23 Lecture 17.03.2010

- Recall definition of k-linear rigid even symmetric categories with one non-self dual generator $\mathcal{G}_k(d)$ and one self-dual generator $\mathcal{O}_k(d)$. We call them the Schur category and the Brauer category, respectively.

- Endomorphism algebras:

$$\begin{aligned} \mathrm{End}_{\mathcal{G}_k(d)}(+^n) &= kS_n, \\ \mathrm{End}_{\mathcal{O}_k(d)}(n) &= B_k(d;n), \\ \mathrm{End}_{\mathcal{G}_k(d)}(+^n-^m) &= WB_k(d;n,m). \end{aligned}$$

  (Notice that, in view of the symmetry, for every $w \in \{+,-\}^*$ we have $\mathrm{End}_{\mathcal{G}_k(d)}(w) \cong WB_k(d;n,m)$ where $n = \#+(w), m = \#-(w)$.) The first statement is an easy consequence of the definition. $B_k(d;n)$ is the Brauer algebra defined by Brauer in 1937. $WB_k(d;n,m)$ is the "walled Brauer algebra", discovered independently by various people, the first probably Turaev (1989). For our purposes, the last to identities above will be taken as the definitions of the (walled) Brauer algebra.

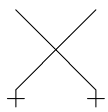
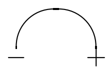
  Explain name *walled* Brauer algebra.

- The symmetric group algebras and the (walled) Brauer algebras are examples of "diagram algebras". Thus we might call $\mathcal{G}_k(d), \mathcal{O}_k(d)$ diagram categories. There are many other examples, in particular the Temperley-Lieb algebra/category $\mathcal{TL}_k(d)$. Its definition is obtained from that of the Brauer category by forbidding diagrams with crossing lines. The category thus obtained is just a k-linear tensor category, and it has an obvious universal property. Similarly, one can consider an oriented version.

- For k algebraically closed of characteristic zero, the group algebra kS_n is semisimple (Maschke). The Brauer algebra and walled Brauer algebra fail to be semisimple for certain combinations $(d;n)$ or $(d;n,m)$ of parameters! In particular, $B_k(d;n)$ is not semisimple when $d, n \in \mathbb{N}, n < d$. Thus $\mathcal{G}_k(d), \mathcal{O}_k(d)$ are not always semisimple!

- $\mathcal{G}_k(d)$ has the following universal property. (We assume $\mathrm{char}(k) = 0$ and algebraic closedness throughout.)

  23.1 PROPOSITION *Let $\mathcal{C}$ be k-linear rigid even semisimple with* $\mathrm{End}\,\mathbf{1} = k$ *and let $X \in \mathcal{C}$. Then there is a symmetric monoidal k-linear functor $F : \mathcal{G}_k(d) \to \mathcal{C}$ such that $F(+) = X$ and*

  F :  $\mapsto c_{X,X}$,  $\mapsto d_X$, *etc.*

  F is unique up to monoidal natural isomorphism

  Proof. (Sketch) For a word $w = (s_1 \cdots s_n) \in \{+,-\}^*$, we define $F(w) = X^{s_1} \otimes \cdots \otimes X^{s^n}$, where $X^+ = X$ and $X^- = \overline{X}$. On morphisms, F is defined by writing a morphism $s : w \to w'$ in terms of generators, i.e. as a composite of transpositions and duality morphisms for $+$ or $-$. That this is well defined is more or less clear. ■

- Now, the generating object $1 \in \mathcal{O}_k(d)$ is self-dual. This raises the question whether $\mathcal{O}_k(d)$ is universal for pairs $(\mathcal{C}, X)$ where $X \in \mathcal{C}$ is self-dual. It turns out that $(\mathcal{C}, X)$ must satisfy one more condition!

