Failures to weaken list colouring through prescribed separation

Ross J. Kang

Radboud University Nijmegen

STRUÇO Workshop Paris, 5/2019
Imagine *adversaries* to colouring

- that issue arbitrary lists of allowable colours per vertex
- but must give at least $\ell$ per list

What is least $\ell$ for which colouring is always possible? (Necessarily $\ell \geq \chi$)
Imagine *adversaries* to colouring

- that issue arbitrary lists of allowable colours per vertex
- but must give at least $\ell$ per list

What is least $\ell$ for which colouring is always possible? (Necessarily $\ell \geq \chi$)

Called *list chromatic number* or *choice number* or *choosability* $\text{ch}$
ch is not bounded by any function of $\chi$

**Theorem (Erdős, Rubin, Taylor 1980)**

$\text{ch}(K_{d,d}) \sim \log_2 d$ (*and* $\text{ch}(K_{d+1}) = d + 1$)
ch is not bounded by any function of $\chi$

Theorem (Erdős, Rubin, Taylor 1980)
$\text{ch}(K_{d,d}) \sim \log_2 d$ (and $\text{ch}(K_{d+1}) = d + 1$)

Rather, more closely related to density

$\text{ch}(G) \gtrsim \log_2 \delta$ for any $G$ of minimum degree $\delta$
ch is not bounded by any function of $\chi$

**Theorem (Erdős, Rubin, Taylor 1980)**

$\text{ch}(K_{d,d}) \sim \log_2 d$ (*and* $\text{ch}(K_{d+1}) = d + 1$)

Rather, more closely related to density


$\text{ch}(G) \gtrsim \log_2 \delta$ *for any* $G$ *of minimum degree* $\delta$

Still poorly understood

**Conjecture (Alon & Krivelevich 1998)**

$\text{ch}(G) \lesssim \log_2 \Delta$ *for any* bipartite $G$ *of maximum degree* $\Delta$
Separation makes it "easier"?

What if lists connected by edge are all disjoint?

Theorem (Kratochvíl, Tuza, Voigt 1998)

\[ \text{ch}^{\text{sep}}(K_d + 1) \sim \sqrt{d} \]

Theorem (Füredi, Kostochka, Kumbhat 2014)

\[ \text{ch}^{\text{sep}}(K_d, d) \sim \log_2 d \]

Theorem (Esperet, Kang, Thomassé 2019)

\[ \text{ch}^{\text{sep}}(G) = \Omega(\log \delta) \] for any bipartite \( G \) of minimum degree \( \delta \)

Question: Does \( \text{ch}^{\text{sep}} \) grow in \( \delta \)?

Problem: Almost-disjointness of lists is not monotone under edge-addition!
Separation makes it “easier”? 

What if lists connected by edge are all *almost* disjoint?
What if lists connected by edge are all *almost* disjoint, so 1 common colour?
Separation makes it “easier”? 

What if lists connected by edge are all *almost* disjoint, so 1 common colour?

Call the corresponding least $\ell$ separation choosability $ch_{\text{sep}}$
Separation makes it “easier”?  

What if lists connected by edge are all \textit{almost} disjoint, so 1 common colour?

Call the corresponding least $\ell$ separation choosability $\text{ch}_{\text{sep}}$

\textbf{Theorem (Kratochvíl, Tuza, Voigt 1998)}

\[ \text{ch}_{\text{sep}}(K_{d+1}) \sim \sqrt{d} \]
Separation makes it “easier”? 

What if lists connected by edge are all *almost* disjoint, so 1 common colour?

Call the corresponding least \( \ell \) separation choosability \( \text{ch}_{\text{sep}} \)

**Theorem (Kratochvíl, Tuza, Voigt 1998)**

\[ \text{ch}_{\text{sep}}(K_{d+1}) \sim \sqrt{d} \]

**Theorem (Füredi, Kostochka, Kumbhat 2014)**

\[ \text{ch}_{\text{sep}}(K_{d,d}) \sim \log_2 d \]
Separation makes it “easier”? 

What if lists connected by edge are all *almost* disjoint, so 1 common colour? 

