Global graph structure derived from local sparsity

Ross J. Kang*

Radboud University Nijmegen

QMUL Combinatorics Study Group

London (via zoom) 30/10/2020

*With Davies, Hurley, de Joannis de Verclos, Pirot, Sereni. Support from NWO.

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Asymptotically, what global graph structure forms if no edge is induced in any neighbourhood?

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- Mantel (1907), Turán (1941)
- Ramsey (1930), Erdős & Szekeres (1935)
- Zykov (1949), Ungar & "Blanche Descartes" (1954)

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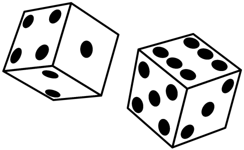
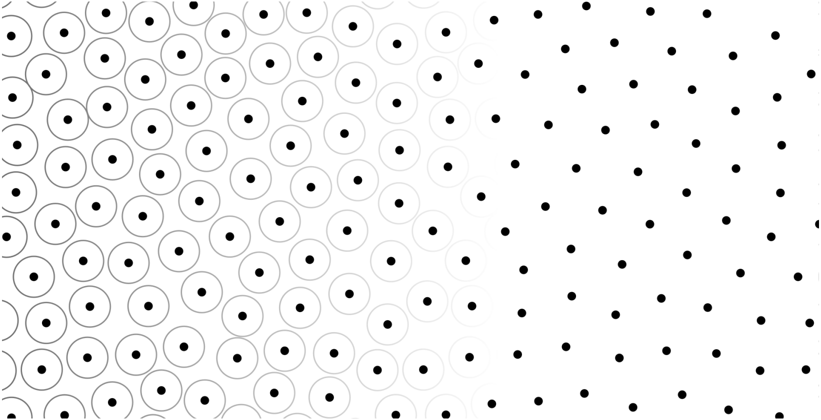
Elegant, modern challenges!



If a random object has desired property with positive probability, then there exists *at least one* object with that property

Hard-core model[†]



[†] More fully, the lattice gas with hard-core self-repulsion and nearest-neighbour exclusion.
Picture credit: Wikipedia/Grap-wh

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‡

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A sequence of structural parameters

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In general, each inequality can be strict‡

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We'll first focus on triangle-free...

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Off-diagonal Ramsey numbers[§]

[§] Picture credit: Soifer 2009

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$R(3,k)$: largest n such that there is red/blue-edge-coloured K_{n-1}
with no red triangle and no blue K_k

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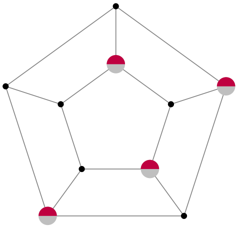
¶Yes, cf. Davies, Jenssen, Perkins, Roberts 2018...

Fractional chromatic number



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fractional vertex-colouring : allow "fractions" of independent sets

fractional chromatic number χ_f : least "amount" needed

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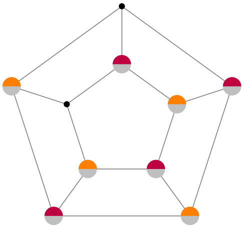

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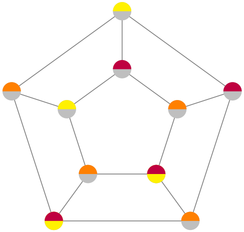

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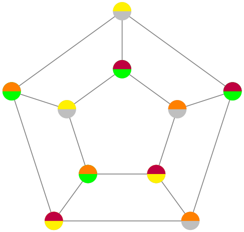

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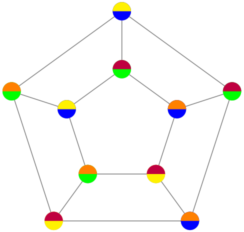

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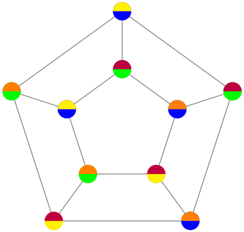

