On a question of Alon and Mohar

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Graphs and Matroids

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*Based on joint work with François Pirot (ENSL $\rightarrow$ Nijmegen/Lorraine).

A class of colouring problems

Fix a pair of positive integers t and γ.

Let G be a graph of maximum degree d with no cycle of fewer than γ edges.

Consider the t^{th} power G^t of G, with all edges between vertices at distance $\leq t$.

How large can the chromatic number $\chi(G^t)$ of G^t be in terms of d?

Colouring ($t = 1$, $\gamma = 3$)

Let G be a graph of maximum degree d.

$$\chi(G) \leq d + 1.$$

More precisely, because of odd cycles and cliques,

$$\sup_{\Delta(G)=d} \chi(G) = d + 1.$$

More crudely,

$$\sup_{\Delta(G)=d} \chi(G) \sim d \text{ as } d \to \infty.$$

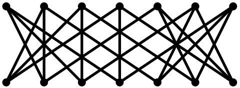
Colouring squares ($t = 2$, $\gamma = 3$)

Let G be a graph of maximum degree d.

$$\chi(G^2) \leq \Delta(G^2) + 1 \leq d + d(d-1) + 1 = d^2 + 1.$$

This is attained for $d \in \{2, 3, 7\}$ and maybe $d = 57$ (Hoffman & Singleton).

This is *nearly* attained for d one less than a prime power.
The point-line incidence graph of a finite projective plane of order $d - 1$ is a d-regular graph of order $2(d^2 - d + 1)$ whose square is covered by two cliques.



$$\implies \sup_{\Delta(G)=d} \chi(G^2) \sim d^2 \text{ as } d \to \infty.$$

Colouring powers ($t \geq 3$, $\gamma = 3$)

Let G be a graph of maximum degree d.

$$\chi(G^t) \leq \Delta(G^t) + 1 \leq d \sum_{i=1}^{t-1} (d-1)^i + 1 \leq d^t + 1.$$

For d even, the De Bruijn graph of dimension t on alphabet $[d/2]$ is a d-regular graph of order $d^t/2^t$ whose t^{th} power is a clique.

$$\implies \lim_{d\to\infty} \sup_{\Delta(G)=d} \frac{\chi(G^t)}{d^t} \in \left[\frac{1}{2^t}, 1\right] \quad \implies \sup_{\Delta(G)=d} \chi(G^t) = \Theta(d^t).$$

Remarks:

- Better bounds through degree-diameter problem (e.g. $\geq (5/8)^t$), but the value remains unknown for all $t \geq 6$.
- A conjecture of Bollobás would imply it is 1 for all t large enough.

$\gamma > 3$

If G has girth[†] at least γ, can we get better bounds on $\chi(G^t)$?

[†] girth $=$ length of a shortest cycle in the graph

Colouring and girth ($t = 1$)

If G has girth at least γ, can we get better bounds on $\chi(G^t)$?

For $t = 1$, this essentially was a question of Vizing from the 1960s, settled asymptotically with the semi-random method.

Theorem (Johansson 1996)

$$\sup_{\substack{\Delta(G)=d \\ \text{girth}(G)\geq 4}} \chi(G) = \Theta\left(\frac{d}{\log d}\right) \text{ as } d \to \infty.$$

For $t = 1$, asymptotically YES if and only if $\gamma > 3$.

Colouring squares and girth ($t = 2$)

If G has girth at least γ, can we get better bounds on $\chi(G^t)$?

For $t = 2$, this was settled asymptotically by Alon and Mohar.

Theorem (Alon & Mohar 2002)

$$\sup_{\substack{\Delta(G)=d \\ \mathrm{girth}(G)\geq 7}} \chi(G^2) = \Theta\left(\frac{d^2}{\log d}\right) \text{ as } d \to \infty.$$

The point-line incidence graphs of projective planes are of girth 6, so

$$\sup_{\substack{\Delta(G)=d \\ \mathrm{girth}(G)\geq 6}} \chi(G^2) \sim d^2 \text{ as } d \to \infty.$$

For $t = 2$, asymptotically YES if and only if $\gamma > 6$.

Colouring powers and girth ($t \geq 3$)

If G has girth at least γ, can we get better bounds on $\chi(G^t)$?

Problem (Alon & Mohar 2002)

For $t \geq 3$, is there some γ_t such that as $d \to \infty$

$$\sup_{\substack{\Delta(G)=d \\ \text{girth}(G)>\gamma_t}} \chi(G^t) = \Theta\left(\frac{d^t}{\log d}\right) \quad \textit{and} \quad \sup_{\substack{\Delta(G)=d \\ \text{girth}(G)\leq\gamma_t}} \chi(G^t) = \Theta(d^t)?$$

They noted $3 \leq \gamma_t \leq 3t$ (if it exists) and speculated it to be the upper bound (since that holds for $t = 1$ and $t = 2$).

For $t \geq 3$, asymptotically YES if and only if $\gamma > 3t$?

Our results

Problem (Alon & Mohar 2002)

For $t \geq 3$, is there some γ_t such that as $d \to \infty$

$$\sup_{\substack{\Delta(G)=d \\ \text{girth}(G)>\gamma_t}} \chi(G^t) = \Theta\left(\frac{d^t}{\log d}\right) \quad \textit{and} \quad \sup_{\substack{\Delta(G)=d \\ \text{girth}(G)\leq\gamma_t}} \chi(G^t) = \Theta(d^t)?$$

Ours is the first investigation of this problem. Summarising, if γ_t exists, then

- $\gamma_t \leq 2t + 2$,
- $\gamma_t \geq 4$ for $t = 3$,
- $\gamma_t \geq 6$ for $t \geq 4$, and
- $\gamma_t \geq 8$ for $t = 9$ and $t \geq 11$.

