldots, \phi_n(p))$ is a chart around p for N. That the charts obtained in this way are mutually compatible and thus define a differentiable structure for N is obvious. The same holds for the last statement. ■

III.2.3 Exercise Prove that if $N \subset M$ is a submanifold (and thus itself a manifold) and $P \subset N$ is a submanifold then P is a submanifold of M. $\square$

III.2.4 Exercise Prove the following claims:

1. $N \subset M$ is a zero dimensional submanifold iff it is a discrete subset of M (i.e. N is discrete w.r.t. the subset topology).
2. $N \subset M$ is an m-dimensional submanifold iff N is an open subset of M (with the induced differentiable structure).

$\square$

III.2.5 Exercise For $m < n$ the following are submanifolds: $\mathbb{R}^m \subset \mathbb{R}^n$, $S^m \subset S^n$, $O(m) \subset O(n)$, $U(m) \subset U(n)$. Find further examples. $\square$

III.2.6 Remark Note that a submanifold $N \subset M$ need not be an open or closed subset: $\{(x,0) \mid x \in \mathbb{R}\} \subset \mathbb{R}^2$ is a closed submanifold, $(0,1) \subset \mathbb{R}$ is an open submanifold, and $\{(x,0) \mid x \in (0,1)\} \subset \mathbb{R}^2$ is neither open nor closed. $\square$

III.2.7 Exercise If $N \subset M$ is a submanifold then the inclusion $\iota : N \to M$ is an injective immersion and $T_p\iota(T_pN) \subset T_pM$ is a subspace of dimension $\dim N$ for every $p \in N$. We will usually identify T_pN with its image in T_pM: $T_pN \subset T_pM$. $\square$

*********** define submanifolds of manifolds with boundary ************

If $N \subset M$ is a submanifold and N and M both have a boundary, the relation between the boundaries can be quite complicated. We give some examples:

$\partial M = \partial N = \emptyset$	$\mathbb{R}^n \subset \mathbb{R}^m,\ S^n \subset S^m,\ O(n) \subset O(m),\ n < m.$
$\partial M = \emptyset,\ \partial N \neq \emptyset$	$\mathbb{R}^n_+ \subset \mathbb{R}^m,\ n \leq m;\ D^n \subset \mathbb{R}^n.$
$\partial M \neq \emptyset,\ \partial N = \emptyset$	$\mathbb{R}^m_+ \subset \mathbb{R}^n_+,\ m < n,$ where $(x_1,\dots,x_m) \mapsto (0,\dots,0,x_1,\dots,x_m).$
$\partial M \neq \emptyset,\ \partial N \neq \emptyset$	$D^n \subset \mathbb{R}^n_+.\ \mathbb{R}^m_+ \subset \mathbb{R}^n_+,\ m < n,$ where $(x_1,\dots,x_m) \mapsto (x_1,\dots,x_m,0,\dots,0).$

III.2.8 Definition *A submanifold N of a manifold M with boundary is neat if $\partial N = N \cap \partial M$.*

Of the above examples of submanifolds, those in the first row are trivially neat since $\partial M = \partial N = \emptyset$, whereas no submanifold of the second type can be neat. $D^n \subset \mathbb{R}^n_+$ (where the disc sits in the interior of $\mathbb{R}^n_+$) is not neat.

The following result provides a (rather special) way to construct submanifolds of a given manifold M. More widely applicable methods will be studied later.

III.2.9 Theorem *Let M be connected without boundary and let $f : M \to M$ be a smooth map such that $f \circ f = f$. Then $f(M)$ is a closed submanifold of M.*

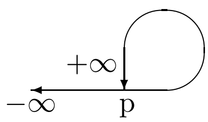
Proof. The image $f(M)$ equals the fixpoint set $\{p \in M \mid f(p) = p\}$, thus it is closed. It is easy to see that $f(M)$ is connected. Now it is sufficient to consider the map f in a neighborhood of a point p. By the chain rule, $f \circ f = f$ implies $T_{f(q)}f \circ T_qf = T_qf$ for all $q \in M$, in particular $T_pf \circ T_pf = T_pf$ for every $p \in f(M)$. Thus

$$\operatorname{im} T_pf = \{v \in T_pM \mid T_pf(v) = v\} = \ker(\mathrm{id}_{T_pM} - T_pf)$$

for all $p \in f(M)$. This implies $\dim M = \operatorname{rank} T_pf + \operatorname{rank}(\mathrm{id}_{T_pM} - T_pf)$, and both ranks can only increase in a neighborhood of p, we conclude that $\operatorname{rank} T_pf$ is locally constant on $f(M)$, thus constant since $f(M)$ is connected. Let $r = \operatorname{rank} T_p f$ for some $p \in f(M)$ be this constant. Then there is an open neighborhood U of $f(M)$ such that $\operatorname{rank} T_q f \geq r$ for all $q \in U$. Now $\operatorname{rank} T_q f = \operatorname{rank}(T_{f(q)} f \circ T_q f) \leq \operatorname{rank} T_{f(q)} f = r$, thus $\operatorname{rank} T_q f$ is constant on U. Therefore the Rank Theorem III.1.3 applies. Let $p \in f(M)$, thus $f(p) = p$. By the rank theorem there are charts (U, ϕ) and (V, ψ) around $p = f(p)$ such that $\psi \circ f \circ \phi^{-1} : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_r, 0, \ldots, 0)$ and $\phi(f(U)) = \phi(U) \cap \mathbb{R}^r$, thus $f(M)$ is a submanifold. ■

III.3 Embeddings

Given a smooth map $f : M \to N$ it is a natural question whether $f(M) \subset N$ is a submanifold. In this generality, however, the question is too difficult. We therefore limit ourselves to the more restricted question: When is $f(M) \subset N$ a submanifold such that $f : M \to f(M)$ is a diffeomorphism? Clearly, f must be injective and immersive (by Exercise II.3.6). This is, however, not sufficient. Consider a map $f : \mathbb{R} \to \mathbb{R}^2$ whose image looks like



f can easily be made injective and immersive, but $f(\mathbb{R}) \subset \mathbb{R}^2$ is not a submanifold near the point p. (With a view to Lemma II.8.6 we note that $\lim_{x \to \infty} f(x) = p$ is finite and that the image of the closed set $[x, +\infty)$ is not closed if $x > f^{-1}(p)$.)

III.3.1 PROPOSITION *Let $f : M \to N$ be a smooth map of manifolds. Then the following are equivalent:*

(i) $f(M) \subset N$ is a closed submanifold and $f : M \to f(M)$ is a diffeomorphism.

(ii) $f(M) \subset N$ is closed, f is an immersion and $f : M \to f(M)$ is a homeomorphism.

(iii) f is a proper injective immersion.

Proof. (i)⇒(ii). The diffeomorphism $f : M \to f(M)$ is a fortiori a homeomorphism. By Exercise II.3.6, $T_p f : T_p M \to T_{f(p)}(f(M))$ is invertible, thus the composition $T_p M \to T_{f(p)}(f(M)) \hookrightarrow T_{f(p)} N$ is injective.

(ii)⇒(i). Since f is an immersion we have $\operatorname{rank}(T_p f) = m$ at all $p \in M$, cf. Remark III.1.5. Thus the Rank Theorem III.1.3 applies and, for every $p \in M$, provides charts $(U, \phi), (V, \psi)$ around p and $q = f(p) \in N$, respectively, such that $f(U) \subset V$ and $\widetilde{f} : \psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$ is given by $x \mapsto (x, 0_{n-m})$, where 0_{n-m} denotes the zero of $\mathbb{R}^{n-m}$. Here $\phi(U) \subset \mathbb{R}^m$ and $\psi(V) \subset \mathbb{R}^n$ are open neighborhoods of zero. Since $m \leq n$ we can find open neighborhoods $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^{n-m}$, both containing 0, such that $A \times B \subset \psi(V)$. Then $U' = \phi^{-1}(A \cap \phi(U)) \subset M$ is open and we have $\phi(U') \times B \subset \psi(V)$. Finally, defining $V' = \psi^{-1}(\phi(U') \times B)$ we have a map $\tilde{f}$ from $\phi(U') \subset \mathbb{R}^m$ to $\psi(V') = \phi(U') \times B \subset \mathbb{R}^n$.

Since $f : M \to f(M)$ is a homeomorphism, $f(U')$ is open in $f(M)$, thus there is an open $W \subset N$ such that $f(U') = W \cap f(M)$, thus $U' = f^{-1}(W)$. Now, $(V' \cap W, \psi \upharpoonright V' \cap W)$ clearly is a chart around $q \in N$. In view of $\psi(V') = \phi(U') \times B \subset \mathbb{R}^n$, a point $y \in V' \cap W$ satisfies $\psi(y) \subset \mathbb{R}^m$ (i.e. $\psi_i(y) = 0$ for all $i > m$) iff there is $x \in U'$ such that $\psi(y) = (\phi(x), 0_{n-m})$. This is equivalent to $\psi(f(M) \cap V' \cap W) = \psi(V' \cap W) \cap \mathbb{R}^m$. Hence $f(M) \subset N$ is a submanifold and $f : M \to f(M)$ is locally smoothly invertible, thus a diffeomorphism.

(ii)⇔(iii). This follows from Lemma II.8.6. ■

III.3.2 DEFINITION *When the equivalent conditions of Proposition III.3.1 are satisfied, the map $f : M \to N$ is called an embedding and $f(M) \subset N$ an embedded submanifold.*

III.3.3 COROLLARY *Let M be compact and $f : M \to N$ an injective immersion. Then $f(M) \subset N$ is a submanifold and $f : M \to f(M)$ is a diffeomorphism.*

Proof. Let $Z \subset N$ be compact, thus closed. By continuity, $f^{-1}(Z) \subset M$ is closed, thus compact by compactness of M. Thus f is proper. ■

In Section III.5 we will show that every n-manifold admits an embedding into $\mathbb{R}^{2n+1}$. The proof requires some further preparations which will be the subject of the next section.

III.4 Measure zero in manifolds: The easy case of Sard's theorem

In this section we will develop some rudiments of measure theory in manifolds where we will only need the notion of measure zero.

III.4.1 DEFINITION *By a cube of edge λ in $\mathbb{R}^n$ we mean a product $D = \prod_{i=1}^n [a_i, b_i]$ of n intervals in $\mathbb{R}$ with $|a_i - b_i| = \lambda$ for all i. We write $|D| = \lambda^n$. Now, a set $C \subset \mathbb{R}^n$ has measure zero if for every $\varepsilon > 0$ there exists a sequence of cubes $\{D_i \subset \mathbb{R}^n\}_{i\in\mathbb{N}}$ such that*

$$C \subset \bigcup_{i=1}^{\infty} D_i \qquad \text{and} \qquad \sum_{i=1}^{\infty} |D_i| < \varepsilon.$$

III.4.2 REMARK 1. It is important to understand that measure zero is a relative notion. The interval $I = [0, 1] \subset \mathbb{R}$ has non-zero measure, but $I \times 0 \subset \mathbb{R}^2$ has measure zero!

2. We would arrive at the same notion of measure zero if we replace closed by open cubes. Since the ratio of the volumes of a cube and the circumscribed ball depends only on n, $U \subset \mathbb{R}^n$ has measure zero iff it can be covered by countably many balls of arbitrarily small total volume. Similarly, one could use rectangles, etc. □

III.4.3 EXERCISE If $U \subset \mathbb{R}^n$ has measure zero then any $V \subset U$ has measure zero. If $m < n$ then $\mathbb{R}^m \cong \mathbb{R}^m \times 0 \subset \mathbb{R}^n$ has measure zero. □

III.4.4 LEMMA *Let $(C_i \subset \mathbb{R}^n)_{i\in\mathbb{N}}$ be a sequence of sets of measure zero. Then $\bigcup_i C_i$ has measure zero.*

Proof. Let $\varepsilon > 0$. Since C_i has measure zero we can pick a sequence $\{D_i^j, j \in \mathbb{N}\}$ of cubes such that $C_i \subset \bigcup_j D_i^j$ and $\sum_j |D_i^j| < 2^{-i}\varepsilon$. Then $\{D_i^j, i, j \in \mathbb{N}\}$ is a countable cover of $\bigcup_i C_i$ and we have $\sum_{i,j} |D_i^j| < \varepsilon \sum_i 2^{-i} = \varepsilon$. ■

III.4.5 LEMMA *Let $U \subset \mathbb{R}^m$ be open and $f : U \to \mathbb{R}^m$ differentiable (C^1). If $C \subset U$ has measure zero then $f(C) \subset \mathbb{R}^m$ has measure zero.*

Proof. Let $\|\cdot\|$ be the euclidean norm on $\mathbb{R}^m$. Every $p \in U$ belongs to an open ball $B \subset U$ such that $\|T_q f\|$ is uniformly bounded on B, say by $\kappa > 0$. Then

$$\|f(x) - f(y)\| \le \kappa \|x - y\|$$

for all $x, y \in B$. Thus, if $C \subset B$ is an m-cube of edge λ then $f(C)$ is contained in an m-cube of edge less than $\sqrt{m}\kappa\lambda$. It follows that $f(C)$ has measure zero if C has measure zero. Writing U as a countable union of such C, the claim follows by Lemma III.4.4. ■

The preceding lemma shows that the following definition has a coordinate independent sense:

III.4.6 DEFINITION *A subset C of a manifold M has measure zero iff $\phi(U \cap C)$ has measure zero in $\mathbb{R}^n$ for every chart (U, ϕ) in the maximal atlas of M.*

III.4.7 EXERCISE 1. Use Lemma III.4.5 to show that $C \subset M$ has measure zero iff $\phi(U \cap C) \subset \mathbb{R}^m$ has measure zero for every chart (U, ϕ) in *some* atlas compatible with the differential structure of M.

2. If $C \subset M$ has measure zero then $M - C$ is dense in M. □

III.4.8 LEMMA *Let $U \subset \mathbb{R}^m$ be open and $f : U \to \mathbb{R}^n$ differentiable, where $n > m$. Then $f(U) \subset \mathbb{R}^n$ has measure zero.*

Proof. Define $\widehat{f} : U \times \mathbb{R}^{n-m} \to \mathbb{R}^n$ by $\widehat{f}(x, y) = f(x)$. Since $U \times \{0\} \subset \mathbb{R}^n$ has measure zero, Lemma III.4.5 implies that $f(U) = \widehat{f}(U \times \{0\}) \subset \mathbb{R}^n$ has measure zero. ■

Now we can state the easy case of Sard's theorem:

III.4.9 PROPOSITION *Let $f : M \to N$ a smooth map of manifolds, where $\dim M < \dim N$. Then $f(M)$ has measure zero in N.*

Proof. Let $(U_i, \phi_i), (V_j, \psi_j)$ be countable atlasses for M and N, respectively. Then

$$\psi_j(f(M) \cap V_j) = \bigcup_i \psi_j(f(U_i) \cap V_j) = \bigcup_i (\psi_j \circ f \circ \phi_i^{-1}) \left(\phi_i(U_i \cap f^{-1}(V_j))\right).$$

Now, $\psi_j \circ f \circ \phi_i^{-1}$ is a smooth map from an open subset of $\mathbb{R}^m$ to $\mathbb{R}^n$, thus the measure of its image is zero by Corollary III.4.8. Thus $\psi_j(f(M) \cap V_j)$ has measure zero by Lemma III.4.4, and this is what is required by Definition III.4.6. ■

The general version of Sard's theorem will be given in Section IV.2. We give a first application of Proposition III.4.9.

III.4.10 DEFINITION *Two smooth maps $f, g : M \to N$ are smoothly homotopic if there is a smooth map $h : M \times [0, 1] \to N$ such that $h_0 = f$ and $h_1 = g$, where we write $h_t = h(\cdot, t)$.*

III.4.11 THEOREM *If M is a manifold of dimension $m < n$ then any smooth map $f : M \to S^n$ is smoothly homotopic to a constant map.*

Proof. By Proposition III.4.9, $f(M) \subset S^n$ has measure zero, thus there is a point $q \in S^n$ not in the image of f. Therefore f maps into the $X = S^n - \{q\}$, which is smoothly homeomorphic to $\mathbb{R}^n$, and therefore contractible (to wit, there is a smooth map $r : X \times [0, 1] \to X$ such that $r(x, 0) = x$ for all $x \in X$ and $x \mapsto r(x, 1)$ is a constant map). Composing f with such a contraction gives the desired homotopy. ■

For later purposes we prove that smooth homotopies behave similarly to continuous homotopies.

III.4.12 LEMMA *Smooth homotopy is an equivalence relation. (The set of smooth homotopy classes of smooth maps $X \to Y$ will be denoted by $[X, Y]_s$.)*

Proof. Symmetry and reflexivity are obvious, but transitivity requires proof. Let $\varphi : [0, 1] \to [0, 1]$ be a smooth function such that $\varphi(t) = 0$ for $t < 1/3$ and $\varphi(t) = 1$ for $t > 2/3$. (For example, let $\varphi(t) = \lambda(t - 1/3)/(\lambda(t - 1/3) + \lambda(2/3 - t))$ where $\lambda(t) = 0$ for $t \leq 0$ and $\lambda(t) = e^{-1/t}$ for $t > 0$.) If now h is a smooth homotopy between f and g, define $h'(x, t) = h(x, \varphi(t))$. Then h' is a smooth homotopy between f and g that is constant as a function of t for $t < 1/3$ and $t > 2/3$. The usual (double speed) composite of two such modified homotopies is a smooth homotopy. ■

III.5 Whitney's embedding theorem

III.5.1 Lemma *Let M be a compact manifold (with or without boundary). Then there exists an embedding $\Psi : M \to \mathbb{R}^n$ for some $n \in \mathbb{N}$.*

Proof. As usual let $m = \dim M$. Let $\{(U_i, \phi_i)\}$ be an atlas. By compactness finitely many of the U_i suffice to cover M, thus we may assume the atlas to be finite with $i = 1, \dots, k$, and by Lemma II.9.5 we can also find sets V_i still covering M such that $\overline{V_i} \subset U_i$. Furthermore, Proposition II.10.3 provides smooth functions $\lambda_i : M \to \mathbb{R}$ which are 1 on $\overline{V_i}$ and have support in U_i. Defining $\psi_i : M \to \mathbb{R}^m$ to be $\lambda_i(p)\phi_i(p)$ for $p \in U_i$ and zero otherwise, ψ_i is smooth. Now define $\Psi : M \to (\mathbb{R}^m)^k \times \mathbb{R}^k$ by $\Psi = (\psi_1, \dots, \psi_k, \lambda_1, \dots, \lambda_k)$. We claim that Ψ is injective: If $\Psi(p) = \Psi(q)$ then $\lambda_i(p) = \lambda_i(q)$ for all i. Now, $p \in V_j$ for some j, thus $\lambda_j(p) = 1$. Since also $\lambda_j(q) = 1$, we have $q \in U_j$, and $\phi_j(p) = \lambda_j(p)\phi_j(p) = \psi_j(p) = \psi_j(q) = \lambda_j(q)\phi_j(q) = \phi_j(q)$ implies $p = q$ since $\phi_i : U_j \to \mathbb{R}^n$ is injective. Next we show that Ψ is an immersion, i.e. $\Psi_* = T_p\Psi$ is injective for all $p \in M$. Again, $p \in V_j$ for some j and thus $\lambda_j = 1$ on a neighborhood of p. Now $\psi_j = \phi_j$, and $\psi_{j*} = \phi_{j*}$ is injective since ϕ_j is a chart.

Thus $\Psi : M \to \mathbb{R}^{k(m+1)}$ is an injective immersion. Since M is compact, Corollary III.3.3 applies and Ψ is an embedding. ■

III.5.2 Proposition *Let M be a compact manifold of dimension m. Then there exists an embedding $\Psi : M \to \mathbb{R}^{2m+1}$.*

Proof. We know already that there exists an embedding $\Psi : M \to \mathbb{R}^n$ for some n. The proposition thus follows by induction if we can show that n can be reduced by one provided $n > 2m+1$. For any non-zero $a \in \mathbb{R}^n$ we let π_a be the orthogonal projection onto the orthogonal complement $a^\perp \cong \mathbb{R}^{n-1}$. (Thus $\pi_a(x) = x - a(a,x)/(a,a)$, where $(\cdot,\cdot)$ is some scalar product on $\mathbb{R}^n$.) We write $\Psi_a = \pi_a \circ \Psi$ and claim that there exists $a \neq 0$ such that Ψ_a is an embedding.

