Acyclic and frugal colourings of graphs

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1 Introduction

In this paper, a *(vertex) colouring* of a graph $G = (V, E)$ is any map $f : V \to \mathbb{Z}^+$. The *colour classes* of a colouring f are the preimages $f^{-1}(i), i \in \mathbb{Z}^+$. A colouring of a graph is *proper* if adjacent vertices receive distinct colours; however, in this paper, we will devote considerable attention to colourings that are not necessarily proper, but that satisfy another condition. A colouring of G is t*-frugal* if no colour appears more than t times in any neighbourhood. The notion of frugal colouring was introduced by Hind, Molloy and Reed [5]. They considered proper t-frugal colourings as a way to improve bounds related to the Total Colouring Conjecture (cf. [6]). In Section 2, we study t-frugal colourings for graphs of bounded maximum degree.

In Section 3, we impose an additional condition that is well-studied in the graph colouring literature (cf. [3]). A colouring of V is *acyclic* if each of the bipartite graphs consisting of the edges between any two colour classes is acyclic. In other words, a colouring of G is acyclic if G contains no *alternating cycle* (that is, an even cycle that alternates between two distinct colours). For graphs of bounded maximum degree, the study of acyclic proper colourings was instigated by Erdős (cf. [2]) and more or less settled asymptotically by Alon, McDiarmid and Reed [3]. Extending the work of Alon *et al.*, Yuster [9] investigated acyclic proper 2-frugal colourings. In Section 3, we expand this study to different values of t and colourings that are not necessarily proper.

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Let us outline our notation. As usual, the *chromatic number* $\chi(G)$ (resp. *acyclic chromatic number* $\chi_a(G)$) denotes the least number of colours needed in a proper (resp. acyclic proper) colouring. Analogously, for $t \geq 1$, we define the *t-frugal chromatic number* $\varphi^t(G)$, *proper t-frugal chromatic number* $\chi_\varphi^t(G)$, *acyclic t-frugal chromatic number* $\varphi_a^t(G)$ and *acyclic proper t-frugal chromatic number* $\chi_{\varphi,a}^t(G)$. We have designated φ as a mnemonic for frugal. We are interested in graphs G of bounded degree, so let $\chi(d)$ denote the maximum possible value of $\chi(G)$ over all graphs G with $\Delta(G) = d$. We analogously define $\chi_a(d)$; $\varphi^t(d)$, $\chi_\varphi^t(d)$, $\varphi_a^t(d)$ and $\chi_{\varphi,a}^t(d)$ for $t \geq 1$. The *square* of a graph G, i.e. the graph formed from G by adding edges between any two vertices at distance two, is denoted G^2. Note the following basic observations.

Proposition 1 *For any graph G and any $t \geq 1$, the following hold:*

(i) $\chi_\varphi^1(G) = \chi_{\varphi,a}^1(G) = \chi(G^2)$;
(ii) $\varphi^t(G) \leq \chi_\varphi^t(G)$, $\varphi_a^t(G) \leq \chi_{\varphi,a}^t(G)$; $\varphi^t(G) \leq \varphi_a^t(G)$, $\chi_\varphi^t(G) \leq \chi_{\varphi,a}^t(G)$;
(iii) $\varphi^{t+1}(G) \leq \varphi^t(G)$, $\chi_\varphi^{t+1}(G) \leq \chi_\varphi^t(G)$, $\varphi_a^{t+1}(G) \leq \varphi_a^t(G)$, $\chi_{\varphi,a}^{t+1}(G) \leq \chi_{\varphi,a}^t(G)$; *and*
(iv) $\varphi^t(G) \geq \Delta(G)/t$.

We may invoke basic probabilistic tools such as the Lovász Local Lemma, details of which can be found in various references, e.g. Molloy and Reed [7].

2 Frugal colourings

As a way to improve bounds for total colouring (cf. [6]), Hind *et al.* [5], showed that $\chi_\varphi^{(\ln d)^5}(d) \leq d + 1$ for sufficiently large d. Recently, this was improved.

Theorem 2 (Molloy and Reed [8]) $\chi_\varphi^{50 \ln d / \ln \ln d}(d) \leq d + 1$ *for sufficiently large d.*

Since $\chi_\varphi^t(K_{d+1}) \geq d + 1$, it follows that $\chi_\varphi^t(d) = d + 1$ for $t = t(d) \geq 50 \ln d / \ln \ln d$. For smaller frugalities, Hind *et al.* [5] also showed the following.

Theorem 3 (Hind *et al.* [5]) *For any $t \geq 1$ and sufficiently large d,* $\chi_\varphi^t(d) \leq \max\left\{(t+1)d, \left\lceil e^3 d^{1+1/t}/t \right\rceil\right\}$.

By Proposition 1(i), $\chi_\varphi^1(d) \sim d^2$. We note that an example based on projective geometries due to Alon (cf. [5]), to lower bound $\chi_\varphi^t(d)$, is also valid for $\varphi^t(d)$.

Proposition 4 *For any $t \geq 1$ and any prime power n,* $\varphi^t(n^t + \cdots + 1) \geq (n^{t+1} + \cdots + 1)/t$.

