Circular choosability of planar graphs

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21 July 2006
Horizons of Combinatorics
Balatonalmádi

circular chromatic number

Introduced by Vince (1988).

Optimises over all (p, q)-*colourings*:

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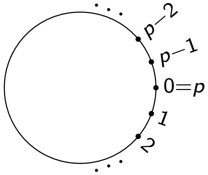
$V(G) \longmapsto$



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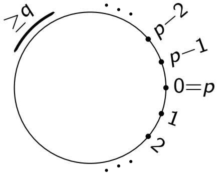

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Optimises over all (p, q)-*colourings*:

$$\chi_c(G) := \inf \left\{ \frac{p}{q} : G \text{ admits a } (p, q)\text{-colouring} \right\}$$

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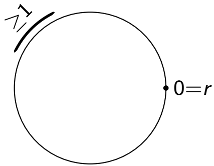

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Optimises over all *r-circular colourings*:

$$\chi_c(G) = \inf \{r : G \text{ admits an } r\text{-circular colouring}\}$$

circular chromatic number

Close connections to homomorphisms, fractional chromatic number, traffic lights, ...

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always rational

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Many interesting questions; consult survey by Zhu (2001).

circular choosability

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$$\mathrm{cch}(G) := \inf\{t \geq 1 : G \text{ is circularly } t\text{-choosable}\}$$

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circular choosability: Zhu (2005)

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upper bound for planar graphs

Define

$$\tau := \sup\{\text{cch}(G) : G \text{ is planar }\}$$

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Mohar asked the following: is $4 \leq \tau \leq 5$?

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Recall the (now) classical theorem of Thomassen (1994).

Theorem

Every planar graph is 5-choosable.

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Proposition

Let G be a near triangulation with outer face C. Let L be a list-assignment such that

$$|L(v)| \geq \begin{cases} 3 & \text{if } v \in C \\ 5 & \text{otherwise} \end{cases}.$$

Then any precolouring of two adjacent vertices of C can be extended to a colouring of G.

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upper bound for planar graphs

Proposition

Let G be a near triangulation with outer face C. Let L be a (p, q)-list-assignment such that

$$|L(v)| \geq \begin{cases} 4q - 1 & \textit{if } v \in C \\ 8q - 3 & \textit{otherwise} \end{cases} .$$

Then any L-(p, q)-precolouring of two adjacent vertices of C can be extended to a L-(p, q)-colouring of G.

upper bound for planar graphs

Theorem

Every planar graph is circularly 8-choosable, i.e. $\tau \leq 8$.

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planar graphs with high cch

Voigt (1993) described a non-4-choosable planar graph.
We show that there exist *circularly* non-$(6-\varepsilon)$-choosable graphs.

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For any $n \geq 2$, there exists planar G_n with $\text{cch}(G_n) \geq 6 - \frac{1}{n}$.

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Our examples are relatively simple.

In the next frames, we denote $t = 6 - \frac{1}{n}$.

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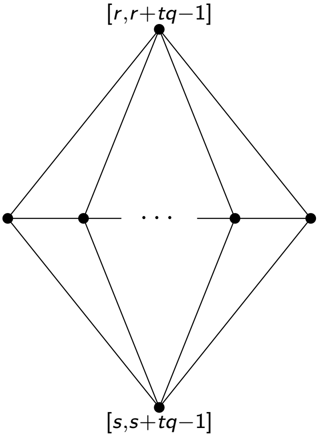

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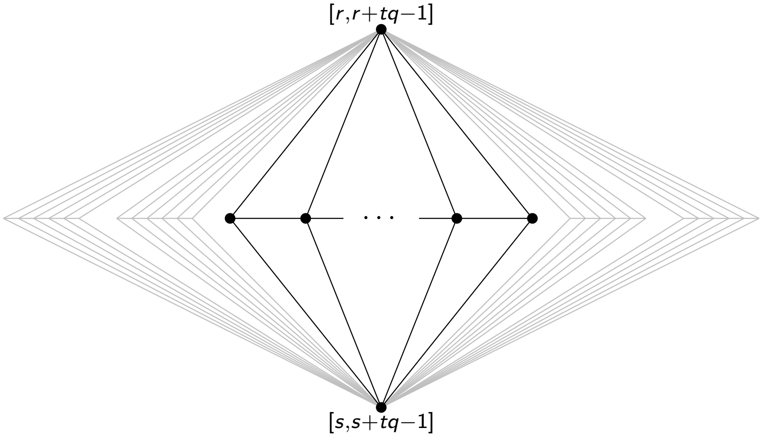