To prove this we define $h : M \times M \times \mathbb{R} \to \mathbb{R}^n$ by $h(p,q,t) = t(\Psi(p) - \Psi(q))$ and $g : TM \to \mathbb{R}^n$ by $g(p,v) = T_p\Psi(v)$. In view of $\dim M \times M \times \mathbb{R} = 2m+1$, $\dim TM = 2m$ and our assumption $n > 2m+1$, the (easy case of) Sard's theorem, cf. Proposition III.4.9, implies that $\operatorname{im} h \cup \operatorname{im} g$ has measure zero, thus there exists a point $a \in \mathbb{R}^n - \operatorname{im} h - \operatorname{im} g$. Note that $a \neq 0$ since 0 belongs to both images. Now assume $\Psi_a(p) = \Psi_a(q)$, which is equivalent to $\Psi(p) - \Psi(q) = \lambda a$. By assumption Ψ is an embedding, thus injective. Assuming $p \neq q$ we therefore have $\lambda \neq 0$ and we can write $a = \lambda^{-1}(\Psi(p) - \Psi(q)) = h(p,q,\lambda^{-1})$. This is in contradiction with our choice of $a \notin \operatorname{im} h$, thus Ψ_a is injective.

Next, suppose $T_p\Psi_a(v) = 0$ for some $v \in T_pM$. By definition of Ψ_a this is equivalent to $T_p\Psi(v) = \lambda a$. Again, assuming $v \neq 0$ we have $T_p\Psi(v) \neq 0$ since Ψ is an immersion. Thus $\lambda \neq 0$ and $a = \lambda^{-1}T_p\Psi(v) = T_p(\lambda^{-1}v)$ in contradiction with $a \notin \operatorname{im} g$. Thus Ψ_a is an immersion. By Corollary III.3.3, Ψ_a is an embedding. ■

III.5.3 Remark 1. We have actually proven a bit more than stated: On the one hand, it is clear from the proof that every compact n-manifold admits an immersion into $\mathbb{R}^{2n}$. On the other hand, if a not necessarily compact manifold is already given as a submanifold of some $\mathbb{R}^n$, the preceding arguments provide an immersion into $\mathbb{R}^{2n}$ and an injective immersion into $\mathbb{R}^{2n+1}$. In the non-compact case there are two problems: in Lemma III.5.1 we cannot always find a finite atlas, and in Proposition III.5.2, an injective immersion need not be an embedding. Nevertheless, we will prove that Theorem III.5.4 generalizes to non-compact manifolds. In fact, every n-manifold, whether compact or not, admits an embedding into $\mathbb{R}^{2n}$, but this is more difficult to prove (Whitney 1944).

2. If one asks for the lowest n such that every m-manifold admits an immersion, not necessarily injective, into $\mathbb{R}^n$ the answer is $n = 2m - \alpha(m)$, where $\alpha(m)$ is the number of non-zero digits in the binary representation of m. The proof (1985) is very difficult. □

III.5.4 Theorem (Whitney) *Every manifold of dimension m admits an embedding into $\mathbb{R}^{2m+1}$.*

Proof. Since M is locally compact we can find an cover $(U_i,\, i \in I)$ of M by open sets with compact closures. Since M is paracompact this cover admits a locally finite refinement $(V_j,\, j \in J)$. Obviously also the V_j have compact closures. By Proposition II.1.6, we can replace J by a countable subset J_0. We index this cover by $\mathbb{N}$ and call it $(V_j, j \in \mathbb{N})$. Let (λ_j) be a partition of unity subordinate to (V_j). We index $(V_j), (\lambda_j)$ by the natural numbers and define $\eta(x) = \sum_{i\in\mathbb{N}} i\lambda_i(x)$. By local finiteness, this is a smooth map, and it is proper since $\eta^{-1}([1, N]) \subset \bigcup_{i=1}^N V_i$. Now let $U_i = \eta^{-1}\left((i - \frac{2}{3}, i + \frac{2}{3})\right)$ and $C_i = \eta^{-1}\left([i - \frac{3}{4}, i + \frac{3}{4}]\right)$. Then U_i is open, C_i is compact and $\overline{U_i} \subset C_i^0$. Furthermore, all the C_i with even indeces are mutually disjoint and the same holds for the odd indexes. The above methods gives us smooth maps $\Psi_i : M \to \mathbb{R}^{2m+1}$ that are embeddings on $\overline{U_i}$ and map the complement of C_i to 0. (Note: While every open subset U of a manifold is a manifold, this does not need to be true for $\overline{U}$: The closure of an open square in $\mathbb{R}^2$ has corners. However, Lemma III.5.1 and Proposition III.5.2 still apply to $\overline{U}$ as is clear from their proofs.) Composing with a diffeomorphism of $\mathbb{R}^{2m+1}$ to an open ball in $\mathbb{R}^{2m+1}$ we may assume that the images of all Ψ_i are contained in the same bounded subset. Now define $\Psi_e = \sum_i \Psi_{2i}$, $\Psi_o = \sum_i \Psi_{2i-1}$ and $\Psi = (\Psi_e, \Psi_o, \eta) : M \to \mathbb{R}^{2m+1} \times \mathbb{R}^{2m+1} \times \mathbb{R}$. If $\Psi(x) = \Psi(y)$ then $\eta(x) = \eta(y)$, thus x, y are in the same U_i. If i is odd (even) then Ψ_o (Ψ_e) is an embedding on U_i, implying $x = y$. Thus Ψ is an injective immersion. Since η is proper, Ψ is proper and thus an embedding. By construction, $\Psi(M) \subset K \times \mathbb{R}$ with $K \subset \mathbb{R}^{2(2m+1)}$ compact. As remarked before, the cut down argument of Proposition III.5.2 works also for non-compact M and provides a projection $\pi : \mathbb{R}^{2(2m+1)+1} \to \mathbb{R}^{2m+1}$ onto a hyperplane such that $\Psi' = \pi \circ \Psi$ is an injective immersion. π can be chosen such that its kernel does not contain the last coordinate axis. Then Ψ' is still proper, thus an embedding by Proposition III.3.1 and Lemma II.8.6. ■

III.5.5 Remark The theorem says that every manifold (smooth, finite dimensional) is a submanifold of some $\mathbb{R}^n$. This might be compared with the result that every finite group and every compact Lie group is a matrix group, i.e. a subgroup of $GL(N, \mathbb{C})$ for some N. Thus one could in principle dispense with the abstract notion of a manifold in the sense of Definition II.1.9 and consider only embedded manifolds. (This is in fact the approach of [19, 12].) There are however good reasons for not doing so: On the one hand the abstract perspective keeps the focus on the relevant intrinsic properties, the manifold or group structure and not the embedding. More importantly, many constructions produce only the abstract manifold or group structure, but no embedding. E.g., the automorphism group of some structure, even when finite or compact Lie, does not usually come with an embedding into $GL(N, \mathbb{C})$, and similarly the Riemann surface constructed from a germ of a holomorphic function is an abstract manifold without given embedding into $\mathbb{R}^N$. Thus the supposedly more concrete embedded approach would make life much more difficult. □

III.5.6 Remark Every embedding $\Phi : M \to \mathbb{R}^d$ of a manifold gives rise to a riemannian metric on M: Let $(\cdot, \cdot)$ be the scalar product $a \times b \mapsto \sum_i a_i b_i$ on $\mathbb{R}^d$ and define

$$\langle X, Y\rangle_p = (T_p\Phi(X(p)), T_p\Phi(Y(p))).$$

(The easy verification that this is a metric is left to the reader.) Together with Theorem III.5.4 this provides an alternative proof of Lemma II.11.4. It is natural to ask whether all riemannian metrics arise in this way, or equivalently whether every (smooth) riemannian manifold can be embedded isometrically into $\mathbb{R}^d$. This was indeed proven by Nash, first for C^1-manifolds and then in the smooth case [46]. □

III.5.7 Remark If M is an manifold M *with* boundary the preceding arguments still give an embedding Ψ into some $\mathbb{R}^m$. Let us show that one can fine an embedding $\Psi' : M \to \mathbb{R}^{m+1}$ such that $\Psi(M)$ is a neat submanifold, i.e. $\partial\Psi(M) = \Psi(M) \cap \partial\mathbb{R}_+^{m+1}$. Let $\alpha : U \to \partial M \times [0, 1)$ be a collar of ∂M and write $\alpha_1 : U \to [0,1)$ for the second component. Let $\beta : [0,1] \to [0,1]$ be a smooth function satisfying $\beta^{-1}(0) = \{0\}$, $\beta \restriction [1-\varepsilon, 1] = 1$ for some $\varepsilon > 0$, and $\beta'(0) > 0$. Now define $\phi : M \to [0,1]$ to be $\beta \circ \alpha_1$ on U and the constant function 1 on $M - U$. This function is smooth and satisfies $\phi^{-1}(0) = \partial M$. Clearly $\Psi' = (\phi, \Psi) : M \to \mathbb{R}^{m+1}_+$ is an injective immersion, and by Proposition II.8.4 it is proper, thus an embedding. Since α_1, thus ϕ vanishes precisely on ∂M we have $\Psi'^{-1}(\partial \mathbb{R}^{m+1}) = \partial M$. Since $\alpha_1 = 0$ on ∂M but $T_p\alpha_1 \neq 0$, also ϕ has non-zero derivative on ∂M, thus the embedding is neat. □

III.5.8 Remark It is no exaggeration to say that the four most important technical *tools* in differential topology are (i) partitions of unity, (ii) the rank theorem, (iii) Sard's theorem and (iv) flows arising from suitable vector fields. There is no non-trivial proof that does not use at least one of these tools. This is nicely illustrated by the neat embedding theorem for manifolds with boundary, which relies on the rank theorem via Proposition III.3.1, on partitions of unity via Lemma III.5.1, on Sard's theorem via Proposition III.5.2 and on flows via Theorem **??**. However, the most important *concept* of differential topology is probably that of transversality or general position. In the next chapter we will consider the simplest version in the guise of the theory of regular values. The general theory will be discussed later. □

III.6 Digression: The tangent groupoid

In the proof of Proposition III.5.2 we considered maps from $h : M \times M \times \mathbb{R} \to \mathbb{R}^n$ and $h : TM \to \mathbb{R}^n$ associated with an embedding $\Phi : M \to \mathbb{R}^n$. In this section we show that there is a manifold $\mathfrak{T}M$, called the 'tangent groupoid', of dimension $2m+1$ constructed from $M \times M \times \mathbb{R}$ and TM. The significance of $\mathfrak{T}M$ derives from its applications in quantization theory which we cannot go into. We just give its definition as another example of a canonically constructed manifold.

Let $I \subset \mathbb{R}$ be connected and containing zero. Then as a set, $\mathfrak{T}_I M$ is defined as the disjoint union

$$\mathfrak{T}_I M = M \times M \times (I - \{0\}) \coprod TM.$$

Chapter IV

Transversality Theory I: The degree and its applications

IV.1 Inverse images of smooth maps

Consider a map $f : M \to N$ and a subset $L \subset N$. In this section we ask which subsets of M can appear as inverse image $f^{-1}(L)$ and when this is a submanifold. Our first result shows any closed subset $A \subset M$ appears as zero set of a smooth $\mathbb{R}$-valued function.

IV.1.1 PROPOSITION (WHITNEY) *Let M be a manifold and $A \subset M$ a closed subset. Then there exists a smooth function $f : M \to \mathbb{R}$ such that $A = f^{-1}(0)$.*

IV.1.2 LEMMA *Let $A \subset U \subset \mathbb{R}^n$ with A closed and U open. Then there exists a smooth function $\psi : U \to \mathbb{R}$ such that $A = f^{-1}(0)$.*

Proof. Since the open balls are a basis for the topology of $\mathbb{R}^n$ we can write the open set $U - A$ as a countable unit of open balls $(B_i = B(q_i, R_i), i \in \mathbb{N})$. We choose smooth functions $\psi_i : U \to [0, \infty)$ such that

(a) $\psi_i(x) > 0$ iff $x \in B_i$.

(b) ψ_i and all its derivatives up to order i are uniformly bounded by 2^{-i}.

(To satisfy condition (a) let $\psi_i(p) = F(|p - q_i|/R_i)$, where $F : \mathbb{R} \to \mathbb{R}$ is as in Lemma II.10.1. Since ψ_i and all derivatives are bounded, Condition (b) can be enforced by multiplying ψ_i by a sufficiently small positive number.) We now write $\psi = \sum_i \psi_i$. In view of $\|\psi_i^{(n)}\|_\infty \leq 2^{-i}\ \forall n \leq i$ we have

$$\left|\sum_{i=1}^{\infty} \psi_i^{(n)}\right| = \left|\sum_{i=1}^{n-1} \psi_i^{(n)} + \sum_{i=n}^{\infty} \psi_i^{(n)}\right| \leq \left|\sum_{i=1}^{n-1} \psi_i^{(n)}\right| + \left|\sum_{i=n}^{\infty} \psi_i^{(n)}\right| \leq \left|\sum_{i=1}^{n-1} \psi_i^{(n)}\right| + \sum_{i=n}^{\infty} 2^{-i},$$

thus the sum converges uniformly on all of U. Thus ψ is a smooth function. In view of (a), $\psi(x) > 0$ iff $x \in B_i$ for some i, thus $\psi(x) > 0$ iff $x \notin A$. ■

Proof of the proposition. Let (U_i, ϕ) be a locally finite atlas and $(\lambda_i)_{i \in I}$ subordinate partition of unity with $\operatorname{supp} \lambda_i \subset U_i$. Then $A \cap \operatorname{supp} \lambda_i$ is a closed subset of U_i and using the homeomorphism $\phi_i : U_i \to \phi(U_i)$ and the lemma, we find a smooth function $\eta_i : U_i \to [0, \infty)$ such that $\eta_i(p) = 0$ iff $p \in A \cap \operatorname{supp} \lambda_i$. We extend η_i by declaring it to be zero on $M - U_i$ and define $\eta = \sum_i \lambda_i \eta_i$. (This is well defined since the partition is locally finite.) If $x \in A$ then $\eta_i(x) = 0$ for all i, thus $\eta(x) = 0$. If $x \notin A$ then $\lambda_i(x) > 0$ for some i, and $x \notin A \cap \operatorname{supp} \lambda_i$. Thus $\eta_i > 0$ and $\eta(x) \geq \lambda_i(x)\eta_i(x) > 0$. ■

The above proposition is not very useful in practice, precisely because it is so general. Its main significance lies in showing that in order for $A = f^{-1}(L)$ to be a submanifold we need to impose requirements on the function f and the subset $L \subset N$. We begin with the special case where $L = \{q\}$.

IV.1.3 DEFINITION *Given $f : M \to N$, a point $p \in M$ is called regular if $T_p f : T_p M \to T_{f(p)} N$ is surjective, i.e., f is submersive at p, and critical otherwise. A point $q \in N$ is called a regular value iff every $p \in f^{-1}(q)$ is a regular point. Otherwise it is a critical value.*

IV.1.4 LEMMA *Let $f : M \to N$ be a smooth map of manifolds without boundary and let $q \in N$ a regular value. If $f^{-1}(q)$ is non-empty then $W = f^{-1}(q) \subset M$ is a submanifold of dimension $\dim M - \dim N$. For $p \in W$ we have $T_p W = \{v \in T_p M \mid T_p f(v) = 0\}$.*

Proof. If $f(p) = q$ then $\operatorname{rank} T_p f = \dim N = n$ in a neighborhood of p, thus by the rank Theorem III.1.3 there are charts (U, ϕ), (V, ψ) around p and q, respectively, such that $f(U) = V$ and $\psi \circ f \circ \phi^{-1}$ is of the form $(x_1, \dots, x_n, x_{n+1}, \dots, x_m) \mapsto (x_1, \dots, x_n)$. Since $\psi(q) = 0 \in \mathbb{R}^n$, we have $\phi(f^{-1}(q) \cap U) = \phi(U) \cap \mathbb{R}^{m-n}$, where $\mathbb{R}^{m-n}$ sits in $\mathbb{R}^m$ as $0_{\mathbb{R}^n} \times \mathbb{R}^{m-n}$. This is precisely the definition of a submanifold. The above local description of $W \subset M$ also implies the claim on $T_p W$. ■

IV.1.5 REMARK If $\dim M < \dim N$ then every $p \in M$ is critical, thus the set of critical values coincides with the image $f(M) \subset N$. Therefore Lemma IV.1.4 is empty if $\dim M < \dim N$. □

IV.1.6 EXERCISE Show that p is a regular point of $f : M \to \mathbb{R}$ iff there exists a chart (U, ϕ) around p such that the partial derivatives $\partial(f \circ \phi^{-1}(x_1, \dots, x_m))/\partial x_i, i = 1, \dots, m$ do not all vanish at $x = 0$. □

IV.1.7 EXAMPLE Let $M = \mathbb{R}^n$ and $f : M \to \mathbb{R}$ given by $(x_1, \dots, x_n) \mapsto x_1^2 + \cdots + x_n^2$. We claim that every $a \neq 0$ is a regular value: If $a < 0$ then $f^{-1}(a) = \emptyset$. If $a > 0$ then $f(x) = a$ implies that some x_1 is non-zero. Then $\partial f / \partial x_i = 2x_i \neq 0$, thus x is a regular point. We have thus shown that the sphere $f^{-1}(a)$ is a submanifold of $\mathbb{R}^n$.

IV.1.8 EXERCISE Let $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ denote the set of real $n \times n$ matrices. Show that the orthogonal group $O(n) = \{A \in M_n(\mathbb{R}) \mid AA^T = \mathbf{1}\}$ consisting of orthogonal matrices is a submanifold of dimension $n(n-1)/2$. □

We now generalize Lemma IV.1.4 to the situation where M has a boundary.

IV.1.9 LEMMA *Let M be a manifold without boundary and let $f : M \to \mathbb{R}$ be a smooth function with zero as regular value. Then the subset $S = \{x \in M \mid f(x) \geq 0\}$ is a manifold with boundary $\partial S = \{x \in M \mid f(x) = 0\}$.*

Proof. The set where $f > 0$ is open in M and therefore a submanifold of the same dimension as M. Suppose $f(x) = 0$. Since f is regular at x, by the rank Theorem III.1.3 it is locally equivalent to the canonical submersion $(x_1, \dots, x_m) \mapsto x_1$. But for the latter, the lemma is obvious. ■

IV.1.10 PROPOSITION *Let $f : M \to N$ be smooth with $\partial N = \emptyset$ and let $q \in N$ a regular value for f and $\partial f = f \restriction \partial M$. If $f^{-1}(q)$ is non-empty then $f^{-1}(q) \subset M$ is a neat submanifold (i.e. $\partial(f^{-1}(q)) = f^{-1}(q) \cap \partial M$) of dimension $\dim M - \dim N$.*

Proof. Let m, n be the dimensions of M, N, respectively. $M - \partial M$ and ∂M are manifolds without boundary and by the regularity assumptions on q, Lemma IV.1.4 implies that $f^{-1}(q) \cap (M - \partial M)$ and $f^{-1}(q) \cap \partial M$ are submanifolds (without boundary) of dimensions $m-n$ and $m-n-1$, respectively, and we must show that their union is a manifold with boundary. Since this is a local property it suffices to consider the case where $M = \mathbb{R}^m_+$. Let $p \in \partial\mathbb{R}^m_+ \cap f^{-1}(q)$ and let $U \subset \mathbb{R}^m$ be an open neighborhood of p. One can find a smooth map $g : U \to N$ coinciding with f on $U \cap \mathbb{R}^m_+$. Replacing U by a smaller neighborhood if necessary, we may assume that g has no critical points. Thus $g^{-1}(q) \subset \mathbb{R}^m$ is a smooth manifold of dimension $m - n$. Now, the tangent space at p of $g^{-1}(q)$ is the kernel of the map $T_p g = T_p f : T_p\mathbb{R}^m_+ \to T_q N$, and the hypothesis that q is a regular value of $\partial f = f \restriction \partial\mathbb{R}^m_+$ implies that this kernel cannot completely be contained in $0 \times \mathbb{R}^{m-1}$. Therefore zero is a regular value of the coordinate projection $\pi : g^{-1}(q) \to \mathbb{R}$, $(x_1, \ldots, x_m) \to x_1$, and Lemma IV.1.9 implies that $g^{-1}(q) \cap \mathbb{R}^m_+$ is a manifold with boundary $\pi_1^{-1}(0)$. Since the latter two sets coincide with $f^{-1}(q) \cap U$ and $f^{-1}(q) \cap U \cap \partial\mathbb{R}^m_+$, respectively, we are done. ■

In some applications, like Example IV.1.7 and Exercise IV.1.8, one needs to show that a specific value $q \in N$ is regular. In many other applications, some of which will be considered soon, it is sufficient to show that a regular value of $f : M \to N$ exists. That regular values always exist (and in fact are dense) is the content of Sard's theorem which we will now prove in its general form.

