The Annals of Applied Probability
2012, Vol. 22, No. 3, 931–970
DOI: 10.1214/11-AAP761

INVASION PERCOLATION ON THE POISSON-WEIGHTED INFINITE TREE

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We study invasion percolation on Aldous' Poisson-weighted infinite tree, and derive two distinct Markovian representations of the resulting process. One of these is the $\sigma \to \infty$ limit of a representation discovered by Angel et al. [*Ann. Appl. Probab.* **36** (2008) 420–466]. We also introduce an exploration process of a randomly weighted Poisson incipient infinite cluster. The dynamics of the new process are much more straightforward to describe than those of invasion percolation, but it turns out that the two processes have extremely similar behavior. Finally, we introduce two new "stationary" representations of the Poisson incipient infinite cluster as random graphs on $\mathbb{Z}$ which are, in particular, factors of a homogeneous Poisson point process on the upper half-plane $\mathbb{R} \times [0, \infty)$.

1. Introduction. Invasion percolation (or *Prim's algorithm* [20]) was first introduced by Jarńik [14] as a procedure for constructing the minimum weight spanning tree of a connected, weighted, finite graph. The procedure, however, may be applied to many infinite graphs without modification. Given a connected graph $G = (V, E)$, a starting node $v_0 \in V$ and an injective weight function $w : E \to \mathbb{R}$, the algorithm grows a component from the root inductively, adding at each step the lowest weight edge leaving the current component.

For each $i = 0, 1, \ldots$:
1. If $\{v_0, \ldots, v_i\} = V$, then stop.
2. Otherwise, let $e = uv \in E$ be the smallest weight edge for which $u \in \{v_0, \ldots, v_i\}$, $v \notin \{v_0, \ldots, v_i\}$.
3. Let $v_{i+1} = v$, and let $e_{i+1} = uv$.

(Throughout the paper, the graphs and weight functions we consider will be such that step 2, above, is well defined; i.e., the infimum of the weights of all edges from $\{v_0, \ldots, v_i\}$ to the rest of the graph is attained.) If $|V| < \infty$, the resulting

Received December 2009.
[1]Supported by an NSERC Discovery Grant during this research.
[2]Supported by an NSERC Postdoctoral Fellowship during this research.
MSC2010 subject classifications. Primary 60C05; secondary 60G55.
Key words and phrases. Invasion percolation, Prim's algorithm, Poisson-weighted infinite tree, percolation, random trees.

graph with vertex set $\{v_0, \dots, v_{|V|-1}\}$ and edge set $\{e_1, \dots, e_{|V|-1}\}$ is the unique minimum weight spanning tree of G. However, in general, for an infinite graph, this procedure does not necessarily build a spanning subgraph of G. In particular, if there is an infinite path leaving v_0 and containing only edges of weight at most h, for some $h \in \mathbb{R}$, then no vertex v for which $\inf_{e \ni v} w(e) > h$ will ever be explored.

Prim's algorithm was rediscovered under the name of invasion percolation in the 1980s [5, 16]. The strong connection between invasion percolation and critical percolation was immediately recognized—a particularly nice example of this connection is contained in the fact that invasion percolation on $\mathbb{Z}^d$ occupies an asymptotically zero proportion of the vertices of $\mathbb{Z}^d$ if and only if the percolation probability at the critical point $p_c(\mathbb{Z}^d)$ is zero (see Newman [19], page 24).

The well-known heuristic that percolation-style processes on $\mathbb{Z}^d$ should behave like percolation on a regular tree when d is large led Angel, Goodman, den Hollander and Slade [3] to study invasion percolation on regular trees. Angel et al. prove far too many results for us to summarize here. Among other topics, they study volume growth and boundary growth, spectral and Hausdorff dimensions for the set of vertices explored by invasion percolation. We hereafter refer to this set—and to the subgraph induced by this set, which will cause no confusion—as the *invasion percolation cluster*. Their results all stem from a Markovian representation of the invasion percolation cluster as—informally—a single infinite path, at each point of which is attached an independent random tree. (These trees are "subcritical Bernoulli percolation clusters" with a parameter which becomes increasingly close to critical the further along the backbone they are attached.) One of the major purposes of our paper is to explore a new approach to this structural representation which applies in some generality, so we take a moment to explain the representation itself in more detail.

For the duration of the introduction, for integers $\sigma \geq 2$, let $\mathcal{T}_\sigma$ denote the infinite rooted σ-regular tree (each node except the root has degree $\sigma + 1$), with each edge e labeled by $U_e \sim \text{Uniform}[0, \sigma]$ independently of all other edges. In general, for a weighted rooted graph G, let $G(p)$ be the connected subgraph of G containing the root when all edges of weight greater than p are discarded. Let $p_1 = \inf\{p : \mathcal{T}_\sigma(p) \text{ is infinite}\}$. Then with probability one, $1 < p_1 < \sigma$, and $\mathcal{T}_\sigma(p_1)$ is infinite and contains precisely one edge e_1 of weight p_1 (this is not hard, and in particular follows from Corollary 22 in Section 2.3). The component of $\mathcal{T}_\sigma(p_1)$ containing the root when e_1 is removed is finite (or else we never would have explored edge e_1). Let $\mathcal{T}_{\sigma,1}$ be the component of $\mathcal{T}_\sigma(p_1)$ *not* containing the root when e_1 is removed; then $\mathcal{T}_{\sigma,1}$, which we view as rooted at its unique vertex which is an endpoint of e_1, is infinite and contains only edges of weight less than p_1. Supposing we have defined $p_1, \dots, p_i$, $e_1, \dots, e_i$, and $\mathcal{T}_{\sigma,1}, \dots, \mathcal{T}_{\sigma,i}$, let $p_{i+1} = \inf\{p : \mathcal{T}_{\sigma,i}(p) \text{ is infinite}\}$. Then with probability one, $1 < p_{i+1} < p_i$, and $\mathcal{T}_{\sigma,i}(p_{i+1})$ is infinite and contains precisely one edge e_{i+1} of weight p_{i+1}, which separates the root of $\mathcal{T}_{\sigma,i}$ from infinity. We define $\mathcal{T}_{\sigma,i+1}$ to be the component of

$\mathcal{T}_{\sigma,i}(p_{i+1})$ not containing the root when e_{i+1} is removed, and root this tree at its unique vertex which is an endpoint of e_{i+1}.