Call the corresponding least $\ell$ separation choosability $\text{ch}_{\text{sep}}$

**Theorem (Kratochvíl, Tuza, Voigt 1998)**

$\text{ch}_{\text{sep}}(K_{d+1}) \sim \sqrt{d}$

**Theorem (Füredi, Kostochka, Kumbhat 2014)**

$\text{ch}_{\text{sep}}(K_{d,d}) \sim \log_2 d$

**Theorem (Esperet, Kang, Thomassé 2019)**

$\text{ch}_{\text{sep}}(G) = \Omega(\log \delta)$ for any bipartite $G$ of minimum degree $\delta$
Separation makes it “easier”?  

What if lists connected by edge are all *almost* disjoint, so 1 common colour? 

Call the corresponding least \( \ell \) separation choosability \( ch_{\text{sep}} \)  

**Theorem (Kratochvíl, Tuza, Voigt 1998)**  
\[ ch_{\text{sep}}(K_{d+1}) \sim \sqrt{d} \]  

**Theorem (Füredi, Kostochka, Kumbhat 2014)**  
\[ ch_{\text{sep}}(K_{d,d}) \sim \log_2 d \]  

**Theorem (Esperet, Kang, Thomassé 2019)**  
\[ ch_{\text{sep}}(G) = \Omega(\log \delta) \text{ for any bipartite } G \text{ of minimum degree } \delta \]  

**Question: Does** \( ch_{\text{sep}} \) **grow in** \( \delta \) **?**  

**Problem: Almost-disjointness of lists is not monotone under edge-addition!**
Theorem (Kratochvıl, Tuza, Voigt 1998)
\[ \text{ch}_{\text{sep}}(K_{d+1}) \sim \sqrt{d} \]

Theorem (Esperet, Kang, Thomassé 2019)
\[ \text{ch}_{\text{sep}}(G) = \Omega(\log \delta) \text{ for any bipartite } G \text{ of minimum degree } \delta \]

Question: Does \( \text{ch}_{\text{sep}} \) grow in \( \delta \)?
Theorem (Kratochvíl, Tuza, Voigt 1998)
\[ \text{ch}_{\text{sep}}(K_{d+1}) \sim \sqrt{d} \]

Theorem (Esperet, Kang, Thomassé 2019)
\[ \text{ch}_{\text{sep}}(G) = \Omega(\log \delta) \text{ for any bipartite } G \text{ of minimum degree } \delta \]

Question: Does \( \text{ch}_{\text{sep}} \) grow in \( \delta \)?

Related question: Does every graph of high minimum degree contain either

- a large clique or
- a large minimum degree bipartite induced subgraph?
Conjecture (Esperet, Kang, Thomassé 2019)

Any triangle-free graph of minimum degree \( \delta \) has a bipartite induced subgraph of minimum degree \( \Omega(\log \delta) \)
Conjecture (Esperet, Kang, Thomassé 2019)

Any triangle-free graph of minimum degree $\delta$ has a bipartite induced subgraph of minimum degree $\Omega(\log \delta)$

- Without triangle-free, trivially false due to cliques
Conjecture (Esperet, Kang, Thomassé 2019)

Any triangle-free graph of minimum degree $\delta$ has a bipartite induced subgraph of minimum degree $\Omega(\log \delta)$

- Without triangle-free, trivially false due to cliques
- Without induced, trivially true with $d/2$ rather than $C \log d$
Bipartite induced density

Conjecture (Esperet, Kang, Thomassé 2019)

Any triangle-free graph of minimum degree $\delta$ has a bipartite induced subgraph of minimum degree $\Omega(\log \delta)$

- Without triangle-free, trivially false due to cliques
- Without induced, trivially true with $d/2$ rather than $C \log d$
- If true, it is sharp up to constant factor
Bipartite induced density

Conjecture (Esperet, Kang, Thomassé 2019)

Any triangle-free graph of minimum degree $\delta$ has a bipartite induced subgraph of minimum degree $\Omega(\log \delta)$

- Without triangle-free, trivially false due to cliques
- Without induced, trivially true with $d/2$ rather than $C \log d$
- If true, it is sharp up to constant factor
- $2$ rather than $\Omega(\log \delta)$ corresponds to presence of an even hole (Radovanović and Vušković ’13)
Conjecture (Esperet, Kang, Thomassé 2019)

Any triangle-free graph of minimum degree $\delta$ has a bipartite induced subgraph of minimum degree $\Omega(\log \delta)$

