Independent sets, graph colouring and the hard-core model

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UU Condensed Matter Physics Seminar 4/2019

Party problem

A host welcomes 6 dinner guests:

Party problem

A host welcomes 6 dinner guests:

3 are mutually acquainted, *or*

Party problem

A host welcomes 6 dinner guests:

3 are mutually acquainted, *or*
3 will meet for the first time

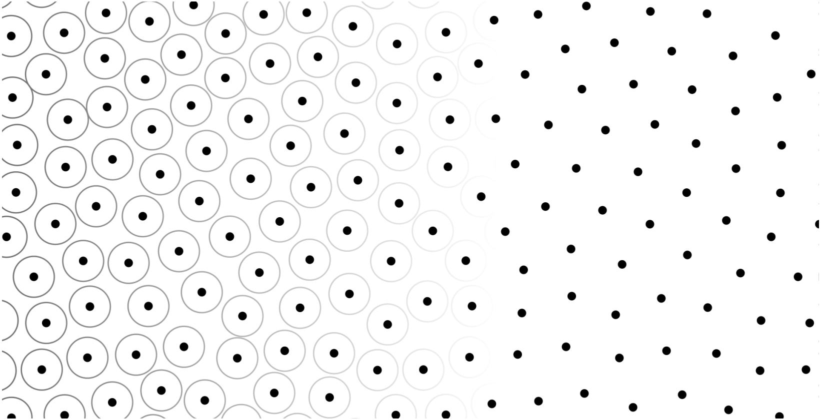
Party problem

A host welcomes 6 dinner guests:

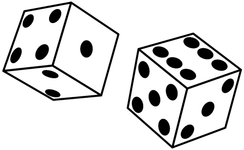
3 are mutually acquainted, *or*
3 will meet for the first time

regardless of who was invited

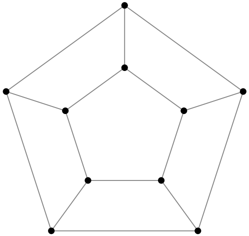
Hard-core model*



* More fully, the lattice gas with hard-core self-repulsion and nearest-neighbour exclusion.
Picture credit: Wikipedia/Grap-wh.



If a random object has desired property with positive probability, then there exists *at least one* object with that property

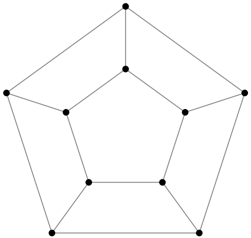


graphs=networks⊇lattices

vertex=node=site
edge=link=bond

adjacent
neighbourhood
degree

Off-diagonal Ramsey numbers



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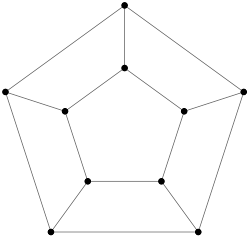
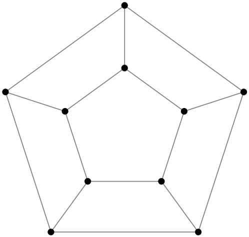


triangle : 3 pairwise adjacent vertices



no 

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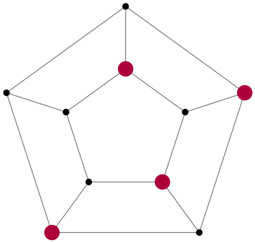

triangle : 3 pairwise adjacent vertices

independent set : set of pairwise non-adjacent vertices

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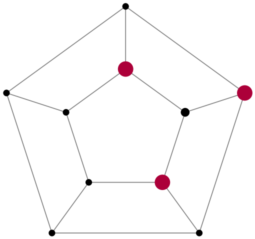
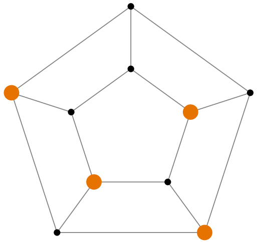


triangle : 3 pairwise adjacent vertices

independent set : set of pairwise non-adjacent vertices

NB : each neighbourhood in graph with no triangles is independent (!)

