Bounded palette list colouring

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*Joint work with Marthe Bonamy (Montpellier). Support from NWO.

List colouring

A classic notion in graph theory from 1970's.

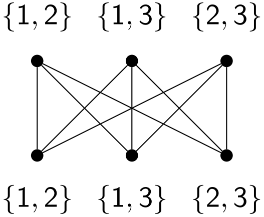
Aim is to properly colour vertices from individual lists.

Lists chosen by adversary subject to uniform minimum list size.

The optimal minimum so we can always colour is the *choosability*, or *list chromatic number* or *choice number*, denoted $\mathrm{ch}(\cdot)$.

List colouring examples

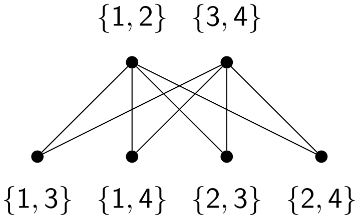
The standard example, $\mathrm{ch}(K_{3,3}) > 2 (= \chi(K_{3,3}))$:



More generally, $\mathrm{ch}(K_{n,n}) \sim \log_2 n$ as $n \to \infty$.

List colouring examples

Another standard example, $\mathrm{ch}(K_{2,4}) > 2 (= \chi(K_{2,4}))$:



More generally, $\mathrm{ch}(K_{n,n^n}) \geq n+1$ for all n.

Bounded palette

What if adversary's ground set of colours pre-determined?

We call this the *palette* and denote it $[s] = \{1, \ldots, s\}$.

How much easier is it be to list colour?

Bounded palette list colouring

$G = (V, E)$ is a simple undirected graph.

$[s] = \{1, \ldots, s\}$ is the *palette*.

Any $L : V \to \binom{[s]}{k}$ is a (k, s)-*list-assignment* of G.

$\forall L$, $c : V \to [s]$ is an *L-colouring* if $c(v) \in L(v)\ \forall v \in V$.

G is (k, s)-*choosable* if for any such L there is a proper L-colouring.

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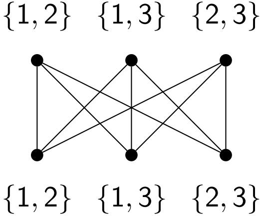
G is (k, s)-*choosable* if for any such L there is a proper L-colouring.

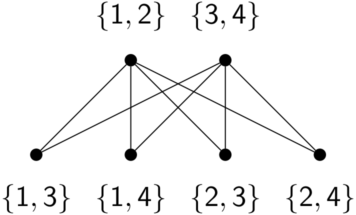
- G is k-choosable iff it is (k, s)-choosable for every $s \geq k$. $\mathrm{ch}(G)$ is the least such k.
- G is k-colourable iff it is (k, k)-choosable.

Bounded palette list colouring

$K_{3,3}$ is not $(2,3)$-choosable.



$K_{2,4}$ is not $(2,4)$-choosable.



TCS 'overflow' question

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How many distinct colors are needed to lower-bound the choosability of a graph?

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12

A graph is k-choosable (also known as k-list-colorable) if, for every function f that maps vertices to sets of k colors, there is a color assignment c such that, for all vertices v, $c(v) \in f(v)$, and such that, for all edges vw, $c(v) \neq c(w)$.

Now suppose that a graph G is not k-choosable. That is, there exists a function f from vertices to k-tuples of colors that does not have a valid color assignment c. What I want to know is, how few colors in total are needed? How small can $\bigcup_{v \in G} f(v)$ be? Is there a number $N(k)$ (independent of G) such that we can be guaranteed to find an uncolorable f that only uses $N(k)$ distinct colors?