Now let $P = \{f_i\}_{i=1}^{\infty}$ be the unique path starting from the root of $\mathcal{T}_\sigma$ and passing through all of $\{e_j\}_{j=1}^{\infty}$ (so P is only a.s. defined). This path is called the *backbone* of the invasion percolation cluster. The components of the invasion percolation cluster when all edges in $\{e_j\}_{j=1}^{\infty}$ are removed are called *ponds*; Angel et al. also study the sizes of these ponds. There is further interesting recent work on invasion percolation: on the sizes of ponds for invasion percolation in $\mathbb{Z}^2$ [6, 7] and on rescaled invasion percolation on trees [2]. For integers $n \geq 1$, let $W_n = W_n(\mathcal{T}_\sigma) = \sup_{j \geq n} U_{f_j}$. Angel et al. term the process $\{W_n\}_{n=1}^{\infty}$ the *backbone forward maximal process* of $\mathcal{T}_\sigma$. W_n is nonincreasing and has $\lim_{n\to\infty} W_n = 1$. Note that $W_n > W_{n+1}$ only when f_n is one of the edges e_j, in which case $W_n = U_{f_n} = p_j$. Angel et al. prove that $\{W_n\}_{n=1}^{\infty}$ is a Markov process and specify both its transition probabilities and its large-n rescaled behavior.

The removal of the vertices and edges of P separates the cluster into components of finite size. Suppose T is one such cluster and that its neighbor on the path P has distance n from the root. Then Angel et al. show that T is distributed as $\mathcal{T}_\sigma(W_n)$ conditioned to stay finite, independently of all other components. This fact and the results about the backbone forward maximal process mentioned in the preceding paragraph form the heart of their structural results.

In this paper we introduce a new mechanism for studying invasion percolation on randomly weighted trees, which can in particular give a new perspective on the structural results of Angel et al. The methodology works in some generality—in fact, parts of it are most easily formulated as statements about invasion percolation on graphs with deterministic weights. To apply such results, one then needs to check that the hypotheses hold a.s. in a randomly weighted tree under consideration (which in practice is always a trivial matter). We have chosen to present our results in the setting where they are the most simple and striking, which is that of the Poisson-weighted infinite tree, or PWIT.

Informally, the PWIT can be described as follows. The root r has a countably infinite number of children $v_1, v_2, \dots$. The edges $rv_1, rv_2, \dots$ are assigned weights: for each $i \geq 1$ the edge rv_i is weighted with the position of the ith point of a homogeneous Poisson process of rate 1 on $[0, \infty)$. [Equivalently, starting from an infinite sequence of independent Exponential(1) random variables $E_1, E_2, \dots,$ for each i the edge rv_i is given weight $E_1 + \cdots + E_i$.] This construction is repeated independently and recursively at each child of the root. We may view the nodes of the PWIT as labeled by $\bigcup_{i=0}^{\infty} \mathbb{N}^i$, so that the root has label $\varnothing$ and in general, node $n_1 n_2 \cdots n_k$ has parent $n_1 n_2 \cdots n_{k-1}$ and children $\{n_1 n_2 \cdots n_k n\}_{n \in \mathbb{N}}$; however, this labeling will not play a major role in the paper.

The PWIT shows up as a standard large-n limit for combinatorial optimization problems on the complete graph K_n; see the excellent survey paper by Aldous and Steele [1] for details of how. Our case is no exception; as one consequence of our study, we obtain novel proofs of the main results of [17], about the early behavior of Prim's algorithm on K_n with i.i.d. uniform weights. Our main results, however, link invasion percolation on the PWIT with the Poisson *incipient infinite cluster*—IIC, for short—constructed for general critical branching processes by Kesten [15], but earlier in the Poisson case by Grimmett [10]. The Poisson IIC is, informally, a critical Poisson Galton–Watson tree—PGW(1), for short—conditioned to be infinite. There are at least two natural ways to formalize this statement, but they both yield the same limiting construction, which we now describe. Start with a single, one-way infinite path, and then make each node of the path the root of an independent copy of PGW(1). The resulting infinite tree is the Poisson IIC, which we denote by $\mathcal{T}_{\mathrm{IIC}}$.

For the remainder of the introduction, let $\mathcal{T}_0$ be a random weighted tree with the distribution of the subgraph of the PWIT explored by invasion percolation, with vertices $\{v_0, v_1, \ldots\}$ in order of exploration, and let $\{W_i\}_{i=1}^{\infty}$ be its forward maximal process. (We have not yet proved that $\mathcal{T}_0$ *has* a forward maximal process, although the proof is straightforward—in particular, this fact follows from Corollary 22 in Section 2.3.) Also, for any tree T and vertex v of T, let $T^{(v)}$ denote T re-rooted at v. For two rooted random graphs G, H, we write $G \stackrel{d}{=} H$ to mean G and H have the same distribution in the local weak sense (i.e., neighborhoods of finite order of the root have the same distribution in both graphs; see [1], Section 2, for more details). Similarly, we write $G_n \stackrel{d}{\to} G$ to denote local weak convergence of a sequence $\{G_n\}$ of rooted random graphs to a limiting random graph G. (This notion of convergence in distribution deals only with the topological structure of the graph, so in particular ignores any edge weights of the graphs under consideration.)

Let $\mathcal{P}$ be a homogeneous Poisson process of rate 1 in the upper half-plane $\mathbb{R} \times [0, \infty)$. Given two random variables X and Y, we say a random variable X is a *factor* of Y if almost surely $X = f(Y)$ for some deterministic function f. (Usage of this term has not been fully standardized; ours agrees with that of [13].) The first main theorem of our paper is the following.

THEOREM 1. *There exist two $\mathcal{P}$-a.s. distinct random trees $T = T(\mathcal{P})$, $T' = T'(\mathcal{P})$ with vertex set $\mathbb{Z}$ such that*:

(a) *in T there is a unique infinite rightward path from each vertex $\mathcal{P}$-a.s.*;
(b) *in T' there is a unique infinite leftward path from each vertex $\mathcal{P}$-a.s.*;
(c) *neither T nor T' is a factor of the other.*

Furthermore, *setting $U = T$ or $U = T'$*, *we have*:

(d) *for any $n \in \mathbb{Z}$, $U(\mathcal{P} + n) = U(\mathcal{P}) + n$*;
(e) *for any $n \in \mathbb{Z}$, $U^{(n)}$ is distributed as $\mathcal{T}_{\mathrm{IIC}}$.*