IV.2 Sard's theorem: The general case

Sard's theorem, is one of the cornerstones of differential topology – most of the subsequent developments will rely on it. As in most other treatments, our proof essentially is the one of [19].

IV.2.1 THEOREM *The set of critical values of a smooth map $f : M \to N$ has measure zero in N.*

IV.2.2 REMARK 1. Thus the regular values are dense in N.

2. The theorem is blatantly wrong if one replaces 'critical values' by 'critical points'! E.g., if $f : M \to N$ is a constant map then all $p \in M$ are critical, thus the critical points have non-zero measure.

3. If $\dim M < \dim N$ it follows from Remark IV.1.5 that the theorem reduces to Proposition III.4.9. □

The proof will use Fubini's lemma, to be proven first. We denote by $\mathbb{R}^{n-1}_t$ the subset $\mathbb{R}^{n-1} \times t \subset \mathbb{R}^n$.

IV.2.3 LEMMA *An open cover of the interval $[0, 1]$ contains a finite subcover by intervals $I_j, j = 1, \ldots, k$ such that $\sum_{j=1}^k |I_j| \leq 2$.*

Proof. By compactness a finite subcover $I_j, j = 1, \ldots, k$ exists, and we may assume that it minimal, i.e. none of the I_j may be omitted. Then every point p of $[0, 1]$ is contained in at most two of the I_j: Assume $p \in I_1 \cap I_2 \cap I_3$ and let $s = \min(I_1 \cup I_2 \cup I_3)$, $t = \max(I_1 \cup I_2 \cup I_3)$. Now one of the intervals, say I_1, contains $[s, p]$ and another one, say I_2, contains $[p, t]$. But now $I_1 \cup I_2 = [s, t]$ and I_3 is superfluous, contradicting the minimality of the covering. Thus the I_j cover $[0, 1]$ at most twice and the claim follows. ■

IV.2.4 PROPOSITION (FUBINI'S LEMMA) *Let C be a a countable union of compact subsets of $\mathbb{R}^n$ such that $C_t = C \cap \mathbb{R}^{n-1}_t$ has measure zero in $\mathbb{R}^{n-1}_t \cong \mathbb{R}^{n-1}$ for each $t \in \mathbb{R}$. Then C has measure zero.*

Proof. We give the proof assuming that C is compact, leaving the generalization as an exercise. Then there exists an interval $[a, b] \subset \mathbb{R}$ such that $C \subset [a, b]^n$. Let $\varepsilon > 0$. By assumption, $C_t = C \cap \mathbb{R}^n_t$, considered as a subset of $[a, b]^{n-1}$ has measure zero, thus can be covered by open cubes $W^i_t \subset [a, b]^{n-1}$, $i \in \mathbb{N}$, such that $\sum_i |W^i_t| \leq \varepsilon$. Let $W_t = \cup_i W^i_t \subset [a, b]^{n-1}$. For any fixed $t \in \mathbb{R}$, the map $d_t : \mathbb{R}^n \to \mathbb{R},\ x \mapsto |x_n - t|$ is continuous and vanishes precisely on $\mathbb{R}^{n-1}_t$. Since W_t is open, $([a,b]^{n-1} - W_t) \times [a,b]$ is closed, thus $C - (W_t \times [a,b])$ is compact, and d_t assumes a minimum $\alpha > 0$ on this set. This implies

$$\{x \in C \mid |x_n - t| < \alpha\} \ \subset\ W_t \times I_t \quad \text{with} \quad I_t = (t - \alpha, t + \alpha). \tag{IV.1}$$

The intervals I_t constructed in this way cover $[0,1]$, thus by Lemma IV.2.3 there is a finite subcover $I_1, \ldots, I_k$ of volume ≤ 2. Here $I_i = I_{t_i}$ for some $t_i \in [0,1]$. In view of (IV.1), the boxes

$$\{W^i_{t_j} \times I_j \mid i \in \mathbb{N},\ j = 1, \ldots, k\}$$

cover C and have total volume $< 2\varepsilon$, whence the claim. ■

IV.2.5 Exercise Conclude the proof by considering the case where C is a countable union of compact sets. (This family includes open sets and closed sets and is stable w.r.t. countable unions and intersections as well as under continuous images.) □

Proof of Sard's theorem. In view of Lemma III.4.5 and the fact that every manifold admits a countable atlas, it suffices to prove that $f(U)$ has measure zero for a smooth map $f : U \to \mathbb{R}^n$ when $U \subset \mathbb{R}^m$ is open. In this situation, let $D \subset U$ be the set of critical points and let D_i denote the set of $p \in U$ at which all partial derivatives of f of order $\le i$ vanish. The D_i form a descending sequence $D_0 \supset D_1 \supset D_2 \supset \ldots$ of closed sets. We will prove

(a) $f(D - D_1)$ has measure zero.

(b) $f(D_i - D_{i+1})$ has measure zero for all i.

(c) $f(D_k)$ has measure zero for sufficiently large k.

The claim then follows by Lemma III.4.4.

We begin with the proof of (c) which is similar to that of Lemma III.4.5. Let $W \subset U$ be a cube of edge a, and let $k > m/n - 1$. We will show that $f(W \cap D_k)$ has measure zero, which is sufficient since U is a countable union of cubes. For $x \in D_k \cap W$ and $x + h \in W$, Taylor's formula gives

$$|f(x+h) - f(x)| \le c|h|^{k+1},$$

where c depends only on f and W. We decompose W into a union of r^m cubes of edge a/r. If W_1 is one of these small cubes containing $x \in D_k$, every point in W_1 is of the form $x+h$ where $|h| \le \sqrt{m}a/r$. Thus by the above estimate, $f(W_1)$ is contained in a cube of edge

$$2 \cdot c \cdot \left(\frac{\sqrt{m} \cdot a}{r}\right)^{k+1} = \frac{b}{r^{k+1}},$$

where the constant b depends only on f and W but not on r. The union of these cubes has total volume $s \le r^m \cdot b^n / r^{n(k+1)}$, and this expression tends to zero as $r \to \infty$, provided $n(k+1) > m$. Thus the volume sum can be made arbitrarily small by choosing a sufficiently fine subdivision of W.

We now turn to the proof of (a). Note first that if $n = 1$ then $D = D_1$ (recall Exercise IV.1.6), thus the claim is trivially true. We may thus assume that $n \ge 2$. Claim (a) will now be proven by induction over m. To begin the induction note that for $m = 1, n \ge 2$, $f(U)$ has measure zero by the easy case of Sard's theorem proven earlier, cf. Proposition III.4.9. We now assume that (a) holds for every $f : \mathbb{R}^{m-1} \supset U \to \mathbb{R}^n$ and it for maps $f : \mathbb{R}^m \supset U \to \mathbb{R}^n$. Around each $x \in D - D_1$ we will find an open set V such that $f(V \cap D)$ has measure zero. Since $D - D_1$ is covered by countably many such neighborhoods, this proves that $f(D - D_1)$ has measure zero. If $x \in D - D_1$ then there is a partial derivative that does not vanish, say $\partial f_1/\partial x_1 \ne 0$. Then the map $h : (x_1, \ldots, x_m) \mapsto (f_1(x), x_2, \ldots, x_m)$

has non-singular Jacobian at x, thus by the inverse function theorem it maps a neighborhood V of x diffeomorphically onto an open set V'. The transformed map $g = f \circ h^{-1} : V' \to \mathbb{R}^n$ has the form

$$g : (z_1, \ldots, z_m) \mapsto (z_1, g_2(z), \ldots, g_n(z))$$

around $h(x)$. This clearly maps the hyperplane $\{z \mid z_1 = t\}$ into the hyperplane $\{y \mid y_1 = t\}$. Note that the set D' of critical points of g is precisely $h(V \cap D)$. Denoting by

$$g^t : (t \times \mathbb{R}^{m-1}) \cap V' \to t \times \mathbb{R}^{n-1}$$

the restriction of g, a point in $(t \times \mathbb{R}^{m-1}) \cap V'$ is critical for g iff it is critical for g^t since the Jacobian is given by

$$Dg = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline ? & Dg^t \\ \hline \end{array}$$

which is non-singular iff Dg^t is non-singular. By the induction assumption, the set of critical values of g^t has measure zero in $t \times \mathbb{R}^{m-1}$, thus the set of critical values of g has measure zero intersection with each hyperplane $\{y \mid y_1 = t\}$. Since $g(D')$ is a countable union of compact sets, Fubini's lemma applies and we conclude that $g(D') = f(V \cap D)$ has measure zero.

The proof of (b) is similar, in that it also works by induction over m. For every $x \in D_k - D_{k+1}$ there is a $(k+1)$-th derivative that does not vanish at x. We may assume $\partial^{k+1} f_r / \partial x_1 \partial x_{\nu_1} \cdots \partial x_{\nu_k} \neq 0$. Let $w : U \to \mathbb{R}$ be the function $w = \partial^k f_r / \partial x_{\nu_1} \cdots \partial x_{\nu_k}$. Then $w(x) = 0$, $\partial w / \partial x_1(x) \neq 0$, and as above the map $h : x \mapsto (w(x), x_2, \ldots, x_n)$ is a diffeomorphism $h : V \to V'$ for some neighborhood V of x, and $h(D_k \cap V) \subset 0 \times \mathbb{R}^{n-1} \subset \mathbb{R}^n$. Considering again the transformed map $g = f \circ h^{-1} : V' \to \mathbb{R}^m$ and its restriction $g^0 : (0 \times \mathbb{R}^{m-1}) \cap V' \to \mathbb{R}^n$, the set of critical values of g^0 has measure zero by induction assumption. But each point of of $h(D_k \cap V)$ is critical for g^0 since all partial derivatives of g at that point, thus also of g^0, of order $\leq k$ vanish. Thus also $f(D_k \cap V) = g \circ h(D_k \cap V)$ has measure zero. Since $D_k - D_{k+1}$ is covered by a countable union of such sets, it follows that $f(D_k - D_{k+1})$ has measure zero. ■

IV.3 Retractions onto boundaries and Brouwer's fixpoint theorem

In this section we combine Proposition IV.1.10 with the general form of Sard's theorem to prove (the smooth version of) a classical result of algebraic topology.

IV.3.1 PROPOSITION *If M is a compact manifold with boundary, there is no (smooth) retraction $f : M \to \partial M$. (A retraction is a map $f : M \to \partial M$ such that $f \restriction \partial M = \mathrm{id}_{\partial M}$.)*

Proof. [Hirsch] Suppose a smooth retraction f exists. By Sard's theorem there is a regular value $q \in \partial M$ for f. Obviously, q is also a regular value for the identity map $\partial f = \mathrm{id}_{\partial M}$. Thus Proposition IV.1.10 applies and $f^{-1}(q)$ is a submanifold of M such that $\partial(f^{-1}(q)) = f^{-1}(q) \cap \partial M = \{q\}$. (The last equality follows from $f \restriction \partial M = \mathrm{id}$.) The codimension of $f^{-1}(q)$ is equal to $\dim \partial M = \dim M - 1$, thus $\dim f^{-1}(q) = 1$. Furthermore, $f^{-1}(q) \subset M$ is closed, thus compact and by Corollary II.12.3 it must have an even number of (distinct) boundary points. This is a contradiction. ■

We can now prove the smooth version of Brouwer's fixpoint theorem. By D^n we denote the closed unit ball in $\mathbb{R}^n$.

IV.3.2 Theorem (Brouwer) *Any smooth map $f : D^n \to D^n$ has a fixpoint.*

Proof. Suppose $f : D^n \to D^n$ has no fixpoint, thus $f(x) \neq x$ for all $x \in D^n$. Consider the ray (=half line) through x starting at $f(x)$. Let $g(x)$ be its unique intersection with the boundary $\partial D^n = S^{n-1}$. Clearly, $g : D^n \to \partial D^n$ is a retraction, thus the theorem follows from Proposition IV.3.1 provided we can show g to be smooth. Since $x, f(x), g(x)$ lie on a line, we have $g(x) - f(x) = t(x - f(x))$ where $t \geq 1$. On the other hand, $|g(x)|^2 = 1$. Combining these equations we get $|tx + (1-t)f(x)|^2 = 1$ or

$$t^2|x - f(x)|^2 + 2tf(x) \cdot (x - f(x)) + |f(x)|^2 - 1 = 0.$$

The standard formula for the solutions of a quadratic equation shows that the unique positive root t, and therefore $g(x) = tx + (1-t)f(x)$, of this equation depends smoothly on x. ■

IV.3.3 Remark In Section V.4 we will prove the continuous version of Brouwer's fixpoint theorem by reducing it to the above result. □

IV.4 The mod 2 degree

If M and N are manifolds of the same dimension and $q \in N$ is a regular value then $f^{-1}(q) \subset M$ is zero dimensional, thus discrete by Exercise III.2.4. If M is compact, $f^{-1}(q)$ is finite. We are interested in the cardinality of this set.

IV.4.1 Lemma *Let M, N be manifolds of the same dimension with M compact and $f : M \to N$ a smooth map. Let $R \subset N$ be the set of regular values of f. Then R is open and the function $R \ni q \mapsto \#f^{-1}(q)$ is locally constant. (I.e. every $q \in R$ has a neighborhood $V \subset N$ such that $\#f^{-1}(p) = \#f^{-1}(q)$ for all $p \in V$.)*

Proof. That R is open was proven in Proposition III.1.1. Now let $p_1, \ldots, p_k$ be the points of $f^{-1}(q)$. By Corollary II.3.8 we can find pairwise disjoint open neighborhoods $U_1, \ldots, U_k$ of these points that are mapped diffeomorphically to open neighborhoods $V_1, \ldots, V_k$ in N. Since M is compact, $f : M \to N$ is a closed map, thus $f(M - U_1 \cdots - U_k)$ is closed. Therefore

$$V = V_1 \cap V_2 \cap \cdots \cap V_k - f(M - U_1 \cdots - U_k)$$

is open. Now, every $p \in V$ has one preimage in each of the U_i and no others, implying $\#f^{-1}(q) \restriction V \equiv k$. ■

IV.4.2 Lemma *Let M, N be manifolds of the same dimension with M compact without boundary. If $f, g : M \to N$ are smoothly homotopic and q is a regular value for f and g then*

$$\#f^{-1}(q) \equiv \#g^{-1}(q) \pmod 2.$$

Proof. Let $h : M \times [0,1] \to N$ be a smooth homotopy. Assume first that q is a regular value for h. Then by Proposition IV.1.10, and using Exercise II.6.11, we have

$$\begin{aligned}
\partial(h^{-1}(q)) &= h^{-1}(q) \cap \partial(M \times [0,1]) \\
&= h^{-1}(q) \cap (M \times 0 \cup M \times 1) \\
&= f^{-1}(q) \times 0 \cup g^{-1}(q) \times 1,
\end{aligned}$$

thus $\#\partial(h^{-1}(q)) = \#f^{-1}(q) + \#g^{-1}(q)$. Since $h^{-1}(q)$ is a compact 1-manifold, its boundary has an even number of points by Corollary II.12.3, whence $\#f^{-1}(q) \equiv g^{-1}(q) \pmod 2$.

Now suppose that q is not a regular value of h. By Lemma IV.4.1 there is a neighborhood $V \subset N$ of q consisting of regular values for f such that $\#f^{-1}(q') = \#f^{-1}(q)$ for all $q' \in V$. Similarly, there is a neighborhood $V' \subset N$ of q consisting of regular values for g such that $\#g^{-1}(q') = \#g^{-1}(q)$ for all $q' \in V$. By Sard's theorem, $V \cap V'$ contains a regular value q' for h. Now

$$\#f^{-1}(q) = \#f^{-1}(q') \equiv \#g^{-1}(q') = \#g^{-1}(q) \pmod 2$$

gives the desired equality. ■

IV.4.3 Definition *Let M be a manifold and $h : M \times [0,1] \to M$ a smooth map. Then h is a diffeotopy if $h_t = h(\cdot, t) : M \to M$ is a diffeomorphism for every $t \in [0,1]$.*

IV.4.4 Proposition *Let M be connected. Then for all $p, q \in M$ there is a diffeotopy $h : M \times [0,1] \to M$ such that $h_0 = \mathrm{id}_M$ and $h_1(p) = q$. (In particular, the diffeomorphism group of M acts transitively.) h_1 can be chosen to act identically outside a compact set.*

Proof. We call two points p, q isotopic if the statement is true for them. This clearly defines an equivalence relation. Now the result follows from connectedness of M provided we can show that the equivalence classes are open. It suffices to show for every $p \in M$ that all points in a neighborhood $\tilde{U}$ are isotopic to p. This neighborhood can be chosen small enough to be contained in the domain of a coordinate chart (U, ϕ). Thus everything follows if we prove the following claim: Let q be contained in the open unit ball B in $\mathbb{R}^n$. Then there exists a diffeotopy $h : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ leaving the complement of B pointwise stable and such that $h_1(0) = q$. There are various ways of doing this; we will follow [19].

Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function satisfying $\phi(x) > 0$ if $|x| < 1$ and $\phi(x) = 0$ if $|x| \geq 1$. (E.g., let $\phi(x) = \lambda(1 - |x|^2)$, where $\lambda(t) = e^{-1/t}$ for $t > 0$ and $\lambda(t) = 0$ otherwise.) Let $c \in \mathbb{R}^n$ be a unit vector and $x \in \mathbb{R}^n$. Since ϕ has compact support, the system

$$\frac{dy_i}{dt} = c_i \phi(x_1, \ldots, x_n), \quad i = 1, \ldots, n$$

of differential equations has a unique solution $y_x(t)$, defined for all $t \in \mathbb{R}$, and satisfying the initial condition $y(0) = x$. We write $\alpha_t(x) = y_x(t)$. It is clear that $\alpha_t(x)$ is defined for all $t \in \mathbb{R}, x \in \mathbb{R}^n$ and smooth in both variables. Furthermore, $\alpha_0(x) = x$ and $\alpha_s \circ \alpha_t(x) = \alpha_{s+t}(x)$. Thus $t \mapsto \alpha_t(\cdot)$ is a one-parameter group of diffeomorphisms that acts trivially on the complement of B. If $q \in B, q \neq 0$, the choice $c = q/|q|$ clearly implies that $\alpha_t(0) = q$ for some $t > 0$. Now $x \times t \mapsto \alpha_t(x)$ is the desired diffeotopy. ■

IV.4.5 Exercise Combine Proposition IV.4.4 with an inductive argument to show that one can find a compactly localized diffeotopy sending any finite set $\{x_1, \ldots, x_r\}$ to any other set $\{y_1, \ldots, y_r\}$. □

IV.4.6 Remark Proposition IV.4.4 and Exercise IV.4.5 are special cases of a much more general result, proven e.g. in [7, §9]: If $h : N \times [0,1] \to M$ is an isotopy, i.e. a smooth map such that $h_t : N \to M$ is an embedding for all $t \in [0,1]$, then there exists a diffeotopy $v : M \times [0,1] \to M$ such that $v_t \circ h_0 = h_t$. (One says, the isotopy has been embedded into a diffeotopy.) The proof uses similar ideas, namely the diffeomorphism group associated to a flow generated by a suitable vector field. For more on the latter concepts see Section II.5. □

IV.4.7 Proposition *Let M, N be manifolds of the same dimension with M compact and N connected. Then $\#f^{-1}(p) \equiv \#f^{-1}(q) \pmod 2$ for all regular values $p, q \in N$. This common value $\deg_2 f \in \{0, 1\}$ depends only on the smooth homotopy class of f.*

Proof. Choose a diffeotopy h as in IV.4.4, thus h_1 is a diffeomorphism such that $h_1(p) = q$. Thus q is a regular value of $h_1 \circ f$. Since h_1 is smoothly homotopic to the identity h_0, Lemma IV.4.2 implies $\#f^{-1}(q) \equiv \#(h_1 \circ f)^{-1}(q) = \#(f^{-1} \circ h_1^{-1}(q)) = \#f^{-1}(p)$. Denote this element of $\mathbb{Z}_2$ by $\deg_2 f$. If g is smoothly homotopic to f, by Sard's theorem there is a regular value p for f and g. Then

$$\deg_2 f = \#f^{-1}(p) \equiv \#g^{-1}(p) = \deg_2 g \pmod 2,$$

as claimed. ■

IV.4.8 Exercise Let $f : M \to N$ be smooth, where M be compact and N connected of the same dimension. If $\deg_2 f \neq 0$ then f is surjective. □

IV.4.9 Exercise Let M, N be manifolds, where M is compact, N is connected without boundary and $\dim M = \dim N + 1$. Show that $\deg_2 \partial f = 0$. *Hint:* Show that there is a regular value q for f and ∂f. Then use $(\partial f)^{-1}(q) = f^{-1}(q) \cap \partial M = \partial(f^{-1}(q))$. □

IV.5 Applications of the mod 2 degree (Unfinished!)