- Without triangle-free, trivially false due to cliques
- Without induced, trivially true with $d/2$ rather than $C \log d$
- If true, it is sharp up to constant factor
- $2$ rather than $\Omega(\log \delta)$ corresponds to presence of an even hole (Radovanović and Vušković ’13)
- True with “semi-bipartite” instead of bipartite
Conjecture (Esperet, Kang, Thomassé 2019)

Any triangle-free graph of minimum degree $\delta$ has a bipartite induced subgraph of minimum degree $\Omega(\log \delta)$

- Without triangle-free, trivially false due to cliques
- Without induced, trivially true with $d/2$ rather than $C \log d$
- If true, it is sharp up to constant factor
- $2$ rather than $\Omega(\log \delta)$ corresponds to presence of an even hole (Radovanović and Vušković ’13)
- True with “semi-bipartite” instead of bipartite
- True with $\Omega\left(\frac{\log \delta}{\log \log \delta}\right)$ (Kwan, Letzter, Sudakov, Tran 2018+)
Suppose minimum degree $\delta$ and there is a proper $k$-colouring
Suppose minimum degree $\delta$ and there is a proper $k$-colouring.

Each of $\sim \frac{k^2}{2}$ pairs of colour classes induces a bipartite graph with at least $\frac{n\delta}{2}$ edges distributed across these classes.

By pigeonhole, one has $\gtrsim \frac{n\delta}{k^2}$ edges.
Suppose minimum degree $\delta$ and there is a proper $k$-colouring

Each of $\sim \frac{k^2}{2}$ pairs of colour classes induces a bipartite graph

$\geq \frac{n\delta}{2}$ edges are distributed across these

By pigeonhole, one has $\gtrsim \frac{n\delta}{k^2}$ edges

So it has minimum degree $\Omega\left(\frac{\delta}{k}\right)$ if the colouring is balanced...
Suppose minimum degree $\delta$ and there is a proper $k$-colouring
Each of $\sim \frac{k^2}{2}$ pairs of colour classes induces a bipartite graph
$\geq \frac{n\delta}{2}$ edges are distributed across these
By pigeonhole, one has $\gtrsim \frac{n\delta}{k^2}$ edges
So it has minimum degree $\Omega(\frac{\delta}{k})$ if the colouring is balanced...

**Theorem (Esperet, Kang, Thomassé 2019)**

*Any graph with fractional chromatic number at most $k$ and minimum degree $\delta$ has a bipartite induced subgraph of minimum degree at least $\frac{\delta}{2k}$.***
Suppose minimum degree $\delta$ and there is a proper $k$-colouring

Each of $\sim \frac{k^2}{2}$ pairs of colour classes induces a bipartite graph

$\geq \frac{n\delta}{2}$ edges are distributed across these

By pigeonhole, one has $\gtrsim \frac{n\delta}{k^2}$ edges

So it has minimum degree $\Omega\left(\frac{\delta}{k}\right)$ if the colouring is balanced . . .

Theorem (Esperet, Kang, Thomassé 2019)

Any graph with fractional chromatic number at most $k$ and minimum degree $\delta$

has a bipartite induced subgraph of minimum degree at least $\frac{\delta}{2k}$.

Conjecture (Harris 2019)

Any triangle-free graph with degeneracy $\delta^*$ has fractional chromatic number

$O\left(\frac{\delta^*}{\log \delta^*}\right)$
Imagine *adversaries* to colouring

- that issue arbitrary matchings specifying pairwise conflicts of colours
- between lists of size $\ell$ on vertices joined by an edge

What is least $\ell$ for which colouring is always possible? (Necessarily $\ell \geq \text{ch}$)
Correspondence colouring

Imagine *adversaries* to colouring

- that issue arbitrary matchings specifying pairwise conflicts of colours
- between lists of size $\ell$ on vertices joined by an edge

What is least $\ell$ for which colouring is always possible? (Necessarily $\ell \geq \chi$)

Called *correspondence chromatic number* or *DP-chromatic number* $\chi_{DP}$
Correspondence colouring
Correspondence even “harder”

Or rather, it is much more closely linked with density


\[ \chi_{DP}(G) \gtrsim \frac{\delta}{\log \delta} \text{ for any } G \text{ of minimum degree } \delta \]
Correspondence even “harder”