The relevance to CS is that, if $N(k)$ exists, we can test k-choosability for constant k in singly-exponential time (just try all $\binom{N(k)}{k}^n$ choices of f, and for each one check that it can be colored in time $k^n n^{O(1)}$) whereas otherwise something more quickly growing like n^{kn} might be required.

graph-theory co.combinatorics graph-colouring exp-time-algorithms

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edited Dec 19 '10 at 20:18 Peter Shor 13.7k 1 49 84

asked Nov 2 '10 at 18:58 David Eppstein 26.1k 2 79 157

7 Excellent question! – RJK Nov 2 '10 at 19:18

1 Is there an example when N(k)>2k-1? – Yaroslav Bulatov Nov 4 '10 at 22:35

1 My first thought is to try to lower bound the number of colors required in the standard example that bipartite graphs can have arbitrarily-high list-chromatic number. However, the number of colors in the list in this construction is exponential to the achieved k. I didn't take enough time to prove the lower bound, however (so this isn't an answer...yet). – Derrick Stolee Dec 19 '10 at 18:08

1 It might be worth posting this excellent question on MathOverflow too... – François G. Dorais Feb

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TCS‘overflow’ question

Question (Eppstein, November 2010, TCS‘overflow’)

For any k, is there some $s_k \geq k$ such that,
if G is (k, s_k)-choosable, then it is k-choosable?

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For any k, is there some $s_k \geq k$ such that,
if G is (k, s_k)-choosable, then it is k-choosable?

In words, is there $s_k \geq k$ (independent of the graph) such that k-choosability is guaranteed by just checking palettes of size s_k?

Motivation: a positive answer guarantees a singly-exponential algorithm for k-CHOOSABILITY, a Π_2^P-complete problem.

TCS'overflow' answer

cstheory.stackexchange.com/questions/2661/how-many-distinct-colors-are-needed-to-lower-bound-the-choosability-of-a-graph — co.combinatorics – How many di... Reader

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1 Answer

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Daniel Král and Jiří Sgall answered your question to the negative. From the abstract of their paper:

> A graph G is said to be (k,ℓ)-choosable if its vertices can be colored from any lists $L(v)$ with $|L(v)| \geq k$, for all $v \in V(G)$, and with $|\bigcup_{v\in V(G)} L(v)| \leq \ell$. For each $3 \leq k \leq \ell$, we construct a graph G that is (k,ℓ)-choosable but not $(k,\ell+1)$-choosable.

So, $N(k)$ does not exist if $k \geq 3$. Král and Sgall also show that $N(2) = 4$. Of course, $N(1) = 1$.

Daniel Král, Jiří Sgall: Coloring graphs from lists with bounded size of their union. Journal of Graph Theory 49(3): 177-186 (2005)

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answered Feb 11 '11 at 12:45
Serge Gaspers
3,421 ● 16 ● 31

Wow. This settles the question, although negatively. Thank you @Serge! And I wish I could give thanks to Daniel and Jiří too! – Hsien-Chih Chang 張顯之 Feb 11 '11 at 13:27

I would also have preferred a positive answer to the question. – Serge Gaspers Feb 11 '11 at 14:34

Your Answer

B I

18 Coloring Planar Graphs

34 Conjectures implying Four Color Theorem

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23 Coloring complexity of graphs

18 Reasons for which a graph may be not k colorable?

3 How can we derive this lower bound of a special cut in a graph?

11 Small graph with gap between chromatic and vector chromatic number?

An earlier counterexample

It turns out question was (asked and) answered five years earlier:

Theorem (Král' & Sgall, 2005)

For all $s \geq k \geq 3$, *there exists* $G_{k,s}$ *that is* (k, s)*-choosable but not* $(k, s+1)$*-choosable.*

$\implies$ No, mostly.

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$\implies$ No, mostly.

- On the other hand, $(2, 4)$-choosability implies 2-choosability.
- $|V(G_{k,s})| = O(s^2)$; uses precolouring (non)extension.

Follow-up questions

(k, s)-choosability doesn't guarantee k-choosability in general.

Question (1)

Does it imply C-choosability for some large $C = C(k, s)$?

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Question (1)

Does it imply C-choosability for some large $C = C(k, s)$?

Question (2)

Does it imply $(k+1)$-choosability if $s = s_{k+1}(k)$ is large?

