Partition of random graphs into subgraphs of bounded component order

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Connections in Discrete Mathematics
Simon Fraser University
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Component colouring

Graph $G = (V, E)$

t-component set = subset of V, no induced component more than t vertices

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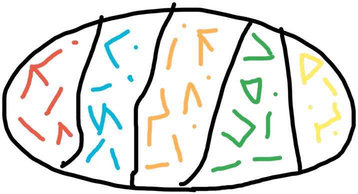

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$t = 1 \implies$ ordinary chromatic number

larger $t \implies$ "more error" allowed and possibly fewer colours needed

Warm up

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$$\frac{\chi(G)}{t} \leq \chi_c^t(G) \leq \min\left\{\chi(G), \left\lceil \frac{|V(G)|}{t} \right\rceil\right\}$$

Previous study

Dynamic/evolving databases:
cf. Kleinberg, Motwani, Raghavan, Venkatasubramanian 1997

H-minor free and bounded maximum degree graphs:

- Alon, Ding, Oporowski, Vertigan 2003
- Berke and Szabó 2007
- Kawarabayashi and Mohar 2007
- Linial, Matoušek, Sheffet, Tardos 2008
- Esperet and Joret 2014
- Liu and Oum 2015+

Random graphs $G_{n,p}$

Binomial random graph $G_{n,p}$, championed by Erdős and Rényi 1959/60:

$V(G_{n,p})$: $[n] = \{1, \ldots, n\}$

$E(G_{n,p})$: edges included independently with probability $p = p(n)$

Want properties holding *asymptotically almost surely* (a.a.s.), i.e. with prob. $\to 1$ as $n \to \infty$

History of $\chi_c^1(G_{n,p})$

Problem of finding $\chi(G_{n,p})$ mentioned by Erdős and Rényi 1959/60 and conjecture made by Grimmett and McDiarmid 1979

Theorem (Bollobás 1988, Matula 1987, Łuczak 1991)

Suppose $np \to \infty$ and p bounded from 1 as $n \to \infty$. Letting $b = 1/(1-p)$,

$$\chi(G_{n,p}) \sim \frac{n}{2\log_b np} \text{ a.a.s.}$$

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… fixed np remains open, but close to settled: Coja-Oghlan and Vilenchik 2013

A rough view of $\chi_c^t(G_{n,p})$

Proposition (Broutin and Kang)

Suppose $np \to \infty$ and p bounded from 1 as $n \to \infty$. Let $b = 1/(1-p)$.

- *If $t = o\left(\log_b np\right)$, then $\chi_c^t(G_{n,p}) = \Theta\left(\frac{n}{\log_b np}\right)$ a.a.s.*
- *If $t = \omega\left(\log_b np\right)$ and $t = o(n)$, then $\chi_c^t(G_{n,p}) \sim \frac{n}{t}$ a.a.s.*

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Compare with $\chi_c^t(G) \leq \min\left\{\chi(G), \left\lceil\frac{|V(G)|}{t}\right\rceil\right\}$.
Recall $\chi(G_{n,p}) \sim \frac{n}{2\log_b np}$ a.a.s. and note $\left\lceil\frac{|V(G_{n,p})|}{t}\right\rceil \sim \frac{n}{t}$ if $t = o(n)$.

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I.e. partition into asymptotically fewer than $\chi(G_{n,p})$ parts yields a monochromatic component of at least average part size.

A rough view of $\chi_c^t(G_{n,p})$, p fixed

Slightly better focus when p fixed as $n \to \infty$:

Fix $0 < p < 1$. *Let* $b = 1/(1-p)$.

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- *If* $t = \Theta\left(\log_b np\right)$, *then* $\ldots$
- *If* $t = \omega\left(\log_b np\right)$ *and* $t = o(n)$, *then* $\chi_c^t(G_{n,p}) \sim \dfrac{n}{t}$ *a.a.s.*

A finer view of $\chi_c^t(G_{n,p})$, p fixed

Theorem (Broutin and Kang)

Fix $0 < p < 1$. *Let* $b = 1/(1-p)$.

