Colouring powers of graphs with one cycle length forbidden*

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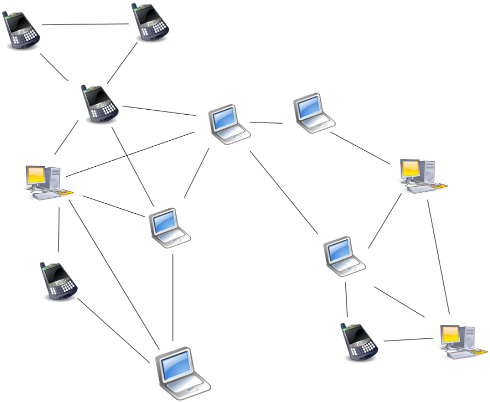
CWI Networks & Optimization seminar 1/2017

* Joint work with François Pirot.

Example 1: ad hoc frequency assignment†



† Image credit: Mesoderm/Wikipedia

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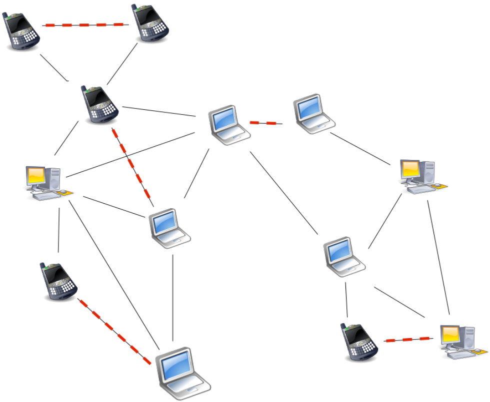

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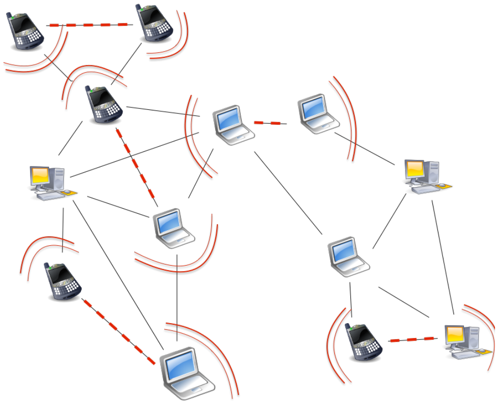

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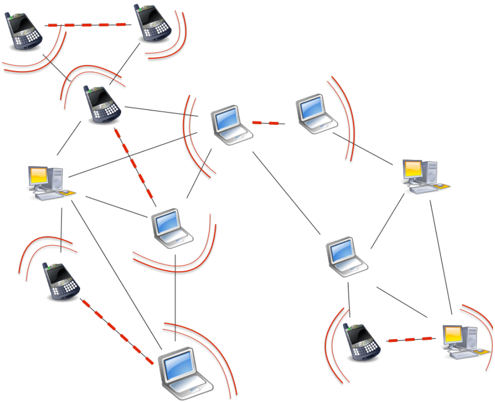
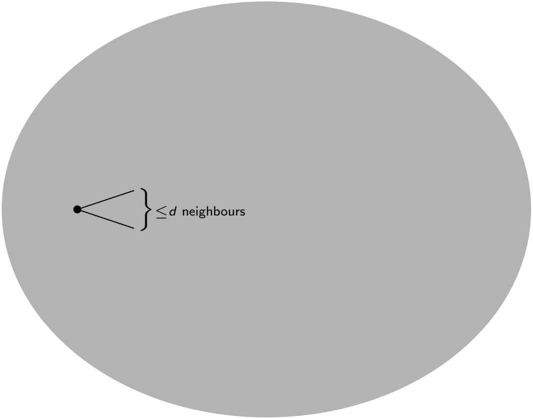


How many channels needed?