The mod 2 degree of a map is a rather weak invariant since it can assume only two values. There are, however, various situations where this is no drawback at all due to an intrinsic $\mathbb{Z}_2$ structure of the problem. We will consider two of them.

IV.5.1 The Borsuk-Ulam theorem

IV.5.1 Lemma *The following statements are equivalent:*

(i) If $f : S^n \to S^n$ satisfies $f(-x) = -f(x)$ then $\deg_2 f = 1$.

(ii) If $f : S^n \to S^m$ satisfies $f(-x) = -f(x)$ then $n \leq m$.

(iii) Let $f : S^n \to \mathbb{R}^n$ be a smooth map. Then there exists $x \in S^n$ such that $f(x) = f(-x)$.

Proof. (i)⇒(ii). Assume $f : S^n \to S^m$ satisfies $f(-x) = -f(x)$ and $m < n$. Composing f with an inclusion $S^m \hookrightarrow S^n$ we get a map that satisfies $f(-x) = -f(x)$ and $\deg_2 f = 0$ (by Exercise IV.4.8), contradicting (i).

(ii)⇒(i). Assume $f : S^n \to S^n$ satisfies $f(-x) = -f(x)$ and $\deg_2 f = 0$. Then we construct a map $g : S^n \to S^{n-1}$ satisfying $f(-x) = -f(x)$. This is in contradiction with (ii). ************************

(ii)⇒(iii). If (iii) does not hold then $\phi(x) = (f(x)-f(-x))/|f(x)-f(-x)|$ defines a map $S^n \to S^{n-1}$ satisfying $\phi(-x) = -\phi(x)$ contradicting (ii).

(iii)⇒(ii). Assume $f : S^n \to S^m$ satisfies $f(-x) = -f(x)$ and $m < n$. Then composing with the inclusion $S^m \hookrightarrow \mathbb{R}^{m+1} \hookrightarrow \mathbb{R}^n$ we obtain a map $S^n \to \mathbb{R}^n$ satisfying $f(-x) = -f(x) \neq 0$, contradicting (iii). ■

IV.5.2 Theorem (Borsuk-Ulam) *The equivalent statements of Lemma IV.5.1 are true.*

Proof. To be written. For the time being, see [12, Chapter 2, §6]. ■

IV.5.3 Corollary *At any given time there are two antipodal places on earth having exactly the same weather (in the sense of having the same temperature and air pressure).*

Proof. Follows from (iii) above. ■

IV.5.2 The Jordan-Brouwer theorem

The classical Jordan curve theorem says that a closed connected curve C in $\mathbb{R}^2$ divides $\mathbb{R}^2 - C$ into two connected components. In algebraic topology one proves the generalization according to which every subset $X \subset \mathbb{R}^n$ homeomorphic to S^{n-1} divides $\mathbb{R}^n - X$ into two connected components, exactly one of which is bounded, cf. e.g. [6, p. 234]. Using the mod 2 degree, we prove a smooth version which is more general in that X need not be homeomorphic to S^{n-1}. Consider e.g. $X \subset \mathbb{R}^3$, where X is a compact connected surface of genus g.

IV.5.4 THEOREM (JORDAN-BROUWER) *Let $X \subset \mathbb{R}^n$ be a compact connected hypersurface, i.e. a submanifold of dimension $n-1$. Then*

1. *The complement of X consists of two connected components, the "outside" D_0 and the "inside" D_1. Furthermore, $\overline{D_1}$ is a compact manifold with boundary $\partial\overline{D_1} = X$.*

2. *Let $z \in \mathbb{R}^n - X$. Then $z \in D_1$ iff any ray r emanating from z and transversal to X intersects X in an odd number of points.*

Proof. To be written. For the time being, see [12, Chapter 2, §5]. ■

IV.6 Oriented manifolds

In order to define a $\mathbb{Z}$-valued homotopy invariant of smooth maps (between manifolds of the same dimension) we need the notion of an orientation of a manifold. The latter concept is important in many other contexts as well.

IV.6.1 DEFINITION *Let $B = \{x_1, \dots, x_n\}, B' = \{x'_1, \dots, x'_n\}$ be bases of $\mathbb{R}^n$. We consider them as equivalent if the matrix M defined by $Mx_i = x'_i$ for all i has determinant > 0. An equivalence class of bases on $\mathbb{R}^n$ is called an orientation.*

It is clear that $\mathbb{R}^n, n \geq 1$ has precisely two orientations, called ± 1, and we choose the basis $(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, \dots, 0, 1)$ to represent $+1$. By decree, the zero dimensional vector space also admits orientations ± 1.

IV.6.2 DEFINITION *An orientation of the m-manifold M consists of a choice of an orientation of T_pM for every $p \in M$. If $m > 0$ we require that, for every chart (U, ϕ), the invertible linear map $T_p\phi : T_pM \to T_{\phi(p)}\mathbb{R}^m$ maps the orientation of T_pM to the same orientation of $\mathbb{R}^m$ for all $p \in U$. A manifold is orientable if it admits an orientation. An oriented manifold is a manifold together with a choice of an orientation. If M is an oriented manifold then $-M$ denotes the same manifold with the opposite orientation.*

Not every manifold is orientable. A counterexample is provided by the well known Möbius strip.

IV.6.3 EXERCISE A manifold M is orientable if there exists an atlas $\mathcal{A}$ compatible with the given differentiable structure such that

$$\det\left(\frac{\partial \phi'_i \circ \phi^{-1}(x_1, \dots, x_n)}{\partial x_j}\right) > 0$$

for any $(U, \phi), (U', \phi') \in \mathcal{A}_0$ and $x \in \phi(U \cap U')$. □

An orientation for M defines an orientation for the boundary ∂M as follows. Assume $\dim M > 1$. For $x \in \partial M$ choose a basis $(v_1, \ldots, v_m)$ of T_xM that is (1) positively oriented, i.e. represents the given orientation of M, (2) the $v_i, i > 1$, are tangent to ∂M, i.e. in the image of $T_x\iota$, where $\iota : \partial M \to M$ is the inclusion, and (3) v_1 points outside of M. Now the orientation of ∂M at x is declared to be defined by $(v_2, \ldots, v_m)$. If $\dim M = 1$, the orientation of ∂M at $x \in \partial M$ declared to be $+1$ or -1 if the given orientation of M at x points outward or inward, respectively. With this definition and the standard orientation on $[0, 1]$ we have $\partial[0, 1] = \{(0, -), (1, +)\}$.

If M, N are oriented manifolds an orientation of the product $M \times N$ arises canonically from the isomorphism $T_{(x,y)}(M \times N) = T_xM \oplus T_yN$. To wit, if $(e_1, \ldots, e_m), (f_1, \ldots, f_n)$ are positively oriented bases of T_xM, T_yN, respectively, we define the basis $\{(e_1, 0), \ldots, (e_m, 0), (0, f_1), \ldots (0, f_n)\}$ of $T_xM \oplus T_yN$ to be positively oriented. In particular, let M be a manifold without boundary and $I = [0, 1]$. Then $\partial(M \times I) = (-M) \times 0 \cup M \times 1$.

IV.6.4 Exercise The obvious diffeomorphism $\sigma : M \times N \to N \times M$ is orientation reversing iff both M and N have odd dimension. □

IV.6.5 Exercise Let M be a manifolds where $\partial N = \emptyset$. Then

$$\partial(M \times N) = \partial M \times N, \qquad \partial(N \times M) = (-1)^{\dim N} N \times \partial M.$$

□

IV.7 The Brouwer degree

Now we turn to the discussion of the degree of a smooth map between oriented manifolds of the same dimension.

IV.7.1 Definition *Let M, N be oriented manifolds of the same dimension, where M is compact. For a smooth map $f : M \to N$ and a regular value $q \in N$ we define $\deg(f, q) \in \mathbb{Z}$ by*

$$\deg(f, q) = \sum_{p \in f^{-1}(q)} \operatorname{sign} T_pf,$$

where $\operatorname{sign} T_pf = 1$ if the image of the orientation of T_pM under $T_pf : T_pM \to T_{f(p)}N$ coincides with the orientation of $T_{f(p)}N$, and -1 otherwise. Depending on $\operatorname{sign} T_pf$ we call p a point of positive or negative type.

IV.7.2 Exercise By Lemma IV.4.1, the set $R \subset N$ of regular values of f is open. Show that the map $R \mapsto \mathbb{Z}$, $q \mapsto \deg(f, g)$ is locally constant. □

IV.7.3 Lemma *Let M, N be oriented manifolds, where M is compact, N is connected without boundary and $\dim M = \dim N + 1$. Then $\deg(\partial f, q) = 0$ for every regular value q of $\partial f = f \restriction \partial M$.*

Proof. Assume first that q is a regular value for f and ∂f. Then $f^{-1}(q)$ is a compact 1-dimensional submanifold of M, thus by Section II.12 it consists of finitely many circles and closed arcs. By Proposition IV.1.10, the endpoints of these arcs are on ∂M, and $f^{-1}(q) \cap \partial M$ consists precisely of these endpoints. Let $A \subset f^{-1}(q)$ be one of these arcs and $\partial A = \{a, b\}$. We will show that $\operatorname{sign} T_a\partial f + \operatorname{sign} T_b\partial f = 0$. Since $\deg(\partial f)$ is the sum over $\operatorname{sign} T_a\partial f$ for all endpoints of the said arcs, this implies $\deg(\partial f) = 0$, as claimed.

The orientations for M, N determine an orientation for A as follows: Given $p \in A$, let $(v_1, \ldots, v_m)$ be a positively oriented basis for T_pM such that v_1 is tangent to A. If T_pf carries $(v_2, \ldots, v_m)$ into a positively oriented basis for T_qN then we declare the orientation of A to be given by v_1, otherwise by $-v_1$. Let $v_1(p)$ be the positively oriented unit vector tangent to A at p. Clearly $v_1(p)$ a smooth function and $v_1(p)$ points inward at one boundary point and outward at the other. This implies $\operatorname{sign} T_a\partial f = -\operatorname{sign} T_b\partial f$, as claimed.

Now assume that q is a regular value only of ∂f. By Exercise IV.7.2, $q \mapsto \deg(\partial f, q)$ is locally constant, thus there exists an open neighborhood $U \subset R$ of q such that $\deg(\partial f, q') = \deg(\partial f, q)$ for all $q' \in U$. By Sard's theorem, U contains a regular value q' of f **and** ∂f. Then the above implies $\deg(\partial f, q') = 0$ and thus $\deg(\partial f, q) = 0$. ■

IV.7.4 Lemma *Let M, N be oriented manifolds of the same dimension with M compact without boundary. If $f, g : M \to N$ are smoothly homotopic and q is a regular value for f and g then*

$$\deg(f, q) = \deg(g, q).$$

Proof. By assumption we have a smooth map $h : M \times [0, 1] \to N$ such that $h_0 = f, h_1 = g$. Now $\partial(M \times [0, 1]) = (-M) \times 0 \cup M \times 1$. Thus the degree of $h \restriction \partial(M \times [0, 1])$ is equal to $\deg g - \deg f$, and this must vanish by Lemma IV.7.3. ■

IV.7.5 Proposition *Let $f : M \to N$ be a smooth map of oriented manifolds of the same dimension with M compact and N connected. Then $\deg(f, p) = \deg(f, q)$ for all regular values p, q. This common value $\deg f$ depends only on the smooth homotopy class of f.*

Proof. The proof now proceeds exactly as the one of Proposition IV.4.7. We only need to remark that the diffeomorphism h_1, being diffeotopic to the identity, is orientation preserving. Thus orientations are preserved throughout the argument. ■

IV.7.6 Remark Clearly, $-1 \equiv 1 \pmod 2$. Thus if $f : M \to N$ satisfies the assumptions of Proposition IV.7.5 then we have $\deg f \equiv \deg_2 f \pmod 2$. □

IV.7.7 Exercise Show that the map $f : S^1 \to S^1$, $e^{i\phi} \mapsto e^{im\phi}$, where $m \in \mathbb{Z}$, has degree m. □

IV.7.8 Exercise Show that two smooth maps $f, g : S^1 \to S^1$ are (smoothly) homotopic iff they have the same degree. *Hint:* Consider the lifts $\hat{f}, \hat{g} : S^1 \to \mathbb{R}$ known from covering space theory. □

IV.8 Applications of the degree

IV.8.1 The fundamental theorem of algebra

We begin by showing that the degree can be used for a proof of the fundamental theorem of algebra. Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $m > 0$ that is monic (i.e. $a_m = 1$). The family $P_t(z) = tP(z) + (1 - t)z^m = z^m + ta_{m-1}z^{m-1} + \cdots + ta_0$, where $t \in [0, 1]$, defines a smooth homotopy between $z \mapsto z^m$ and P. In view of

$$\frac{P_t(z)}{z^m} = 1 + t\left(a_{m-1}\frac{1}{z} + a_{m-2}\frac{1}{z^2} + \cdots + a_0\frac{1}{z^m}\right)$$

and the fact that the expression in the bracket goes to zero as $|z| \to \infty$, we see that for sufficiently large R, none of the P_t has a zero of absolute value $\geq R$. Writing $D = \{z \in \mathbb{C} \mid |z| \leq R\}$, we get a family of maps

$$\phi_t : S^1 \to S^1 : \ z \mapsto \frac{P_t(Rz)}{|P_t(Rz)|}.$$

By Exercise IV.7.7, $\phi_0(z) = (z/|z|)^m : S^1 \to S^1$ has degree m. By homotopy invariance of the degree, $\deg \phi_1 = \deg(P/|P|) = m$. If now P has no zeros in the interior of D, then $\phi_1 = P/|P|$ extends to D, thus has degree zero by Lemma IV.7.3. This is a contradiction.

IV.8.2 Vector fields on spheres

IV.8.1 EXERCISE The map $S^n \to S^n$, $(x_1, \ldots, x_{n+1}) \mapsto (-x_1, x_2, \ldots, x_{n+1})$ has degree -1. Thus $S^n \to S^n$, $x \mapsto -x$ has degree $(-1)^{n+1}$. Show that for even n there is no smooth homotopy between the identity of S^n and the reflection $x \mapsto -x$. □

The preceding facts can be profitably applied to the classical subject of vector fields on spheres. We consider the embedded manifold $S^n = \{x \in \mathbb{R}^{n+1} \mid |x|^2 = 1\}$. Then we may identify $T_x S^n = \{y \in \mathbb{R}^{n+1} \mid x \cdot y = 0\}$, thus a vector field on S^n just is a smooth map $v : S^n \to \mathbb{R}^{n+1}$ such that $x \cdot v(x) = 0$ for all x. If n is odd, a nowhere vanishing vector field on S^n is given by the formula

$$v(x_1, \ldots, x_{2k}) = (x_2, -x_1, x_4, -x_3, \ldots, x_{2k}, -x_{2k-1}).$$

IV.8.2 EXERCISE Show that the sphere S^n does not admit a nowhere vanishing vector field iff n is even. *Hints:* 1. Show that a nowhere vanishing vector field v gives rise to a smooth map $v' : S^n \to S^n$. 2. Consider the map $h : S^n \times \mathbb{R} \to \mathbb{R}^{n+1}$ defined by

$$h(x, \theta) = \cos\theta \, x + \sin\theta \, v'(x).$$

Show that h maps into S^n. 3. Consider the homotopy $h : S^n \times [0, \pi] \to S^n$ and use Exercise IV.8.1. □

IV.8.3 The Hopf theorem on maps into spheres

In Theorem III.4.11 we have seen that all smooth maps $f : M \to S^n$ are homotopic if $\dim M < n$. Elucidating the case $\dim M = n$ for compact M will be our third application of the degree.

IV.8.3 LEMMA *Let M be a compact oriented connected n-manifold. For every $d \in \mathbb{Z}$ there exists a smooth map $f : M \to S^n$ of degree d.*

Proof. A constant map has degree zero. For $d \in \mathbb{N}$ let $(U_i, \phi_i), i = 1, \ldots, d$ be charts of disjoint support where each $\phi_i : U_i \to \mathbb{R}^n$ is orientation preserving and surjective. Let $s : \mathbb{R}^n \to S^n$ be a smooth orientation preserving map that maps all x with $|x| \geq 1$ to a point s_0 and the open unit ball diffeomorphically to $S^n - \{s_0\}$. (E.g., let $s(x) = h^{-1}(x/\lambda(|x|^2))$, where $h : S^n - \{s_0\} \to \mathbb{R}^n$ is the stereographic projection from s_0, and λ is a smooth monotone decreasing function with $\lambda(t) > 0$ for $t < 1$ and $\lambda(t) = 0$ for $t \geq 1$.) Now define $f : M \to S^n$ by

$$f(p) = \begin{cases} s \circ \phi_i(p) & p \in U_i \\ s_0 & p \in M - \bigcup U_i \end{cases}$$

Then f is smooth. Now, every $q \in S^n - \{s_0\}$ is a regular value and has exacly d inverse images. By construction, $T_p f$ is orientation preserving for all $p \in f^{-1}(q)$, thus f has degree d. In order to obtain degree $-d$, choose all ϕ_i be orientation reversing. ■

IV.8.4 PROPOSITION *Let M be a connected oriented compact manifold of dimension $n+1$ with $\partial M \neq \emptyset$. Let $f : \partial M \to S^n$ be a smooth map. Then f extends to a smooth map $M \to S^n$ iff $\deg f = 0$.*

Proof. The $\Rightarrow$ direction has been shown in Lemma IV.7.3. The proof of the 'if' direction requires some tools that have not been introduced yet. The first half of the argument will be given in the next subsection, while the remaining part is postponed until Section VI.2. ■

IV.8.5 THEOREM (HOPF) *Let M be a connected oriented compact n-manifold without boundary. Let $f, g : M \to S^n$ be smooth maps. Then f and g are smoothly homotopic iff $\deg f = \deg g$. For every $d \in \mathbb{Z}$ there is a map of degree d. (Thus $[M, S^n]_s \cong \mathbb{Z}$.)*

Proof. The argument for the first statement is the same as in Lemma IV.7.4: The pair f, g is the same as a map $(-M) \times 0 \cup M \times 1 \to N$ of degree $\deg g - \deg f$, and a lift of this map to $M \times [0,1]$ is the same as a smooth homotopy between f and g. Now the first claim follows from Proposition IV.8.4 and the second from Lemma IV.8.3. ■

IV.8.6 Remark There are versions of the preceding results where 'oriented' is replaced by 'unorientable'. Just replace the degree by the mod 2 degree in the conclusions of Proposition IV.8.4 and Theorem IV.8.5. (Thus there are exactly two homotopy classes of smooth maps $M \to S^n$.) When $\partial M \neq \emptyset$, all maps $M \to S^n$ are homotopic, whether M is orientable or not. See [13, Chapter 5] for proofs. □

IV.8.7 Corollary *The degree establishes a bijective correspondence between $\mathbb{Z}$ and the smooth homotopy classes of smooth maps $S^n \to S^n$.*

In Section V.4 we will use smooth approximation of continuous maps to prove $\pi_n(S^n) \cong \mathbb{Z}$.

