Largest sparse subgraphs of random graphs

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The stability of random graphs

Notation:

- $G_{n,p}$ — Erdős-Rényi random graph on n vertices, $0 < p < 1$.
- A property holds *asymptotically almost surely (a.a.s.)* if it holds with probability tending to one as $n \to \infty$.
- Denote $b = \frac{1}{1-p}$. (Note $\log b \to p$ if $p \to \infty$.)
- $\chi(G)$ denotes chromatic number of G.
- $\alpha(G)$ denotes the stability of G.

The stability of random graphs

Theorem (Bollobás and Erdős 1976, Matula 1976)

If

$$\alpha_{0,p}(n) = 2\log_b n - 2\log_b \log_b(np) + 2\log_b(\tfrac{e}{2}) + 1,$$

then

$$\lfloor \alpha_{0,p}(n) - \delta \rfloor \le \alpha(G_{n,p}) \le \lfloor \alpha_{0,p}(n) + \delta \rfloor \text{ a.a.s.}$$

The chromatic number of random graphs

Theorem (Bollobás 1988, Matula and Kučera 1990)

$$\chi(G_{n,p}) = (1 + o(1))\frac{n}{2\log_b n} \text{ a.a.s.}$$

Two-point concentration of $\chi(G_{n,p})$: Łuczak 1991, Alon and Krivelevich 1997, and Achlioptas and Naor 2004.

Extensions to more general parameters

A graph property $\mathscr{P}$ is *hereditary* if it is closed under taking induced subgraphs.
The $\mathscr{P}$-stability $\alpha_{\mathscr{P}}(G)$ of G is the order of a largest vertex subset of G that induces a subgraph which satisfies $\mathscr{P}$.

The *t-stability* $\alpha^t(G)$ of G is the order of a largest vertex subset of G that induces a subgraph of maximum degree at most t.
The *t-sparsity* $\hat{\alpha}^t(G)$ of G is the order of a largest vertex subset of G that induces a subgraph of average degree at most t.
Note $\alpha^0(G) = \hat{\alpha}^0(G) = \alpha(G)$.

Extensions to more general parameters

Theorem (Bollobás and Thomason 2000)

For fixed $0 < p < 1$ and non-trivial hereditary $\mathscr{P}$, there exists $c_{p,\mathscr{P}}$ such that a.a.s.

$$(c_{p,\mathscr{P}} - \delta)\log n \le \alpha_{\mathscr{P}}(G_{n,p}) \le (c_{p,\mathscr{P}} + \delta)\log n.$$

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Indeed, a.a.s.

$$\chi_{\mathscr{P}}(G_{n,p}) = \frac{n}{(c_{p,\mathscr{P}} + o(1))\log n}.$$

Extensions to more general parameters

Theorem (K and McDiarmid 2010)

For fixed $0 < p < 1$*, there exists* $\kappa_p(\tau)$*, continuous, strictly increasing for* $\tau \in [0, \infty)$*, with* $\kappa_p(0) = \frac{2}{\log b}$ *and* $\kappa_p(\tau) \sim \frac{\tau}{p}$ *as* $\tau \to \infty$ *such that a.a.s.*

$$(\kappa_p(\tfrac{t}{\log n}) - \delta)\log n \le \alpha_t(G_{n,p}) \le \hat{\alpha}_t(G_{n,p}) \le (\kappa_p(\tfrac{t}{\log n}) + \delta)\log n$$

if $t(n) = o(n)$*.*

Extensions to more general parameters

Theorem (K and McDiarmid 2010)

For fixed $0 < p < 1$, there exists $\kappa_p(\tau)$, continuous, strictly increasing for $\tau \in [0,\infty)$, with $\kappa_p(0) = \frac{2}{\log b}$ and $\kappa_p(\tau) \sim \frac{\tau}{p}$ as $\tau \to \infty$ such that a.a.s.

$$(\kappa_p(\tfrac{t}{\log n}) - \delta)\log n \leq \alpha_t(G_{n,p}) \leq \hat{\alpha}_t(G_{n,p}) \leq (\kappa_p(\tfrac{t}{\log n}) + \delta)\log n$$

if $t(n) = o(n)$.

An analogous statement for $\chi_t(G_{n,p})$ and $\hat{\chi}_t(G_{n,p})$.

The *t*-stability of random graphs

Theorem (Fountoulakis, K and McDiarmid 2010)

For fixed $0 < p < 1$, $\delta > 0$ *and* $t \geq 0$, *if*

$$\alpha_{t,p}(n) = 2\log_b n + (t-2)\log_b \log_b np + \log_b(\tfrac{t^t}{t!^2}) + t\log_b(\tfrac{2bp}{e}) + 2\log_b(\tfrac{e}{2}) + 1,$$

then

$$\lfloor \alpha_{t,p}(n) - \delta \rfloor \leq \alpha_t(G_{n,p}) \leq \lfloor \alpha_{t,p}(n) + \delta \rfloor \text{ a.a.s.}$$

The t-sparsity of random graphs

Theorem

For fixed $0 < p < 1$ *and* $t \geq 0$*, if* $\delta = \delta(n) = \frac{(\log\log n)^2}{\log n}$ *and*

$$\hat{\alpha}_{t,p}(n) = 2\log_b n + (t-2)\log_b\log_b np - t\log_b t + t\log_b(2bpe) + 2\log_b(\tfrac{e}{2}) + 1,$$

then

$$\lfloor \hat{\alpha}_{t,p}(n) - \delta \rfloor \leq \hat{\alpha}_t(G_{n,p}) \leq \lfloor \hat{\alpha}_{t,p}(n) + \delta \rfloor \text{ a.a.s.}$$

The difference

$$\hat{\alpha}_{t,p}(n) - \alpha_{t,p}(n) = 2\log_b \frac{t!}{(t/e)^t}$$
$$\sim \log_b(2\pi t) \text{ as } t \to \infty.$$

Concluding remarks

- Rather than analytic techniques, large deviations estimates for both first and second moment are applied to obtain tight bounds.
- These techniques extend modestly to the case where $p \to 0$ as $n \to \infty$, though new ideas may be necessary for very sparse random graphs.
- Some precise bounds for the analogous chromatic numbers have been obtained.