† Image credit: Mesoderm/Wikipedia

Example 2: small-world networks



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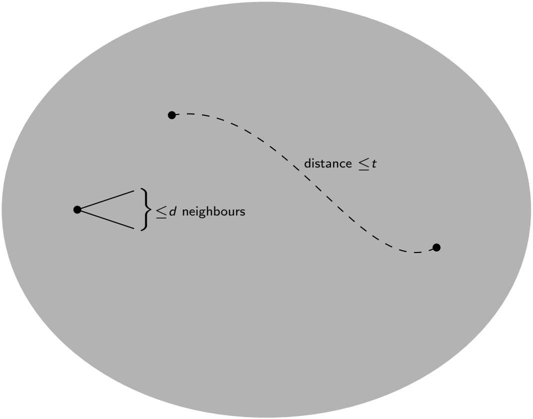

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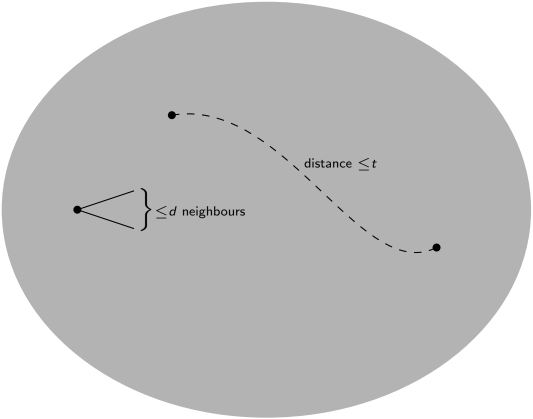
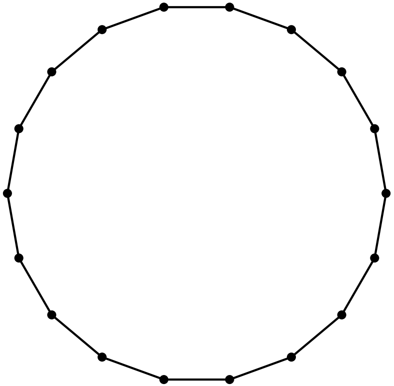


How large can the network be?

Graph powers

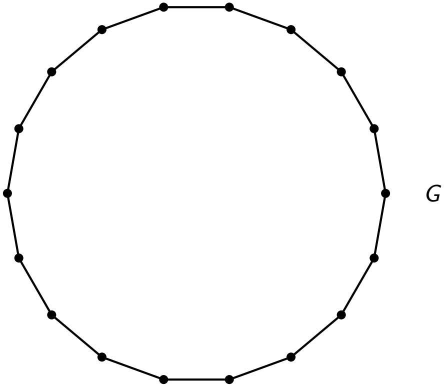
Given a graph G,



G

Graph powers

Given a graph G, the *t-th power* G^t
is formed from G by adding all edges between vertices at distance $\leq t$.



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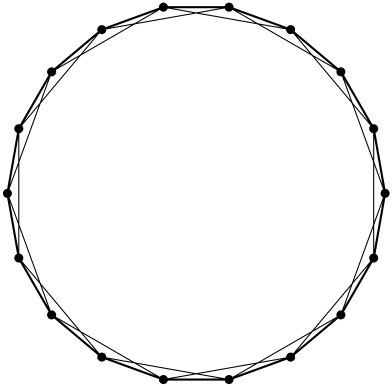

G^2

Graph powers

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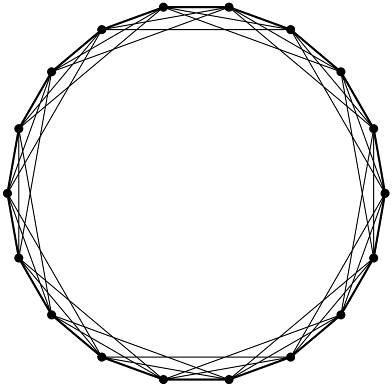

G^3

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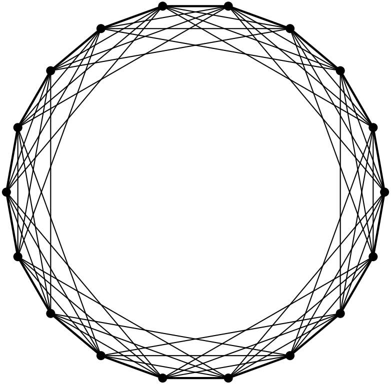

G^4

Example 2: degree–diameter problem[‡]

With maximum degree d, how large can a network of diameter t be?