Off-diagonal Ramsey numbers



$\alpha = 4$

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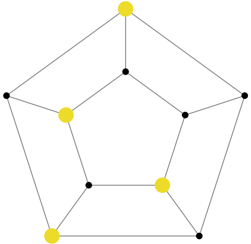
independence number α : size of largest independent set

Off-diagonal Ramsey numbers



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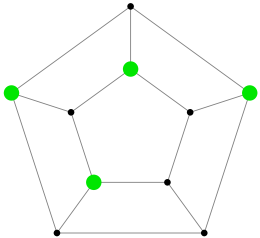

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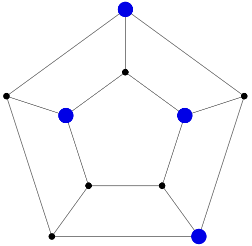

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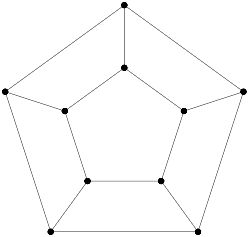

$\alpha = 4$, no triangle

$\implies R(3,5) > 10$

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$R(3,k)$: 1 plus size of largest graph with no triangle and $\alpha < k$

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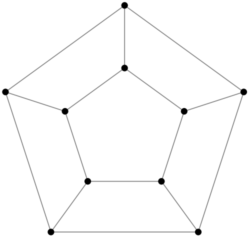

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NB: Party problem is $R(3,3) = 6$

Off-diagonal Ramsey numbers†

cf. "80 Years of $R(3, k)$" (Spencer 2011)

† Picture credit: Soifer 2009.

Off-diagonal Ramsey numbers

$R(3, k)$: 1 plus size of largest graph with no triangle and $\alpha < k$

Off-diagonal Ramsey numbers
i.e. Independence number of triangle-free graphs

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Theorem (Shearer 1983, cf. Ajtai, Komlós, Szemerédi 1980/1)

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Theorem (Bohman, Keevash 2013+, Fiz Pontiveros, Griffiths, Morris 2013+, cf. Kim 1995)

$$R(3,k) \gtrsim \frac{k^2}{4\log k}$$

 Erdős, Suen, Winkler 1995 

- vertices $1, \ldots, n$
- randomly order all $\binom{n}{2}$ potential edges
- one by one in order, add an edge *if it doesn't create a triangle*

‡

Triangle-free process

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Has qualitative properties of random graph[‡], but *much* more difficult analysis

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[‡]a.k.a. mean field theory for unweighted bond percolation

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$$\textit{The triangle-free process adds} \sim \frac{n\sqrt{n\log n}}{2\sqrt{2}} \textit{ edges with high probability}$$

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($R(3,k)$: minimum size of triangle-free graph guaranteeing $\alpha \geq k$)

Take triangle-free graph of size $n \sim \dfrac{k^2}{\log k}$.

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Probabilistic method[§]

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§

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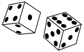
$\alpha \gtrsim \dfrac{n \log \Delta}{\Delta}$ *for n-vertex triangle-free graphs of maximum degree* Δ

Proof with ?

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Does there exist a probability distribution on $\mathscr{I}$ the set of independent sets such that for a random $\mathbf{I}$

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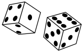
Proof with ?

Does there exist a probability distribution on $\mathscr{I}$ the set of independent sets such that for a random $\mathbf{I}$

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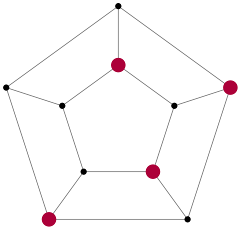
[§] Yes, cf. Davies, Jenssen, Perkins, Roberts 2018

Chromatic number[¶]



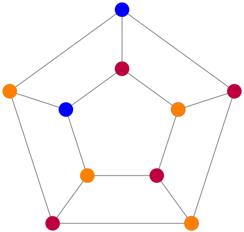
$\alpha = 4$

vertex-colouring : non-intersecting indep sets that cover all vertices

chromatic number χ : least number of independent sets needed

[¶]a.k.a. dilute (?) antiferromagnetic Potts model

Chromatic number[¶]



$\alpha = 4$, $\chi = 3$

$\chi \geq n/\alpha = 5/2$

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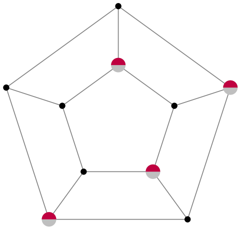
Fractional chromatic number



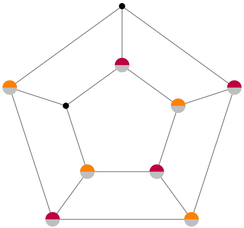
$\alpha = 4$

$\chi_f \geq n/\alpha = 5/2$

fractional vertex-colouring : allow "fractions" of independent sets

fractional chromatic number χ_f : least "amount" needed

Fractional chromatic number



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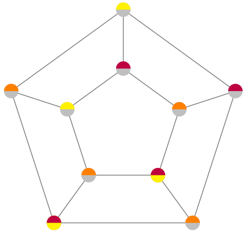

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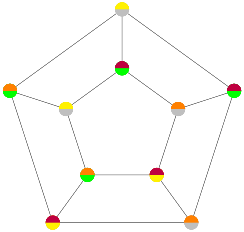