Or rather, it is much more closely linked with density

\[ \chi_{DP}(G) \gtrsim \frac{\delta}{2 \log \delta} \text{ for any } G \text{ of minimum degree } \delta \]

Theorem (Bernshteyn 2019, cf. Molloy 2019)
\[ \chi_{DP}(G) \lesssim \frac{\Delta}{\log \Delta} \text{ for any triangle-free } G \text{ of maximum degree } \Delta \]

NB: This settles correspondence version of conjecture of Alon & Krivelevich
What if lists connected by edge are all *almost* disjoint, so 1 conflict?
A generalisation to multigraphs is natural (also for “adaptable choosability”)

Call the corresponding least $\ell$ least conflict choosability $\text{ch}_{\text{DP1}}$

Theorem (Dvořák, Esperet, Kang, Ozeki 2018+)

$\text{ch}_{\text{DP1}}(G) \lesssim 2\sqrt{\Delta}$ for any (multigraph) $G$ of maximum degree $\Delta$

NB: $\text{ch}_{\text{DP1}}(G) \gtrsim \sqrt{\Delta}$ for a 2-vertex $G$ of multiplicity $\Delta$ (!)

Theorem (Dvořák, Esperet, Kang, Ozeki 2018+)

$\text{ch}_{\text{DP1}}(G) \gtrsim \sqrt{\delta} \log \delta$ for any $G$ of minimum degree $\delta$

An analogue of Heawood’s Formula (roughly of form $\chi = O(\sqrt{g} + 1)$)

Theorem (Dvořák, Esperet, Kang, Ozeki 2018+)

$\text{ch}_{\text{DP1}}(G) = O((g + 1)^{1/4} \log(g + 2))$ for any simple $G$ embeddable on a surface of Euler genus $g$
A generalisation to multigraphs is natural (also for “adaptable choosability”)

Call the corresponding least \( \ell \) least conflict choosability \( \text{ch}_{\text{DP1}} \)

**Theorem (Dvořák, Esperet, Kang, Ozeki 2018+)**

\[ \text{ch}_{\text{DP1}}(G) \lesssim 2\sqrt{\Delta} \text{ for any (multigraph) } G \text{ of maximum degree } \Delta \]

NB: \( \text{ch}_{\text{DP1}}(G) \gtrsim \sqrt{\Delta} \text{ for a 2-vertex } G \text{ of multiplicity } \Delta \) (!)
Correspondence and separation

A generalisation to multigraphs is natural (also for “adaptable choosability”)

Call the corresponding least $\ell$ least conflict choosability $\text{ch}_{\text{DP1}}$

Theorem (Dvořák, Esperet, Kang, Ozeki 2018+)

$\text{ch}_{\text{DP1}}(G) \lesssim 2\sqrt{\Delta}$ for any (multigraph) $G$ of maximum degree $\Delta$

NB: $\text{ch}_{\text{DP1}}(G) \gtrsim \sqrt{\Delta}$ for a 2-vertex $G$ of multiplicity $\Delta$ (!)

Theorem (Dvořák, Esperet, Kang, Ozeki 2018+)

$\text{ch}_{\text{DP1}}(G) \gtrsim \sqrt{\frac{\delta}{\log \delta}}$ for any $G$ of minimum degree $\delta$
**Correspondence and separation**

A generalisation to multigraphs is natural (also for “adaptable choosability”)

Call the corresponding least $\ell$ least conflict choosability $ch_{DP1}$

**Theorem (Dvořák, Esperet, Kang, Ozeki 2018+)**

\[ ch_{DP1}(G) \lesssim 2\sqrt{\Delta} \text{ for any (multigraph) } G \text{ of maximum degree } \Delta \]

NB: \[ ch_{DP1}(G) \gtrsim \sqrt{\Delta} \text{ for a 2-vertex } G \text{ of multiplicity } \Delta \ (!) \]

**Theorem (Dvořák, Esperet, Kang, Ozeki 2018+)**

\[ ch_{DP1}(G) \gtrsim \sqrt{\frac{\delta}{\log \delta}} \text{ for any } G \text{ of minimum degree } \delta \]

An analogue of Heawood’s Formula (roughly of form $\chi = O(\sqrt{g+1})$)