(diameter $\leq t \equiv G^t$ has all possible edges)

[‡] Image credit: Bela_Mulder/Wikipedia

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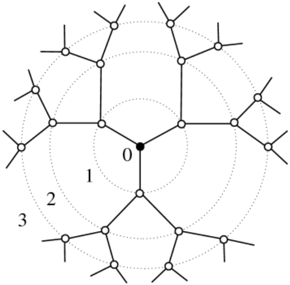

"Moore bound":

$$|G| \leq 1 + d + d(d-1) + \cdots + d(d-1)^{t-1}$$
$$= 1 + d \sum_{i=1}^{t} (d-1)^{i-1}$$

‡Image credit: Bela_Mulder/Wikipedia

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Theorem (Hoffmann & Singleton 1960)

For $t = 2$, there are three or four graphs attaining the Moore bound.
For $t = 3$, there can be only one.

Is there a graph of diameter 2, maximum degree 57, and 3250 vertices?

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Conjecture (Bollobás 1980)

Fix $t \geq 1$. Given $\varepsilon > 0$, for arbitrarily many d, there must be a graph with diameter t, maximum degree d and $\geq (1-\varepsilon)d^t$ vertices.

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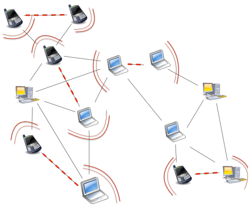
- Known only for $t \in \{1, 2, 3, 5\}$.
- For other choices of t, De Bruijn graphs give 0.5^t instead of $1 - \varepsilon$, and best known constant is 0.625^t (Canale & Gómez 2005).

Example 1: strong edge-colouring

If nearby transmissions interfere, how many channels needed?



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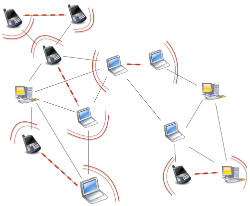

Model ad hoc network with a graph.

Each transmission occurs on an edge.

Represent each channel by a colour.

Interference at distance 2.

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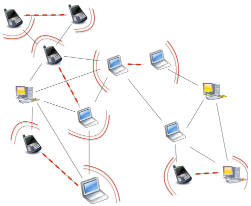

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Problem translates:

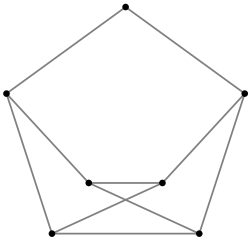
What is the least number of colours required so that edges within distance 2 *must get distinct colours?*

Called *strong chromatic index* of graph.

Example 1: strong edge-colouring

What is the least number of colours required so that edges within distance 2 *must get distinct colours?*

The *line graph* $L(G)$ of a graph G has vertices corresponding to G-edges and edges if the two corresponding G-edges have a common G-vertex.



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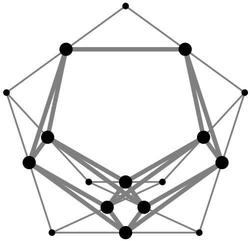

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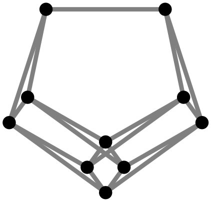

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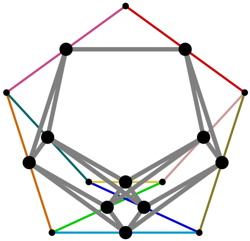

strong edge-colouring in G

strong chromatic index of G

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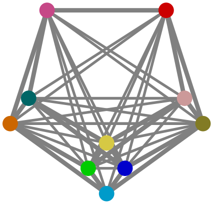

strong edge-colouring in $G \equiv$ vertex-colouring in $(L(G))^2$

strong chromatic index of $G \equiv$ chromatic number of $(L(G))^2$

Erdős–Nešetřil conjecture

With maximum degree d how large can strong chromatic index be?

