Acyclic dominating partitions
(of graphs of maximum degree three)

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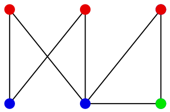
Acyclic chromatic number

Definition

A proper vertex colouring of a graph G is *acyclic* if the subgraph induced by the union of any two colour classes contains no cycles.

Not an acyclic proper colouring



The *acyclic chromatic number* $\chi_a(G)$ of a graph G is the least number of colours needed in a acyclic proper colouring of G.

Acyclic chromatic number

Acyclic colouring was introduced by Grünbaum (1973) and intensively studied for planar graphs over the next few years. For bounded degree graphs, Alon, McDiarmid and Reed (1991) considered the behaviour of

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and, verifying an Erdős conjecture from 1976, showed

Theorem (AMR, 1991)

$\chi_a(d) = O\left(d^{4/3}\right)$.

Theorem (AMR, 1991)

$\chi_a(d) = \Omega\left(d^{4/3}/(\log d)^{1/3}\right)$.

Acyclic improper colourings

In this talk, we extend this study for bounded degree graphs to *acyclic improper colourings*, i.e. where we relax the notion of acyclic colouring by allowing some monochromatic edges.

It is not immediately clear how to define such a relaxation:
What if we maintain that the subgraph induced by the union of any two colour classes contains no cycles?

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What if we maintain that the subgraph induced by the union of any two colour classes contains no cycles?

In a sense, this relaxation is not enough, as the corresponding chromatic number is always at least half of the acyclic chromatic number.

(To see this, observe that each part in such a partition must admit a proper 2-colouring.)

Acyclic t-improper chromatic number

Instead, we maintain that the *bipartite* subgraph induced by *the edges between* any two colour classes contains no cycles — in other words, we forbid *alternating cycles* — and then allow edges within the colour classes in some controlled fashion.

Acyclic t-improper chromatic number

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Definition

We say a vertex colouring (partition) of G is

- *acyclic* if there are no alternating cycles in G and
- *t-improper* if each colour class (part) induces a subgraph of maximum degree at most t.

Acyclic t-improper chromatic number

Definition

The *acyclic t-improper chromatic number* $\chi_a^t(G)$ of G is the least number of colours needed in an acyclic t-improper colouring of G. We let

$$\chi_a^t(d) := \max\{\chi_a^t(G) : \Delta(G) \leq d\}.$$

Note that $\chi_a^0(d) = \chi_a(d)$ and $\chi_a^d(d) = 1$.

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Acyclic t-improper colouring was first considered by Boiron, Sopena and Vignal (1997, 1999), studied more recently for bounded degree graphs by Addario-Berry, Esperet, Kang, McDiarmid and Pinlou (2007) and Addario-Berry, Kang and Müller (2007):

Acyclic improper chromatic number for subcubic graphs

In this talk, we settle the smallest nontrivial case of determining $\chi_a^t(d)$, i.e. the case of graphs of maximum degree $d = 3$ (or *subcubic graphs*).

Theorem (Grünbaum, 1973)

$\chi_a^0(3) = 4$.

Acyclic improper chromatic number for subcubic graphs

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Theorem (Grünbaum, 1973)

$\chi_a^0(3) = 4$.

Theorem (BSV, 1997)

$\chi_a^1(3) = 3$.

Conjecture (BSV, 1997)

$\chi_a^2(3) = 2$.

(Trivially, $\chi_a^3(3) = 1$.)

Acyclic dominating partitions

We affirm the conjecture, but for a slightly stronger notion:

Definition

We say a vertex partition $\mathcal{P}$ of $G = (V, E)$ is *dominating* if for each part A of $\mathcal{P}$, the set $V \setminus A$ is a dominating set, i.e. each vertex has at least one neighbour of a different colour.

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The *acyclic dominating number* $\text{ad}(G)$ of G is the smallest number of parts needed in an acyclic dominating partition of G. We let

$$\text{ad}(d) := \max\{\text{ad}(G) : \Delta(G) \leq d\}.$$

Acyclic dominating partitions

Notes:

- $\chi_a^{\Delta(G)-1}(G) \le \mathrm{ad}(G)$ for any graph G;
- in particular, $\chi_a^{d-1}(d) \le \mathrm{ad}(d)$ and $\chi_a^2(3) \le \mathrm{ad}(3)$.