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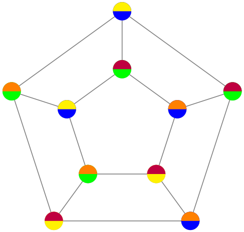

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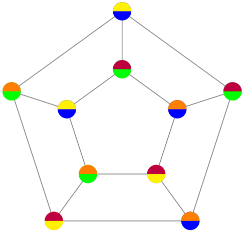

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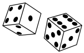

Chromatic number of triangle-free graphs

$$\omega(G) \le \max_{\emptyset \ne H \subseteq G} \frac{|H|}{\alpha(H)} \le \chi_f(G) \le \chi(G) \le \mathrm{ch}(G) \le \Delta(G) + 1$$

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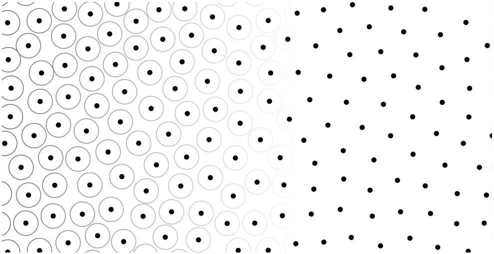
Why? Simple, conceptual, versatile.

A probability distribution over $\mathscr{I}(G)$ the set of independent sets of G



Hard-core model

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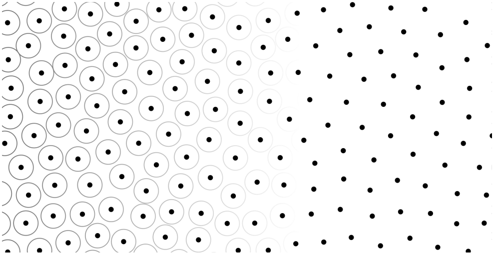
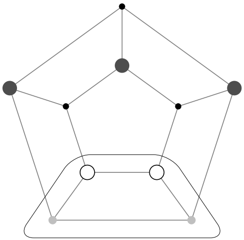


Hard-core model on G at fugacity $\lambda > 0$ is probability distribution over $\mathscr{I}(G)$ such that random $\boldsymbol{I}$ satisfies for all $S \in \mathscr{I}(G)$

$$\mathbb{P}(\boldsymbol{I} = S) = \frac{\lambda^{|S|}}{Z_G(\lambda)}, \quad \text{where } Z_G(\lambda) = \sum_{S \in \mathscr{I}(G)} \lambda^{|S|}$$

Spatial Markov property

For $S \in \mathscr{I}(G)$, call u occupied if $u \in S$ and call u uncovered if $N(u) \cap S = \emptyset$



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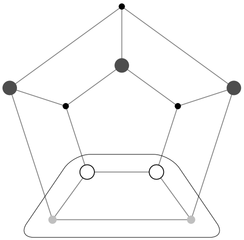

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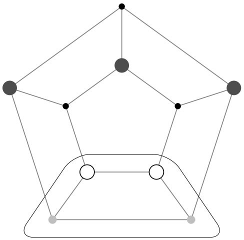

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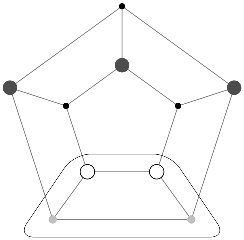

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Then $\boldsymbol{I} \cap X$ is hard-core on $G[\mathrm{U}_X]$ at fugacity λ

Local occupancy method

Distribution $\boldsymbol{I}$ on $\mathscr{I}(G)$ has local (a, b)-occupancy if for every vertex v

$$a \cdot \mathbb{P}(v \in \boldsymbol{I}) + b \cdot \mathbb{E}|N(v) \cap \boldsymbol{I}| \geq 1$$

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$\rightsquigarrow$ analysis to minimise $a + b \cdot \Delta$ $\rightsquigarrow$ $\chi_f(G) \lesssim \dfrac{\Delta}{\log \Delta}$ 

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Idea: greedily add weight/colour to independent sets according to probability distribution induced by $\boldsymbol{I}$ on vertices not yet completely coloured, and iterate

One can think of it as "evening out" the distribution

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Fact 2 $\mathbb{P}(v \text{ uncovered} \mid v \text{ has } j \text{ uncovered neighbours}) = \dfrac{1}{(1+\lambda)^j}$

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(needs triangle-free!)