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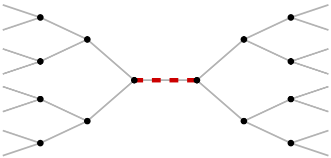
With maximum degree d how large can strong chromatic index be?

Must be $\leq 2d(d-1)+1 = 2d^2 - 2d + 1$. Greedy.



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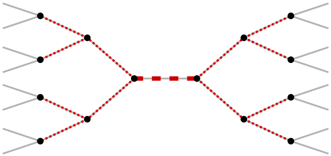

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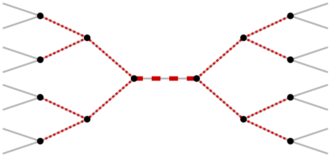


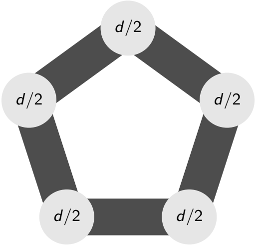
Lower bound examples?

Better upper bound?

Erdős–Nešetřil conjecture

With maximum degree d how large can strong chromatic index be?

Can be $\geq 5d^2/4$, d even:



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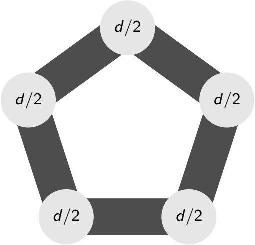

Conjecture (Erdős & Nešetřil 1980s)

Must be $\leq 5d^2/4$.

Theorem (Molloy & Reed 1997)

Must be $\leq (2-\varepsilon)d^2$ *for some absolute* $\varepsilon > 0$. ($\varepsilon \ll 0.002$.)

Colouring powers / distance colouring

Let G have maximum degree d.

Chromatic number of $(L(G))^2$? $\rightsquigarrow$ strong chromatic index $\chi'_s(G)$ of G

Colouring powers / distance colouring

Fix $t \geq 1$.
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Chromatic number of $(L(G))^t$? $\rightsquigarrow$ distance-t chromatic index $\chi'_t(G)$ of G

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Fix $t \geq 1$.
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$|G|$ if G^t is a clique? $\rightsquigarrow$ degree–diameter problem

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Colouring powers / distance colouring

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Clique number of G^t? $\rightsquigarrow$ distance-t clique number of G

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- Greedy: must be $\leq d^t + 1$.

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  Known for $t \in \{1, 2, 3, 5\}$. Can be $\geq 0.625^t \cdot d^t$.

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Chromatic number of $(L(G))^t$? $\rightsquigarrow$ distance-t chromatic index $\chi'_t(G)$ of G

- Greedy: must be $\leq 2d^t$.

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- Kaiser & K (2014): must be $\leq (2 - \varepsilon)d^t$ for some absolute $\varepsilon > 0$.

The main questions

Fix $t \geq 1$.
Let $\Delta(G)$ denote the maximum degree in a graph G.

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That is,

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The main questions

Fix $t \geq 1$. Let C_ℓ denote a cycle of length ℓ.
Let $\Delta(G)$ denote the maximum degree in a graph G.

What is the worst value among those G with $\Delta(G) = d$ and no cycle C_ℓ as a subgraph?

That is,

What is $\chi_{t,\ell}(d) := \sup\{\chi_t(G) = \chi(G^t) \mid \Delta(G) = d, G \not\supset C_\ell\}$*?*

What is $\chi'_{t,\ell}(d) := \sup\{\chi'_t(G) = \chi(L(G)^t) \mid \Delta(G) = d, G \not\supset C_\ell\}$*?*

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We're satisfied with being correct only up to a constant factor.

Chromatic number, triangle-free (χ_1, $\ell = 3$)

What is $\chi_{1,3}(d) := \sup\{\chi_1(G) = \chi(G) \mid \Delta(G) = d, G \not\supset C_3\}$*?*

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This was a question of Vizing from the 1960s (maybe motivated by Grötzsch's), eventually settled asymptotically with the semi-random method.