This theorem seems very similar in spirit to results of Ferrari, Landim and Thorisson [9], on tree and forest factors of Poisson processes in $\mathbb{R}^d \times \mathbb{R}$, $d \geq 1$ (with the final copy of $\mathbb{R}$ viewed as a time dimension). The graph they define is a tree when $d = 1, 2$ and a forest when $d \geq 3$. Some particular similarities of note: Ferrari et al. explain how to use a preorder traversal (or *depth-first search*, a procedure quite similar to invasion percolation) of the points of the Poisson process in order to view their trees as having vertex set $\mathbb{Z}$; their graphs also have only one end (only one infinite path leaving any vertex); their graphs are built by joining each point to its first time-successor within $\mathbb{R}^d$-distance one, yielding a "coalescing random walk" interpretation of the construction, that is, reminiscent of our random-walk description of the forward maximal process in Section 2.3. Ferrari et al. do not explicitly identify the distribution of the graph they define, but it would be very interesting to know if it can be meaningfully interpreted as a higher-dimensional analog of the Poisson IIC. Holroyd and Peres [12] have also studied tree and forest factors of Poisson point processes in $\mathbb{R}^d$, and Holroyd and Peres [12], Timár [23] have studied tree and forest factors of general point process in $\mathbb{R}^d$. Also, factors of one-dimensional Poisson processes that commute with discrete shifts [i.e., as in Theorem 1(d), above] are one of the subjects studied in [11].

As a byproduct of the proof of Theorem 1, we will also obtain the following theorem, which is a "PWIT analog" of [3], Theorem 1.2.

THEOREM 2. *$\mathcal{T}_0^{(v_n)} \xrightarrow{d} \mathcal{T}_{\mathrm{IIC}}$ as $n \to \infty$.*

Before stating our third theorem (in fact, the first two theorems lean heavily on tools introduced in proving the third), we have a few more concepts to introduce. For each edge e of $\mathcal{T}_{\mathrm{IIC}}$, let $X_e \sim \text{Uniform}[0, 1)$, independently of all other edges. Let $e_0 = v_0v_1, e_1 = v_1v_2, \ldots$ be the edges of the unique infinite path (the backbone) in $\mathcal{T}_{\mathrm{IIC}}$, let $M_0 = 0$, and for integers $i \geq 1$, let $M_i = \max_{0 \leq j < i} X_{e_j}$. Now let $\mathcal{T}_{\mathrm{IIC}}^*$ be the subtree of $\mathcal{T}_{\mathrm{IIC}}$ obtained as follows. Let v be a vertex of $\mathcal{T}_{\mathrm{IIC}}$, and let v_i be the nearest vertex of the backbone to v. If any edge of the path from v to v_i has weight greater than M_i, then remove v from the tree. Do this for each $v \in \mathcal{T}_{\mathrm{IIC}}$. Finally, remove v_0 and root at v_1. The resulting subtree of $\mathcal{T}_{\mathrm{IIC}}$ is $\mathcal{T}_{\mathrm{IIC}}^*$.

THEOREM 3. *There is a continuous, strictly decreasing bijective map $q : [1, \infty) \to (0, 1]$ such that $(q(W_1), q(W_2), \ldots) \stackrel{d}{=} (M_1, M_2, \ldots)$, in the sense of finite-dimensional distributions. Furthermore, $\mathcal{T}_0$ conditional on $(W_1, W_2, \ldots)$ is distributed as $\mathcal{T}_{\mathrm{IIC}}^*$ conditional on $(M_1, M_2, \ldots) = (q(W_1), q(W_2), \ldots)$, in the local weak sense.*

It is worth mentioning that the Markovian nature of $(W_1, W_2, \ldots)$ can be immediately deduced from this theorem. Given M_i, M_{i+1} is greater than M_i precisely if $X_{e_{i+1}} \in (M_i, 1]$, in which case $M_{i+1} = X_{e_{i+1}}$. Thus, given M_i, M_{i+1} is equal to M_i with probability $(1 - M_i)$, and otherwise is uniform on $(M_i, 1]$. Translating this to W_i immediately yields the "PWIT analog" of the Markov process construction ([3], Proposition 3.1).

1.1. *The PWIT as a $\sigma \to \infty$ limit of $K_{\sigma+1}$ or of $\mathcal{T}_\sigma$.* We mention in passing that with not much effort, it is possible to prove convergence of invasion percolation on $\mathcal{T}_\sigma$ or on $K_{\sigma+1}$ to invasion percolation on the PWIT, in a stronger sense than the local weak sense. Let $\mathbf{U} = (U_1^*, U_2^*, \dots, U_\sigma^*)$ be the order statistics of σ independent Uniform$[0, \sigma]$ random variables. Then U tends weakly to the vector of points of a homogeneous rate one Poisson process $\mathcal{P}$ on $[0, \infty)$. More importantly for our current purpose, the vector $(U_1, \dots, U_{\lfloor\sqrt{\sigma}\rfloor})$ has total variation distance $O(\sigma^{-1/2})$ from the vector of the first $\lfloor\sqrt{\sigma}\rfloor$ points of $\mathcal{P}$. It follows, in a sense that can easily be made precise, that *the first $o(\sqrt{\sigma})$ steps of invasion percolation on $\mathcal{T}_\sigma$* together have total variation distance $o(1)$ from the same number of steps of invasion percolation on the PWIT. A similar statement holds for the first $o(\sqrt{\sigma})$ steps of invasion percolation on $K_{\sigma+1}$. This in particular yields new proofs of the explicit error bounds derived in [17] for the behavior of the early stages of Prim's algorithm on $K_{\sigma+1}$. The details are straightforward, and we leave them to the interested reader.

1.2. *Outline.* In Section 2 we construct the building blocks on which the remainder of the article rests. In particular, we describe a different way to view invasion percolation, in terms of a "note-taking" procedure that accompanies the invasion percolation procedure, and in the special case of invasion percolation on trees contains all the information required to reconstruct the original procedure. To best understand this note-taking procedure we introduce the "box process" (Definition 6), which gives us a clear picture of the connection mechanism of invasion percolation. The box process also allows for an understanding of a related "two-way infinite" invasion percolation process, which can be seen as describing the behavior of invasion percolation far from the root. Furthermore, with the introduction of the "box graph" in Section 2.2, the box process itself becomes an interesting object of study, and we derive some of its fundamental properties. Throughout Section 2, our studies are in the deterministic setting.

In Section 3 we apply our tools to study $\mathcal{T}_0$. In particular, we prove the PWIT analog of the forward maximal representation of $\mathcal{T}_0$ in more detail. Section 3 also contains some results concerning ballot style theorems, queueing processes and Poisson Galton–Watson duality that are of use in proving Theorems 1–3.

Finally, in Section 4 we prove a number of results concerning the box graph and the stationary process. In particular, we find that these graphs resemble the Poisson IIC locally. Using these results, we deduce Theorems 1–3.

