Improper colouring of unit disk graphs

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Outline

Improper colouring

We consider a generalisation of proper vertex colouring:

Definition

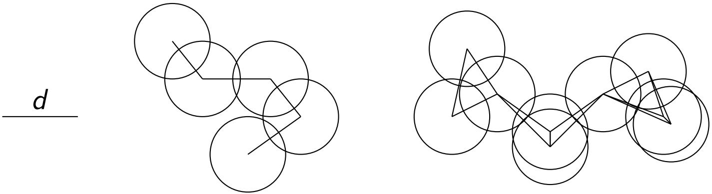
G is k-improper l-colourable if $V(G)$ can be partitioned into at most l colour classes each of which induces a graph with max degree at most k.

$\chi^k(G) \equiv$ smallest l such that G is k-improper l-colourable

Background for improper colouring

1. Cowen, Cowen and Woodall (1986)
   - introduced concept
   - characterisation for planar graphs
2. Cowen, Goddard and Jeserum (1997)
   - studied complexity
   - higher surfaces
3. Eaton and Hull (1999) and Škrekovski (1999)
   - *near* characterisation of improper choosability for planar graphs
   - Q: Is every planar graph 1-improper 4-choosable?

Unit disk graphs



Definition

Given *n* points (in $\mathbb{R}^2$) and $d > 0$, we centre disks of diameter d at each point and connect two points if their disks intersect.

Such graphs are called unit disk graphs (or UDGs).

Background for (colouring) unit disk graphs

1. Hale (1980)
   - linked radio channel assignment and colouring of UDGs
2. Clark, Colbourn and Johnson (1990)
   - tabulated complexity of classical problems (e.g. INDEP SET, DOM SET, CLIQUE)
   - observed links between PLANAR and UD
3. Gräf, Stumpf and Weißenfels (1998)
   - showed that UD l-COLOURABILITY ($l \geq 3$) is NP-C
   - alternative approximation algorithm for colouring UDGs

Summary of complexity for planar v. UD graphs

	planar graphs	UDGs
HAMILTONIAN CIRCUIT	NP-complete	NP-complete
DOMINATING SET	NP-complete	NP-complete
INDEPENDENT SET	NP-complete	NP-complete
MAX CLIQUE	Polynomial	Polynomial
CHROMATIC NUMBER	NP-complete	NP-complete
k-**IMPROPER CHROMATIC NUMBER**	**NP-complete**	**NP-complete**

Improper colouring of UDGs is hard

UD k-IMPROPER CHROMATIC NUMBER

Input: integer l and unit disk graph G
Question: is G k-improper l-colourable?

This problem is NP-complete

- "Easy proof" considers k-improper 3-colourability of weighted induced subgraphs of triangular lattice
- "Hard proof" considers k-improper l-colourability for all possible values k, l

"Easy proof"

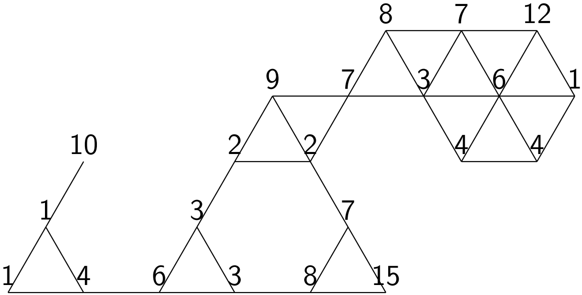


Figure: An example of a weighted induced subgraph of the triangular lattice

"Easy proof"

- McDiarmid and Reed (2000)
  - proper 3-colourability of such graphs is NP-complete
  - reduction from 3-colourability of planar graphs of max degree 4
- For k-improper 3-colourability, multiply each node by $k+1$

"Easy proof"

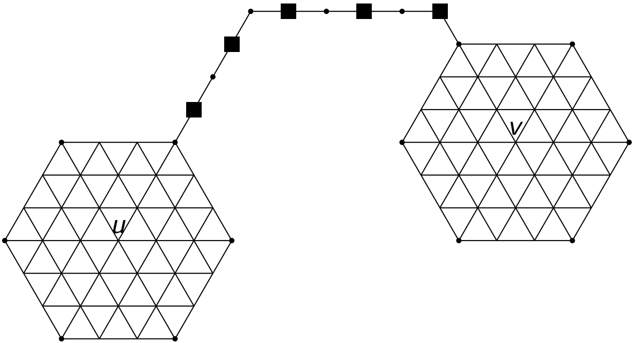


Figure: Gadgets used for "easy proof"

"Hard proof"

The "easy proof" does not give the complete picture:

- k-improper 2-colourability ($k \geq 1$) of UDGs?
  - reduction from planar k-improper 2-colourability
  - box-orthogonal embeddings
- k-improper l-colourability ($k \geq 0$, $l \geq 4$) of UDGs?
  - reduction from l-colourability
  - generalisation of Gräf *et al*

For both of these, the answer is NP-complete.

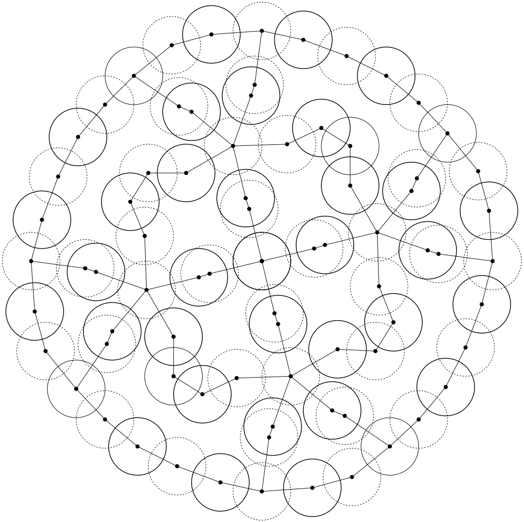


Figure: Gadget for k-improper l-colourability reduction

Further work

Distinct weighted improper colouring

- Given a *weighted* UDG, suppose that the colours assigned to each vertex must all be distinct?

Unit interval graphs

- Restriction of UDGs to $\mathbb{R}$
- Complexity unknown

Approximation algorithms?

- Best known approximation ratio for χ^k is 6
  (by taking vertex of max degree)

References

1. In Google^{TM}, enter "Ross Kang" and hit "I'm Feeling Lucky"
2. Download "transfer paper" (or, if I was diligent, a preprint)