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- in particular, $\chi_a^{d-1}(d) \leq \mathrm{ad}(d)$ and $\chi_a^{2}(3) \leq \mathrm{ad}(3)$.
- $\chi_a^{\Delta(G)-1}(G) = \mathrm{ad}(G)$ for any *regular* graph G;
- however, in an acyclic $(\Delta(G)-1)$-improper colouring of an arbitrary graph G, a vertex with degree less than $\Delta(G)$ is not forbidden from sharing the same colour with all of its neighbours.

Main theorem

Theorem

$\mathrm{ad}(3) = 2$.

That is, any subcubic graph admits a partition into two dominating sets such that the edges between the two parts induce a forest.

The crucial lemma

We can straighforwardly reduce our theorem to the following:

Lemma

If $G = (V, E)$ is a 2-connected, subcubic graph and $\mathcal{P}$ is a dominating bipartition of G such that G has a unique alternating cycle, then G admits an acylic dominating bipartition.

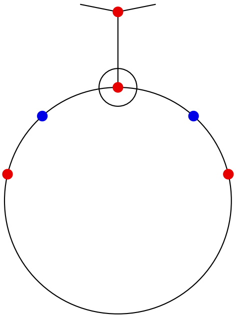
The bulk of the work is in proving the lemma by induction.

The main idea

Starting from a dominating bipartition with a unique alternating cycle C_1, we iterate the following procedure, always maintaining that the partition is dominating:

1. Test for the application of three types of *good* local alterations, each of which, if successfully performed, destroys the alternating cycle on C_i and creates no others.
2. If Step 1 was unsuccessful, then isolate an "almost" alternating cycle C_{i+1} and apply a fourth kind of local alteration such that C_{i+1} becomes the unique alternating cycle.

Some good local alterations



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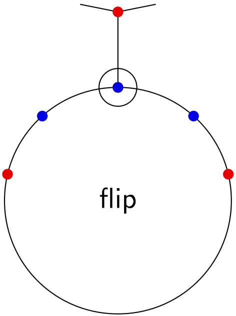

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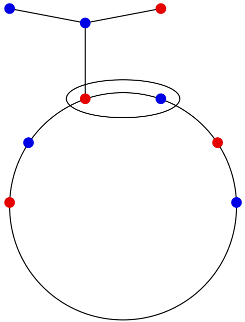

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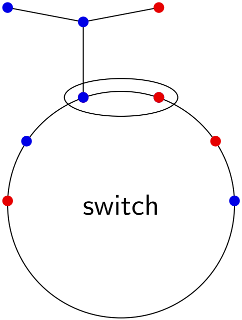

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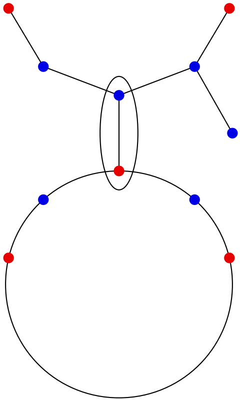

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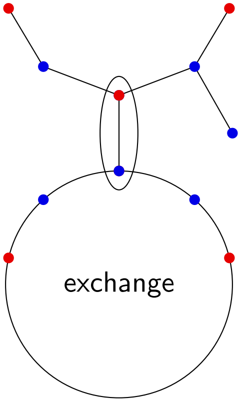
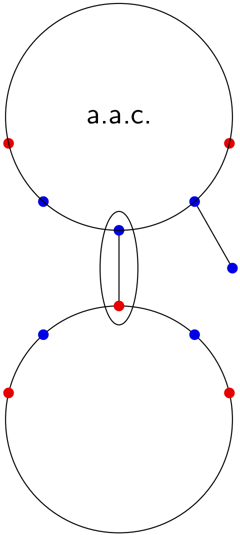


An almost alternating cycle



Finishing the proof

If, in Step 2, we can always identify an almost alternating cycle that contains *new* vertices, then our procedure cannot continue indefinitely.

We then perform a good local alteration on some cycle C_k to produce an acyclic dominating bipartition.

Bonus slide

Theorem (AKM, 2007)

$\chi_a^{d-1}(d) \leq \mathrm{ad}(d) = O(d \ln d)$.

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Applying estimates on $\alpha^t(G_{n,p})$, the following lower bound was shown:

Theorem (AEKMP, 2007)
$\chi_a^t(d) = \Omega\left((d-t)^{4/3}/(\ln d)^{1/3}\right)$, *uniformly over all* $t \leq d - 10\sqrt{d \ln d}$.