Tree-like distance colouring for planar graphs of sufficient girth*

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ICGT, Lyon 7/2018

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* Joint work with Willem van Loon.

What is the effect of fixed girth on planar distance colouring?

Distance 1 vertex-colouring of planar graphs

Appel & Haken '77: $\chi \leq 4$. Sharp for K_4.

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Trees, i.e. with girth ∞, of course are bipartite.

Distance 1 edge-colouring of planar multigraphs

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Distance 1 edge-colouring of planar multigraphs and girth

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Vizing '65:	$\chi' \leq \Delta$ for girth 3 if Δ large. Sharp for trees.

Distance 2 vertex-colouring of planar graphs

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Borodin, Glebov, Ivanova, Neustroeva, Tashkinov '04:
$\chi_2 \leq \Delta + 1$ for girth 7 if Δ large. Sharp for trees.

Distance 2 edge-colouring of planar **graphs**

Faudree, Schelp, Gyárfás, Tuza '90: $\chi'_2 \leq 4\Delta + 4$. Sharp for "hairy" K_4.

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In fact, "hairy" odd cycles show that χ'_2 can be $> 2\Delta - 1$ with girth $\Theta(\Delta)$.
No *fixed* girth achieves the optimal value for trees.

How does the story extend to colouring with larger distance?

Distance t vertex-colouring of planar graphs

For trees:

$$\tau_t(\Delta) := \begin{cases} \frac{1}{\Delta-2}(\Delta(\Delta-1)^{t/2}-2) & 2 \mid t \\ \frac{1}{\Delta-2}(2(\Delta-1)^{(t+1)/2}-2) & 2 \nmid t \end{cases}.$$

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Distance t edge-colouring of planar **multi**graphs

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Merci!