- In a tensor category with 2-sided duals, one has isomorphisms

$$\mathrm{Hom}(X \otimes Y, Z) \cong \mathrm{Hom}(X, Z \otimes \overline{Y}) \cong \mathrm{Hom}(Y, \overline{X} \otimes Z)$$

  for all X, Y, Z. In particular we have:

$$X \text{ is simple} \Leftrightarrow \ \mathrm{End}(X) \cong k\,\mathrm{id}_X \ \Leftrightarrow \ \mathrm{End}(\overline{X}) \cong k\,\mathrm{id}_X \ \Leftrightarrow \ \overline{X} \text{ is simple.}$$

  Furthermore, $\dim \mathrm{Hom}(X, \overline{X}) = \dim \mathrm{Hom}(\mathbf{1}, X \otimes X$, and this is 0 if X is non-self dual and 1 if X is self-dual. In the latter case, let $0 \neq d \in \mathrm{Hom}(\mathbf{1}, X \otimes X)$. Now $c_{X,X} \circ d \in \mathrm{Hom}(\mathbf{1}, X \otimes X)$ must be proportional to d: $c_{X,X} \circ d = \lambda_X\, d$. In view of $c^2_{X.X} = \mathrm{id}$, we have $\lambda_X = \pm 1$. A simple self dual object X is called real (or orthogonal) if $\lambda_X = 1$ and quaternionic (or symplectic) if $\lambda_X = -1$. Non-self dual simple objects are called complex (or unitary, but this is less common).

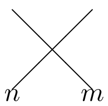
- Let G be a compact group and (H,π) an irreducible representation with character χ_π. Then one can prove
$$\int \chi_\pi(g^2)d\mu(g) = \begin{cases} 1 & \text{if } \pi \text{ is real} \\ 0 & \text{if } \pi \text{ is complex} \\ -1 & \text{if } \pi \text{ is quaternionic} \end{cases}$$
(For a proof cf. [165]. Furthermore, (H,π) is real iff one can chose a basis in H with respect to which $\pi(g)$ is real for all $g \in G$, in other words, π is real-linear. Similarly, a representation (H,π) is quaternionic iff it admits a commuting action of the quaternions.

- A non-simple self-dual object is called real if all simple direct summands are real, and similarly for the quaternionic case. (It is clear that there can be non-simple self-dual objects that are neither real nor quaternionic.)

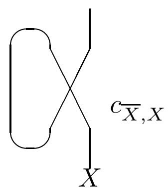
- Returning to the universality of $\mathcal{O}_k(d)$, let $\mathcal{C}$ be k-linear rigid even symmetric etc., and $X \in \mathcal{C}$ self-dual. If X is real then $d_X \circ c_{X,X} = d_X$. This implies that we can consistently define a symmetric tensor functor $F : \mathcal{O}_k(d) \to \mathcal{C}$ such that $F(1) = X$ etc. Thus

  23.2 Proposition *$\mathcal{O}_k(d)$ is universal for the rigid even symmetric k-linear categories and $X \in \mathcal{C}$ real. (More precisely, the pair $(\mathcal{O}_k(d), 1)$ is an initial object in the category of pairs $(\mathcal{C}, X)$ as above, where the morphisms are monoidal isomorphism classes of symmetric tensor functors.)*

  Notice that this doesn't work (without suitable modification) when X is not real.

- Our next goal is to construct a category $\mathcal{SP}_k(d)$ universal for $(\mathcal{C}, X)$ with $X \in \mathcal{C}$ quaternionic. We must turn the generating (simple) object $1 \in \mathcal{O}_k(d)$, which is real, into a quaternionic one. Since $\mathrm{Hom}(\mathbf{1}, X^2)$ with $X = 1$ is one-dimensional, this has nothing to do with the choice of duality morphisms, but rather with the symmetry. We observe that $\mathrm{Hom}(n,m) = 0$ whenever $n+m$ is odd. There is thus a $\mathbb{Z}_2$-grading on the objects, and we use this to define a new braiding by
$$\widetilde{c}_{n,m} = (-1)^{nm}\, c_{n,m} = (-1)^{nm}$$


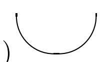
(This follows already from $\widetilde{c}_{1,1} = -c_{1,1}$.) It is clear that the generating object 1 now is quaternionic.