IV.8.4 Winding numbers

The notion of winding number is just a simple, but useful, reinterpretation of the degree of a map. The rationale of its name should be evident from the case $n = 1$ of the following

IV.8.8 Definition *Let M be a compact oriented n-manifold and $f : M \to \mathbb{R}^{n+1} - \{z\}$ a smooth map. Then the winding number $W(f, z)$ is defined as $W(f, z) = \deg \tilde{f}$, where*

$$\tilde{f} : M \to S^n, \qquad x \mapsto \frac{f(x) - z}{|f(x) - z|}.$$

IV.8.9 Lemma *Let $U \subset \mathbb{R}^k$ be open and $f : U \to \mathbb{R}^k$ smooth. Let x be a regular point with $f(x) = z$. If B is a sufficiently small closed ball centered at x and $\partial f = f \restriction \partial B$ then $W(\partial f, z) = 1$ if f preserves the orientation at x and -1 otherwise.*

Proof. By Corollary II.3.8, f restricts to a diffeomorphism between sufficiently small neighborhoods $U \ni x$ and $V \ni z$. We may assume $x = z = 0$. If we choose U connected it is easy to see that $T_p f$ is either orientation preserving for all $p \in U$ or orientation reversing for all p. Let $B \subset U$ be a closed ball. Then $\partial f : \partial B \to f(\partial B)$ is a diffeomorphism with the same orientation behavior as f. Furthermore, $\partial f / |\partial f|$ is homotopic to ∂f, thus also $\partial f / |\partial f| : \partial B \to S^{k-1}$ has the same orientation behavior as f. Therefore $W(f, 0) = \deg(\partial f / |\partial f|) = \pm 1$ according to whether f is orientation preserving at x or not. ■

IV.8.10 Lemma *Let $B \subset \mathbb{R}^k$ be a closed ball and $f : B \to \mathbb{R}^k$ smooth. Let $\partial f = f \restriction \partial B$. If z is a regular value of f without preimages on ∂B then $W(\partial f, z) = \deg(f, z)$, where the right hand side is as defined in Section IV.7.*

Proof. Let $B_i \subset B$ be sufficiently small disjoint closed balls around the preimages $\{x_i\}$ of z, and let $C = \cup_i B_i$. It is clear that $\deg(f \restriction C, z) = \deg(f, z)$, and by Lemma IV.8.9 the left hand side equals $\sum_i W(\partial B_i, z) = W(\partial C, z)$. Now, the map $x \mapsto \frac{f(x)-z}{|f(x)-z}$ is well defined on $B - \cup_i B_i$, thus its restriction to the boundary $\partial(B - \cup_i B_i)$ has degree zero by Lemma IV.7.4. Therefore, $W(\partial B, z) = W(\cup_i B_i, z) = \deg(f, z)$, and we are done. ■

IV.8.11 Exercise If $B \subset \mathbb{R}^k$ is a closed ball and $f : \mathbb{R}^k - \mathrm{Int} B \to Y$ is smooth then f extends to a smooth map $\mathbb{R}^k \to Y$ iff the restriction $\partial f : \partial B \to Y$ is homotopic to a constant map. □

IV.8.12 Proposition *For all $k \geq 1$ we have:*

1. *Any smooth map $f : S^k \to S^k$ of degree zero is homotopic to a constant map.*
2. *Any smooth map $f : S^k \to \mathbb{R}^{k+1} - \{0\}$ with $W(f, 0) = 0$ is homotopic to a constant map.*

Proof. 1⇒2: If $f : S^k \to \mathbb{R}^{k+1} - \{0\}$ satisfies $W(f, 0) = 0$ then $\tilde{f} : S^k \to S^k,\ x \mapsto f(x)/|f(x)|$ has degree zero, thus is homotopic to a constant map by statement 1. Now statement 2 follows from the fact that f and $\tilde{f} = f/|f|$ are homotopic.

Now we prove statement 1 by induction. For $k = 1$ this follows from Exercise IV.7.8. Thus assume claim 1 (and thus 2) has been proven for $k < l$ and consider $f : S^l \to S^l$ with $\deg f = 0$. Let a, b be distinct regular values of f. Pick an open set $U \subset S^l$ such that (i) $f^{-1}(a) \subset U$, (ii) $b \notin f(U)$ and (iii) there exists a diffeomorphism $\alpha : \mathbb{R}^l \to U$. (To see that such U exists, pick an open $U' \subset S^l$ diffeomorphic to $\mathbb{R}^l$ and apply Exercise IV.4.5 to find a diffeomorphism γ of S^l that maps all points of $f^{-1}(a)$ into U' and all points of $f^{-1}(b)$ to $S^l - U'$.) Let $\beta : S^l - \{b\} \to \mathbb{R}^l$ be a diffeomorphism that maps a to 0. Then $g = \beta \circ f \circ \alpha : \mathbb{R}^l \to \mathbb{R}^l$ makes sense and has 0 as a regular value with finite pre-image. Thus $\deg(g, 0)$ is well defined and equal to zero, the latter following easily from $\deg f = 0$. We claim that there exists a smooth map $\tilde{g} : \mathbb{R}^l \to \mathbb{R}^l - \{0\}$ coinciding with g outside a compact set. To see this let B be a ball containing $g^{-1}(0)$ in its interior. By Lemma IV.8.10, the winding number $W(\partial g, 0)$ is zero. Thus statement 2, as already proven for $k = l - 1$, implies that $\partial g : S^{l-1} \to \mathbb{R}^l - \{0\}$ is homotopic to a constant map. Now Exercise IV.8.11 implies that we can extend $g \restriction \mathbb{R}^l - B$ to $\tilde{g} : \mathbb{R}^l \to \mathbb{R}^l - \{0\}$. Clearly $\tilde{g}$ is homotopic to g by a homotopy that is constant outside the compact set $\overline{B}$. It follows that f is homotopic to a map $\tilde{f} : S^l \to S^l - \{b\}$. Since $S^l - \{b\}$ is diffeomorphic to $\mathbb{R}^l$, thus contractible, it follows that f is homotopic to a constant map. ■

IV.8.13 Remark If X is any topological space, the set $[S^n, X]$ of (continuous) homotopy classes of continuous maps $S^n \to X$ (preserving base points) has a group structure, abelian if $n \geq 2$, see any book on homotopy theory or [4]. Restricting to smooth maps, one can show that the assignment $[S^n, S^n]_s \ni [f] \mapsto \deg f$ gives rise to an isomorphism of abelian groups. □

IV.9 Transversality

So far, we have considered inverse images $f^{-1}(q)$ of smooth maps $f : M \to N$. We will now generalize Proposition IV.1.10 to inverse images $f^{-1}(L)$, where $L \subset N$ is a submanifold. This requires the notion of transversality due to Thom. First some linear algebra.

Let V_1, V_2 be linear subspaces of a vector space V. We write $V_1 + V_2 = V$ if every $x \in V$ can be written – not necessarily uniquely – as $x = x_1 + x_2$ where $x_1 \in V_1, x_2 \in V_2$.

IV.9.1 Exercise Let V_1, V_2 be linear subspaces of a vector space V. The following are equivalent:

1. $V_1 + V_2 = V$.
2. The composite map $V_1 \hookrightarrow V \to V/V_2$ is surjective.
3. $\dim V_1 + \dim V_2 = \dim(V_1 \cap V_2) + \dim V$.

□

IV.9.2 Definition *Let $f : M \to N$ be smooth and $L \subset N$ a submanifold. We say that f is transversal to L and write $f \pitchfork L$ iff for every $p \in f^{-1}(L)$ we have $T_pf(T_pM) + T_{f(p)}L = T_{f(p)}N$.*

IV.9.3 Exercise If $f : M \to N \supset L$ satisfies $f \pitchfork L$ and $\dim M + \dim L < \dim N$ then $f(M) \cap L = \emptyset$. □

IV.9.4 Exercise If $f : M \to N$ is submersive then $f \pitchfork L$ for every submanifold $L \subset N$. □

IV.9.5 Theorem *Consider $f : M \to N$ where $L \subset M$ is a submanifold, all manifolds being boundaryless. If $f \pitchfork L$ and $f^{-1}(L)$ is non-empty then $W = f^{-1}(L) \subset M$ is a submanifold whose codimension is equal to that of L in N (thus $\dim M - \dim f^{-1}(L) = \dim N - \dim L$). We have $T_pW = (T_pf)^{-1}(T_{f(p)}L)$ for all $p \in W$.*

Proof. As in Lemma IV.1.4 it suffices to prove the claim locally. Thus let $p \in f^{-1}(L)$ and (U, ϕ) a chart around p. Let (V, ψ) be a chart around $f(p)$ such that $\psi(V) = X \times Y$ and $\psi(V \cap L) = X \times 0$, where X, Y are open neighborhoods of 0 in $\mathbb{R}^\ell$ and $\mathbb{R}^{n-\ell}$, respectively. If we suitably shrink U, the composite $\tilde{f} = \psi \circ f \circ \phi^{-1}$ maps $\tilde{U} = \phi(U)$ into $X \times Y$.

By Exercise IV.9.1, $f \pitchfork L$ is equivalent to surjectivity of $T_pM \to T_{f(p)}N/T_{f(p)}L$ for all $p \in f^{-1}(L)$. In terms of $\tilde{f}$ this is equivalent to the composite map

$$\tilde{g} : \tilde{U} \xrightarrow{\tilde{f}} X \times Y \xrightarrow{\pi} Y$$

having 0 as regular value. Since $\tilde{f}^{-1}(X \times 0) = \tilde{g}^{-1}(0)$, the first claim follows from Lemma IV.1.4.

By Lemma IV.1.4, $T_pW = \{v \in T_pM \mid T_p\tilde{g}\phi(v) = 0\}$. Thus $T_pW = \{v \in T_pM \mid T_p\tilde{f}\phi(v) \in T_{(0,0)}(X \times 0)\}$. Now, $T_{(0,0)}(X \times 0) = T_p\psi(T_pL)$, and the formula for T_pW follows. ■

IV.9.6 Remark In view of Definition IV.9.2, any map f whose image does not meet L is transversal to L. Therefore the condition that $f^{-1}(L)$ be non-empty cannot be dropped (unless we want to consider the empty set as a manifold of any dimension). □

Combining the methods in the proofs of Propositions IV.1.10 and Theorem IV.9.5 one can prove

IV.9.7 Theorem *Consider $f : M \to N$ where $L \subset N$ is a submanifold and $\partial L = \partial N = \emptyset$. If $f \pitchfork L$, $\partial f \pitchfork L$ and $f^{-1}(L)$ is non-empty then $f^{-1}(L) \subset M$ is a neat submanifold (i.e. $\partial(f^{-1}(L)) = f^{-1}(L) \cap \partial M$) whose codimension is equal to that of L in N.*

IV.9.8 Exercise Prove the theorem. (*Hint:* See [12, p. 60-62].) □

The theory of regular values that we have studied in detail is a special case of transversality:

IV.9.9 Exercise If $L = \{q\}$ then $f \pitchfork L$ iff q is a regular value. □

Another important special case of transversality and Theorem IV.9.7 is the following:

IV.9.10 Definition *Let A, B be submanifolds of M. We write $A \pitchfork B$ if $\iota \pitchfork B$, where $\iota : A \to M$ is the inclusion map. Thus $A \pitchfork B$ iff $T_pA + T_pB = T_pM$ for all $p \in A \cap B$. (This is symmetric in A, B.)*

IV.9.11 Corollary *Let A, B be submanifolds of M satisfying $A \pitchfork B$ and $\partial M = \partial B = \emptyset$. If $A \cap B \subset M$ is non-empty it is a submanifold of codimension $\operatorname{codim} A + \operatorname{codim} B$ (i.e. dimension $\dim A + \dim B - \dim M$) and $\partial(A \cap B) = \partial A \cap B$.*

IV.9.12 Exercise Let $A, B \subset M$ be transversal submanifolds. Show that $T_p(A \cap B) = T_pA \cap T_pB$ whenever $p \in A \cap B$. □

IV.9.13 EXERCISE Which of the following linear spaces intersect transversally?

1. The xy plane and the z axis in $\mathbb{R}^3$.
2. The xy plane and the plane spanned by $\{(3,2,0),(0,4,-1)\}$ in $\mathbb{R}^3$.
3. The plane spanned by $\{(1,0,0),(2,1,0)\}$ and the y axis in $\mathbb{R}^3$.
4. $\mathbb{R}^k \times 0_{\mathbb{R}^l}$ and $0_{\mathbb{R}^k} \times \mathbb{R}^l$ in $\mathbb{R}^n$ (depending on k, l, n).
5. $\mathbb{R}^k \times 0_{\mathbb{R}^l}$ and $\mathbb{R}^l \times 0_{\mathbb{R}^k}$ in $\mathbb{R}^n$ (depending on k, l, n).
6. $V \times 0$ and the diagonal in $V \times V$.
7. The skew symmetric ($A^t = -A$) and symmetric ($A^t = A$) matrices in $M_n(\mathbb{R})$.

□

IV.9.14 EXERCISE Show that the ellipses $x^2 + 2y^2 = 3$ and $3x^2 + y^2 = 4$ intersect transversally and that the ellipses $2x^2 + y^2 = 2$ and $(x-1)^2 + 3y^3 = 4$ don't. *Hint:* Draw! □

The crucial ingredient for the definition of the degree and its mod 2 version was Sard's theorem to the effect that regular values always exist. In order to apply Theorem IV.9.7 to situations where we do not a priori have a map $f : M \to N$ and a submanifold $L \subset N$ satisfying $f \pitchfork L$ and $\partial f \pitchfork L$, we need a higher dimensional generalization of Sard's theorem. This is provided by the transversality theorem, one version of which is the following:

IV.9.15 THEOREM *Let $f : M \to N$ be smooth, $L \subset N$ a submanifold such that $\partial L = \partial N = \emptyset$. Then there exists a smooth map $g : M \to N$ smoothly homotopic to f such that $g \pitchfork L$ and $\partial g \pitchfork L$. The map g can be chosen arbitrarily close to f in the C^0-topology.*

Before we can prove such results in Section VI.2, some preparation is needed. Again, the concepts introduced along the way (vector bundles, normal bundles, tubular neighborhoods) are important in many other contexts, like the smooth approximations of continuous maps to be discussed in Section V.4.

Chapter V

More General Theory

V.1 Vector bundles

V.1.1 Vector bundles and their maps

Vector bundles are a natural generalization of the tangent bundle considered earlier. They play a central rôle in all branches of differential geometry (and also in K-theory, which is a branch of algebraic topology). While we will work only with vector bundles over manifolds, we give the general definition.

V.1.1 Definition *A (real) vector bundle over a space B is a space E together with a continuous map $\pi : E \to B$ such that $\pi^{-1}(p)$ is a vector space (over $\mathbb{R}$) for every $p \in B$. Furthermore, every $p \in B$ admits a neighborhood U and a homeomorphism $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\psi"] \arrow[dr, "\pi"'] & U \times \mathbb{R}^n \arrow[d, "p_1"] \\ & U \end{tikzcd}$$

commutes and such that $\psi : \pi^{-1}(p) \to \{p\} \times \mathbb{R}^n$ is an isomorphism of vector spaces for every $p \in U$. A vector bundle $\pi : E \to B$ is smooth if B and E are manifolds, π is smooth and the homeomorphisms ψ are diffeomorphisms.

V.1.2 Remark 1. It is obvious that $p \mapsto \dim \pi^{-1}(p)$ is a locally constant function. If $\dim \pi^{-1}(p) = n$ for all $p \in B$ we say that the vector bundle has rank n.

2. If $\pi : E \to B$ is a continuous vector bundle and M a (smooth) manifold, one can equip E with a manifold structure such that π becomes a smooth map. □

V.1.3 Definition *Let $\pi : E \to B$ and $\pi' : E' \to B'$ be vector bundles and $f : B \to B'$. Then $g : E \to E'$ is a map of vector bundles over f if*

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

commutes and $g_p : \pi^{-1}(p) \to \pi^{-1}(f(p))$ is linear for every $p \in B$. In the case of manifolds we require g to be smooth.

V.1.4 Remark A vector bundle $\pi : E \to B$ should be understood as a family of vector spaces $V_p = \pi^{-1}(p) \cong \mathbb{R}^n$, one for each $p \in B$, where the total space $E = \coprod_{p \in M} V_p$ has a topology (or manifold structure) that locally looks like a direct product $U \times \mathbb{R}^n$. (This property is called local triviality.) In particular, $\pi : M \times \mathbb{R}^n \to M,\ (x, v) \mapsto x$ is a vector bundle. A vector bundle $\pi : E \to B$ is called (globally) trivial if there exists an isomorphism (over id_B) $\phi : E \to B \times \mathbb{R}^n$ of vector bundles. One can show that every vector bundle over a paracompact contractible space is trivial! (Cf. [13, Corollary 2.5] or [3].) □

V.1.5 Example Clearly the tangent bundle $\pi : TM \to M$ of M defined in Section II.4 is a vector bundle over M. For every $f : M \to N$, the map $Tf : TM \to TN$ is a map of vector bundles over f.

V.1.6 Definition *A section of a vector bundle $\pi : E \to B$ is a smooth map $s : B \to E$ such that $\pi \circ s = id_B$. The set of sections of E is denoted by $\Gamma(E)$.*

V.1.2 Some constructions with vector bundles

V.1.7 Definition *If $\pi : E \to B$ is a vector bundle and $A \subset B$ then $\pi : \pi^{-1}(A) \to A$ is a vector bundle, called the restriction $E \restriction A$.*

V.1.8 Lemma *Let $f : M \to N$ be a map (smooth in the case of manifolds) and $p : E \to N$ a vector bundle over N. Then $f^*E = \{(p, e) \in M \times E \mid f(p) = \pi(e)\}$ and $f^*\pi : (p, e) \mapsto p$ define a vector bundle $f^*\pi : f^*E \to M$, the pullback of $\pi : E \to B$ along f. The diagram*

$$\begin{tikzcd} f^*E \arrow[r, "\hat{f}"] \arrow[d, "f^*\pi"'] & E \arrow[d, "\pi"] \\ M \arrow[r, "f"'] & N. \end{tikzcd}$$

commutes, thus $\hat{f} : f^*E \to E,\ (p, e) \mapsto e$ is a map of vector bundles over f.

Proof. For $p \in M$ we have $(f^*\pi)^{-1}(p) = \{(p, e) \in p \times E \mid \pi(e) = f(p)\} \cong \pi^{-1}(f(p))$, which is a vector space. If $p \in M$ and the open neighborhood $U \subset N$ of $f(p)$ and the local trivialization $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ are as in Definition V.1.1, then $f^{-1}(U) \subset M$ is open and

$$(f^*\pi)^{-1}(f^{-1}(U)) = \{(p, e) \in M \times E \mid f(p) = \pi(e) \in U\}.$$

Thus we can define a map $\psi' : (f^*\pi)^{-1}(f^{-1}(U)) \to f^{-1}(U) \times \mathbb{R}^n$ by

$$\begin{array}{ccccc} (f^*\pi)^{-1}(f^{-1}(U)) & \longrightarrow & U \times \mathbb{R}^n & \longrightarrow & f^{-1}(U) \times \mathbb{R}^n \\ (p, e) & \longmapsto & \psi(e) & \longmapsto & (p, \pi_2(\psi(e))). \end{array},$$

where $\pi_2 : U \times \mathbb{R}^n \to \mathbb{R}^n$ is the projection on the second factor. This map has a continuous inverse

$$(p, \pi_2(\psi(e))) \xmapsto{f \times \mathrm{id}} (f(p), \pi_2(\psi(e))) \equiv \psi(e) \xmapsto{p \times \psi^{-1}} (p, e).$$

Thus ψ' is a homeomorphism and f^*E is locally trivial. The consideration of smooth structures in the manifold case is left as an exercise.