Local occupancy method (for triangle-free graphs)

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Distribution $\boldsymbol{I}$ on $\mathscr{I}(G)$ has local (a, b)-occupancy if for every vertex v

$$a \cdot \mathbb{P}(v \in \boldsymbol{I}) + b \cdot \mathbb{E}|N(v) \cap \boldsymbol{I}| \geq 1$$

A Hard-core model on any triangle-free G has local (a, b)-occupancy, for specific a, b depending on fugacity λ and maximum degree Δ

B If there is probability distribution $\boldsymbol{I}$ on $\mathscr{I}(G)$ with local (a, b)-occupancy, then $\chi_f(G) \leq a + b \cdot \Delta$

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Fact 1 and Fact 2 imply

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By B it suffices to find suitable a, b such that $a + b \cdot \Delta \lesssim \dfrac{\Delta}{\log \Delta}$ by minimising

$$a + b \cdot \Delta \quad \text{subject to} \quad \frac{b\lambda(\log((ea/b)\log(1+\lambda)))}{(1+\lambda)\log(1+\lambda)} \geq 1 \rightsquigarrow$$

Barriers for local occupancy method

"90 years of $R(3,k)$":

Theorem (Shearer 1983)

$\alpha(G) \gtrsim \frac{n \log \Delta}{\Delta}$ *for any n-vertex triangle-free G of maximum degree* Δ

Sharp up to factor 2 for the random Δ-regular graphs $G_{n,\Delta}$

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Question (Karp 1976)

Is there a polynomial-time algorithm that with high probability outputs an independent set of $G_{n,1/2}$ *of size* $(1+\varepsilon)\log_2 n$*?*

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Conjecture (Davies, Jenssen, Perkins, Roberts 2018)

$\alpha(G) \gtrsim 2 \cdot \dfrac{Z'_G(1)}{Z_G(1)}$ *for any triangle-free G of minimum degree* δ

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Hard-core model on G has local (a, b)-occupancy if for every vertex v and every induced subgraph F of the neighbourhood subgraph $G[N(v)]$.

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C If hard-core model has local (a, b)-occupancy ($+$ mild conditions), then $\mathrm{ch}(G) \leq a \cdot O(\log \Delta) + (1+\varepsilon) b \cdot \Delta$

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C relies crucially on seminal proofs of Molloy (2019) and Bernshteyn (2019) combined with properties of the hard-core model

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C*, an algorithmic version of C (under additional conditions), merges the hard-core model with framework of Achlioptas, Iliopoulous, Sinclair (2019)

Shearer (1995), Bansal, Gupta, Guruganesh (2018)

Theorem (Davies, Kang, Pirot, Sereni 2020+)

$$\alpha(G) \gtrsim \max\left\{ \frac{n \log \Delta}{\Delta(r-1)\log\log\Delta}, \frac{n\sqrt{\log \Delta}}{2\Delta\sqrt{\log r}} \right\}$$ *for any n-vertex K_r-free G of maximum degree Δ*

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Applications

Shearer (1995), Bansal, Gupta, Guruganesh (2018)

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NB: common derivation via Erdős–Szekeres bounds for Ramsey numbers

Application: locally r-colourable graphs

Theorem (Johansson 1996+, cf. Alon 1996, Molloy 2019, Bonamy, Kelly, Nelson, Postle 2018+)

$\mathrm{ch}(G) = O\left(\log(r+1)\dfrac{\Delta}{\log\Delta}\right)$ *for any G of maximum degree Δ in which every neighbourhood is r-colourable*

