The t-improper chromatic number of random graphs

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Introduction

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- $G_{n,p}$ — random graph with vertex set $[n] = \{1, \dots, n\}$, edges included independently with probability p.
- *t-dependent set* of G — a vertex subset of G which induces a subgraph of maximum degree at most t.
- *t-improper chromatic number* $\chi^t(G)$ of G — fewest colours needed in a t-improper colouring of G, a colouring of the vertices of G in which colour classes are t-dependent sets.

  Note: $\chi^0(G) = \chi(G)$.

Improper colouring background

Cowen, Cowen and Woodall (1986) considered, for fixed $t \geq 0$, the t-improper chromatic number of planar graphs. Combined with FCT, they completely pinned down the behaviour of χ^t:

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Theorem (Cowen, Cowen and Woodall, 1986)

- *Every planar graph is 2-improperly 3-colourable,*
- $\exists$ *planar graph that is not 1-improperly 3-colourable, and*
- $\exists$ *planar graphs that are not t-improperly 2-colourable.*

Improper colouring basics

Proposition

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$$\frac{\chi(G)}{t+1} \leq \chi^t(G) \leq \chi(G).$$

Proposition (Lovász, 1966)

$$\chi^t(G) \leq \left\lceil \frac{\Delta(G)+1}{t+1} \right\rceil.$$

The chromatic number of random graphs

(a very brief history)

We say that a property holds asymptotically almost surely (a.a.s.) if it holds with probability tending to one as $n \to \infty$.

Fix $p > 0$ and let $\gamma = \frac{2}{\ln \frac{1}{1-p}}$.

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Theorem (Grimmett and McDiarmid, 1975)

$$(1-\varepsilon)\frac{n}{\gamma \ln n} \leq \chi(G_{n,p}) \leq (2+\varepsilon)\frac{n}{\gamma \ln n} \text{ a.a.s.}$$

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Theorem (Bollobás, 1988, Matula and Kučera, 1990)

$$\chi(G_{n,p}) \sim \frac{n}{\gamma \ln n} \textit{ a.a.s.}$$

Random improper colouring basics

Proposition

$$(1-\varepsilon)\frac{n}{t\gamma \ln n} \le \chi^t(G_{n,p}) \le (1+\varepsilon)\frac{n}{\gamma \ln n} \text{ a.a.s.}$$

Random improper colouring basics

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$$(1-\varepsilon)\frac{n}{t\gamma\ln n} \leq \chi^t(G_{n,p}) \leq (1+\varepsilon)\frac{n}{\gamma\ln n} \text{ a.a.s.}$$

Proposition

$$\chi^t(G_{n,p}) \leq (1+\varepsilon)\frac{np}{t} \text{ a.a.s.}$$

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The upper bounds of the previous slide give the correct behaviour in nearly all choices of $t = t(n)$:

- if $t(n) \ll \ln n$, then $\chi^t(G_{n,p})$ is near $\chi(G_{n,p})$;
- if $t(n) \gg \ln n$, then $\chi^t(G_{n,p})$ is near $\Delta(G_{n,p})/t$; and
- in the intermediary case, more work is required.

Formally, . . .

Main theorem

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For constant edge probability $0 < p < 1$, the following holds:

(a) *if $t(n) = o(\ln n)$, then $\chi^t(G_{n,p}) \sim \frac{n}{\gamma \ln n}$ a.a.s.;*

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(a) *if $t(n) = o(\ln n)$, then $\chi^t(G_{n,p}) \sim \frac{n}{\gamma \ln n}$ a.a.s.;*

(c) *if $t(n) = \omega(\ln n)$ and $t(n) = o(n)$, then $\chi^t(G_{n,p}) \sim \frac{np}{t}$ a.a.s.;*

(d) *if $t(n)$ satisfies $\frac{np}{t} \to x$, where $0 < x < \infty$ and x is not integral, then $\chi^t(G_{n,p}) = \lceil x \rceil$ a.a.s.*