The following consequence shows (by Proposition 1(ii)) that Theorem 3 is asymptotically tight up to a constant multiple when $t = o(\ln d/\ln\ln d)$.

Corollary 5 *Suppose that $t = t(d) \geq 2$, $t = o(\ln d/\ln\ln d)$, and $\epsilon > 0$ fixed. Then, for sufficiently large d, $\varphi^t(d) \geq (1-\epsilon)d^{1+1/t}/t$.*

Theorems 2 and 3 determine the behaviour of $\chi_\varphi^t(d)$ up to a constant multiple for all t except for the range such that $t = \Omega(\ln d/\ln\ln d)$ and $t \leq 50\ln d/\ln\ln d$. Recall from Proposition 1(iv) that $\varphi^t(d) \geq d/t$. For the case $t = \omega(\ln d)$, we give a tight upper bound for $\varphi^t(d)$.

Theorem 6 *Suppose $t = \omega(\ln d)$ and $\epsilon > 0$ fixed. Then, for sufficiently large d, $\varphi^t(d) \leq \lceil (1+\epsilon)d/t \rceil$.*

PROOF. Let $G = (V, E)$ be a graph with maximum degree d and let $x = \lceil (1+\epsilon)d/t \rceil$. Let f be a random colouring where for each $v \in V$, $f(v)$ is chosen uniformly and independently at random from $\{1, \ldots, x\}$. For a vertex v and a colour $i \in \{1, \ldots, x\}$, let $A_{v,i}$ be the event that v has more than t neighbours with colour i. If none of these events hold, then f is t-frugal. Each event is independent of all but at most $d^2x \ll d^3$ others. By a Chernoff bound,

$$\begin{aligned}\Pr(A_{v,i}) &= \Pr(\mathrm{BIN}(d, 1/x) > t) \leq \Pr(\mathrm{BIN}(d, 1/x) > d/x + ct)\\ &\leq \exp\left(-c^2t^2/(2d/x + 2ct/3)\right)\end{aligned}$$

where $c = \epsilon/(1+\epsilon)$. Thus, $e\Pr(A_{v,i})(d^3+1) = \exp(-\Omega(t))d^3 < 1$ for large enough d, and by the Lovász Local Lemma, f is t-frugal with positive probability for large enough d. $\square$

3 Acyclic frugal colourings

Using the Lovász Local Lemma, Alon *et al.* [3] established a $o(d^2)$ upper bound for $\chi_a(d)$, answering a long-standing question of Erdős (cf. [2]). Using a probabilistic construction, they also showed this upper bound to be asymptotically correct up to a logarithmic multiple.

Theorem 7 (Alon *et al.* [3]) $\chi_a(d) \leq \lceil 50d^{4/3} \rceil$, $\chi_a(d) = \Omega(d^{4/3}/(\ln d)^{1/3})$.

Yuster [9] considered acyclic proper 2-frugal colourings of graphs and showed that $\chi^2_{\varphi,a}(d) \leq \lceil \max\{50d^{4/3}, 10d^{3/2}\} \rceil$. For acyclic frugal colourings, we first consider the smallest cases then proceed to larger values of t. For $t = 1, 2, 3$, notice that Corollary 5, Proposition 1(i) and Yuster's result imply that $\varphi^1_a(d) = \Theta(d^2)$, $\varphi^2_a(d) = \Theta(d^{3/2})$, $\chi^2_{\varphi,a}(d) = \Theta(d^{3/2})$ and $\varphi^3_a(d) = \Omega(d^{4/3})$. Next, we show that $\chi^3_{\varphi,a}(d) = O(d^{4/3})$. This implies that $\chi^t_{\varphi,a}(d) = O(d^{4/3})$ for any $t \geq 3$, a bound that is within a logarithmic multiple of the lower bound implied by Theorem 7. This answers a question of Esperet, Montassier and Raspaud [4].

Theorem 8 $\chi^3_{\varphi,a}(d) \leq \lceil 40.27 d^{4/3} \rceil$.

PROOF. (Outline.) Our proof is an extension of the proof of Theorem 7 in which we add a fifth event to ensure that the random colouring f is 3-frugal:

V For vertices v, v_1, v_2, v_3, v_4 with $\{v_1, v_2, v_3, v_4\} \subseteq N(v)$, let $E_{\{v_1,\dots,v_4\}}$ be the event that $f(v_1) = f(v_2) = f(v_3) = f(v_4)$. $\square$

For acyclic frugal colourings which are not necessarily proper, for larger values of t, we have adapted a result of Addario-Berry *et al.* [1] to show the following.

Theorem 9 *For any* $t = t(d) \geq 1$, $\varphi^t_a(d) = O(d \ln d + (d-t)d)$.

This implies, for instance, that $\varphi^{d-1}_a(d)$ and $\chi^{d-1}_{\varphi,a}(d)$ differ by a multiplicative factor of order at least $d^{1/3}/(\ln d)^{4/3}$. The result is obtained by studying *total k-dominating sets* — given $G = (V, E)$, $\mathcal{D} \subset V$ is total k-dominating if each vertex has at least k neighbours in $\mathcal{D}$.

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