Theorem (Johansson 1996)

$\chi_{1,3}(d) = \Theta(d/\log d)$ *as* $d \to \infty$.

Strong chromatic index, C_4-free (χ'_2, $\ell = 4$)

What is $\chi'_{2,4}(d) := \sup\{\chi'_2(G) = \chi(L(G)^2) \mid \Delta(G) = d, G \not\supset C_4\}$*?*

(Note $\chi'_{1,\ell}(d) \in \{d, d+1\}$ always.)

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Also with the semi-random method:

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- Vu (2002) extended this to hold also for $\ell > 4$ even.
- The complete d-regular bipartite graph satisfies $\chi'_2(K_{d,d}) = d^2$, so cannot hold for any odd ℓ.

Squared chromatic number, C_6-free (χ_2, $\ell = 6$)

What is $\chi_{2,6}(d) := \sup\{\chi_2(G) = \chi(G^2) \mid \Delta(G) = d, G \not\supset C_6\}$*?*

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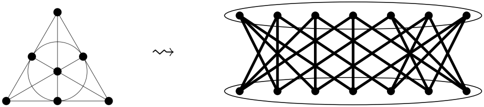
Theorem (K & Pirot 2016, cf. Alon & Mohar 2002)

$\chi_{2,6}(d) = \Theta(d^2/\log d)$ *as* $d \to \infty$*.*

The point-line incidence graph of a finite projective plane of order $d-1$ is a d-regular, girth 6 graph whose square is covered by two (d^2-d+1)-cliques.



So, if $\ell \le 5$, then $\chi_{2,\ell}(d) \sim d^2$ as $d \to \infty$.

Distance vertex-colouring with one forbidden cycle length

What is $\chi_{t,\ell}(d) := \sup\{\chi_t(G) = \chi(G^t) \mid \Delta(G) = d, G \not\supset C_\ell\}$*?*

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Theorem (K & Pirot 2017+)

Fix positive integers t and $\ell \geq 3$. The following hold as $d \to \infty$.

- *For $\ell \geq 2t + 2$ even,* $\chi_{t,\ell}(d) = \Theta(d^t / \log d)$.
- *For t odd and $\ell \geq 3t$ odd,* $\chi_{t,\ell}(d) = \Theta(d^t / \log d)$.
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- *For* t *even and* ℓ *odd,* $\chi_{t,\ell}(d) = \Theta(d^t)$*.*

Last part follows from a "circular unfolding" of the De Bruijn graph.

Distance edge-colouring with one forbidden cycle length

What is $\chi'_{t,\ell}(d) := \sup\{\chi'_t(G) = \chi(L(G)^t) \mid \Delta(G) = d, G \not\supset C_\ell\}$*?*

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(Note $\chi'_{1,\ell}(d) \in \{d, d+1\}$ always.)

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Second part uses a special bipartite graph product operation.

Distance colouring with one forbidden cycle length

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What is $\chi'_{t,\ell}(d) := \sup\{\chi'_t(G) = \chi(L(G)^t) \mid \Delta(G) = d, G \not\supset C_\ell\}$*?*

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- *For t even and ℓ odd, $\chi_{t,\ell}(d) = \Theta(d^t)$.*

If $t \geq 2$, then the following hold as $d \to \infty$.

- *For $\ell \geq 2t$ even, $\chi'_{t,\ell}(d) = \Theta(d^t/\log d)$.*
- *For ℓ odd, $\chi'_{t,\ell}(d) = \Theta(d^t)$.*

Main tool

A "sparse colouring lemma":

Theorem (Alon, Krivelevich & Sudakov 1999)

There exists $c > 0$ such that, if $\hat{G}$ is a graph of maximum degree $\hat{d}$ for which at most $\binom{\hat{d}}{2}/f$ edges span each neighbourhood, then $\chi(\hat{G}) \leq c\hat{d}/\log f$.

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Apply it with $\hat{G} = G^t$ or $L(G)^t$, $\hat{d} = 2d^t$ and $f = \Omega(d^\varepsilon)$ for some fixed $\varepsilon > 0$.