**Theorem (Dvořák, Esperet, Kang, Ozeki 2018+)**

\[ ch_{DP1}(G) = O((g + 1)^{1/4} \log(g + 2)) \text{ for any simple } G \text{ embeddable on a surface of Euler genus } g \]
Theorem (Dvořák, Esperet, Kang, Ozeki 2018+)

$$\text{ch}_{\text{DP}_1}(G) \lesssim 2\sqrt{\Delta} \text{ for any (multigraph) } G \text{ of maximum degree } \Delta$$

NB: $$\text{ch}_{\text{DP}_1}(G) \gtrsim \sqrt{\Delta} \text{ for a 2-vertex } G \text{ of multiplicity } \Delta \text{ (!) }$$
Theorem (Dvořák, Esperet, Kang, Ozeki 2018+)
\[ \text{ch}_{\text{DP1}}(G) \lesssim 2\sqrt{\Delta} \text{ for any (multigraph) } G \text{ of maximum degree } \Delta \]

NB: \[ \text{ch}_{\text{DP1}}(G) \gtrsim \sqrt{\Delta} \text{ for a 2-vertex } G \text{ of multiplicity } \Delta \]

Theorem Redux (Dvořák, Esperet, Kang, Ozeki 2018+)
Given simple \( H \) and a vertex partition \( L : [n] \rightarrow \binom{V(H)}{\ell} \) satisfying
- \( \frac{1}{\ell} \sum_{i \in L(v)} \deg(i) \leq D \) for every \( v \in [n] \)
- \( \ell \gtrsim 4D \),

there is an independent set that is transversal to the partition \( L \)

* Observed in ongoing work with Kelly
Theorem (Dvořák, Esperet, Kang, Ozeki 2018+)
\[ \text{ch}_{\text{DP1}}(G) \lesssim 2\sqrt{\Delta} \text{ for any (multigraph) } G \text{ of maximum degree } \Delta \]

NB: \[ \text{ch}_{\text{DP1}}(G) \gtrsim \sqrt{\Delta} \text{ for a 2-vertex } G \text{ of multiplicity } \Delta \] (!)

Theorem Redux (Dvořák, Esperet, Kang, Ozeki 2018+)

Given simple \( H \) and a vertex partition \( L: [n] \to (V(H)) \) satisfying
\[ \frac{1}{\ell} \sum_{i \in L(v)} \deg(i) \leq D \text{ for every } v \in [n] \]
\[ \ell \gtrsim 4D, \]

there is an independent set that is transversal to the partition \( L \)

So closely related to Haxell 2001 (with instead \( \deg(i) \leq D \) and \( 2D \)) and

Theorem (Bollobás, Erdős, Szemerédi 1975, cf. Szabó & Tardos 2006)
\[ \text{ch}_{\text{DP1}}(G) \gtrsim \sqrt{2\Delta} \text{ for some multigraph } G \text{ of maximum degree } \Delta \]

and also to List Colouring Constants...

*Observed in ongoing work with Kelly*
Conjecture (Esperet, Kang, Thomassé 2019)

Any triangle-free graph of minimum degree $\delta$ has a bipartite induced subgraph of minimum degree $\Omega(\log \delta)$
Conjecture (Esperet, Kang, Thomassé 2019)
Any triangle-free graph of minimum degree $\delta$ has a bipartite induced subgraph of minimum degree $\Omega(\log \delta)$

Conjecture (Cames van Batenburg, de Joannis de Verclos, Kang, Pirot 2018+)
$\chi(G) \lesssim \sqrt{\frac{2n}{\log n}}$ for any triangle-free graph $G$ on $n$ vertices
Conjecture (Esperet, Kang, Thomassé 2019)

Any triangle-free graph of minimum degree $\delta$ has a bipartite induced subgraph of minimum degree $\Omega(\log \delta)$

Conjecture (Cames van Batenburg, de Joannis de Verclos, Kang, Pirot 2018+)

$\chi(G) \lesssim \sqrt{\frac{2n}{\log n}}$ for any triangle-free graph $G$ on $n$ vertices

Conjecture (Cames van Batenburg, de Joannis de Verclos, Kang, Pirot 2018+)

$\text{ch}(G) = O\left(\sqrt{\frac{n}{\log n}}\right)$ for any triangle-free graph $G$ on $n$ vertices