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NB: $r=1$ implies Molloy's and $r=\Delta+1$ matches trivial bound

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Theorem (Molloy 2019, cf. Johansson 1996+, cf. also Bernshteyn 2019)

$\mathrm{ch}(G) \lesssim \frac{\Delta}{\log \Delta}$ *for any triangle-free G of maximum degree Δ*

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NB: $k = \Delta^{o(1)}$ includes both Kim's and Molloy's

Application: locally sparse graphs

Theorem (Alon, Krivelevich, Sudakov 1999, cf. Vu 2002, Achlioptas, Iliopoulos, Sinclair 2019)

$\chi(G) = O\left(\frac{\Delta}{\log(\Delta/\sqrt{T})}\right)$ *for any G of maximum degree Δ with each vertex in $\leq T$ triangles,* $1/2 \leq T \leq \binom{\Delta}{2}$

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NB: $T = \Delta^{o(1)}$ implies Molloy's; bound is sharp throughout up to factor 2 or 4

Locally sparse graphs

Let's awkwardly rephrase:
For $0 < \sigma \leq 1$, G is σ-sparse if every neighbourhood has $\leq (1-\sigma)\binom{\Delta}{2}$ edges

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Graphs of bounded local density

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Easy construction: clique of size $\sqrt{1-\sigma}\Delta$ (plus pendant vertices)

$$\implies \varepsilon \le 1 - \sqrt{1-\sigma} \underset{\sigma\to 0}{\sim} 0.5\sigma + o(\sigma)$$

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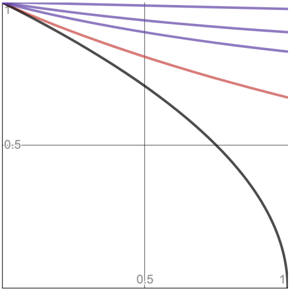

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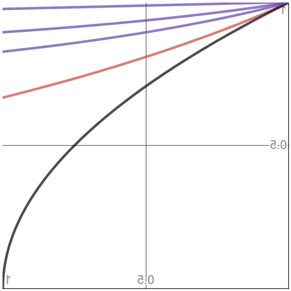

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Naïvely colouring

Take given G and palette $[M] = \{1, \ldots, M\}$.

1. To each vertex v independently assign a uniformly random colour from $[M]$
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Naïvely generating independent sets

Take given Δ-regular graph and fix a parameter $\gamma > 0$.

1. Independently with probability γ/Δ add each vertex to the activated set $\boldsymbol{A}$
2. Assign each activated vertex v a uniformly random priority $\pi(v)$ from $[0, 1]$
3. For any two adjacent activated vertices, remove the one with lower priority:

$$\boldsymbol{I} = \{v \in \boldsymbol{A} \mid \pi(v) > \pi(u) \text{ for every } u \in N(v) \cap \boldsymbol{A}\},$$

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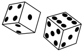
Lemma

For $\iota > 0$, the above process for generating $\boldsymbol{I}$ from a σ-sparse Δ-regular graph satisfies the following for sufficiently large Δ and γ. For every vertex v

$$\left|\mathbb{P}(v \in \boldsymbol{I}) - \frac{1 - e^{-\gamma}}{\Delta}\right| \le \frac{2}{\Delta^2}$$

and, setting $\boldsymbol{I}_v = N(v) \cap \boldsymbol{I}$,

$$\frac{\mathbb{P}(\boldsymbol{I}_v \neq \emptyset)}{\mathbb{E}|\boldsymbol{I}_v|} \le 1 - \frac{\sigma}{2} + \frac{\sigma^{3/2}}{6} + \iota.$$



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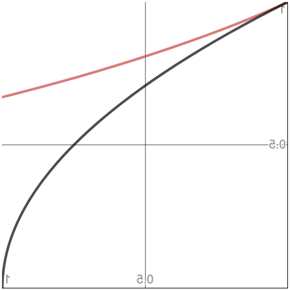

Thanks!