2. Redrawing invasion percolation. In this section we describe a different way to view invasion percolation which is at the heart of most of the results of this paper. First, imagine keeping notes of the local edge landscape we see as we perform invasion percolation, as follows. At step i of invasion percolation, we explore vertex v_i and record the weights of all edges leaving v_i and heading into new territory by putting marks on the vertical half-line $\{i\} \times [0, \infty)$ whose heights are the weights of these edges. (When performing invasion percolation on a rooted tree T, the edges "heading into new territory" are precisely the edges from v_i to its children in T.) Running the invasion percolation process until it terminates (or forever) then yields some set P of points in the positive quadrant.

Formally, suppose $G = (V, E)$ is a weighted graph with all edge weights distinct, and with distinguished vertex v_0. Then the invasion percolation procedure defines an infinite subtree T of G, with vertex set $\{v_0, v_1, \dots\}$. For each $i \geq 0$, let $p_i(1), p_i(2), \dots, p_i(j_i)$ be the weights of the edges from v_i to $V \setminus \{v_0, \dots, v_i\}$, in increasing order of weight. Let $P^i = P^i(G) = \{(i, p_i(j))\}_{j=1}^{j_i}$, and let $P = P(G) = \bigcup_{i=0}^{|V|-1} P^i$.

In general, it is not possible to *reconstruct* the steps taken by invasion percolation by considering only the set P. However, this *is* possible for invasion percolation on trees, and we now explain how. In order to do so, we introduce an inductive procedure for building a tree, given a set of points $P \subset \mathbb{R}^2$ and an interval $\mathcal{I} \subset \mathbb{Z}$ of consecutive integers. We write $n_{\mathcal{I}} = \inf\{n \in \mathcal{I}\} \geq -\infty$ and $m_{\mathcal{I}} = \sup\{n \in \mathcal{I}\} \leq \infty$.

For notational convenience, given $X \subseteq \mathbb{R}^2$, we write $|X|_P$ for $|P \cap X|$. Also, for a point $p \in \mathbb{R}^2$, we write $x(p)$ for the x-coordinate and $y(p)$ for the y-coordinate. Let us assume the following:

1. All points of P lie in the upper half-plane. No bounded set contains unboundedly many points.
2. For any $n \in \mathcal{I}$, there exists $k > 0$ with $n - k > n_{\mathcal{I}} - 1$ for which $|[n-k, n) \times [0, \infty)|_P \geq k$.
3. $|P((-\infty, \infty) \times \{y\})| \leq 1$ for any $y \in \mathbb{R}$.

If P satisfies these three conditions, we say it is *reasonable* (or *$\mathcal{I}$-reasonable*, if $\mathcal{I}$ is not clear from context). (Here, as well as later, we state deterministic requirements for the point set P; these requirements—and therefore, the results derived from them—will hold almost surely for all the random point sets we consider. In particular, the reader will always be safe thinking of P as a Poisson point set of intensity one in the upper half-plane.) We start from an empty set $P_{n_{\mathcal{I}}} = \varnothing$, from which we will build an increasing sequence of subsets of $\mathcal{N}_0$. The following procedure requires $n_{\mathcal{I}} > -\infty$.

For each $i = n_{\mathcal{I}}, n_{\mathcal{I}} + 1, \dots, m_{\mathcal{I}}$:
1. Let $p_{i+1} = p_{i+1}(P, \mathcal{I})$ be the point of $([n_{\mathcal{I}}, i+1) \times [0, \infty)) \cap (P \setminus P_i)$ minimizing $y(p_{i+1})$.
2. Let $P_{i+1} = P_{i+1}(P, \mathcal{I}) = P_i \cup \{p_{i+1}\}$, and let $e_{i+1} = e_{i+1}(P, \mathcal{I}) = (i+1, \lfloor x(p_{i+1}) \rfloor)$.

We refer to this procedure as *point set invasion percolation*. Since P is reasonable, the procedure is well defined. The resulting graph $\mathrm{IPC}(P, \mathcal{I})$ has vertex set $\mathcal{I}$

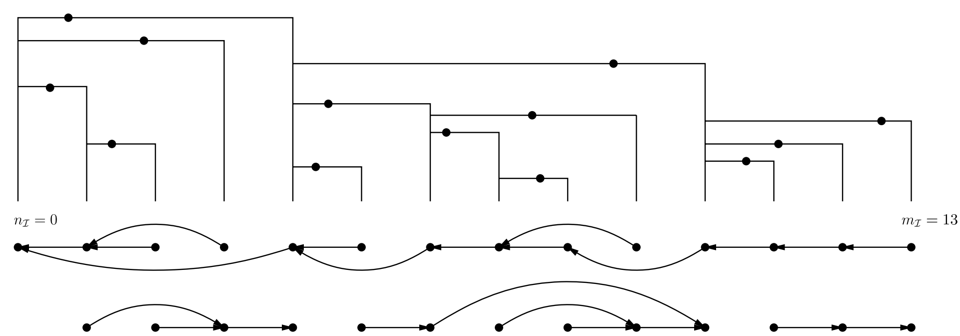

FIG. 1. *Top, an $\mathcal{I}$-reasonable set of points P, with $\mathcal{I} = \{0, \dots, 13\}$, and the corresponding boxes (defined in Definition 6). Middle, the tree* IPC$(P, \mathcal{I})$. *Bottom, the forest* BG$(P, \mathcal{I})$, *defined at the start of Section 2.2. All arrows point from child to parent.*

and edge set $\{e_i : n_{\mathcal{I}} < i < m_{\mathcal{I}} + 1\}$. (We write $i < m_{\mathcal{I}} + 1$ instead of $i \le m_{\mathcal{I}}$ since we may have $m_{\mathcal{I}} = \infty$, but i is always finite.) An example is shown in Figure 1. Note that IPC$(P, \mathcal{I})$ is a tree, which we view as rooted at $n_{\mathcal{I}}$. We often also view IPC$(P, \mathcal{I})$ as a weighted tree in which edge e_i has weight $y(p_i)$. In general in this section we work in the deterministic setting. However, since our eventual aim is to link this work to invasion percolation on randomly weighted trees we briefly discuss how this can be done.

IPC of the PWIT. Now suppose that $\mathcal{T}$ is an instance of the PWIT, and let $\mathcal{T}_0$ be the subtree of $\mathcal{T}$ explored by invasion percolation. The following lemma is then immediate.