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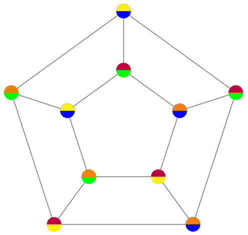

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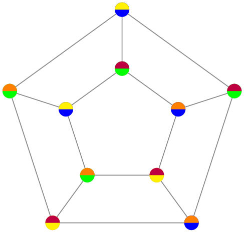

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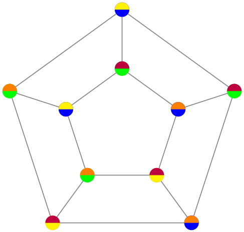

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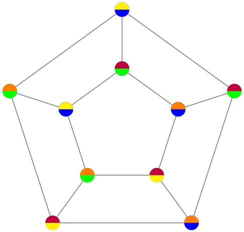

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$$\Longrightarrow \mathbb{E}\,\mathbf{I} \geq n/k$$

Chromatic number of triangle-free graphs

$$\frac{n}{\alpha} \leq \chi_f \leq \chi$$

Theorem (Shearer 1983, cf. Ajtai, Komlós, Szemerédi 1980/1)

$\alpha \gtrsim \dfrac{n \log \Delta}{\Delta}$ *for n-vertex triangle-free graphs of maximum degree* Δ

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Theorem (Molloy 2019, cf. Johansson 1996+)

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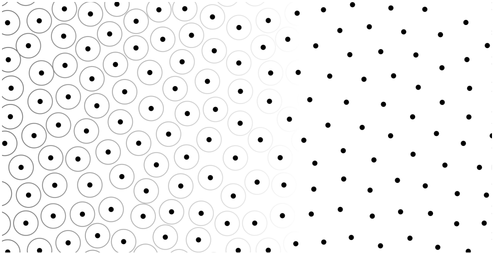
Hard-core model

A probability distribution on $\mathscr{I}$ the set of independent sets



Hard-core model

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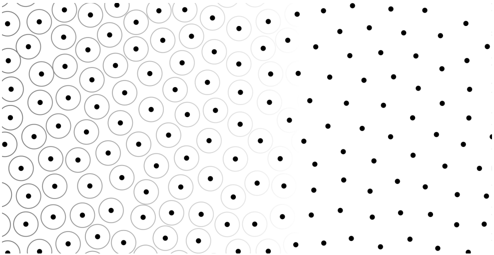

The hard-core model at fugacity $\lambda > 0$ is the probability distribution on $\mathscr{I}$ such that a random $\mathbf{I}_\lambda$ satisfies for all $S \in \mathscr{I}$

$$\mathbb{P}(\mathbf{I}_\lambda = S) = \frac{\lambda^{|S|}}{Z(\lambda)}, \quad \text{where } Z(\lambda) = \sum_{S \in \mathscr{I}} \lambda^{|S|}$$

Local occupancy method

A distribution $\mathbf{I}$ on $\mathscr{I}$ has local (a, b)-occupancy if for every vertex v

$$a \cdot \mathbb{P}(v \in \mathbf{I}) + b \cdot \mathbb{E}|N(v) \cap \mathbf{I}| \geq 1$$

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$\rightsquigarrow$ analysis exercise $\rightsquigarrow$ $\chi_f \lesssim \dfrac{\Delta}{\log \Delta}$ for triangle-free

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For any $S \in \mathscr{I}$, call u uncovered if $N(u) \cap S = \emptyset$ and call u occupied if $u \in S$

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Fact 1 $\mathbb{P}(v \in \mathbf{I} \mid v \text{ uncovered}) = \dfrac{\lambda}{1+\lambda}$

Local occupancy method

A distribution $\mathbf{I}$ on $\mathscr{I}$ has local (a, b)-occupancy if for every vertex v

$$a \cdot \mathbb{P}(v \in \mathbf{I}) + b \cdot \mathbb{E}|N(v) \cap \mathbf{I}| \geq 1$$

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Fact 2 $\mathbb{P}(v \text{ uncovered} \mid v \text{ has } j \text{ uncovered neighbours}) = \dfrac{1}{(1+\lambda)^j}$

Conjecture (Esperet, Kang, Thomassé 2018+)

There is a bipartite induced subgraph of minimum degree $C \log \delta$ *in any triangle-free graph of minimum degree* δ

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Conjecture (Cames van Batenburg, de Joannis de Verclos, Kang, Pirot 2018+)

$$\chi \lesssim \sqrt{\frac{2n}{\log n}} \text{ for triangle-free graphs of size } n$$