The last claim is simply the fact that $f(p) = \pi(e)$, which by definition holds for every $(p, e) \in f^*E$. ■

V.1.9 Exercise Let $\pi : E \to B$ be a vector bundle and $A \subset B$ with inclusion map $\iota : A \hookrightarrow B$. Then the pullback bundle $\iota^*\pi : \iota^*E \to A$ is isomorphic to the restriction $E \restriction A = (\pi : \pi^{-1}(A) \to B)$. □

V.1.10 Proposition *Let $\pi : E \to N$ be a vector bundle and $f : M \to N$ a map. The pullback $f^*\pi : f^*E \to M$ is universal in the following sense. If $\pi' : E' \to M$ is a vector bundle and $g : E' \to E$ a map of vector bundles over f then there is a unique vector bundle map $g' : E' \to f^*E$ over id_M such that $g = \hat{f} \circ g'$, thus the diagram*

$$\begin{tikzcd}
E' \arrow[dr, dotted, "g'"] \arrow[drr, "g"] \arrow[ddr, "\pi'"'] & & \\
& f^*E \arrow[r, "\hat{f}"] \arrow[d, "f^*\pi"] & E \arrow[d, "\pi"] \\
& M \arrow[r, "f"'] & N.
\end{tikzcd}$$

commutes.

Proof. For $e \in E'$ define $g'(e) = (\pi'(e), g(e)) \in M \times E$. It is a trivial matter to verify that the above diagram commutes. The choice of $\pi'(e)$ and $g(e)$ in the two entries of g' is forced by commutativity of the lower and upper triangle, respectively. ■

If V, V' are finite dimensional vector spaces, we obtain new vector spaces $V \oplus V'$, $V \otimes V'$, $\mathrm{Hom}(V, V')$, etc. This generalizes to vector bundles as follows:

V.1.11 Proposition *Let $\pi : E \to B$ and $\pi' : E' \to B$ vector bundles over the base space B. Then there exist vector bundles*

$$\pi_1 : E \oplus E' \to B, \qquad \pi_2 : E \otimes E' \to B, \qquad \pi_3 : Hom(E, E') \to B,$$

over B such that

$$\pi_1^{-1}(p) \cong \pi^{-1}(p) \oplus \pi'^{-1}(p), \quad \pi_2^{-1}(p) \cong \pi^{-1}(p) \otimes \pi'^{-1}(p), \quad \pi_3^{-1}(p) \cong Hom(\pi^{-1}(p), \pi'^{-1}(p)).$$

Proof. We define $E \oplus E' = \{(e, e') \mid \pi(e) = \pi'(e')\}$ and $\pi_1(e, e') = \pi(e)$. Clearly $\pi_1^{-1}(p)$ is a vector space. Let $p \in B$ and U, U' neighborhoods of p over which E, E', respectively, trivialize. Then $U \cap U'$ is a neighborhood of p for which one easily writes down the isomorphism ψ required by Definition V.1.1.

Next we define

$$\begin{aligned}
E \otimes E' &= \coprod_{p \in B} \pi^{-1}(p) \otimes \pi'^{-1}(p), \\
\mathrm{Hom}(E, E') &= \coprod_{p \in B} \mathrm{Hom}(\pi^{-1}(p), \pi'^{-1}(p)).
\end{aligned}$$

The definition of π_2, π_3 and the linear structures on the fibers are obvious. It remains to identify the right manifold structure and to prove local triviality. We consider only $E \otimes E'$, the case of $\mathrm{Hom}(E, E')$ being completely analogous. Let $\{U_i, \psi_i\}$ and $\{U'_i, \psi'_i\}$ be bundle atlasses for E, E', respectively. Then $\{U_i \cap U'_j, \psi_i \otimes \psi'_j\}$ is a bundle atlas for $E \otimes E'$, proving that $E \otimes E'$ is a vector bundle. If B is a manifold it is easy to see that $E \otimes E'$ is a manifold. ■

V.1.12 Remark 1.In fact, every functorial construction with vector spaces generalizes to vector bundles, cf. [3] for the precise formulation and proof.

2. Every vector bundle $\pi : E \to M$ over a 'nice' base space, e.g. a manifold, admits a complement, i.e. a vector bundle $\pi' : E' \to M$ such that the vector bundle $E \oplus E'$ is trivial. This fact is fundamental for K-theory, see e.g. [3]. □

V.1.3 Metrics and orientations

V.1.13 Definition *A (riemannian) metric on a smooth vector bundle $\pi : E \to M$ is a family $\{\langle \cdot, \cdot \rangle_x,\ x \in M\}$ of symmetric positive definite bilinear forms on $E_x = \pi^{-1}(x)$, such that the map $x \mapsto \langle s(x), t(x) \rangle_x$ is smooth for all sections $s, t \in \Gamma(E)$.*

V.1.14 Proposition *Every vector bundle admits a riemannian metric.*

Proof. Let r be the rank of E and et $(U_i, \phi_i), i \in I$ be a bundle atlas for E, i.e. the U_i are an open cover of B such that

$$\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^r.$$

We may assume the cover to be locally finite and choose a subordinate partition of unity $\{\lambda_i, i \in I\}$. For each $i \in I$, let $\langle \cdot, \cdot \rangle_i$ be a positive definite symmetric quadratic form on $\mathbb{R}^n$ and for $X, Y \in \Gamma(TM)$ we define

$$\langle X, Y \rangle_p = \sum_{i \in I} \lambda_i(p)\, \langle p_2 \circ \phi_i(X(p)), p_2 \circ \phi_i(Y(p)) \rangle_i.$$

Here the i-th summand is understood to be zero if $p \notin U_i$. This is well defined by local finiteness of the partition and smooth. Symmetry and positive definiteness are obvious, and positive definiteness follows from $\langle X, X \rangle_p > 0$ which is evident for $X(p) \neq 0$. ■

V.1.15 Definition *An orientation on a vector bundle $\pi : E \to B$ is a choice of an orientation for each vector space $\pi^{-1}(p),\ p \in B$ such that the orientation is locally constant in every bundle chart.*

Clearly, an orientation for the tangent bundle of a manifold M is the same as an orientation of M in the sense of Definition IV.6.2. The orientation of a direct sum $E \oplus E'$ of oriented vector bundles is defined as the product orientation on $T(M \times M')$.

V.2 Normal bundles

Besides the tangent bundles TM, another important class of a vector bundles is provided by the normal bundles NM. As opposed to the former, the latter are not intrinsically defined but depend on an embedding of M into some euclidean space $\mathbb{R}^n$.

V.2.1 Definition *Let $M \subset \mathbb{R}^n$ be a submanifold and write*

$$N_pM = T_pM^\perp = \{v \in T_p\mathbb{R}^n \equiv \mathbb{R}^n \mid \langle v, w \rangle = 0 \ \ \forall w \in T_pM\}.$$

(Here $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$.) Then the normal bundle NM is

$$NM = \{(p, v),\ p \in M,\ v \in N_pM\}$$

with the obvious projection $\pi : NM \to M$.

V.2.2 Proposition *NM admits the structure of a (smooth) manifold of dimension n such that $\pi : NM \to M$ is a submersion and a smooth vector bundle of rank $n - \dim M$.*

Proof. It is clear that each $\pi^{-1}(p)$ is a vector space of dimension $n-m$, where $m = \dim M$. Since $M \subset \mathbb{R}^n$ is a submanifold, for every $p \in M$ we can find a chart $(\tilde{U}, \phi)$ around $p \in \mathbb{R}^n$ such that, writing $U = \tilde{U} \cap M$, we have $\phi(U) = \phi(\tilde{U}) \cap \mathbb{R}^m$. Thus if $\lambda : \mathbb{R}^n \to \mathbb{R}^k$ is the projection onto the last $k = n - m$ coordinates and $\psi = \lambda \circ \phi$, we have $U = \psi^{-1}(0)$. Clearly, ψ is a submersion. We have $NU = NM \cap (U \times \mathbb{R}^n)$, thus NU is open in NM, the latter topologized as a subspace of $M \times \mathbb{R}^n$. For each $p \in M$, the map $T_p\psi : \mathbb{R}^n \to \mathbb{R}^k$ is surjective and its kernel is T_pM. Thus its transpose $(T_p\psi)^t : \mathbb{R}^k \to \mathbb{R}^n$ is injective and its image is N_pM. Therefore the map $\psi' : U \times \mathbb{R}^k \to NU$ defined by $\psi'(p, v) = (p, (T_p\psi)^t v)$ is a bijection and an embedding of $U \times \mathbb{R}^k$ into $M \times \mathbb{R}^n$, thus $(U \times \mathbb{R}^k, \psi')$ is a chart. Since such maps exist for all $p \in M$, NM is a manifold. (Verification of compatibility of these charts is left as an exercise.) Since $\pi \circ \psi' : U \times \mathbb{R}^k \to U$ is just the standard submersion, π is a submersion. That $\pi : NM \to M$ is a vector bundle is now clear, the local trivializations being provided by the inverses of the maps ψ' considered in the proof. ∎

V.2.3 Exercise Show that the map $M \to NM$ given by $p \mapsto (p, 0)$ is an embedding. (Thus M can be considered as submanifold of NM.) □

The above considerations can be generalized to more general submanifolds $M \subset P$, where P is supposed to be equipped with a riemannian metric. (By Remark III.5.6 all metrics arise via pullback from embeddings into some $\mathbb{R}^n$.)

V.2.4 Definition *Let P be a riemannian manifold with metric g and let $M \subset P$ be a submanifold. Then the normal bundle $N(M, P)$ is*

$$N(M,P) = \{(p, v),\ p \in M,\ v \in N_p(M,P)\},$$

where

$$N_p(M,P) = \{v \in T_pP \mid \langle v, w\rangle_p = 0 \ \ \forall w \in T_pM\}$$

with the obvious projection $\pi : N(M,P) \to M$.

V.2.5 Proposition *For any riemannian metric on P, $N(M,P)$ is a manifold of dimension* $\dim P$ *and a vector bundle of rank* $\dim P - \dim M$ *over M. The projection onto M is a submersion.*

Proof. This can be proven with the intrinsic methods of riemannian geometry, cf. e.g. [17, p. 133]. In order to avoid this, but appealing to Nash's difficult embedding theorem instead, we may assume P to be isometrically embedded into some $\mathbb{R}^n$. Then the proof proceeds essentially as that of Proposition V.2.2. ∎

V.2.6 Remark The normal bundle $N(M,P)$ seems to depend on the choice of a riemannian metric on P. Alternatively, one can consider the algebraic normal bundle

$$N^a(M,P) = \{(p, v),\ p \in M,\ v \in T_pP/T_pM\},$$

which can be shown to be a manifold diffeomorphic to $N(M,P)$. (More precisely, one has an isomorphism of vector bundles over id_M.) This implies that, up to diffeomorphism, $N(M,P)$ does not depend on the metric on P. In practice, the more geometric definition of $N(M,P)$ is more useful. □

If M and P are both oriented we define an orientation on $N(M, P)$ by the direct sum

$$N(M, P) \oplus TM = TP \restriction M.$$

(I.e., we choose the orientation of $N(M, P)$ such that the direct sum orientation on $N(M, P) \oplus TM$ coincides with the given orientation on $TP \restriction M$.)

V.2.7 Lemma *Let $M_1, M_2 \subset P$ be transversal submanifolds, i.e. $M_1 \pitchfork M_2$. Then*

$$N_p(M_1 \cap M_2, P) = N_p(M_1, P) \oplus N_p(M_2, P) \quad \forall p \in M_1 \cap M_2.$$

Thus the normal bundle of the submanifold $M_1 \cap M_2$ is given by

$$N(M_1 \cap M_2, P) \;\cong\; (N(M_1, P) \restriction M_1 \cap M_2) \;\oplus\; (N(M_2, P) \restriction M_1 \cap M_2).$$

Proof. By transversality, $M_1 \cap M_2$ is a manifold, and by Exercise IV.9.12, $T_p(M_1 \cap M_2) = T_pM_1 \cap T_pM_2$. Now, let $W_i \subset T_pP$ be subspaces such that $T_pM_i \cong W_i \oplus (T_pM_1 \cap T_pM_2)$ for $i = 1, 2$. By the transversality assumption $T_pM_1 + T_pM_2 = T_pP$ we have $T_pP \cong W_1 \oplus W_2 \oplus (T_pM_1 \cap T_pM_2)$. Thus $N_pM_1 = T_pP \cap T_pM_1^{\perp} = W_2$ and $(1 \leftrightarrow 2)$ and therefore

$$N_p(M_1 \cap M_2, P) = T_pP \cap T_p(M_1 \cap M_2)^{\perp} = W_1 \oplus W_2 = N_pM_2 \oplus N_pM_1.$$

This proves the first claim, and the second is just a reformulation. ■

The preceding lemma is a special case of the following:

V.2.8 Exercise Consider $f : A \to M \supset B$ where $f \pitchfork B$, and let $W = f^{-1}(B)$. If $W \neq \emptyset$ then

$$N_p(W, A) \cong (T_pf)^{-1}(N_{f(p)}(B, M))$$

for all $p \in W$. □

V.2.9 Exercise Let $\Delta = \{(x, x) \mid x \in M\} \subset M \times M$ be the diagonal. Show that the map $TM \to N(\Delta, M \times M)$ defined by $(x, v) \mapsto ((x, x), (v, -v))$ is a diffeomorphism. □

V.3 Tubular neighborhoods

V.3.1 Definition *Let $M \subset \mathbb{R}^n$ be a submanifold and $\varepsilon : M \to (0, \infty)$ a smooth map. Then we define*

$$M^{\varepsilon} = \{p \in \mathbb{R}^n \mid \exists q \in M \; s.th. \; |p - q| < \varepsilon(q)\}.$$

V.3.2 Theorem *Let $M \subset \mathbb{R}^n$ be a submanifold. Define $\theta : NM \to \mathbb{R}^n$ by $\theta(p, v) = p + v$. Then there exists a smooth map $\varepsilon : M \to (0, \infty)$ such that θ restricts to a diffeomorphism between the neighborhood $N^{\varepsilon}M = \{(p, v),\ p \in M, v \in N_pM, |v| < \varepsilon(p)\}$ of $M = \{(p, 0),\ p \in M\}$ in NM and the neighborhood M^{ε} of M in $\mathbb{R}^n$. The latter is called a tubular neighborhood of M.*

Proof. Consider the map $h : NM \to \mathbb{R}^n$ given by $(p, v) \mapsto p + v$. Through every $(p, 0) \in M \times \{0\} \subset NM$ there pass the submanifolds $M \times \{0\}$ and $\{p\} \times N_pM$. The derivative $T_{(p,0)}h$ maps the tangent spaces of these two submanifolds to $T_pM \subset \mathbb{R}^n$ and $N_pM \subset \mathbb{R}^n$, respectively. The latter sum up to $\mathbb{R}^n$, thus $p \times \{0\}$ is a regular point of h. Since NM and $\mathbb{R}^n$ have the same dimension, h is a diffeomorphism of some neighborhood of $M \times \{0\}$ in NM onto a neighborhood $\tilde{M}$ of M in $\mathbb{R}^n$. If M is compact, the latter neighborhood contains M^{ε} for some $\varepsilon > 0$. If M is non-compact then choose an open cover of M by sets $U_i \subset M$ and $v_i > 0$ such that $U_i^{\varepsilon_i} \subset \tilde{M}$. If $\{\lambda_i\}$ is a partition of unity subordinate to $\{U_i\}$ then $\varepsilon(p) = \sum_i \varepsilon_i \lambda_i(p)$ does the job. ■

The theorem permits the following extension which clarifies its geometric meaning:

V.3.3 PROPOSITION *Let $M \subset \mathbb{R}^n$ be compact and let θ and ε be as in Theorem V.3.2. Then for every $p \in M^\varepsilon$ there is a unique closest point $\sigma(p) \in M$. σ is a submersion and the inverse of $\theta : N^\varepsilon M \to M^\varepsilon$ is given by $\theta^{-1} : p \mapsto (\sigma(p), p - \sigma(p))$.*

Proof. Let $p \in M^\varepsilon$ and consider the map $\lambda_p : M \to \mathbb{R}_+,\ q \mapsto |p-q|^2 = (p-q, p-q)_{\mathbb{R}^n}$. Since M is compact, λ_p is bounded and assumes its infimum. Thus there exists $q \in M$ such that $\lambda_p(q) \leq \lambda_p(q')$ for all $q' \in M$. Now the derivative $T_q\lambda_p = 2(p-q, \cdot) : T_qM \to \mathbb{R}$ vanishes, thus $p - q \in T_qM^\perp = N_qM$. Thus $p = q + v = h(q, v)$ with $(q, v) \in NM$. If $q' \in M$ is another point for which $|p-q| = |p-q'|$ then again $p = q' + v' = h(q', v')$ with $(q', v') \in NM$. By Theorem V.3.2, $h : N^\varepsilon M \to M^\varepsilon$ is a diffeomorphism, thus in particular a bijection, implying $(q, v) = (q', v')$. Therefore there is a unique point $\sigma(p) \in M$ closest to $p \in M^\varepsilon$.

If $\pi : NM \to M$ is the canonical projection and $h^{-1} : M^\varepsilon \to N^\varepsilon M$ is the inverse of the diffeomorphism h, it is clear from the preceding reasoning that $\sigma = \pi \circ h^{-1} : M^\varepsilon \to M$. As a composition of a diffeomorphism and a submersion, σ is a submersion. ■

As with normal bundles, the above considerations generalize to arbitrary embeddings $M \subset P$:

V.3.4 DEFINITION *Let P be a riemannian manifold and $M \subset P$ a submanifold. A tubular neighborhood of M is an open neighborhood U of P together with a diffeomorphism $\phi : N(M, P) \to U$ restricting to the identity map on the zero section (where we identify the latter with M).*

V.3.5 THEOREM *Let P be a riemannian manifold and $M \subset P$ a submanifold. Then M has a tubular neighborhood U in P.*

Proof. Again, there is an proof intrinsic to riemannian geometry and avoiding embeddings into euclidean space, cf. e.g. [17, Exercise 8-5]. On the other hand, one can give a more elementary proof assuming an embedding $P \subset \mathbb{R}^n$, cf. [6, Theorem II.11.14]. ■

One can show that all tubular neighborhoods for $M \subset P$ are diffeotopic:

V.3.6 THEOREM *Let P be a riemannian manifold and $M \subset P$ a submanifold. Let U_1, U_2 be tubular neighborhoods of M in P. Then there exists an diffeotopy $\phi : P \times I \to P$ such that $\phi_0 = id$, $\phi_1(U_1) = U_2$ and $\phi_t(p) = p$ for all $p \in M, y \in I$.*

For a proof see [13, Theorem IV.5.3] or [7, Satz 12.13]. We will not use this result.

As an application of tubular neighborhoods we obtain the following result on the topological triviality of Euclidean space:

V.3.7 LEMMA *Let M be a compact manifold with boundary and let $f : \partial M \to \mathbb{R}^n$ be any smooth map. Then f extends to a smooth map $M \to \mathbb{R}^n$.*

Proof. By the embedding theorem, we may consider M as a submanifold of some $\mathbb{R}^k$. Let U be a tubular neighborhood of ∂M in $\mathbb{R}^k$ with projection $\sigma : U \to \partial M$. Then $f \circ \sigma : U \to \mathbb{R}^n$ extends f to U. Let $\rho : U \to \mathbb{R}$ be a smooth function that equals one on ∂M and vanishes outside some compact subset of U. Now we extend f to all of $\mathbb{R}^k$, thus in particular to M, by setting it to be equal to $\rho \cdot f$ on U and 0 elsewhere. ■

V.3.8 EXERCISE Use Exercise V.2.9 and the tubular neighborhood theorem to show that there is a diffeomorphism between a neighborhood of M_0 (the zero section) in TM and a neighborhood of Δ in $M \times M$, extending the usual diffeomorphism $M_0 \to \Delta$, $(x, 0) \mapsto (x, x)$. □

V.4 Smooth approximation

The main motivation for the introduction of normal bundles and tubular neighborhoods was the proof of the transversality theorem. As another application of these tools we will now show that the *smooth* methods of differential topology can be used to prove results about *continuous* maps between manifolds.