LEMMA 4. IPC$(P(\mathcal{T}), \mathbb{N})$ *and $\mathcal{T}_0$ are identical, and for each $i \in \mathbb{N}$, $w(e_i) = y(p_i)$.*

When performing invasion percolation on $\mathcal{T}$, for all i, $P^i(\mathcal{T})$ is a Poisson point process of rate 1 on the vertical half-line $\{i\} \times [0, \infty)$, and $P(\mathcal{T})$ is the union of these point processes.

We remark that since all points in P have integer x-coordinates, the floor in step 2, above, has no effect. The use of the floor is to ensure that if a point $p = (x, y) \in P$ is replaced by a point $p' = (x', y)$, as long as $\lfloor x \rfloor = \lfloor x' \rfloor$, the resulting graph IPC$(P, \mathcal{I})$ will be unchanged. As a result we obtain the following corollary.

COROLLARY 5. *Let $\mathcal{P}$ be a Poisson point process of rate 1 on $[0, \infty) \times [0, \infty)$. Then* IPC$(\mathcal{P}, \mathbb{N})$ *and $\mathcal{T}_0$ are identically distributed.*

PROOF. Associate to each point $p = (x(p), y(p))$ of $P(\mathcal{T})$ an independent uniform U_p, and let p' be the point $(x(p) + U_p, y(p))$. Then $P' = \{p' : p \in P\}$ is a

Poisson point process of rate 1 on $[0, \infty) \times [0, \infty)$, and IPC$(\mathcal{P}', \mathbb{N})$ and IPC$(P, \mathbb{N})$ are identical. The result follows. $\square$

This corollary reduces the study of the distributional properties of $\mathcal{T}_0$ to that of the distributional properties of IPC$(\mathcal{P}, \mathbb{N})$, where $\mathcal{P}$ is a Poisson point process of rate 1 on $[0, \infty) \times [0, \infty)$.

We also demonstrate how the two examples of invasion percolation described in Section 1 can be encoded by suitable point processes.

IPC of an infinite randomly weighted σ-regular tree. Let $\mathcal{T}_\sigma$ be the rooted regular tree with forward degree $\sigma \geq 2$. We can model invasion percolation on $\mathcal{T}_\sigma$ as follows: for each $n \in \mathcal{I} = \mathbb{N}$, choose σ independent, uniformly random points of $[n, n+1) \times [0, \sigma)$ (or of $\{n\} \times [0, \sigma)$). Let P be the union of all these points.

The minimum spanning tree of the complete graph. Let $K_{\sigma+1}$ be the complete graph on $\sigma + 1$ vertices. We may approximately model invasion percolation on a randomly weighted $K_{\sigma+1}$ as follows: for each $n \in \mathcal{I} = \{0, \dots, \sigma\}$, choose $\sigma - n$ independent, uniformly random points from the set $[n, n+1) \times [0, \sigma)$. Let P be the union of all these points.

This representation is not exact due to the cycles in K_n. For example, it is possible that the second least weight leaving the starting vertex is on the edge between the second and third vertices visited by Prim's algorithm. However, the probability of events of this type is asymptotically negligible for the first $o(\sqrt{\sigma})$ steps of the algorithm.

The acyclicity of trees is what allows us to model them by a point process without reference to the order of exploration of vertices. In general—for invasion percolation on $\mathbb{Z}^d$, for example—it may still be possible to use some of the following methodology while jointly constructing the point process P and the exploration process "as we go." However, we have not pursued this avenue of study.

For the remainder of the section, we explore what properties we can derive about the point process invasion percolation procedure with as few restrictions on the point set P as possible. The next definitions and lemma provide an alternative geometric characterization of the connection rule used in the above inductive procedure, one that will be useful throughout the paper.

DEFINITION 6. Given an interval $\mathcal{I}$, with $n_\mathcal{I} > -\infty$, and an $\mathcal{I}$-reasonable point set P, for each $i \in \mathcal{I}$ with $i > n_\mathcal{I}$, let

$$h_i(P, \mathcal{I}) = \inf\{h : \exists j \in \mathcal{I}, n_\mathcal{I} \leq j < i \text{ such that } |[j, i) \times [0, h]|_P \geq i - j\}.$$

Let $\ell_i(P, \mathcal{I})$ be the minimum integer $\ell_i \in [n_\mathcal{I}, i)$ such that $|[\ell_i, i) \times [0, h_{P,\mathcal{I}}(i)]|_P = i - \ell_I$, let $B_i(P, \mathcal{I}) = [\ell_i, i) \times [0, h_i]$ and let $t_i(P, \mathcal{I})$ be the unique point in B_i with $y(t_i) = h_i$.

We often omit reference to the parameters P and $\mathcal{I}$ if the context is clear.

We take a moment to observe that these functions are well defined. It follows from condition 2 that h_i is finite, and from condition 1 that it is positive. The minimality of h_i then implies the existence of a point $p \in P(B_i)$ such that $y(p) = h_i$. The fact there is a unique such point follows from condition 3.

LEMMA 7. *If $n_{\mathcal{I}} > -\infty$ and P is $\mathcal{I}$-reasonable, then for all $n \in \mathcal{I} \setminus \{n_{\mathcal{I}}\}$, we have $t_n = p_n$.*

PROOF. It suffices to show (by condition 3) that $y(t_n) = y(p_n)$. We prove this by induction on n. Clearly, the assertion holds for $n = n_{\mathcal{I}} + 1$. Assume $n > n_{\mathcal{I}} + 1$ and that $t_i = p_i$ for all $n_{\mathcal{I}} + 1 \leq i < n$. First, since $|B_n|_P = n - \ell_n$ and $\bigcup_{i=n_{\mathcal{I}}+1}^{n-1} p_i$ contains at most $n - \ell_n - 1$ points of $P \cap B_n$, the set $(P \setminus P_{n-1}) \cap B_n$ contains at least one point and so $y(p_n) \leq y(t_n)$.

To show that $y(t_n) \leq y(p_n)$, first note that if $x(p_n) \geq n - 1$, then $|[n-1, n) \times [0, y(p_n)]|_P \geq 1$ and so certainly $y(t_n) \leq y(p_n)$. We thus assume that $x(p_n) < n - 1$ and construct a sequence $\{a_i\}_{i=0}^{k}$ inductively as follows:

Let $i = 0$ and let $a_0 = n - 1$.
1. If $a_i \leq x(p_n)$, set $k = i$ and **stop**.
2. Otherwise, let $a_{i+1} = \ell_{a_i}$, then let $i = i + 1$ and return to **1**.