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(a) *if $t(n) = o(\ln n)$, then $\chi^t(G_{n,p}) \sim \frac{n}{\gamma \ln n}$ a.a.s.;*

(b) *if $t(n) = \Theta(\ln n)$, then $\chi^t(G_{n,p}) = \Theta(\frac{n}{\ln n})$ a.a.s.;*

(c) *if $t(n) = \omega(\ln n)$ and $t(n) = o(n)$, then $\chi^t(G_{n,p}) \sim \frac{np}{t}$ a.a.s.;*

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The t-dependence number

We bound a related parameter, the *t-dependence number* $\alpha^t(G)$ of G — the size of a largest t-dependent set in G.
Note: $\alpha^0(G) = \alpha(G)$.

Proposition

$$\chi^t(G) \geq \frac{|V(G)|}{\alpha^t(G)}.$$

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$$\chi^t(G_{n,p}) \geq \frac{n}{\alpha^t(G_{n,p})}.$$

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"$\leq$" follows from $\chi^t \leq \chi$ and
"$\geq$" uses $\chi^t \geq \frac{n}{\alpha^t}$ and a first moment estimate of α^t.

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- Let $k = k(n) = \left\lceil \frac{1}{1-\varepsilon} \gamma \ln n \right\rceil$ and let X be the number of t-dependent sets of size k in $G_{n,p}$.
- We show that $\mathbf{E}(X) \to 0$.

Proof sketch: $t(n) = o(\ln n)$

The crucial estimate is as follows:

- Let $g(k,t)$ be the number of graphs on $[k] = \{1,\ldots,k\}$ with ***average*** degree at most t. The expected number of t-dependent k-sets is at most

$$\binom{n}{k}(1-p)^{\binom{k}{2}-\frac{tk}{2}}g(k,t)$$

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- Since a graph on k vertices with average degree d' has $kd'/2$ edges,

$$g(k,t) \leq \sum_{s=0}^{tk/2}\binom{\binom{k}{2}}{s}.$$

Proof sketch: $t(n) = \Omega(\ln n)$

Theorem

If $t(n) = \Theta(\ln n)$, then there exist constants $C, C' > 0$ such that $C\frac{n}{\ln n} \le \chi^t(G_{n,p}) \le C'\frac{n}{\ln n}$ a.a.s.

Theorem

If $t(n) = \omega(\ln n)$ and $\varepsilon > 0$ fixed, then $(1-\varepsilon)\frac{np}{t} \le \chi^t(G_{n,p}) \le \left\lceil (1+\varepsilon)\frac{np}{t} \right\rceil$ a.a.s.

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For both of these results,

- "$\leq$" follows from $\chi^t \leq \lceil(\Delta+1)/(t+1)\rceil$ and
- "$\geq$" the first moment estimate of α^t relies on passing from maximum to average degree as well as a Chernoff bound.

$t(n) = \Theta(\ln n)$

Suppose $t(n) \sim \tau \ln n$ and $k(n) = \kappa \ln n$ for constants $\tau, \kappa > 0$.

By large deviation techniques (cf. Dembo and Zeitouni (1998)), we can better estimate the expected number of t-dep. k-sets:

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Lemma

The expected number of average t-dependent k-sets is at most

$$\exp\left(k \ln n \left(1 - \frac{\kappa}{2}\Lambda^*\left(\frac{\tau}{\kappa}\right) + o(1)\right)\right)$$

where $\Lambda^*(x) = x \ln \frac{x}{p} + (1-x) \ln \frac{1-x}{1-p}$.

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The expected number of t-dependent k-sets is

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where $\Lambda^*(x) = x \ln \frac{x}{p} + (1-x) \ln \frac{1-x}{1-p}$.