The first follow-up

Does (k,s)-choosability imply C-choosability for some C?

No, if $s < 2k-1$.

All bipartite graphs $(k, 2k-2)$-choosable. (Halve palette.)

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Yes, if $s \geq 2k - 1$.

Theorem (Král' & Sgall, 2005, cf. K., 2013)

For $k \geq 2, s \geq 2k - 1$, there exists $C = C(k, s)$ such that, if G is (k, s)-choosable, then it is C-choosable.

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For $k \geq 2, s \geq 2k - 1$, there exists $C = C(k, s)$ such that, if G is (k, s)-choosable, then it is C-choosable.

- The choice of C satisfies $4^{(1+o(1))k} \leq C \leq 16^{(1+o(1))k}$.

How is this C so large?

For $k \geq 2, s \geq 2k-1$, there exists $C = C(k,s)$ such that, if G is (k,s)-choosable, then it is C-choosable.

- *The choice of C satisfies $4^{(1+o(1))k} \leq C \leq 16^{(1+o(1))k}$.*

Probabilistic proof built upon connection between choosability and degeneracy established by Alon (1993/2000). It relies on the choice

$$C = 12M^2 \ln M \ln k,$$

where $M = M(k,s)$ is an extremal parameter for 'Property B'.

Property B

Bernstein, 1908: A family $\mathcal{F}$ of sets has *Property B* if $\exists$ set B that meets every set of $\mathcal{F}$ but contains no set of $\mathcal{F}$.

(Or $\exists$ bipartition of $\bigcup \mathcal{F}$ where no set is contained in one part. Property B is equivalent to weak 2-colourability of hypergraphs.)

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$M(k, s)$ is size of smallest $\mathcal{F} \subseteq \binom{[s]}{k}$ **without** Property B.

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- e.g. $\mathcal{F} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ does not have Property B, but any proper subfamily of $\binom{[3]}{2}$ does. So $M(2, 3) = 3$.
- For $k \geq 2$, $M(k, 2k-1) = \binom{2k-1}{k}$, while $M(k, 2k-2) = \infty$.

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$\mathrm{ch}(K_{n,n}) \sim \log_2 n$ (Erdős, Rubin & Taylor, 1980).

Bounds on M (and C)

Clearly, $M(k, s)$ is non-increasing in s.

$\implies M(k,s) \leq M(k, 2k-1) = \binom{2k-1}{k} < 2^{2k-1}$

$\implies C(k,s) = 12M(k,s)^2 \ln M(k,s) \ln k \leq 16^{(1+o(1))k}$

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Theorem (Erdős, 1969, "On a combinatorial problem III")

There is some algebraic decreasing function $f : [2,\infty) \to \mathbb{R}$ *satisfying* $\lim_{c\downarrow 2} f(c) = 4$ *and* $\lim_{c\to\infty} f(c) = 2$ *such that, if* $s \ge 2k-1$ *and* $s \sim ck$ *as* $k \to \infty$, *then* $M(k,s) = f(c)^{(1+o(1))k}$.

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$M(k) = \inf_{s \ge 2k-1} M(k,s)$. Radakrishnan & Srinivasan (2000).

Does C have to be so large?

For $k \geq 2, s \geq 2k-1$, there exists $C = C(k,s)$ such that, if G is (k,s)-choosable, then it is C-choosable.

Question (Kráľ & Sgall, 2005)

Must $C(k, 2k-1)$ be exponentially large in k?

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Yes.

Theorem (Bonamy & K.)

For $k \geq 2, s \geq 2k-1$, there exists $R = R(k,s) \geq \exp((k-1)^2/s)$ s.t. $K_{R-1,(R-1)^{R-1}}$ is (k,s)-choosable but not $(R-1)$-choosable.

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$\implies$ alternative to Král' & Sgall construction when $R(k,s) > k$.