- Recall that our definition of an *even* rigid symmetric tensor category was that a left duality $X \mapsto (\overline{X}, d_X, e_X)$ is given and right duality morphisms $d'_X : \mathbf{1} \to \overline{X} \otimes X$, $e'_X : X \otimes \overline{X} \to \mathbf{1}$ are defined by
$$d'_X = c_{X,\overline{X}} \circ d_X, \quad e'_X = e_X \circ c_{X,\overline{X}}.$$
This convention implies that the *twist* $\Theta(X) \in \mathrm{End}\, X$ defined by
$$\Theta(X) =$$


is trivial, i.e. $\Theta(X) = \mathrm{id}_X\ \forall X$. This is appropriate in categories like $\mathrm{Vect}_k, \mathrm{Rep}\, G$ etc., but not in all.

  define twist $\Theta(X)$ in STC with left and right duality. trivial twist means that right duality d', e' is defined in terms of left duality d, e and symmetry.

- note: the twist is trivial in $\mathcal{G}_k(d), \mathcal{O}_k(d)$, Hilb, Vect_k and $\mathrm{Rep}\, G$.

- we want the twist to remain trivial in our new category, thus we define right duality in terms of left duality and *the new* symmetry:
$$d'_1 = (-1)$$


- def: $\widetilde{\mathcal{O}}_k(d) = \mathcal{O}_k(d)$ as tensor category, braiding and right duality are modified by $(-1)^n$.
- new problem: in $\widetilde{\mathcal{O}}_k(d)$, we have $d(1) = -d$, where $d \in k$ is the parameter. therefore we define

$$\mathcal{SP}_k(d) = \widetilde{\mathcal{O}}_k(-d).$$

- thm: $\mathcal{SP}_k(d)$ is universal for pairs $(\mathcal{C}, X)$ where X is quaternionic and $d = d(X)$.
- claim: the categories $\mathcal{G}_k(d), \mathcal{O}_k(d), \mathcal{SP}_k(d)$ are closely related to representation categories of the classical groups $GL(d,k), O(d,k), Sp(d,k)$, but we are not quite there: semisimplicity?
- The following is an easy fact that is nevertheless often gotten wrong:

23.3 Lemma *If $\mathcal{C}$ is k-linear semisimple with two-sided duals and simple unit and $X \in \mathcal{C}$ is simple (NB: zero objects are not considered as simple!) then $d(X) \neq 0$.*

Proof. By definition of a category with 2-sided duals, there are an object $\overline{X} \in \mathcal{C}$ and morphisms $e : \overline{X} \otimes X \to \mathbf{1}, d : \mathbf{1} \to X \otimes \overline{X}, e' : X \otimes \overline{X} \to \mathbf{1}, d' : \mathbf{1} \to \overline{X} \otimes X$ satisfying the usual identities. Since $X, \overline{X}$ are non-zero, this implies that e, d, e', d' are non-zero morphisms that provide bases for the 1-dimensional (by simplicity of X and $\overline{X}$) vector spaces $\mathrm{Hom}(\overline{X} \otimes X, \mathbf{1})$ etc. Semisimplicity of $\mathcal{C}$ and the existence of a non-zero morphism $\mathbf{1} \to X \otimes \overline{X}$ imply that $X \otimes \overline{X} \cong \mathbf{1} \oplus Y$ for some Y. Thus there are morphisms $u : \mathbf{1} \to \overline{X} \otimes X$ and $u' : \overline{X} \otimes X \to \mathbf{1}$ such that $u' \circ u = \mathrm{id}_{\mathbf{1}}$. By $\mathbf{1} \not\cong 0$ and one dimensionality of the above hom-spaces, we have $u = \lambda_1 d'$ and $u' = \lambda_2 e$ with $\lambda_1, \lambda_2 \in k^*$. This implies

$$d(X) = e \circ d' = (\lambda_1 \lambda_2)^{-1} u' \circ u = (\lambda_1 \lambda_2)^{-1} \mathrm{id}_{\mathbf{1}},$$

thus $d(X) \neq 0$. ■

23.4 Corollary *If $\mathcal{C}$ is k-linear semisimple with two-sided duals and simple unit and $X, Y \in \mathcal{C}$ are simple then the trace-pairing $\mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,X) \to k$ is non-degenerate.*