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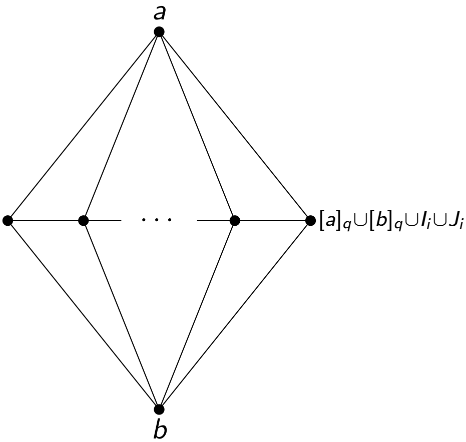

planar graphs of prescribed girth

$$\tau(k) := \sup\{\mathrm{cch}(G) : G \text{ is planar and has girth} \geq k\}.$$

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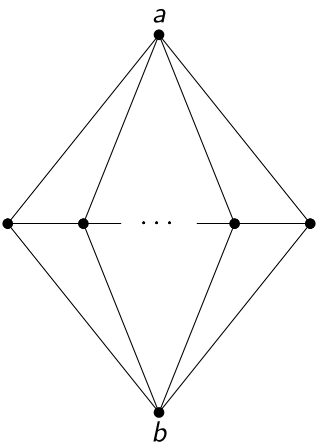
girth	3	4	5	6	7	8	9	10	$k \geq 11$
upper	8	6	$4\frac{1}{2}$	4	4	$3\frac{1}{3}$	3	3	$2 + \frac{4}{2\lfloor (k-2)/4 \rfloor}$
lower	6	4	$3\frac{1}{3}$	3	$2\frac{4}{5}$	$2\frac{2}{3}$	$2\frac{4}{7}$	$2\frac{1}{2}$	$2 + \frac{4}{k-2}$

Table: Bounds for $\tau(k)$.

planar graphs of prescribed girth with high cch



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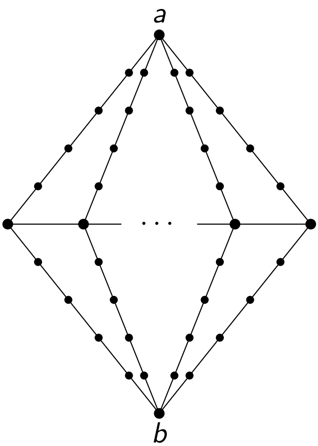

planar graphs of prescribed girth with high cch

Theorem

$\tau_o(k) \geq 2 + \frac{4}{k-2}$ for all integers $k \geq 3$.

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$\tau_o(k) \geq 2 + \frac{4}{k-2}$ *for all integers* $k \geq 3$.

Recall

$$\mathrm{Mad}(G) := \max\left\{\frac{2|E(H)|}{|V(G)|} : H \subset G\right\}$$

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Euler's formula and girth $k \implies \mathrm{Mad} < 2 + \frac{4}{k-2}$

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$$\tau(k) := \sup\{\text{cch}(G) : G \text{ is planar and has girth} \geq k\}.$$

girth	3	4	5	6	7	8	9	10	$k \geq 11$
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Table: Bounds for $\tau(k)$.

outerplanar graphs with prescribed girth

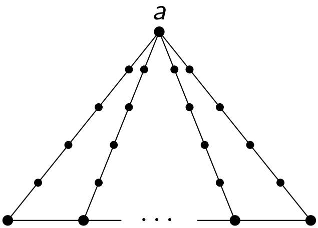
Define

$$\tau_o(k) := \sup\{\text{cch}(G) : G \text{ is outerplanar and has girth } \geq k\}.$$

Theorem

$\tau_o(k) = 2 + \frac{2}{k-2}$ for all integers $k \geq 3$.

outerplanar graphs with prescribed girth



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