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- For $k \geq 2$, $R(k, 2k - 1) = \binom{2k-1}{k}$, while $R(k, 2k - 2) = \infty$.
- $R(k, k^2) \leq k$ by taking arbitrary k-partition of $[k^2]$ as $\mathcal{F}$.

Property K and bipartite graphs

Theorem (Bonamy & K.)

For $2 \leq k \leq s$*, if* G *admits a bipartition* $V = V_1 \cup V_2$ *with* $|V_1| < R(k,s)$*, then* G *is* (k,s)*-choosable.*

$\implies K_{R-1,(R-1)^{R-1}}$ is (k,s)-choosable but not $(R-1)$-choosable.

Property K and bipartite graphs

(Property K(k,s): $\exists$ set $K \in \binom{[s]}{k-1}$ that meets every set of $\mathcal{F}$
$R(k,s)$ is size of smallest $\mathcal{F} \subseteq \binom{[s]}{k}$ not having Property K(k,s).)

Theorem (Bonamy & K.)

For $2 \leq k \leq s$, if G admits a bipartition $V = V_1 \cup V_2$ with $|V_1| < R(k,s)$, then G is (k,s)-choosable.

Proof.

$\forall$ (k,s)-list-assignment L, $\{L(v) : v \in V_1\}$ has Property K(k,s).

$\exists K \in \binom{[s]}{k-1}$ such that $L(u) \cap K \neq \emptyset$ for all $v_1 \in V_1$.

Since $|K| = k-1$, $L(v) \setminus K \neq \emptyset$ for all $v_2 \in V_2$. $\square$

Bounds on R

Theorem (Bonamy & K.)

For $k \geq 2, s \geq 2k-1$,

$$\frac{s!(s-2k+1)!}{(s-k)!(s-k+1)!} \leq R(k,s) < \frac{s!(s-2k+1)!}{(s-k)!(s-k+1)!} \ln \binom{s}{k-1}.$$

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Proof by probabilistic method.

Lower bound: fix $\mathcal{F} \subseteq \binom{[s]}{k}$ then choose $K \in \binom{s}{k-1}$ u.a.r. ...

Upper bound: fix $K \in \binom{s}{k-1}$ then choose $\mathcal{F} \subseteq \binom{[s]}{k}$ u.a.r. ... $\square$

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Upper bound: fix $K \in \binom{s}{k-1}$ then choose $\mathcal{F} \subseteq \binom{[s]}{k}$ u.a.r. ... $\square$

Stirling's $\implies R(k,s) \geq \exp((k-1)^2/s)$

The second follow-up

Let f be an increasing positive integer function f.
Does $(k, s_{f(k)})$-choosability imply $f(k)$-choosability for some large enough $s_{f(k)} = s_{f(k)}(k)$?

The second follow-up

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Does $(k, s_{f(k)})$-choosability imply $f(k)$-choosability for some large enough $s_{f(k)} = s_{f(k)}(k)$?

s_k does not exist except $s_2(2) = 4$.

If G is $(2,3)$-choosable, then it is 3-choosable: $s_3(2) = 3$.

For any polynomial $f(k)$, $s_{f(k)} = \Omega\left(\frac{k^2}{\ln k}\right)$ if it exists.

$s_{4.01^k}$ exists and $s_{16.01^k} = 2k - 1$ for k large enough.

Further aspects

Every planar graph is $(5, s)$-choosable (Thomassen, 1994), but there is a non-$(4, 5)$-choosable one (Mirzakhani, 1996).

We view bounded palette as way to refine list colouring problems (which are often quite hard).

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Theorem (Bonamy & K.)

Any planar graph with max degree 7 *is* $(8, 9)$*-edge-choosable.*

Weak List Colouring Conjecture, restricted to planar graphs:

Any planar graph with max degree Δ is $(\Delta + 1)$-edge-choosable
if $\Delta \leq 4$ (Vizing (1976), Juvan, Mohar & Škrekovski (1999)) and
if $\Delta \geq 8$ (Bonamy (2013+), Borodin (1991)).