Proof. If $X \not\cong Y$ then the two hom spaces are zero. Otherwise we may assume $Y = X$, in which case $s, t \in \mathrm{Hom}(X,X)$ implies $s = \lambda_1 \mathrm{id}_X, t = \lambda_2 \mathrm{id}_X$, thus $Tr(t \circ s) = \lambda_1 \lambda_2 d(X)$. Now the claim follows from the preceding lemma to the effect that $d(X) \neq 0$. ■

The following now follows easily by semisimplicity:

23.5 Corollary *Let $\mathcal{C}$ semisimple k-linear with $\mathrm{End}\,\mathbf{1} = k$, $X, Y \neq 0$. Then the pairing of $\mathrm{Hom}(X,Y)$ and $\mathrm{Hom}(Y,X)$ given by trace is non-degenerate.*

- If $\mathcal{C}$ is a k-linear tensor rigid category we define, for every $X, Y \in \mathcal{C}$,

$$I(X,Y) = \{s \in \mathrm{Hom}(X,Y) \mid Tr(t \circ s) = 0 \ \forall\, t \in \mathrm{Hom}(Y,X)\}.$$

We claim that $I = (I(X,Y))_{X,Y}$ is a tensor ideal in $\mathcal{C}$ in the sense that composing or tensoring a morphism in I with any morphism we obtain a morphism in I.

Proof. Assume $s : X \to Y$ is such that $\mathrm{Tr}(t \circ s) = 0$ for every $t : Y \to X$. Let $u : Y \to Z$ and consider $s' = u \circ s$. If now $t' : Z \to X$ then $Tr(t' \circ s') = Tr(t' \circ (u \circ s)) = Tr((t' \circ u) \circ s) = 0$. Thus $u \circ s \in I(X,Z)$, and a similar argument works for composites $u \circ s$. Now let $s \in I(X,Y)$ and $t : Z \to T$. Let $s' = s \otimes t$. In order to show $s' \in I(X \otimes Z, Y \otimes T)$, let $t' : Y \otimes T \to X \otimes Z)$. Then

$$Tr_{X \otimes Z}(t' \circ s') = Tr_X(u \circ s) \quad \text{where} \quad u = (\mathrm{id} \otimes Tr_Z)(t' \circ (\mathrm{id}_Y \otimes t)) \in \mathrm{Hom}(Y \to X).$$

This is zero by $s \in I(X,Y)$. ■

- If $\mathcal{C}$ is a k-linear tensor category and $I = (I(X,Y))$ an ideal, we can define a new tensor category $\mathcal{C}' = \mathcal{C}/I$ with the same objects and hom-sets given by $\mathrm{Hom}(X,Y)/I(X,Y)$. We have a k-linear tensor functor $Q : \mathcal{C} \to \mathcal{C}'$, and the image under Q of a symmetry or rigid structure is again a symmetry or rigid structure, respectively. NB: If $I(X,X) = \mathrm{End}(X)$ then $Q(X)$ is a zero-object and does not have a dual. We thus might as well consider the full subcategory of $\mathcal{C}'$ containing only those $X \in \mathcal{C}$ such that $Q(X) \neq 0$.

  The traces on $\mathcal{C}'$ to which the image under Q of the rigidity morphisms give rise are non-degenerate. This is easy to verify.

