On dense bipartite induced subgraphs*

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* Joint work with Louis Esperet and Stéphan Thomassé.

What guarantees a bipartite induced subgraph of large min degree?

Conjecture (Esperet, Kang, Thomassé)

There is $C > 0$ *such that any triangle-free graph of min degree* d *contains a bipartite induced subgraph of min degree* $C \log d$*.*

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- True with 2 rather than $C \log d$ (Radovanović and Vušković '13).

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- If true, it is sharp.
- True with 2 rather than $C \log d$ (Radovanović and Vušković '13).
- True with "semi-bipartite" instead of bipartite.

A naïve(?) intuition

Suppose min degree d and there is a proper colouring with k colour classes.

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Each of $\sim k^2/2$ pairs of colour classes induces a bipartite graph.
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By pigeonhole, there is one with $\gtrsim nd/k^2$ edges:
it will have average degree $\Omega(d/k)$ **if** the colouring is balanced...

Colouring (fractionally) **is** helpful

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Any graph with fractional chromatic number k and min degree d has a bipartite induced subgraph of average degree at least d/k.

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Conjecture (Harris '16)

There is $C > 0$ such that any d-degenerate triangle-free graph has chromatic number at most $Cd/\log d$. $\implies$ *Conjecture is true.*

(NB: as Jan Volec mentioned, his open problem on Hall ratio is related.)

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A probability distribution $\mathcal{S}$ over the stable sets of G satisfies property Q_r^* if $\mathbb{P}(v \in \mathbf{S}) \geq r$ for every vertex v and $\mathbf{S}$ sampled from $\mathcal{S}$.

The *fractional chromatic number* $\chi_f(G)$ of G is the least k such that there exists $\mathcal{S}$ satisfying property $\mathrm{Q}_{1/k}^*$.

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Proof on board.

Closing remarks

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Thank you!