

Guarantees of sparse or dense subgraphs

Ross J. Kang*

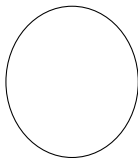
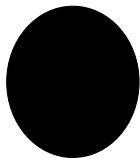


Radboud University Nijmegen

Randomness and Graphs : Processes and Structures
TU Eindhoven 9/2017

*The talk covers joint works with Eoin Long, Janos Pach, Viresh Patel and Guus Regts.

α and ω



A clique has all possible edges and a stable set has none.

The *clique number* ω is the size of a largest clique.

The *stability number* α is the size of a largest stable set.

α_c and ω_c

Consider sets “close” to cliques or stable sets, tuned by a parameter[†] $c \in [0, 1]$.

[†]Note (or foreshadowing): for now consider c as fixed.

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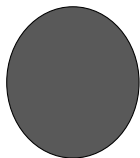
A vertex subset with ℓ vertices

of minimum degree $\geq c(\ell - 1)$

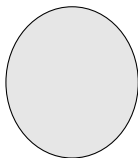
of maximum degree $\leq (1 - c)(\ell - 1)$

is called a c -clique;

is called a c -stable set.



ω_c is size of a largest c -clique.



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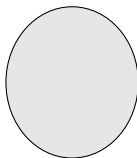
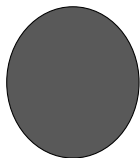
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How does the behaviour change as we tune c between 0 and 1?

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Ramsey numbers[‡]



Ramsey (1930) proved the existence of

$$R(k) = \min \{n : |V(G)| = n \Rightarrow \max\{\alpha(G), \omega(G)\} = k\}.$$

[‡]Picture borrowed from the cover of Soifer (2009).

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Theorem (Erdős 1947, Erdős and Szekeres 1935)

$$\sqrt{2}^{k+o(k)} \leq R(k) \leq 4^{k-o(k)} \text{ as } k \rightarrow \infty.$$

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[‡]Pictures borrowed from the cover of Soifer (2009) and homepages.

Quasi-Ramsey numbers



Ramsey (1930) still implies, for any $c \in [0, 1]$, the existence of

$$R_c^*(k) = \min \{n : |V(G)| = n \Rightarrow \max\{\alpha_c(G), \omega_c(G)\} = k\} \text{ and}$$

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Note that $R_c^*(k) \geq R_c(k)$ always, and both parameters are monotone in c .

Moreover, $R_0(k) = R_0^*(k) = k$ and $R_1(k) = R_1^*(k) = R(k) = \exp(\Theta(k))$.

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Proposition (Erdős and Pach 1983)

Fix $c \in [0, 1]$.

- If $c < 1/2$, then $R_c^*(k) = \Theta(k)$.
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Consider $c = 1/2 + \varepsilon$ where $\varepsilon = \varepsilon(\ell)$ is a real function tending to 0 as $\ell \rightarrow \infty$.

Variable sharp threshold

The “variable” quasi-Ramsey numbers

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- If $\nu = \Theta(1)$ as $\ell \rightarrow \infty$, then $R_c(k) = k^{\Theta(1)}$ as $k \rightarrow \infty$.
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Theorem (Kang, Long, Patel and Regts 2016+)

For some $C > 0$, in any graph on $Ck \log k$ vertices there is a set of k vertices inducing minimum degree $k/2 + \Omega(\sqrt{k/\log k})$ in the graph or complement.

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Call a subset of ℓ vertices *excessive* if it induces minimum degree $\geq \frac{1}{2}(\ell - 1) + \zeta$ for some excess $\zeta \geq 0$.

Beginning with $Ck \log k$ vertices, here are the rough proof ingredients.

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3. Partition of an excessive set into two parts of prescribed size at least one of which is excessive.

Proof ingredient 1: graph discrepancy

Lemma (1)

For fixed $\nu \geq 0$, let $c = 1/2 + \nu/\sqrt{\ell - 1}$. Then $R_c(k) = O(k \log k)$.