V.4.1 THEOREM *Let $f : M \to N$ be a continuous map of manifolds without boundary. Let $f \restriction U$ be smooth where $C \subset U \subset M$ with C closed and U open. Then there exists a smooth map $g : M \to N$ such that $g \restriction C = f \restriction C$. The map g can be chosen homotopic to f and arbitrarily close to f in the C^0-topology.*

Proof. We first consider the case where $N = \mathbb{R}^n$. There exists a locally finite open cover $\{U_i, i \in I\}$ of M subordinate to the open cover $\{U, M - C\}$, which we may assume indexed by $\mathbb{Z}$ such that $U_i \subset U$ if $i < 0$ and $U_i \subset M - C$ if $i \geq 0$. Given a smooth function $\varepsilon : M \to (0, \infty)$, the cover $\{U_i\}$ and vectors $f_i \in \mathbb{R}^n, i \geq 0$ can be chosen such that $|f(p) - f_i| < \varepsilon(p)$ for all $p \in U_i, i \geq 0$. Let $\{\lambda_i, i \in \mathbb{Z}\}$ be a partition of unity with $\operatorname{supp} \lambda_i \subset U_i$. Consider

$$g(p) = f(p) \sum_{i<0} \lambda_i(p) + \sum_{i \geq 0} f_i \lambda_i(p).$$

g is smooth since $\lambda_i(p)$ vanishes for $i < 0$ and $p \in M - C$, and $g \restriction C = f \restriction C$ since for $p \in C$ we have $\lambda_i(p) = 0 \ \forall i \geq 0$, implying $\sum_{i<0} \lambda_i(p) = 1$. The indicated choice of the f_i guarantees that can be chosen in any C^0-neighborhood of f. Being $\mathbb{R}^n$-valued functions, f and g are clearly homotopic.

In the general case, choose an embedding $\Psi : N \to \mathbb{R}^n$ and a tubular neighborhood $\Psi(N)^\varepsilon \supset \Psi(N)$ with projection σ. Let $G = \{(x, \Psi \circ f(x)),\ x \in M\}$ be the graph of $\Psi \circ f$ in $M \times \mathbb{R}^n$, let W be a neighborhood of G and

$$Q = \{(x, y) \in M \times T \mid (x, \sigma(y)) \in W\}.$$

Choosing a smooth map $g : M \to \mathbb{R}^n$ whose graph lies in Q, $\sigma \circ g$ is a smooth map with values in $\Psi(M)$, thus $\Psi^{-1} \circ \sigma \circ g : M \to N$ is smooth, coincides with f on C and is homotopic and arbitrarily close to f. ∎

V.4.2 COROLLARY *Let M, N be smooth manifolds. There are bijections between (i) the (continuous) homotopy classes of continuous maps $M \to N$, (ii) continuous homotopy classes of smooth maps and (iii) smooth homotopy classes of smooth maps.*

Proof. By Theorem V.4.1, every continuous map $f : M \to N$ is (continuously) homotopic to a smooth map $\tilde{f} : M \to N$. This proves the bijection (i)$\leftrightarrow$(ii). Let $f, g : M \to N$ be smooth and let $h : M \times [0, 1] \to N$ be a continuous homotopy. We may assume that $h_t : M \to N$ is independent of t on $[0, \varepsilon)$ and $(1 - \varepsilon, 1]$. Thus $h \restriction U$ with $U = M \times ([0, \varepsilon) \cup (1 - \varepsilon, 1])$ is smooth and the smoothing theorem gives a smooth homotopy between f and g, proving the bijection (ii)$\leftrightarrow$(iii). ∎

We immediately have the following 'continuous' corollaries of our earlier 'smooth' results in Theorem III.4.11 and Corollary IV.8.7:

V.4.3 COROLLARY *If $\dim M < n$ then every continuous map $f : M \to S^n$ is homotopic to a constant map. In particular, $\pi_m(S^n) = 0$ if $0 \leq m < n$.*

V.4.4 COROLLARY *Let M be a connected oriented compact n-manifold without boundary. Then the degree establishes a bijective correspondence between the set $[M, S^n]$ of homotopy classes of continuous maps $M \to S^n$ and $\mathbb{Z}$. In particular, $\pi_n(S^n) \cong \mathbb{Z}$ for all $n \geq 1$.*

The smooth version of Theorem IV.3.2 requires only slightly more work.

V.4.5 Corollary *Any continuous map $f : D^n \to D^n$ has a fixpoint.*

Proof. Suppose the continuous map $f : D^n \to D^n$ has no fixpoint. By the same argument as in Theorem IV.3.2 we deduce the existence of a (continuous) retraction $r : D^n \to \partial D^n = S^{n-1}$. It is easy to change r into a continuous map r' which is a retraction of D onto a neighborhood U of ∂D^n. Thus r' is the identity and therefore smooth on U, and applying smooth approximation to r' we obtain a smooth retraction $r'' : D^n \to \partial D^n$, which cannot exist by Proposition IV.3.1. ∎

V.4.6 Remark We have seen that methods from differential topology, combined with smooth approximations, can be used to compute the homotopy groups $\pi_m(S^n),\ m \leq n$ of spheres. There is a theory, due to Pontryagin and Thom, which establishes a bijection between certain homotopy groups, like $\pi_m(S^n),\ m \geq n$, and 'cobordism classes' of certain manifolds. (The result that $\pi_n(S^n) = \mathbb{Z}$ can be obtained as a very special case of this formalism, cf. [19, §7].) Originally, this theory was intended for the computation of $\pi_m(S^n)$ where $m > n$, and in fact this has been done for $m - n \leq 3$. Unfortunately, the difficulties soon become insurmountable. There are, however, other, more algebraic ways of computing $\pi_m(S^n)$, and Pontryagin-Thom theory can be used the other way round to prove otherwise inaccessible results about smooth manifolds! We refer to [19] for a lucid introduction to the relatively easy theory of 'framed cobordism' and to [13] for 'oriented' and 'unoriented' cobordism. □

Chapter VI

Transversality II: Intersection Theory

VI.1 Parametric transversality

VI.1.1 Proposition *Let $F : M \times S \to N$ be a smooth map and $L \subset N$ a submanifold, where S, N, L are boundaryless. For $s \in S$ we write $F_s = F(\cdot, s) : M \to N$. If $F \pitchfork L$ and $\partial F \pitchfork L$ then $F_s \pitchfork L$ and $\partial F_s \pitchfork L$ for all $s \in S$ but a set of measure zero.*

Proof. By $F \pitchfork L$ and Theorem IV.9.5, $F^{-1}(L) \subset M \times S$ is a submanifold. Consider the projection $M \times S \to S$. We claim, for any $s \in S$, that $F_s \pitchfork L$ iff s is a regular value of $\pi : F^{-1}(L) \to S$, and $\partial F_s \pitchfork L$ iff s is a regular value of $\partial\pi : \partial F^{-1}(L) \to S$. This clearly implies the proposition since by Sard's theorem the union of the sets of critical values of $\pi : F^{-1}(L) \to S$ and of $\partial\pi : \partial F^{-1}(L) \to S$, respectively, has measure zero. It remains to prove the claim, which is a purely algebraic matter.

By the assumption $F \pitchfork L$ we have

$$T_{(a,s)}F\,[T_{(a,s)}(M \times S)] + T_{F(a,s)}L = T_{F(a,s)}N \qquad \forall (a,s) \in F^{-1}(L).$$

In view of $T_{(a,s)}(M \times S) \cong T_aM \oplus T_sS$ this is equivalent to

$$T_sF^a(T_sS) + T_aF_s(T_aM) + T_{F(a,s)}L = T_{F(a,s)}N \qquad \forall (a,s) \in F^{-1}(L), \tag{VI.1}$$

where $F_s = F(\cdot, s)$ as before, and $F^a = F(a, \cdot)$. On the other hand, $F_s \pitchfork L$ means

$$T_aF_s(T_aM) + T_{F(a,s)}L = T_{F(a,s)}N \qquad \forall a \in F_s^{-1}(L). \tag{VI.2}$$

Given (VI.1), the stronger condition (VI.2) follows for a certain $s \in S$ iff we have

$$T_sF^a(T_sS) \subset T_aF_s(T_aM) + T_{F(a,s)}L \qquad \forall a \in F_s^{-1}(L). \tag{VI.3}$$

If $u \in T_aM,\ v \in T_sS$ we have $T_{(a,s)}F(u \oplus v) = T_aF_s(u) + T_sF^a(v)$. Thus (VI.3) holds iff for every $a \in F_s^{-1}(L)$ and $v \in T_sS$ there exists $u \in T_aM$ such that $T_{(a,s)}F(u \oplus v) \in T_{F(a,s)}L$. On the other hand, for the projection $\pi : F^{-1}(L) \to S$ we have $T_{(a,s)}\pi(u \oplus v) = v$. Thus $s \in S$ is a regular value of π iff for every $a \in F_s^{-1}(L)$ and $v \in T_sS$ there exists $u \in T_aM$ such that $u \oplus v \in T_{(a,s)}(F^{-1}(L))$. By Theorem IV.9.5, a vector $u \oplus v \in T_{(a,s)}(M \times S)$ is in $T_{(a,s)}(F^{-1}(L))$ iff $T_{F(a,s)}F(u \oplus v) \in L$, thus the two conditions are equivalent, proving the claim.

The argument for $\partial F_s : M \to N$ and $\partial\pi : F^{-1}(L) \to S$ is exactly the same as (and in fact a special case of) the preceding one. ■

VI.1.2 Corollary *Let $f : M \to \mathbb{R}^n$ be a smooth map and $L \subset \mathbb{R}^n$ a submanifold. For $s \in \mathbb{R}^n$ write $f_s : x \mapsto f(x) + s$. Then $f_s \pitchfork L$ for all $s \in B_1(0)$ but a set of measure zero.*

Proof. Let S be the open unit ball around $0 \in \mathbb{R}^n$ and define $F(x,s) = f(x) + s$. It is clear that $s \mapsto F(x,s)$ is a submersion for any fixed x. A fortiori, $F : M \times S \to \mathbb{R}^n$ is a submersion, thus $F \pitchfork L$. By Proposition VI.1.1, $f_s \pitchfork L$ for almost all $s \in S$. ■

The functions f and $f_s = f + s$ are obviously homotopic. We have thus proven Theorem IV.9.15 in the case where $N = \mathbb{R}^n$. For an arbitrary target manifold N we can choose an embedding $\Phi : N \to \mathbb{R}^n$ for suitable n. Corollary VI.1.2 then implies that there is a map $g : M \to \mathbb{R}^n$ arbitrarily close to Φf such that $g \pitchfork \Phi(L)$. The image $g(M)$ lies in some neighborhood U of $\Phi(N) \subset \mathbb{R}^n$, and all we need is a projection π of U onto $\Phi(N)$ such that $\pi g \pitchfork L$. This requires some preparation, which will be the subject of the next subsections.

VI.2 Transversality theorems

Using tubular neighborhoods it is now easy to prove our first general transversality theorem.

VI.2.1 PROPOSITION *Let $f : M \to N$ be a smooth map, $L \subset N$ a submanifold, where $\partial N = \partial L = \emptyset$. Then there exists a smooth map $g : M \to N$ such that $g \simeq f$ and $g \pitchfork L$, $\partial g \pitchfork L$.*

Proof. Let $\Psi : N \to \mathbb{R}^n$ be an embedding and let S be the unit ball in $\mathbb{R}^n$. Let $\varepsilon : N \to \mathbb{R}_+$ and $\sigma : \Psi(N)^\varepsilon \to N$ as in Theorem V.3.2. We define

$$F : M \times S \to N, \quad F(x,s) = \Psi^{-1}\sigma[\Psi f(x) + \varepsilon(f(x))s].$$

Since $\sigma : \Psi(N)^\varepsilon \to \Psi(N)$ restricts to the identity map on $\Psi(N)$, we have $F(x,0) = f(x)$. Obviously, $s \mapsto \psi \circ f(x) + \varepsilon(f(x))s : M \to \Psi(M)^\varepsilon$ is a submersion for every $x \in M$. Therefore $s \mapsto F(x,s)$ is a composition of two submersions, thus a submersion. It clearly follows that $F : M \times S \to N$ is a submersion. Thus $F \pitchfork L$ for any submanifold $L \subset N$, and Theorem VI.1.1 implies that $F_s \pitchfork L$ and $\partial F_s \pitchfork L$ for almost all $s \in S$. Let $g = F_s$ for such an $s \in S$. Finally, $M \times I \to N, (x,t) \mapsto F(x,ts)$ is a homotopy between $f = F_0$ and $g = F_s$. ■

For the purposes of intersection theory we need a version where g can be taken to coincide with f on a subset on which it is already transversal.

VI.2.2 THEOREM *Let $f : M \to N$ be a smooth map, $L \subset N$ a submanifold, where $\partial N = \partial L = \emptyset$. Let $C \subset N$ be closed, and assume that $(f \restriction C) \pitchfork L$ and $(\partial f \restriction C \cap \partial M) \pitchfork L$. Then there exists a smooth map $g : M \to N$ such that $g \simeq f$, $g \pitchfork L$, $\partial g \pitchfork L$ and g coincides with f on a neighborhood of C.*

Proof. We claim that there is an open set U with $C \subset U \subset M$ such that $(f \restriction U) \pitchfork L$. On the one hand, if $p \in C - f^{-1}(L)$ then, since L is closed, $X = C - f^{-1}(L)$ is an open neighborhood of p such that $(f \restriction X) \pitchfork L$ holds trivially. If, on the other hand, $p \in f^{-1}(L)$, pick an open neighborhood W of $f(p)$ and a submersion $\phi : W \to \mathbb{R}^k$ such that $f \pitchfork L$ at a point $q \in f^{-1}(L \cap W)$ iff $\phi \circ f$ is regular at q. By assumption $\phi \circ f$ is regular at p, thus it is regular in a neighborhood of p. This proves the claim.

Now let C' be any closed set contained in U and containing C in its interior, and let $\{\lambda_i\}$ be a partition of unity subordinate to the open cover $\{U, M - C'\}$. Defining γ to be the sum of those λ_i that vanish outside of $M - C'$ we obtain a function $\gamma : M \to [0,1]$ that is one outside U and zero on some neighborhood of C.

Defining $\tau = \gamma^2$ we have $T_p\tau = 2\gamma(p)T_p\gamma : T_pM \to \mathbb{R}$, thus $T_p\gamma = 0$ whenever $\gamma(p) = 0$. Let $F : M \times S \to N$ be the map considered in the proof of Proposition VI.2.1 and define $G : M \times S \to N$ by $G(x,s) = F(x,\tau(x)s)$. We claim $G \pitchfork L$. To see this, let $(x,s) \in G^{-1}(L)$ and suppose, to begin with, $\tau(x) \neq 0$. Then the map $S \to M$, $r \mapsto G(x,r)$ is a composition of the diffeomorphism $r \mapsto \tau(x)r$ and the submersion $r \mapsto F(x,r)$, thus it is a submersion. Thus (x,s) is a regular point of G and, a fortiori, $G \pitchfork L$ at (x,s). It remains to consider the case $\tau(x) = 0$. We write $G = F \circ H$, where

$H : M \times S \to M \times S$ is given by $(x, s) \mapsto (x, \tau(x)s)$. Then, for $(v, w) \in T_xM \times T_sS = T_xM \times \mathbb{R}^n$, we have

$$T_{(x,s)}G(v,w) = (T_{H(x,s)}F \circ T_{(x,s)}H)(v,w) = T_{(x,\tau(x)s)}F((v, \tau(x)w + T_x\tau(v)s)) = T_{(x,0)}F(v,0),$$

where we have used $\tau(x) = 0$ and $T_x\tau(v) = 0$. Since $F(x, 0) = f(x)$, we have

$$T_{(x,s)}G(v,w) = T_xf(v),$$

thus $T_{(x,s)}G$ and T_xf have the same images. Furthermore, $\tau(x) = 0$ implies $x \in U$, thus $f \pitchfork L$ at x and therefore $G \pitchfork L$ at (x, s), as claimed.

Similarly on shows $\partial G \pitchfork L$. By Proposition VI.1.1 we can find $s \in S$ such that $F_s \pitchfork L$ and $\partial F_s \pitchfork L$. Then $g = F_s$ is homotopic to f and if p belongs to the neighborhood of C on which $\tau(p) = 0$ then $g(x) = G(x, s) = F(x, 0) = f(x)$, as desired. ■

VI.2.3 Corollary *Let $L \subset N$ be a submanifold where $\partial N = \partial L = \emptyset$. If $f : M \to N$ is such that $\partial F : \partial M \to N$ is transversal to L then there exists $g : M \to N$ such that $g \pitchfork L$, $g \simeq f$ and $\partial g = \partial f$.*

VI.2.4 Remark In our proof of the transversality Theorem VI.2.2 and its preliminaries we followed the approach of [12], which has the virtues of being elementary and of exhibiting very clearly the use of Sard's theorem via Proposition VI.1.1. There are more elegant proofs that use either 'jet-transversality', cf. [13, 9], or some more vector bundle theory (the fact that every vector bundle on a manifold admits an 'inverse'), cf. [7, 6]. □

Now we are in a position to finish the proof of Hopf's theorem on maps into spheres:

Proof of Proposition IV.8.4. We are given a map $f : \partial M \to S^n \subset \mathbb{R}^{n+1}$. By Lemma V.3.7, f may be extended to a smooth map $F : M \to \mathbb{R}^{n+1}$. Since f has its image in S^n, $0 \in \mathbb{R}^{n+1}$ is trivially a regular value, thus $f \pitchfork \{0\}$. By the transversality extension Theorem VI.2.2 we can pick F such that $F \pitchfork \{0\}$. Thus 0 is a regular value of F and $F^{-1}(0) \subset M$ is a finite set. Let $U \subset M - \partial M$ be an open set for which there exists a diffeomorphism $\gamma : \mathbb{R}^{n+1} \to U$. By Exercise IV.4.5 we may suppose that $F^{-1}(0)$ is contained in U. Let $B \subset \mathbb{R}^{n+1}$ be an open ball such that $F^{-1}(0) \subset \gamma(B)$. Then $F/|F|$ extends to $M - \gamma(B)$. Since $F \upharpoonright \partial M = f$ has degree zero by assumption, it follows that $F \upharpoonright \partial\gamma(B) \to \mathbb{R}^{n+1} - \{0\}$ has winding number zero. Thus, by part II of Proposition IV.8.12, the restriction $F : \partial\gamma(B) \to \mathbb{R}^{n+1}$ is homotopic to a non-zero constant map, in other words we can change F on $\gamma(B)$ such that it avoids the value zero. Let F' be this function. Then the desired extension of $f : \partial M \to S^n$ to $\hat{f} : M \to S^n$ is given by $\hat{f} = F'/|F'|$. ■

VI.3 Mod-2 Intersection theory

In this section all manifolds are without boundary.

The theories of the degree and the mod 2 degree were concerned with maps $f : M \to N$ between manifolds of the same dimension. Intersection theory, of which again there is an unoriented (mod 2) and an oriented version, is a generalization to the situation where one has a map $f : M \to N$ and a submanifold $L \subset N$ subject to the condition $\dim M + \dim L = \dim N$. (This contains the case where $L = \{q\}$ and $\dim M = \dim N$.) The condition that q be a regular value is replaced by the requirement $f \pitchfork L$, so that Theorem IV.9.5 implies that $f^{-1}(L)$ is a discrete subset of M. We begin our considerations with the unoriented mod 2 intersection theory which contains the formalism of the mod 2 degree as a spectial case. (In fact the latter seems to be the only interesting application of mod 2 intersection theory! If we still consider the mod 2 theory in some detail, it is because it is considerably easier to set up than the oriented theory.)

VI.3.1 Definition *Consider* $f : M \to N \supset L$ *where* M *is compact,* $\dim M + \dim L = \dim N$ *and* $f \pitchfork L$*. Then we define the mod 2 intersection number* $I_2(f, L) \in \{0, 1\}$ *by*

$$I_2(f, L) \equiv \# f^{-1}(L) \pmod 2.$$

VI.3.2 Proposition *Let* $f, g : M \to N$ *be smoothly homotopic and both transversal to* L*. Assuming the conditions of Definition VI.3.1 we have* $I_2(f, L) = I_2(g, L)$*.*

Proof. Let $F : M \times I \to N$ be a homotopy. By the assumption $f \pitchfork L$, $g \pitchfork L$ we have $\partial F \pitchfork L$. By Theorem VI.2.2 there is $G : M \times I \to N$ such that $G \pitchfork L$, $G \simeq F$ and $\partial G = \partial F$. Now $G^{-1}(L)$ is a neat one-dimensional submanifold of $M \times I$, thus

$$\partial(G^{-1}(L)) = G^{-1}(L) \cap (M \times \{0, 1\}) = f^{-1}(L) \times 0 \cup g^{-1}(L) \times 1.$$

By Corollary II.12.3, $\partial(G^{-1}(L))$ has an even number of points, thus $\# f^{-1}(L) \equiv \# g^{-1}(L) \pmod 2$. ■

For an arbitrary map $f : M \to N$ we pick a homotopic map $\tilde{f} : M \to N$ such that $\tilde{f} \pitchfork L$ and define $I_2(f, L) = I_2(\tilde{f}, L)$. The preceding proposition implies that this is well defined, i.e. independent of the choice of $\tilde{f}$, and it is clear that if $f \simeq g$ then $I_2(f, L) = I_2(g, L)$.