For each $0 \leq i \leq k$ for which a_i is defined, if $a_i > x(p_n)$, then $y(p_{a_i}) < y(p_n)$ or else the point p_n was a better choice for p_{a_i}. By the inductive hypothesis, $p_{a_i} = t_{a_i}$. By construction, $|B_{a_i}|_P = a_i - \ell_{a_i} = a_i - a_{i+1}$ for all $i < k$. Since $B_{a_i} \cap B_{a_j} = \varnothing$ for all $i \neq j$, we conclude that $\bigcup_{i=0}^{k-1} B_{a_i}$ has $n-1-a_k$ points of P. Thus, $|[a_k, n) \times [0, y(p_n)]|_P \geq |\{p_n\} \cup \bigcup_{i=0}^{k-1} B_{a_i}|_P \geq n - a_k$. By the choice of h_n minimum, it follows that $h_n = y(t_n) \leq y(p_n)$ as required. □

The structure of the containment relations among the boxes B_i turns out to be interesting in its own right, and we explore aspects of it here as well as later in the paper.

LEMMA 8. *If $n_{\mathcal{I}} > -\infty$ and P is $\mathcal{I}$-reasonable, then for $n \in \mathcal{I} \setminus \{n_{\mathcal{I}}\}$, either $h_{\ell_n} > h_n$ or $\ell_n = n_{\mathcal{I}}$.*

PROOF. Assume $\ell_n \neq n_{\mathcal{I}}$, suppose $h_{\ell_n} \leq h_n$ and write $m = \ell_n$. Then both B_m and B_n are contained in $[\ell_m, n) \times [0, h_n]$, so $|[\ell_m, n) \times [0, h_n]|_P \geq n - \ell_m$. This contradicts either the choice of h_n or the choice of ℓ_n. □

LEMMA 9. *If $n_{\mathcal{I}} > -\infty$ and P is $\mathcal{I}$-reasonable, then for any $i, j \in \mathcal{I} \setminus \{n_{\mathcal{I}}\}$ with $i < j$, either $B_i \cap B_j = \varnothing$ or $B_i \subseteq B_j$.*

PROOF. Suppose that $B_i \cap B_j \neq \phi$. In particular this implies $\ell_j < i$. We prove that $B_i \subseteq B_j$ by proving that $h_i < h_j$ and $\ell_j \leq \ell_i$.

The minimality of h_j implies that $|[\ell_j, j) \times [0, h_j)|_P = j - \ell_j - 1$ and that $|[i, j) \times [0, h_j)|_P \leq j - i - 1$. Thus $|[\ell_j, i) \times [0, h_j)| \geq i - \ell_j$, which immediately implies that $h_i < h_j$.

We now prove $\ell_j \leq \ell_i$. Suppose that $\ell_i < \ell_j$. Then, by reasoning as above, and using the fact that $h_j > h_i$ we have that $|[\ell_i, \ell_j) \times [0, h_j)|_P \geq |[\ell_i, \ell_j) \times [0, h_i)|_P \geq \ell_j - \ell_i$. This implies that $|[\ell_i, j) \times [0, h_j)|_P \geq j - \ell_i$, which contradicts the definition of ℓ_j. □

LEMMA 10. *If $n_{\mathcal{I}} > -\infty$ and P is $\mathcal{I}$-reasonable, then for any $i, j \in \mathcal{I} \setminus \{n_{\mathcal{I}}\}$ such that $\ell_j < i \leq j$, there is a path in* IPC$(P, \mathcal{I})$ *between i and ℓ_j.*

PROOF. Observe that since $\ell_j < i$, $B_i \subset B_j$ by Lemma 9. We apply induction on $i - \ell_j$. If $i - \ell_j = 1$, then we must have $\lfloor x(p_i) \rfloor = \ell_j$, so $e_i = (i, \ell_j)$, verifying the claim.

For larger values of $i - \ell_j$, first note that since $B_i \subset B_j$, we must have $\ell_j \leq \ell_i \leq \lfloor x(p_i) \rfloor < i$. If $\ell_j = \lfloor x(p_i) \rfloor$, then e_i is a path from i to ℓ_j. Otherwise, $\lfloor x(p_i) \rfloor - \ell_j < i - \ell_j$, so by induction there is a path from $\lfloor x(p_i) \rfloor$ to ℓ_j, which together with edge e_i yields a path from i to ℓ_j. □

2.1. *Point process invasion percolation in the upper half-plane.* For suitable point sets P, we may hope to define a version of the invasion percolation procedure in which $\mathcal{I} = \mathbb{Z}$ (or more generally when $n_{\mathcal{I}} = -\infty$). This is indeed possible, and the resulting infinite graph can be said to capture the behavior of invasion percolation "very far from the root." A direct inductive description of the graph seems difficult, and so we define the object IPC$(P, \mathbb{Z})$ as the limit of IPC$(P, \mathbb{Z} \cap [m, \infty))$ as $m \to -\infty$. Later, we shall also see how the alternative characterization of the connection rule given by Definition 6 and Lemma 7 can be used to define this extension of the invasion percolation procedure.

As before, we desire as few restrictions on P as possible. In this section, we suppose we are given a set of points $P \subset \mathbb{R}^2$ and an interval $\mathcal{I}$ with $n_{\mathcal{I}} \geq -\infty$ and $m_{\mathcal{I}} \leq \infty$. We say that P is *seemly* (or $\mathcal{I}$-*seemly*, if $\mathcal{I}$ is not clear from context) if P satisfies conditions 1–3 and additionally either (a) $n_{\mathcal{I}} > -\infty$, or (b) $n_{\mathcal{I}} = -\infty$ and P satisfies conditions 4 and 5, below.

4. For any $n \in \mathcal{I}$, there are infinitely many $m \in \mathcal{I} \cap (-\infty, n)$ such that $|[m, n) \times [0, 1]|_P > n - m$.
5. If $\lambda < 1$, then for any $n \in \mathcal{I}$ there are at most finitely many $m \in \mathcal{I} \cap (-\infty, n)$ such that $|[m, n) \times [0, \lambda]|_P \geq n - m$.

The reader can verify that the following two examples almost surely produce seemly point sets.

Stationary limit of IPC on $\mathcal{T}_\sigma$. Let P be defined by choosing σ independent, uniformly random points in the set $[n, n+1) \times [0, \sigma)$ for each $n \in \mathcal{I} = \mathbb{Z}$.

Stationary limit of the Poisson IPC. Let P be a Poisson point process of intensity 1 in the upper half plane, and let $\mathcal{I} = \mathbb{Z}$.

The following lemma essentially states that for $\mathcal{I}$-seemly point sets with $n_{\mathcal{I}} = -\infty$, all edges have weight less than 1.