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Theorem (Erdős and Spencer 1972)

For n large any graph $G = (V, E)$ with $|V| = n$ has

$$\max_{S \subseteq V} \left| e(S) - \frac{1}{2} \binom{|S|}{2} \right| = \Omega(n^{3/2}).$$

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Theorem (Erdős and Spencer 1974)

For n large and $\frac{1}{2} \log_2 n < t \leq n$ any graph $G = (V, E)$ with $|V| = n$ has

$$\max_{S \subseteq V, |S| \leq t} \left| e(S) - \frac{1}{2} \binom{|S|}{2} \right| = \Omega \left(t^{3/2} \sqrt{\log(n/t)} \right).$$

Proof ingredient 2: set system discrepancy

Lemma (2)

*Suppose X is of size $\ell = Dk$, $D > 1$, inducing minimum degree $\geq \ell/2 + \nu\sqrt{\ell}$.
Some $X' \subseteq X$ of size k induces minimum degree $\geq k/2 + (\nu/\sqrt{D} - 15\sqrt{D})\sqrt{k}$.*

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Theorem (Spencer 1985, Lovász, Spencer and Vesztergombi 1986)

*For $A_1, \dots, A_n \subseteq [n]$ and $p \in [0, 1]$, there exists $Y \subseteq [n]$ such that for all i
 $||A_i \cap Y| - p|A_i|| \leq 6\sqrt{n}$.*

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Apply Theorem to A_1, \dots, A_ℓ with $p = (k + 1 + 6\sqrt{\ell})/\ell$ (done if $p > 1$)
to produce $Y \subseteq [\ell]$ such that $||A_i \cap Y| - p|A_i|| \leq 6\sqrt{\ell}$.

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$i \in [\ell - 1] \implies |N(i) \cap Y| \geq p(\ell/2 + \nu\sqrt{\ell}) - 6\sqrt{\ell} \geq k/2 + \nu k/\sqrt{\ell} + 1 - 3\sqrt{\ell}$.

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to produce $Y \subseteq [\ell]$ such that $||A_i \cap Y| - p|A_i|| \leq 6\sqrt{\ell}$.

$$i \in [\ell - 1] \implies |N(i) \cap Y| \geq p(\ell/2 + \nu\sqrt{\ell}) - 6\sqrt{\ell} \geq k/2 + \nu k/\sqrt{\ell} + 1 - 3\sqrt{\ell}.$$

$$i = \ell \implies k + 1 = p\ell - 6\sqrt{\ell} \leq |Y| \leq p\ell + 6\sqrt{\ell} = k + 1 + 12\sqrt{\ell}.$$

Proof ingredient 2: set system discrepancy

Lemma (2)

Suppose X is of size $\ell = Dk$, $D > 1$, inducing minimum degree $\geq \ell/2 + \nu\sqrt{\ell}$.
Some $X' \subseteq X$ of size k induces minimum degree $\geq k/2 + (\nu/\sqrt{D} - 15\sqrt{D})\sqrt{k}$.

Theorem (Spencer 1985, Lovász, Spencer and Vesztergombi 1986)

For $A_1, \dots, A_n \subseteq [n]$ and $p \in [0, 1]$, there exists $Y \subseteq [n]$ such that for all i
 $||A_i \cap Y| - p|A_i|| \leq 6\sqrt{n}$.

"Six standard deviations suffice."

Proof of Lemma (2).

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Take $X' \subseteq [\ell - 1]$ arbitrary with $|X'| = k$. By the above, for all $i \in X'$

$$|N(i) \cap X'| \geq k/2 + \nu k/\sqrt{\ell} - 15\sqrt{\ell} = k/2 + (\nu/\sqrt{D} - 15\sqrt{D})\sqrt{k}. \quad \square$$

Proof ingredient 3: greedy swaps

Lemma (3)

Suppose X is of size $\ell = \ell_1 + \ell_2$ inducing minimum degree $\geq \delta = \delta_1 + \delta_2$.

Then there exists $X_1, X_2 \subseteq X$ with $|X_1| = \ell_1$ and $|X_2| = \ell_2$ such that either X_1 induces minimum degree $\geq \delta_1$ or X_2 induces minimum degree $\geq \delta_2$

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Proof.

Start with X_1, X_2 an arbitrary partition of X with $|X_1| = \ell_1$ and $|X_2| = \ell_2$.

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The number of edges in X_1 increases by at least

$$\deg_{X_1}(b) - \deg_{X_1}(a) - 1 \geq \delta - \deg_{X_2}(b) - \deg_{X_1}(a) - 1 \geq \delta - \delta_2 - \delta_1 + 1 = 1$$

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At some point we cannot find two vertices to swap, but then we are done. \square

Proof ingredients

Theorem (Kang, Long, Patel and Regts 2016+)

For some $C > 0$, in any graph on $Ck \log k$ vertices there is a set of k vertices inducing minimum degree $k/2 + \Omega(\sqrt{k/\log k})$ in the graph or complement.

Call a subset of ℓ vertices *excessive* if it induces minimum degree $\geq \frac{1}{2}(\ell - 1) + \zeta$ for some excess $\zeta \geq 0$.