- does non-degeneracy imply semisimplicity? NO!
- Remark: A non-degenerate trace does not imply semisimplicity: Consider the algebra $A = k[x]/x^2$ with trace $\tau(a + bx) = b$. The latter is non-degenerate since $Tr((a + bx)(a' + b'x)) = Tr(aa' + (ab' + a'b)x) = ab' + a'b$, but the algebra has the nilpotent ideal xA.
- in [36, 5.8 Mise en garde] Deligne categorifies this example, producing a pseudo-abelian rigid symmetric category with non-degenerate traces that fails to be semisimple.
- Lemma: If τ is trace on fin-dim k-algebra A such that $\mathrm{Rad}(A) \subset \ker\tau$ then A is semisimple.

  Proof. Trivial: Let $0 \neq a \in Rad(A)$. By non-degeneracy of τ, there is $b \in A$ such that $\tau(ab) \neq 0$. But then $ab \in Rad(A) \subset \ker\tau$, which is a contradiction. ■

- Deligne [36, Corollaire 3.6]: If $\mathcal{C}$ is rigid symmetric with non-degenerate traces <u>and abelian</u> then $\mathrm{Rad}(\mathrm{End}\, X) \subset I(X,X)$. Thus, non-degeneracy of the traces does imply semisimplicity!
- We would like to show that the categories $\mathcal{G}_k(d), \mathcal{O}_k(d), \mathcal{SP}_k(d)$ become semisimple upon dividing out the ideal $I = \ker Tr$. Since the quotient categories have non-degenerate traces, by Deligne's result it would be enough to show abelianness. Unfortunately, all we know is pseudo-abeliannness, which is not enough.
- Alternative approach: Show directly that $\mathrm{Rad}(\mathrm{End}(X)) \subset \ker Tr_X$. Then it is clear that the quotient categories are semisimple.
- *** discuss known results on semisimplicity of (walled) Brauer algebras.
- Universal property of $\widehat{\mathcal{G}_k(d)}$ for *semisimple* k-linear rigid even symmetric categories with an object X such that $d(X) = d$. Similarly for $\widehat{\mathcal{O}_k(d)}, \widehat{\mathcal{SP}_k(d)}$ in real and quaternionic cases.
- State duality theorems of Doplicher/Roberts and Deligne (Deligne's thm involves condition that for every $X \in \mathcal{C}$ there is $n \in \mathbb{N}$ such that $\Lambda^n X = 0$. This is automatic in the presence of a positive $*$-operation.)
- Universal properties of $\mathrm{Rep}\, U(N), \mathrm{Rep}\, SO(N), \mathrm{Rep}\, Sp(N)$ as proven by Baez [4]:

  23.6 Theorem *Let $\mathcal{C}$ be an even symmetric tensor $*$-category with direct sums, subobjects, conjugates and $\mathrm{End}\,\mathbf{1} = \mathbb{C}$, and let $X \in \mathcal{C}$ with $d(X) = N \in \mathbb{N}$. Then there is a symmetric tensor $*$-functor $F : \mathrm{Rep}\, U(N) \to \mathcal{C}$ such that $F((\mathbb{C}^N, \mathrm{id})) \cong X$, where $(\mathbb{C}^N, \mathrm{id})$ is the defining representation of $U(N)$. F is unique up to symmetric monoidal natural isomorphism, and it is faithful.*

  Similar statements hold for $\mathrm{Rep}\, O(N)$ and $\mathrm{Rep}\, Sp(N)$ if one requires $X \in \mathcal{C}$ to be self-dual and real or quaternionic, respectively.

- Since universal objects are unique up to isomorphism (in the respective category), we have

$$\begin{aligned} \widehat{\mathcal{G}_k(d)} &\simeq \mathrm{Rep}\, GL(d,k), \\ \widehat{\mathcal{O}_k(d)} &\simeq \mathrm{Rep}\, O(d,k), \\ \widehat{\mathcal{SP}_k(d)} &\simeq \mathrm{Rep}\, Sp(d,k), \end{aligned}$$

  Notice that the proof hardly used any representation theory of the classical groups!