LEMMA 11. *If $n_{\mathcal{I}} = -\infty$ and P is $\mathcal{I}$-seemly, then for any $n \in \mathcal{I}$ there exists $m_0 \in \mathcal{I}$ such that $h_n(P, \mathcal{I} \cap [m, \infty)) < 1$ for all integers $m \le m_0$.*

PROOF. By condition 4, $|[m, n) \times [0, 1]|_P > n - m$ for infinitely many integers $m < n$; therefore, $h_n(P, \mathcal{I} \cap [m, \infty)) < 1$ for infinitely many integers $m < n$. But $h_n(P, \mathcal{I} \cap [m, \infty)) = y(p_n(P, \mathcal{I} \cap [m, \infty)))$, and $y(p_n(P, \mathcal{I} \cap [m, \infty)))$ is nonincreasing as m decreases, so by condition 3 $y(p_n(P, \mathcal{I} \cap [m, \infty))) < 1$ for all m small enough. □

We next consider the family of intervals $\mathcal{I} \cap [m, \infty)$ for $m \in \mathbb{Z}$, and show that as $m \to -\infty$, each vertex only changes its parent a finite number of times. This allows us to consistently define the limiting object IPC$(P, \mathcal{I})$.

LEMMA 12. *If P is $\mathcal{I}$-seemly, then for any $n \in \mathcal{I}$, there exists $m_0 > -\infty$ such that $p_n(P, \mathcal{I} \cap [m, \infty)) = p_n(P, \mathcal{I} \cap [m_0, \infty))$ for all $m \in \mathcal{I} \cap (-\infty, m_0]$.*

PROOF. The lemma is obvious if $n_{\mathcal{I}} > -\infty$ so assume $n_{\mathcal{I}} = -\infty$. Fix $n \in \mathcal{I}$ and suppose the assertion of the lemma fails for this n. Then there exists a strictly decreasing integer sequence $\{m_i\}_{i=0}^{\infty}$ and a sequence $\{q_i\}_{i=0}^{\infty}$ of distinct points in P such that $p_n(P, \mathcal{I} \cap [m_i, \infty)) = q_i$ for all $i \in \mathbb{N}$, whose y-coordinates decrease strictly as i increases. By Lemma 11, there exists some i_0 such that $y(q_i) < 1$ for all $i \ge i_0$. But then for all $i \ge i_0$, $B_n(P, \mathcal{I} \cap [m_i, \infty)) \subset [\ell_n(P, \mathcal{I} \cap [m_i, \infty)), n) \times [0, y(q_i)]$, and so for such i,

$$\begin{aligned}\big|\big[\ell_n\big(P, \mathcal{I} \cap [m_i, \infty)\big), n\big) \times [0, y(q_i)]\big|_P &\ge \big|B_n\big(P, \mathcal{I} \cap [m_i, \infty)\big)\big|_P \\ &\ge n - \ell_n\big(P, Z \cap [m_i, \infty)\big).\end{aligned}$$

This is a contradiction to condition 5. □

For a seemly point set P, we now define IPC$(P, \mathcal{I})$ to be the graph with vertex set $\mathcal{I}$ and such that for each $n \in \mathcal{I}$, $e_n = e_n(P, \mathcal{I}) = \lim_{m \to -\infty} e_n(P, \mathcal{I} \cap [m, \infty))$. This limit is well defined by the preceding lemma. We likewise define $p_n(P, \mathcal{I})$, $\ell_n(P, \mathcal{I})$, $h_n(P, \mathcal{I})$ and $B_n(P, \mathcal{I})$. By a limiting argument Lemmas 8, 9 and 10 are also valid with respect to IPC$(P, \mathcal{I})$ when $n_{\mathcal{I}} = -\infty$. We therefore obtain the following theorem.

THEOREM 13. *If P is $\mathcal{I}$-seemly, then* IPC$(P,\mathcal{I})$ *is a tree.*

PROOF. Since it is clearly acyclic, we just need to show that IPC$(P,\mathcal{I})$ is connected. Suppose $i, j \in \mathcal{I}$, $i < j$. Let $\ell_j^0 = \ell_j$ and for $t \geq 1$, $t \in \mathbb{N}$, let $\ell_j^t = \ell_{\ell_j^{t-1}}$. Then there must exist $t \in \mathbb{N}$ such that $i \in B_{\ell_j^t}$. By Lemma 10, there is a path between j and ℓ_j^{t+1}, and there is a path between i and ℓ_j^{t+1}. As i and j were arbitrary, this completes the proof. □

An advantage of the current formulation of invasion percolation is that we can equivalently define the limit process via conditions on the numbers of points in boxes $[m,n) \times [0,h]$. More precisely, the following lemma is easily verified.

LEMMA 14. *Suppose P is $\mathcal{I}$-reasonable. Fix k, n with $n_{\mathcal{I}} < k < m_{\mathcal{I}} + 1$ and $0 < n < (k - n_{\mathcal{I}}) + 1$, and $y > 0$. In order that $\ell_k = k - n$ and that $h_0 = y$, it is necessary and sufficient that the following three conditions hold*:

- $|[k-n,k) \times [0,y]|_{\mathcal{P}} = n$ *and* $|[k-n,k) \times [0,y)|_{\mathcal{P}} = n-1$ [*call this condition* $E = E(k-n,k,y,P)$].
- *For all* $0 < m \leq n$, $|[k-m,k) \times [0,y)|_{\mathcal{P}} < m$ [*call this condition* $F = F(k-n,k,y,P)$].
- *For all* $m \in \mathbb{N}$, $|[k-n-m,k-n) \times [0,y]|_{\mathcal{P}} < m$ [*call this condition* $G = G(k-n,y,P)$].

In this case, $B_k = [k-n,k] \times [0,y]$, p_n is the unique point $p \in P$ with $y(p) = y$, and $e_n = (n, \lfloor x(p_n) \rfloor)$.

We will sometimes have use for the condition $G(k-n, y^-)$, which is the same as the condition G above but with $[0,y]$ replaced by $[0,y)$. The next lemma provides a condition under which we can determine the behavior to the right of a given integer n without further reference to the behavior of P to the left of n. Its proof is obvious and is omitted.