Beginning with $Ck \log k$ vertices, here are the rough proof ingredients.

1. A “variable” quasi-Ramsey bound to produce a set of $\ell \geq 2k$ vertices that is $\Omega(\sqrt{\ell})$ excessive either in the graph or complement.
2. Reduction from a $\Omega(\sqrt{\ell})$ excessive set of Dk vertices, $D > 1$ fixed, to an excessive set of exactly k vertices.
3. Partition of an excessive set into two parts of prescribed size at least one of which is excessive.

First apply 1.

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First apply 1. If $\ell \not\equiv 0 \pmod{k}$, then apply 3 to lop off a possibly excessive piece of size Dk , with $Dk \equiv \ell \pmod{k}$ and $D > 1$ fixed, then possibly apply 2.

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First apply 1. If $\ell \not\equiv 0 \pmod{k}$, then apply 3 to lop off a possibly excessive piece of size Dk , with $Dk \equiv \ell \pmod{k}$ and $D > 1$ fixed, then possibly apply 2. Otherwise apply 3 repeatedly to partition an excessive set of size $\equiv 0 \pmod{k}$ into roughly equal parts of size $\equiv 0 \pmod{k}$, one excessive.

Summary and open questions

For the quasi-Ramsey numbers

$$R_c^*(k) = \min \{n : |V(G)| = n \Rightarrow \max\{\alpha_c(G), \omega_c(G)\} = k\} \text{ and}$$

$$R_c(k) = \min \{n : |V(G)| = n \Rightarrow \max\{\alpha_c(G), \omega_c(G)\} \geq k\},$$

- identified a sharp transition for $R_c(k)$ at $c = 1/2 + \Theta(\sqrt{\log \ell / \ell})$, and
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- Hypergraphs? (See next page.)

Thank you!

- [KPR16+] Ross J. Kang, Viresh Patel, and Guus Regts.
Discrepancy and large dense monochromatic subsets.
Submitted, 14 pp. arXiv:1610.06359
- [KLPR16+] Ross J. Kang, Eoin Long, Viresh Patel, and Guus Regts.
On a Ramsey-type problem of Erdős and Pach.
To appear in *Bulletin of the London Mathematical Society*, 9 pp.
- [KPPR15] Ross J. Kang, János Pach, Viresh Patel, and Guus Regts.
A precise threshold for quasi-Ramsey numbers.
SIAM Journal on Discrete Mathematics, 29(3): 1670-1682, 2015.
- [ErPa83] Paul Erdős and János Pach.
On a quasi-Ramsey problem.
Journal of Graph Theory 7(1): 137-147, 1983.

Heterogeneously weighted random graph

Theorem (Erdős and Pach 1983, cf. Kang, Pach, Patel and Regts 2015)

$$R_{1/2}(k) = \Omega(k \log k / \log \log k).$$

i.e. there is some graph on $Ck \log k / \log \log k$ vertices such that any set of $\ell \geq k$ vertices is excessive in neither the graph nor complement.

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Let $z = \frac{\zeta \log k}{\log \log k}$ for some suitably chosen fixed $\zeta > 0$.

Let $V = V_1 \cup \dots \cup V_z$ where $|V_1| = \dots = |V_z| = \left(1 - \frac{1}{2z}\right) k$.

Generate E randomly for any $v_i \in V_i$ and $v_j \in V_j$ by

$$\mathbb{P}(v_i v_j \in E) = \begin{cases} \frac{1}{2} - (2z)^{-4(i+j)-1} & \text{if } i \neq j; \\ \frac{1}{2} + (2z)^{-8i} & \text{if } i = j. \end{cases}$$

There is a chance the graph $G = (V, E)$ has the desired properties.

Proof ingredient 2: set system discrepancy

Theorem (Spencer 1985, Lovász, Spencer and Vesztergombi 1986)

For $A_1, \dots, A_n \subseteq [n]$ and $p \in [0, 1]$, there exists $Y \subseteq [n]$ such that for all i

$$||A_i \cap Y| - p|A_i|| \leq 6\sqrt{n}.$$

For $\mathcal{H} = \{A_1, \dots, A_n\} \subseteq 2^{[n]}$, define the *discrepancy* of \mathcal{H} as

$$\text{disc}(\mathcal{H}) := \min_{\chi \in \{-1, 1\}^V} \max_{S \in \mathcal{H}} \sum_{i \in S} \chi(i).$$

Spencer showed that $\text{disc}(\mathcal{H}) \leq 6\sqrt{n}$ for any such \mathcal{H} .