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- If $1 - \frac{\kappa}{2}\Lambda^*\left(\frac{\tau}{\kappa}\right) < 0$, then $\alpha^t(G_{n,p}) \leq \kappa \ln n$ and $\chi^t(G_{n,p}) \geq \frac{n}{\kappa \ln n}$ a.a.s.
- If $1 - \frac{\kappa}{2}\Lambda^*\left(\frac{\tau}{\kappa}\right) > 0$, then the expected number of t-dependent k-sets goes to infinity;

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- If $1 - \frac{\kappa}{2}\Lambda^*\left(\frac{\tau}{\kappa}\right) > 0$, then the expected number of t-dependent k-sets goes to infinity; *moreover*, setting $j(n) \sim \frac{n}{\kappa \ln n}$, the expected number of t-improper j-colourings goes to infinity.

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- If $1 - \frac{\kappa}{2}\Lambda^*\left(\frac{\tau}{\kappa}\right) < 0$, then $\alpha^t(G_{n,p}) \le \kappa \ln n$ and $\chi^t(G_{n,p}) \ge \frac{n}{\kappa \ln n}$ a.a.s.
- If $1 - \frac{\kappa}{2}\Lambda^*\left(\frac{\tau}{\kappa}\right) > 0$, then the expected number of t-dependent k-sets goes to infinity; *moreover*, setting $j(n) \sim \frac{n}{\kappa \ln n}$, the expected number of t-improper j-colourings goes to infinity.

Conjecture

Let κ be the unique value satisfying $\kappa > \tau/p$ and $\frac{\kappa}{2}\Lambda^*\left(\frac{\tau}{\kappa}\right) = 1$. Then $\chi^t(G_{n,p}) \sim \frac{n}{\kappa \ln n}$ a.a.s.

[It is routine to check that there exists a unique $\kappa > \tau/p$ such that $\frac{\kappa}{2}\Lambda^*\left(\frac{\tau}{\kappa}\right) = 1$.]

χ^t for sparse random graphs

Theorem (Łuczak, 1991)

Suppose $0 < p(n) < 1$ *and* $p(n) = o(1)$. *Set* $d(n) = np(n)$.

If $\varepsilon > 0$, *then there exists constant* d_0 *such that, if* $d(n) \geq d_0$, *then* $(1-\varepsilon)\frac{d}{2\log d} \leq \chi(G_{n,p}) \leq (1+\varepsilon)\frac{d}{2\log d}$ *a.a.s.*

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Suppose $0 < p(n) < 1$ *and* $p(n) = o(1)$. *Set* $d(n) = np(n)$.

(a) *If* $\varepsilon > 0$ *and* $t(n) = t_0$ *for some fixed* $t_0 \geq 0$, *then there exists constant* d_0 *such that, if* $d(n) \geq d_0$, *then* $(1-\varepsilon)\frac{d}{2\log d} \leq \chi(G_{n,p}) \leq (1+\varepsilon)\frac{d}{2\log d}$ *a.a.s.*

(b) *If* $d(n) = \omega(1)$ *and* $t(n) = o(\ln d)$, *then* $\chi^t(G_{n,p}) \sim \frac{d}{2\ln d}$ *a.a.s.*

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(b) *If* $d(n) = \omega(1)$ *and* $t(n) = o(\ln d)$, *then* $\chi^t(G_{n,p}) \sim \frac{d}{2\ln d}$ *a.a.s.*

Furthermore, if $d(n) = \omega(\sqrt{\ln n})$, *then the following hold:*

(c) *if* $t(n) = \Theta(\ln d)$, *then* $\chi^t(G_{n,p}) = \Theta\left(\frac{d}{\ln d}\right)$ *a.a.s.;*

(d) *if* $t(n) = \omega(\ln d)$ *and* $t(n) = o(d)$, *then* $\chi^t(G_{n,p}) \sim \frac{d}{t}$ *a.a.s.;*

(e) *if* $t(n)$ *satisfies* $\frac{d}{t} \to x$, *where* $0 < x < \infty$ *and* x *is not integral, then* $\chi^t(G_{n,p}) = \lceil x \rceil$ *a.a.s.*