If $t \sim \tau \log_b np$ *and* κ *is unique root of* $\iota(\tau, \kappa) = 0$, *then* $\chi_c^t(G_{n,p}) \sim \dfrac{n}{\kappa \log_b np}$ *a.a.s.*

A finer view of $\chi_c^t(G_{n,p})$, p fixed

For $\tau, \kappa > 0$, define

$$\iota(\tau,\kappa) = \frac{1}{2}\left(\left(\kappa - \tau\left\lfloor\frac{\kappa}{\tau}\right\rfloor\right)\left(\kappa - \tau\left\lfloor\frac{\kappa}{\tau}\right\rfloor - \tau\right) - \kappa(\kappa - \tau - 2)\right).$$

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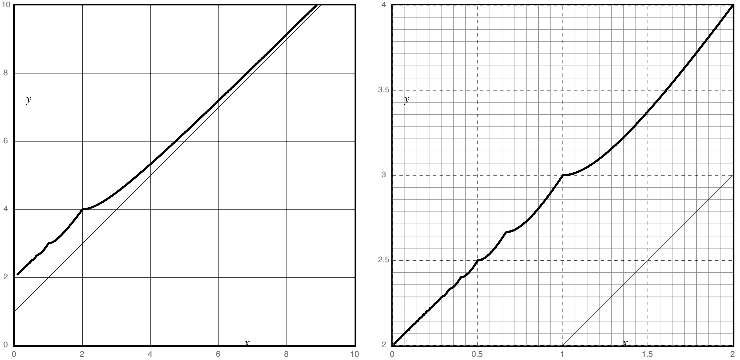
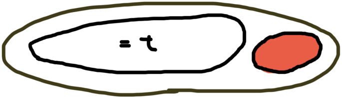
Plot of $\iota(x, y)$



First moment

Non-smooth behaviour governed by expected number of t-component k-sets

$\tau > 2$: two components, one with exactly t vertices



$\tau \leq 2$: more than two components, all but at most one with exactly t vertices

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Lemma

If $t \sim \tau \log_b np$, κ *is unique root of* $\iota(\tau, \kappa) = 0$, *and* $k \sim (\kappa - \varepsilon) \log_b n$,
then $\mathbb{P}(\alpha_c^t(G_{n,p}) < k) \leq \exp(-n^2/(\log n)^5)$.

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Start with $S' = [n]$.
While $|S'| > n/(\log n)^2$, extract a t-component k-set from S' as a new colour.
For each of remaining $n/(\log n^2)$ vertices, use a new colour. □

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Requires an upper bound on

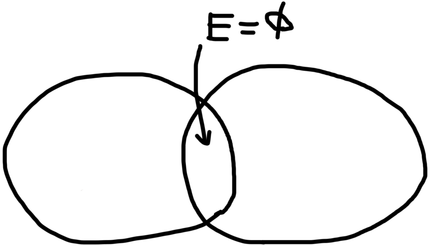
$$\Delta = \sum_{A,B\subseteq[n],2\le|A\cap B|<k} \mathbb{P}(A, B \text{ are } t\text{-comp. } k\text{-sets})$$
$$= \sum_{2\le\ell<k} \binom{n}{k}\binom{k}{\ell}\binom{n-k}{k-\ell} p(k,\ell)$$

where $p(k,\ell)$ is prob. two k-sets, sharing exactly ℓ vertices are t-comp. sets.

Want Δ to be much smaller than squared expected number of t-comp. k-sets.

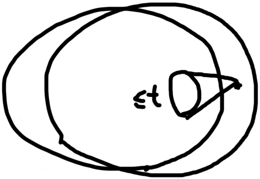
Δ_1

When number of common vertices ℓ is small, we reduce to when the graph of the intersection $A \cap B$ has no edges:



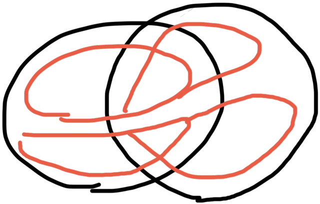
Δ_3

When number of common vertices ℓ is large, we use the fact that the maximum degree from $B \setminus A$ to B is at most t if B is a t-component set:



Δ_2

An intermediate case consideration needed only when $\tau \geq 2$, so that we basically only need consider when A and B have two components:



Ramsey-type parameter

The *t-component Ramsey number* is the smallest $R^t(k)$ such that any $R^t(k)$-vertex graph or its complement has a t-component k-set.

$R^1(k)$ coincides with diagonal 2-colour Ramsey numbers so exponential in k, while $R^k(k)$ is trivially k.

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Proposition

$R^{k-\Omega(k)}(k) = 2^{\Omega(k)}$. $R^{k-\omega(\log k)}(k) = k^{\omega(1)}$.

Proposition

$R^{k-O(\log k)}(k) = k^{O(1)}$.

Further study

Theorem (Broutin and Kang)

Fix $0 < p < 1$. *Let* $b = 1/(1-p)$.

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What about *edge* partitions? (Bohman, Frieze, Krivelevich, Loh, Sudakov 2011 and Spöhel, Steger, Thomas 2010)

Thank you!