An important special case is that of transverse submanifolds.

VI.3.3 Definition *Let* M *be compact and let* $A, B \subset M$ *be transverse submanifolds, i.e.* $A \pitchfork B$*, such that* $\dim A + \dim B = \dim M$*. Then we define*

$$I_2(A, B) = I_2(\iota, B),$$

where $\iota : A \hookrightarrow M$ *is the canonical embedding map. If we want to emphasize the ambient manifold* M *we write* $I_2(A, B; M)$*.*

By its definition, together with Proposition VI.4.2, $I_2(A, B)$ is stable w.r.t. deformations of A. In order to show that $I_2(A, B)$ is stable also under perturbations of B and to understand the relation between $I_2(A, B)$ and $I_2(B, A)$ we generalize our approach somewhat:

VI.3.4 Definition *Let* $f : A \to M$, $g : B \to M$ *be smooth maps between compact manifolds without boundary. We say* $f \pitchfork g$ *if* $T_p f(T_p A) + T_q g(T_q B) = T_r M$ *whenever* $f(p) = g(q) = r$*.*

Assuming $f \pitchfork g$ we would like to define $I_2(f, g)$ by

$$I_2(f, g) \equiv \#\{(p, q) \in A \times B \mid f(p) = g(q)\} \pmod 2.$$

The problem is that it is not evident that the set $\{\ldots\}$ is finite.

VI.3.5 Lemma *Let* U, V *be subspaces of the vector space* W*. Then* $W = U \oplus V$ *(i.e.* $U + V = W$ *and* $U \cap V = \{0\}$*) iff* $(U \times V) \oplus \Delta = W \times W$*, where* $\Delta = \{x \times x, x \in W\}$*.*

Proof. Clearly $U \cap V = \{0\}$ is equivalent to $(U \times V) \cap \Delta = \{0\}$. Under these equivalent assumptions, $U \oplus V = W$ and $(U \times V) \oplus \Delta = W \times W$ are equivalent to $\dim U + \dim V = \dim W$ and $\dim U \cdot \dim V + \dim W = 2 \dim W$, respectively, which in turn are equivalent. ■

VI.3.6 Proposition *In the situation* $A \xrightarrow{f} B \xleftarrow{g} B$ *with* M *compact,* $f \pitchfork g$ *iff* $(f \times g) \pitchfork \Delta$*, where* Δ *now is the diagonal in* $M \times M$*. Under these (equivalent) conditions*

$$I_2(f, g) = I_2(f \times g, \Delta).$$

Proof. The first claim is an immediate consequence of the lemma, taking $U = T_pf(T_pA)$, $V = T_qg(T_qB)$, $W = T_rM$ for $f(p) = g(q) = r$. Assume these equivalent transversality conditions are satisfied. Now the set $\{(p,q) \in A \times B \mid f(p) = g(q)\}$ is just $(f \times g)^{-1}(\Delta)$, and this is finite by $(f \times g) \pitchfork \Delta$. ∎

VI.3.7 Proposition *If $f' \simeq f$, $g' \simeq g$ then $I_2(f',g') = I_2(f,g)$.*

Proof. If f_t, g_t are homotopies from f to f' and from g to g', respectively, then $f_t \times g_t$ is a homotopy from $f \times g$ to $f' \times g'$. ∎

VI.3.8 Corollary *Let A, B, M be compact and $\dim A + \dim B = \dim M$. If $B \subset M$ is a submanifold and ι the inclusion map then $I_2(f,B) = I_2(f,\iota)$ for any $f : A \to M$.*

Proof. If $f \pitchfork B$ then this is trivial. Otherwise find $f' \simeq f$ such that $f \pitchfork B$. Then we have $I_2(f,B) = I_2(f',B) = I_2(f',\iota) = I_2(f,\iota)$. ∎

Thus perturbing the embedding ι by a homotopy does not change the mod 2 intersection number, which is the desired stability w.r.t. B. In particular, it turns out that the theory of the mod 2 degree is a special case of intersection theory:

VI.3.9 Corollary *Let M be compact and N connected with $\dim M = \dim N$. Then $I_2(f,\{q\})$ is independent of q and coincides with $\deg_2 f$.*

Proof. Since N is connected the inclusion maps i, i' of $q, q' \in N$ into N are homotopic. Thus $I_2(f,\{q\}) = I_2(f,i) = I_2(f,i') = I_2(f,\{q'\})$. Picking q to be a regular value of f it is clear that $I_2(f,\{q\}) = \#f^{-1}(q) \pmod 2 = \deg f$. ∎

VI.3.10 Corollary *Under the same assumptions as above, $I_2(A,B) = I_2(B,A)$.*

Proof. If $A \pitchfork B$ this is obvious since then $I_2(A,B) = I_2(B,A) = \#(A \cap B) \pmod 2$. In the general case it follows from $I_2(A,B) = I_2(f,g)$ where $f : A \to M, g : B \to M$ satisfy $f \pitchfork g$ and are homotopic to the inclusion maps. But it is clear that $I_2(f,g) = I_2(g,f)$. ∎

VI.4 Oriented intersection theory

We now turn to the more interesting case, where the manifolds M, N, L come with orientations. Some preliminary considerations are in order. In connection with the condition $\dim M + \dim L = \dim N$ on the dimensions, the transversality condition $T_pf(T_pM) + T_{f(p)}L = T_{f(p)}N$ becomes

$$T_pf(T_pM) \oplus T_{f(p)}L = T_{f(p)}N \quad \forall p \in M. \tag{VI.4}$$

(This follows from the equivalence $1 \Leftrightarrow 3$ in Exercise IV.9.1.) Furthermore, $T_pf : T_pM \to T_{f(p)}N$ is an isomorphism, thus it defines an orientation for its image. For $p \in f^{-1}(L)$ we define $\operatorname{sign} T_pf = 1$ if (VI.4) holds as an equation of *oriented* vector spaces, i.e. the given orientation of $T_{f(p)}N$ coincides with the direct sum orientation of $T_pf(T_pM) \oplus T_{f(p)}L$, and $\operatorname{sign} T_pf = -1$ otherwise. (Recall that the direct sum of oriented vector spaces is not commutative in general!)

VI.4.1 Definition *Consider $f : M \to N \supset L$ where all manifolds are oriented, M is compact, $\dim M + \dim L = \dim N$ and $f \pitchfork L$. Then we define the oriented intersection number $I(f,L) \in \mathbb{Z}$ by*

$$I(f,L) = \sum_{p \in f^{-1}(L)} \operatorname{sign} T_pf.$$

VI.4.2 Proposition *Let M, N, L be as in Definition VI.4.1 and let $f, g : M \to N$ be homotopic maps satisfying $f \pitchfork L, g \pitchfork L$. Then $I(f, L) = I(g, L)$.*

Proof. Similar to the proof of Lemma IV.7.4, but the details are quite tedious. See [12, Section II.3]. ■

Again, if $f : M \to N$ is any smooth map, not necessarily transversal to L, Theorem IV.9.15 allows us to find $g \simeq f$ such that $g \pitchfork L$. Then Proposition VI.4.2 implies that the definition $I(f, L) := I(g, L)$ makes sense. As in the unoriented case, given two submanifolds A, B of a compact oriented manifold M we define $I(A, B) = I(\iota, B)$, where $\iota : A \hookrightarrow M$ is the canonical embedding.

If $f : M \to N$ satisfies $f \pitchfork \{q\}$, equivalently q is a regular value of f, we find

$$I(f, \{q\}) = \sum_{p \in f^{-1}(q)} \mathrm{sign} T_p f = \deg f.$$

By the definitions of $I(f, \{q\})$ and $\deg f$ it follows that this equality holds for all $q \in N$.

If we have maps $A \xrightarrow{f} M \xleftarrow{g} B$ with M compact satisfying $f \pitchfork g$ we define $I(f, g)$ as the sum over the pairs $(p, q) \in A \times B,\ f(p) = g(q) = r$ of numbers ± 1, depending on whether the (given) orientation on $T_r M$ coincides with the direct sum orientation on $T_r M$ induced from the orientations on $T_p A, T_q B$ by the isomorphism $T_p f(T_p A) \oplus T_q g(T_q B) \cong T_r M$.

VI.4.3 Proposition *In the situation $A \xrightarrow{f} M \xleftarrow{g} B$ with M compact, $f \pitchfork g$ iff $(f \times g) \pitchfork \Delta$, where Δ is the diagonal in $M \times M$. Under these (equivalent) conditions*

$$I(f, g) = (-1)^{\dim B} I(f \times g, \Delta).$$

As a consequence, $I(f, g)$ is homotopy invariant w.r.t. f and g, and $I(A, B)$ is stable under small perturbations of A, B.

Proof. The first half has been proven in the preceding subsection. The statement on the orientations is left as an exercise. (For the solution see [12, p. 113].) ■

We conclude our general study of intersection theory by examining the behavior of the intersection number under exchange of A and B.

VI.4.4 Lemma *Let A, B, M be compact with $\dim A + \dim B = \dim M$. Then*

$$I(f, g) = (-1)^{\dim A \cdot \dim B} I(g, f)$$

for any $f : A \to M$ and $g : B \to M$.

Proof. Follows from $V \oplus U = (-1)^{\dim U \cdot \dim V}(U \oplus V)$. ■

VI.4.5 Corollary $I(A, B) = (-1)^{\dim A \cdot \dim B} I(B, A)$.

VI.5 The Euler number and vector fields

VI.5.1 Euler numbers and Lefshetz numbers

Still, all manifolds are assumed boundaryless. For a manifold M let $\Delta = \{x \times x,\ x \in M\} \subset M \times M$ be the diagonal. Clearly, this is an m-dimensional submanifold of the $2m$-dimensional manifold $M \times M$.

VI.5.1 Definition *Let M be compact connected oriented (and boundaryless). Then the Euler number of M is defined by*

$$\chi(M) = I(\Delta, \Delta; M \times M).$$

VI.5.2 Corollary *Let M be compact connected oriented and odd dimensional. Then $\chi(M) = 0$.*

Proof. Follows from Corollary VI.4.5 since $\dim \Delta = \dim M$ is odd, thus $I(\Delta, \Delta) = -I(\Delta, \Delta)$. ■

VI.5.3 Lemma *Let M, N be compact connected oriented. Then $\chi(N \times M) = \chi(N)\chi(M)$.*

Proof. As a consequence of $T_{p\times q}(M \times N) = T_pM \oplus T_qN$ one has $f_1 \times f_2 \pitchfork B_1 \times B_2$ for $f_i : A_i \to M_i \supset B_i$, $i = 1, 2$ satisfying $f_i \pitchfork B_i$. Similarly, $(f_1 \times f_2)^{-1}(B_1 \times B_2) = f_1^{-1}(B_1) \times f_2^{-1}(B_2)$ implies $I(f_1 \times f_2, B_1 \times B_2) = I(f_1, B_1)I(f_2, B_2)$. The claim follows by observing that also orientations behave as expected. ■

The Euler number of a manifold is a fundamental invariant. Later on, it will be interpreted in terms of de Rham cohomology and CW-decompositions. For the time being, our only efficient way of computing the Euler number will be via its relation to vector fields with finitely many zeros. The following will be used later.

VI.5.4 Lemma *Let M be as in Definition VI.5.1. Then*

$$\chi(M) = I(M_0, M_0; TM),$$

where M_0 is the zero section of TM.

Proof. Let $\iota : M \to TM$ be the inclusion map of the zero section. The transversality theorem allows us to choose $\iota' : M \to TM$ such that $\iota' \pitchfork M_0$ and such that $\iota'(M)$ is contained in any given neighborhood of M_0. Now the claim follows from Exercise V.3.8. ■

VI.5.5 Remark A generalization of the Euler number is provided by the Lefshetz number. As before, let M be a compact oriented manifold without boundary and let $f : M \to M$ be a smooth map. Then the Lefshetz number of f is defined as the intersection number $L(f) = I(G(f), \Delta; M \times M)$, where $G(f) = \{(x, f(x)),\ x \in M\}$ is the graph of f. Clearly, $\chi(M) = G(\mathrm{id}_M)$, and one shows that $L(f)$ depends only on the smooth homotopy class of f. The relevance of the Lefshetz number derives from the Lefshetz fixpoint theorem: If $L(f) \neq 0$ then f has a fixpoint. (The proof is trivial: If f has no fixpoint then $G(f) \cap \Delta = \emptyset$, thus $I(G(f), \Delta; M \times M) = 0$.) When $G(f) \pitchfork \Delta$ one has a nice explicit formula for $L(f)$ in terms of the behavior of f near its fixpoints, see [12, Section III.4]. □

VI.5.2 Euler numbers and vector fields (Unfinished!!!)

VI.5.6 Definition *Let $v : M \to TM$, $v' : N \to TN$ be vector fields and $f : M \to N$ a smooth map. We say that v' corresponds to v under f if*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "v"] & N \arrow[d, "v'"] \\ TM \arrow[r, "Tf"'] & TN \end{tikzcd}$$

commutes.

If f is a diffeomorphism, f and Tf are invertible, thus we can use the formulae

$$\begin{aligned} v' &= Tf \circ v \circ f^{-1}, \\ v &= (Tf)^{-1} \circ v' \circ f \end{aligned}$$

to transport vector fields from M to N or conversely.

VI.5.7 DEFINITION *Let $v \in \Gamma(TM)$ be a vector field on M and let $p \in M$ be an isolated zero of v. Let (U, ϕ) be a chart around p. Then the map $r : U \to \mathbb{R}^n$ defined by*

$$\mathbb{R}^n \supset \phi(U) \xrightarrow{\phi^{-1}} U \xrightarrow{v} TU \xrightarrow{T\phi} T\phi(U) = \phi(U) \times \mathbb{R}^n \xrightarrow{p_2} \mathbb{R}^n$$

has zero as a regular value. Then we define $\mathrm{ind}_p v = \#r^{-1}(0)$. *Equivalently, let $\varepsilon > 0$ be such that 0 is the only zero of r in the ball $B(ve, 0) \subset \mathbb{R}^n$. Then* $\mathrm{ind}_p v$ *is equal to the degree of the map $S^{n-1} \to S^{n-1}$ defined by*

$$x \mapsto \frac{r(\varepsilon x)}{\|r(\varepsilon x)\|}.$$

VI.5.8 EXERCISE Prove that $\mathrm{ind}_p s$ is well defined. (One must show that the choice of another chart (U', ϕ') and of $\varepsilon' > 0$ gives a map α'_p that is homotopic to α_p and thus has the same degree.) □

VI.5.9 THEOREM (POINCARÉ-HOPF) *Let M be a compact connected oriented manifold without boundary and let $s \in \Gamma(TM)$ be a smooth vector field with finitely many zeros. Then*

$$\chi(M) = \sum_{p \in s^{-1}(0)} \mathrm{ind}_p s.$$

Proof. Let $v : M \to TM$ be a vector field, to wit a section of the tangent bundle. We write $v(x) = (x, v_x)$, where $v_x \in T_x M$. In view of $\pi \circ v = \mathrm{id}_M$ it is clear that v is an injective immersion. If $S \subset TM$ then $v^{-1}(S) = \pi(S)$, implying that $v : M \to TM$ is a proper map. Thus v is an embedding of M into TM. For $\lambda \in [0, 1]$ we denote by $v_\lambda : M \to TM$ the vector field $x \mapsto (x, \lambda v_x)$. The family (v_λ) is a homotopy between v and the zero section $x \mapsto (x, 0)$, thus

$$I(v, M_0) = I(M_0, M_0; TM) = I(\Delta, \Delta; M \times M) = \chi(M),$$

where the second equality is Lemma VI.5.4. Now, $v(M) \cap M_0 = \{(x, 0) \mid v(x) = 0\}$, and we have $v \pitchfork M_0$ iff

****

■

VI.5.10 COROLLARY $\chi(S^n) = 0$ *for odd n.*

Proof. As seen in Section IV.8, the odd dimensional spheres admit vector fields that vanish nowhere. In view of Theorem VI.5.9 the conclusion is immediate. ■

VI.5.11 PROPOSITION *If n is even then $\chi(S^n) = 2$.*

Proof. We will later give an elegant proof using Morse theory. Therefore, here we limit ouselves to a sketch proof. It is clear that on any sphere S^n (whether n is even or odd) one can find a vector field X as follows: The only zeros are at the North and South poles, the former being a source and the latter a sink of the associated flow. The direction of the flow is along the great circles from the North to the South pole. Let's look at this flow locally, in a neighborhood of a source or sink. Then, w.r.t. suitable coordinates, a source looks like the vector field $x \to x$ from a neighborhood U of 0 to $\mathbb{R}^n$, whereas a sink is given by $x \to -x$. The corresponding maps from S^{n-1} to S^{n-1} are the identity map and the map $x \to -x$, respectively. The former has degree one, the latter degree $(-1)^n$ by Exercise IV.8.1. Thus

$$\chi(S^n) = \sum_{p \in X^{-1}(0)} \mathrm{ind}_p X = 1 + (-1)^n,$$

from which the claim and a new proof of the preceding corollary follow. ■

We conclude this chapter with a result that uses almost everything developed so far:

VI.5.12 Theorem *A compact connected oriented manifold M without boundary admits a nowhere vanishing vector field iff $\chi(M) = 0$.*

Proof. The 'only if' part is immediate by Theorem VI.5.9. Thus assume $\chi(M) = 0$. We begin by constructing a vector field on M with finitely many zeros, all non-degenerate. To this purpose choose an embedding $M \subset \mathbb{R}^n$ and consider the map $\rho : M \times \mathbb{R}^n \to TM$ which maps (x, v) to the image of $v \in \mathbb{R}^n$ under the orthogonal projection to $T_xM \subset T_x\mathbb{R}^n \equiv \mathbb{R}^n$. It is easy to see that ρ is a submersion, thus transversal to the zero section $M_0 \subset TM$. We can therefore apply Proposition VI.1.1 on parametric transversality to conclude that there exists $v \in \mathbb{R}^n$ such that the vector field $\rho_v : x \mapsto \rho(x, v)$ is transversal to M_0, in particular it has finitely many zeros.

Now pick an open set $U \subset M$ for which a diffeomorphism $\phi : U \to \mathbb{R}^m$ exists. By Exercise IV.4.5 there exists a diffeomorphism $\alpha : M \to M$ mapping the zeros of ρ_v into U. Now the vector field $\rho' = T\alpha \circ \rho_v \circ \alpha^{-1}$ has all its zeros in U, and $\rho'' = T\phi \circ \rho' \circ \phi^{-1}$ is the pullback of $\rho' \restriction U$ to $\mathbb{R}^n$, which we now consider as a map $f : \mathbb{R}^n \to \mathbb{R}^n$. Theorem VI.5.9 implies that the sum over the indices of the zeros of ρ' is zero, and diffeomorphism invariance of the index together with the fact that all zeros of ρ' lie in U implies that also the sum over the indices of the zeros of ρ'' is zero. Now, the vector fields ρ, ρ', ρ'' are non-degenerate, thus transversal to the zero section, implying that $f : \mathbb{R}^n \to \mathbb{R}^n$ has zero as regular value. Furthermore, $\deg f$ equals the sum over the indices of the zeros, thus $\deg f = 0$. Picking a ball $B \subset \mathbb{R}^n$ containing all zeros of f in its interior, the considerations of Subsection IV.8.4 imply that there exists a map $f' : \mathbb{R}^m \to \mathbb{R}^m - \{0\}$ coinciding with f on the complement of B. Considering f' again as a vector field on $\mathbb{R}^n$ and replacing $\rho' \restriction U$ by the pullpack to U of the latter, we obtain a vector field on M without zeros. ■

VI.5.13 Remark The results of this section can be generalized without difficulty to the situation where TM is replaced by any oriented vector bundle of rank $\dim M$ over M. Also boundaries can be taken into account. See [13, Chapter 5]. □

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