Upper bounds

If γ_t exists, then $\gamma_t \leq 2t+2$:

Theorem (K & Pirot)

As $d \to \infty$,

$$\sup_{\substack{\Delta(G)=d \\ G \not\supseteq C_6}} \chi(G^2) = \Theta\left(\frac{d^2}{\log d}\right) \quad \textit{and} \quad \sup_{\substack{\Delta(G)=d \\ G \not\supseteq C_8}} \chi(G^3) = \Theta\left(\frac{d^3}{\log d}\right)$$

and, if $t \geq 3$,

$$\sup_{\substack{\Delta(G)=d \\ G \not\supseteq C_\ell \ \textit{for}\ \ell \in \{8,10,\ldots,2t+2\}}} \chi(G^t) = \Theta\left(\frac{d^t}{\log d}\right).$$

Upper bounds

A "sparsity lemma" (that Alon & Mohar also used):

Theorem (Alon, Krivelevich & Sudakov 1999)

There exists $c > 0$ such that, if $\hat{G}$ is a graph of maximum degree $\hat{d}$ for which at most $\binom{\hat{d}}{2}/f$ edges span each neighbourhood, then $\chi(\hat{G}) \le c\hat{d}/\log f$.

(Remark: this relies on the aforementioned result of Johansson.)

We apply it with $\hat{G} = G^t$, $\hat{d} = d^t$ and $f = \Omega(d)$, so it suffices to show there are $O(d^{2t-1})$ edges spanning any neighbourhood in G^t.

Upper bounds

We apply it with $\hat{G} = G^t$, $\hat{d} = d^t$ and $f = \Omega(d)$, so it suffices to show there are $O(d^{2t-1})$ edges spanning any neighbourhood in G^t.

Fix $u \in G$. Let A_i be the vertices at distance exactly i from u.
It is enough to count the t-paths both endpoints of which are in A_t.

We can show and use the lack of certain paths at specific distance from u.
For instance, the induced subgraph $G[A_t \cup A_{t+1}]$ contains no 6-path every even vertex of which is in A_t. (This intuits the forbidden even cycles type condition.)

The forbidden 6-path claims guarantee certain *bottlenecks* in most t-paths:

- a vertex $v \in A_t$ where v has at least four neighbours in A_{t-1},
- an edge vw, $v \in A_t$, $w \in A_{t+1}$ where v has at least four neighbours in A_t,
- an edge vw in $G[A_t]$.

Lower bound examples

Theorem (K & Pirot)

As $d \to \infty$, there are constructions certifying

$$\sup_{\substack{\Delta(G)=d \\ \text{girth}(G)\le g}} \chi(G^t) \gtrsim \frac{d^t}{2^t}$$

- *if $t = 3$ and $\gamma = 4$,*
- *if $t \in \{4, 5, 6, 7, 8, 10\}$ and $\gamma = 6$, and*
- *if $t = 9$ or $t \geq 11$ and $\gamma = 8$*

(plus miscellaneous factor 2, 3 or 5 improvements for all $t \geq 3$).

Lower bound examples

One of our constructions relies on the "circular unfolding" of a De Bruijn graph.

1. Consider $[d/2]^t$, the words of length t on alphabet $[d/2]$.
2. The vertex set is t copies $U^0, \dots, U^{t-1}$ placed around a circle.
3. Join $u_0^i u_1^i \dots u_{t-1}^i$ and $u_0^{i+1 \bmod t} u_1^{i+1 \bmod t} \dots u_{t-1}^{i+1 \bmod t}$ by an edge if latter is one left cyclic shift of former, i.e. $u_j^{i+1 \bmod t} = u_{j+1}^i \ \forall j \in [t-2]$.

A full turn of the circle shifts through all t coordinates,
ensuring each U^i induces a clique in the t^{th} power.

Lower bound examples

Problem: there are many, many C_4s between U^j and $U^{j+1 \bmod t}$.

Solution: substitute segments of the circle using high girth "conduits".

A balanced bipartite graph with parts A and B is called a *good conduit* with parameters (τ, D, Γ, c) if

- there is a path of $\leq \tau$ edges between any $a \in A$ and any $b \in B$,
- it is D-regular and has girth Γ,
- $|A| = |B| \geq cD^\tau$ (so it is of the optimal order).

Our constructions are limited by the existence of such structures:

- $(1, D, 4, 1)$: complete bipartite graphs $K_{D,D}$;
- $(3, q+1, 8, 1)$: point-line incidence of symplectic (q, q)-quadrangle;
- $(5, q+1, 12, 1)$: point-line incidence of split Cayley (q, q)-hexagons;
- no generalised (q, q)-octagon, no other even n-gons exist (Feit & Higman).

Further investigations

Just the $\gamma = 3$ case is interesting (hard) if we ask for the asymptotic constants.

We took well-known good conduits from generalised polygons, but are there slightly less efficient ones, i.e. with constant $c < 1$ or girth $\Gamma < 2\tau + 2$?

A related Ramsey-type question: what is the least stability number $\alpha(G^t)$ of G^t taken over all G with $\Delta(G) \leq d$ and girth$(G) \geq \gamma$?

The Alon–Mohar problem remains very open. If it exists, is γ_t unbounded in t?

Thank you!