So it suffices to show $O(d^{2t-\varepsilon})$ edges span any neighbourhood in G^t or $L(G)^t$ (under the assumed cycle length restriction for G).

Vertex-colouring with one forbidden cycle length

What is $\chi_{1,\ell}(d) := \sup\{\chi_1(G) = \chi(G) \mid \Delta(G) = d, G \not\supset C_\ell\}$*?*

Proposition

Fix $\ell \geq 3$*. Then* $\chi_{1,\ell}(d) = \Theta(d/\log d)$ *as* $d \to \infty$*.*

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Fix k*. The maximum number of edges in a graph on* n *vertices with no path* P_k *of length* k *as a subgraph is at most* $(k-1)n/2$*.*

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Then $|G[N(x)]]| \leq d$, $G[N(x)] \not\supset P_{\ell-2}$, so $|E(G[N(x)])| \leq (\ell-3)d/2$ by EG.

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Apply AKS with $\hat{G} = G$, $\hat{d} = d$, $f = 2d/(\ell-3)$. $\square$

Proving sparsity without even cycles

When excluding even C_ℓ, another classic Turán-type result is useful.

Theorem (Bondy & Simonovits 1974)

Fix ℓ even. The maximum number of edges in a graph on n vertices with no cycle C_ℓ as a subgraph is $O(n^{1+2/\ell})$ as $n \to \infty$.

We in fact also prove and apply a special version of it suited to our needs.

Circular unfolding of De Bruijn graph

One of the graph constructions from K & Pirot 2016:

1. Consider $[d/2]^t$, the words of length t on alphabet $[d/2]$.
2. The vertex set is t copies $U^0, \ldots, U^{t-1}$ placed around a circle.
3. Join $u_0^i u_1^i \ldots u_{t-1}^i$ and $u_0^{i+1 \bmod t} u_1^{i+1 \bmod t} \ldots u_{t-1}^{i+1 \bmod t}$ by an edge if latter is one left cyclic shift of former, i.e. $u_j^{i+1 \bmod t} = u_{j+1}^i \; \forall j \in [t-2]$.

A full turn of the circle shifts through all t coordinates, ensuring each U^i induces a clique in t^{th} power.

Each U^i has $0.5^t \cdot d^t$ vertices.

The graph is d-regular and has no odd cycles if t is even.

Conclusion and open problems

What is $\chi_{t,\ell}(d) := \sup\{\chi_t(G) = \chi(G^t) \mid \Delta(G) = d, G \not\supset C_\ell\}$*?*

What is $\chi'_{t,\ell}(d) := \sup\{\chi'_t(G) = \chi(L(G)^t) \mid \Delta(G) = d, G \not\supset C_\ell\}$*?*

For each t, we settled these up to a constant factor except for finitely many ℓ.

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For each t, we settled these up to a constant factor except for finitely many ℓ.

Despite no manifest monotonicity, the following are natural open questions.

1. For $t \geq 1$, is there a critical ℓ_t^{e} so that $\chi_{t,\ell}(d) = \Theta(d^t)$ if $\ell < \ell_t^{\mathrm{e}}$ even, while $\chi_{t,\ell}(d) = \Theta(d^t/\log d)$ if $\ell \geq \ell_t^{\mathrm{e}}$ even?
2. For $t \geq 1$ odd, is there a critical ℓ_t^{o} so that $\chi_{t,\ell}(d) = \Theta(d^t)$ if $\ell < \ell_t^{\mathrm{o}}$ odd, while $\chi_{t,\ell}(d) = \Theta(d^t/\log d)$ if $\ell \geq \ell_t^{\mathrm{o}}$ odd?
3. For $t \geq 2$, is there a critical ℓ'_t so that $\chi'_{t,\ell}(d) = \Theta(d^t)$ if $\ell < \ell'_t$ even, while $\chi'_{t,\ell}(d) = \Theta(d^t/\log d)$ if $\ell \geq \ell'_t$ even?

We proved these hypothetical critical values are at most linear in t.

Thank you!