- A "philosophical" observation. There are three series of classical compact groups: $U(N), O(N), Sp(N)$. (Here we neither count the minor variations $SU(N), SO(N)$ nor the complexifications $GL(N), O(N,\mathbb{C}), Sp(N,\mathbb{C})$.) On the other hand, the classification of simple Lie algebras/groups tells us that there are actually four series, since $O(2n)$ and $O(2n+1)$ are quite different. (This is seen without complicated machinery: While $Z(O(N)) = \{\pm 1\}$ for all N, we have $-1 \in Z(SO(N))$ only when N is even. This leads to $Z(SO(2n)) = \{1, -1\}$, whereas $SO(2n+1)$ has trivial center. Furthermore, $O(2n+1) \cong SO(2n+1) \times \mathbb{Z}_2$, whereas $O(2n) \cong SO(2n) \rtimes \mathbb{Z}_2$ is a non-trivial semidirect product.)

  But from the point of view of our preceding constructions, $\mathcal{O}_k(d)$ and $\mathcal{SP}_k(d)$ are so similar that some authors talk about the ortho-symplectic group or category. Thus from the point of view of invariant theory, there are only two types of classical groups...

- Determinant: n-th symmetrization and antisymmetrization of object in semisimple k-linear rigid even symmetric category. Idempotents in $\mathrm{End}(X^n)$:

$$S_n^X = \frac{1}{n!}\sum_{\sigma\in S_n}\Pi_n^X(\sigma), \quad A_n^X = \frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)\Pi_n^X(\sigma).$$

- Lemma: If $\Theta(X) = \mathrm{id}$ then $Tr_{X^{\otimes n}}(\Pi_n^X(\sigma)) = d(X)^{\#\sigma}\ \forall\sigma\in S_n$, where $\#\sigma$ is the number of cycles in the permutation σ.

- Lemma: For any $n\in\mathbb{N}$ and $z\in\mathbb{C}$, we have

$$\sum_{\sigma\in P_n}\mathrm{sgn}(\sigma)\, z^{\#\sigma} = z(z-1)(z-2)\cdots(z-n+1).$$

- Let $\Lambda^X \prec X^{\otimes n}$ be the image of the projection $A_n^X \in \mathrm{End}\, X^{\otimes n}$. Then

$$\begin{aligned} d(\Lambda^n X) = Tr_{X^{\otimes n}}(A_n^X) &= \frac{1}{n!}\sum_{\sigma\in P_n}\mathrm{sgn}(\sigma)\, Tr_{X^{\otimes n}}\,\Pi_n^X(\sigma) = \frac{1}{n!}\sum_{\sigma\in P_n}\mathrm{sgn}(\sigma)\, d(X)^{\#\sigma} \\ &= \frac{d(X)(d(X)-1)(d(X)-2)\cdots(d(X)-n+1)}{n!}. \end{aligned}$$

- universal properties of $\mathrm{Rep}\, SU(N)$ and $\mathrm{Rep}\, SO(N)$:

  23.7 THEOREM *Let $\mathcal{C}$ be an even symmetric tensor $*$-category with direct sums, subobjects, conjugates and* $\mathrm{End}\,\mathbf{1} = \mathbb{C}$, *and let $X\in\mathcal{C}$ with $d(X) = N\in\mathbb{N}$ and* $\det X \cong \mathbf{1}$. *Then there is a symmetric tensor $*$-functor $F : \mathrm{Rep}\, SU(N)\to\mathcal{C}$ such that $F((\mathbb{C}^N, \mathrm{id}))\cong X$, where $(\mathbb{C}^N, \mathrm{id})$ is the defining representation of $SU(N)$. F is unique up to symmetric monoidal natural isomorphism, and it is faithful.*

  Similarly for $\mathrm{Rep}\, SO(N)$.

- A question arises: How are the representation categories of $U(N)$ and of $SU(N)$ related? Similarly for $O(N)$ and $SO(N)$. Can they be obtained from each other in a categorical way? Yes, but we need more preparations.