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If A is the incidence matrix of \mathcal{H} , i.e. A is the $n \times n$ matrix given by

$$A_{ij} = \begin{cases} 1 & \text{if } j \in A_i, \\ 0 & \text{otherwise.} \end{cases},$$

then the *linear discrepancy* is

$$\text{lindisc}(\mathcal{H}) := \max_{c \in [0, 1]^V} \min_{x \in \{0, 1\}^V} \|A(x - c)\|_\infty.$$

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Apply this result with c the all p vector.

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Lemma (1)

For fixed $\nu \geq 0$, let $c = 1/2 + \nu/\sqrt{\ell - 1}$. Then $R_c(k) = O(k \log k)$.

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Outline proof of Lemma (1).

Let $G = (V, E)$ be a graph on $Ck \log k$ vertices and define for any $X \subseteq V$

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Then $(\frac{5}{6})^{(m-1)/3} D(X_{i_1}) \geq D(X_{i_m}) \geq 1 \Rightarrow m - 1 \leq 3 \log_{6/5} D(X_{i_1}) \leq 6 \log_{6/5} k$.

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Pigeonhole guarantees some $|X_{i_j}| \geq \frac{|V| \log(6/5)}{25 \log k} = \frac{C \log(6/5)}{25} k \geq k$ if $C \geq \frac{25}{\log(6/5)}$.

Proof ingredient 1: graph discrepancy

Lemma (1)

For fixed $\nu \geq 0$, let $c = 1/2 + \nu/\sqrt{\ell - 1}$. Then $R_c(k) = O(k \log k)$.

Outline proof of Lemma (1).

Let $G = (V, E)$ be a graph on $Ck \log k$ vertices and define for any $X \subseteq V$

$$D_\nu(X) := |D(X)| - \nu|X|^{3/2}.$$

Let $V_0 = V$. Form V_{i+1} in step $i + 1$ by letting $X_i \subseteq V_i$ maximise $D_\nu(X_i)$ and $V_{i+1} = V_i \setminus X_i$. Stop after step $t + 1$ if $|V_{t+1}| < \frac{1}{2}|V|$.

Let $\{i_1, \dots, i_m\} \subseteq [t]$ be those i with $D(X_i) > 0$. $\forall \log \sum_{j \in [m]} |X_{i_j}| \geq \frac{1}{4}|V|$.

Claim 1 For any $j \in [m]$, H_{i_j} has minimum degree $\geq \frac{1}{2}(|X_{i_j}| - 1) + \nu\sqrt{|X_{i_j}| - 1}$.

Claim 2 For any $\ell \in [m - 3]$, $D(X_{i_{\ell+3}}) \leq \frac{5}{6}D(X_{i_\ell})$.

Then $(\frac{5}{6})^{(m-1)/3} D(X_{i_1}) \geq D(X_{i_m}) \geq 1 \Rightarrow m - 1 \leq 3 \log_{6/5} D(X_{i_1}) \leq 6 \log_{6/5} k$.

Pigeonhole guarantees some $|X_{i_j}| \geq \frac{|V| \log(6/5)}{25 \log k} = \frac{C \log(6/5)}{25} k \geq k$ if $C \geq \frac{25}{\log(6/5)}$.

Claim 1 implies that X_{i_j} is the desired subset. \square

Claim 1 For any $j \in [m]$, H_j has minimum degree $\geq \frac{1}{2}(|X_j| - 1) + \nu\sqrt{|X_j| - 1}$.

Proof of Claim 1.

If not there exists $x \in X_j$ with $\deg_{H_j}(x) < \frac{1}{2}(|X_j| - 1) + \nu\sqrt{|X_j| - 1}$.

Let $X'_j = X_j \setminus \{x\}$. Since $D(X_j) > 0$,

$$\begin{aligned} D_\nu(X'_j) &= e(X'_j) - \frac{1}{2} \binom{|X'_j| - 1}{2} - \nu(|X'_j| - 1)^{3/2} \\ &> e(X_j) - \frac{1}{2} \binom{|X_j|}{2} - \nu\sqrt{|X_j| - 1} - \nu(|X_j| - 1)^{3/2} \\ &> e(X_j) - \frac{1}{2} \binom{|X_j|}{2} - \nu|X_j|^{3/2} = D_\nu(X_j) \end{aligned}$$

(since $n^{3/2} > \sqrt{n-1} + (n-1)^{3/2}$), contradicting maximality of $D_\nu(X_j)$. \square