LEMMA 15. *Suppose P is $\mathcal{I}$-reasonable. Fix $n_{\mathcal{I}} < n < m_{\mathcal{I}} + 1$ and $y > 0$, let $Q = \{p \in P : x(p) \geq n, y(p) \leq y\}$ and let $\mathcal{J} = \mathcal{I} \cap \{n, \dots, \infty\}$. If $G(n, y^-)$ holds and Q is $\mathcal{J}$-reasonable, then for all m with $n < m < m_{\mathcal{I}} + 1$, $\ell_m(P,\mathcal{I}) = \ell_m(Q,\mathcal{J})$, $h_m(P,\mathcal{I}) = h_m(Q,\mathcal{J})$ and $p_n(P,\mathcal{I}) = p_n(Q,\mathcal{J})$.*

We will also have use of the following sufficient condition for Q to be reasonable. (Again, the proof is straightforward and is omitted.)

LEMMA 16. *Let $\mathcal{I}$, P, k, n and y be as in Lemma 14, let $\mathcal{J} = \{k-n, \dots, k\}$ and let $Q = P \cap ([k-n,k] \times [0,y])$. If E, F and G all hold, then $Q = P \cap B_k$ and Q is $\mathcal{J}$-reasonable.*

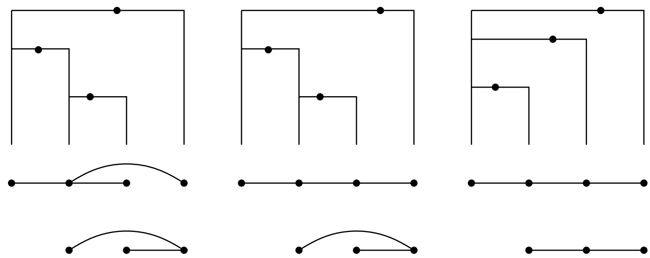

FIG. 2. *The leftmost and middle sets of points have the same* BG *graphs but different* IPC *graphs. The middle and rightmost sets of points have the same* IPC *graph but different* BG *graphs.*

2.2. *Box graphs.* As we saw above, the boxes B_n play a useful role in our study of invasion percolation. The boxes can also be seen to capture information about the structure of the point process invasion percolation procedure itself. For example, it is easily checked that if the procedure explores some edge e lying within a box B_n, then it will explore all other edges lying within B_n before exploring any edges with an endpoint outside of B_n. (Of course, the procedural interpretation does not exist when $\mathcal{I} = \mathbb{Z}$, but in this case we can still think of the boxes as capturing information about the process behavior "far from the root.")

In this section, we introduce a graph which characterizes the containment relation among the boxes. Given an $\mathcal{I}$-reasonable point set P, we define $\mathrm{BG}(P, \mathcal{I})$ to be the graph with vertex set $\mathcal{I} \setminus \{n_\mathcal{I}, +\infty\}$ and such that, for $i < j$, i and j are joined by an edge if and only if $B_i(P, \mathcal{I}) \subseteq B_j(P, \mathcal{I})$ and $B_i(P, \mathcal{I}) \not\subseteq B_{j'}(P, \mathcal{I})$ for any $i < j' < j$. Also, for $i \in \mathcal{I} \setminus \{n_\mathcal{I}, m_\mathcal{I}\}$, we write $a_i(P, \mathcal{I})$ for the parent of i in $\mathrm{BG}(P, \mathcal{I})$.

The examples shown in Figure 2 demonstrate that between the graphs $\mathrm{IPC}(P, \mathcal{I})$ and $\mathrm{BG}(P, \mathcal{I})$, neither is determined by the other. [Theorem 1(c) is essentially a consequence of this fact.]

Clearly, $\mathrm{BG}(P, \mathcal{I})$ is acyclic for any $\mathcal{I}$. We shall show that $\mathrm{BG}(P, \mathbb{Z})$ is a tree (i.e., connected) under the additional assumption of the "rightward version" of condition 5.

6. If $\lambda < 1$, then for any $m \in \mathcal{I}$, there are at most finitely many $n \in \mathbb{Z} \cap (m, \infty)$ such that $|[m, n) \times [0, \lambda]|_P \geq n - m$.

If P satisfies conditions 1–6 with $\mathcal{I} = \mathbb{Z}$, we say that P is *exemplary*. Both examples of the last subsection are almost surely exemplary point sets.

LEMMA 17. *Suppose $m_\mathcal{I} = \infty$ and P is an $\mathcal{I}$-reasonable point set that satisfies condition 6. Choose any $m \in \mathcal{I} \setminus \{n_\mathcal{I}, +\infty\}$ for which $h_m < 1$ and for which there is no $m' \in \mathbb{Z} \cap (m, \infty)$ such that $B_m \subseteq B'_m$ and $h'_m \geq 1$. Then there are infinitely many $n \in \mathbb{Z} \cap (m, \infty)$ such that $B_m \subseteq B_n$.*

PROOF. Suppose m is as in the statement of the lemma but that there are only finitely many $n > m$ such that $B_m \subseteq B_n$. Then by replacing B_m with the tallest box that contains it, we may assume that in fact there is no $n > m$ such that $B_m \subseteq B_n$. By condition 6, we may choose $n > m$ for which $|[m, n) \times [0, h_m]|_P < n - m$. Thus, there must be $i \in \{m + 1, \ldots, n\}$ for which $h_i > h_m$, so take i minimum such that this holds. By Lemma 8, we must then have $\ell_i < m$, and so by Lemma 9 we must have $B_m \subset B_i$, a contradiction. □

Before showing that $\mathrm{BG}(P, \mathbb{Z})$ is a tree, let us first use the lemma to confirm the basic property of exemplary point sets that every point of P under the line $y = 1$ lies along the top of some box B_n.

PROPOSITION 18. *If P is exemplary, then for all $p \in P \cap [n_{\mathcal{I}}, +\infty) \times [0, 1)$, we have $p = p_n$ for some $n \in \mathcal{I}$.*

PROOF. Let $p \in P$ have $y(p) < 1$. We first note that if $p \in B_m$ for some m, then $p = p_n$ for some $\lceil x(p) \rceil \le n \le m$. Also, there must be some integer $k \le x(p)$ for which $h_k > y(p)$, or else $|[\lceil x(p) \rceil - i, \lceil x(p) \rceil] \times [0, y(p)]|_P \ge i$ for infinitely many integers $i > 0$, which contradicts condition 5.

By Lemma 11, $h_m < 1$ for all $m \in \mathbb{Z}$, so by Lemma 17, there are infinitely many $m \in \mathbb{Z}$ for which $B_k \subseteq B_m$. One of these boxes contains p, so $p = p_n$ for some n, as claimed. □

THEOREM 19. *If P is exemplary, then* $\mathrm{BG}(P, \mathbb{Z})$ *is a tree.*

PROOF. It suffices to show that $\mathrm{BG}(P, \mathbb{Z})$ is connected. Recall that $h_n < 1$ for any $n \in \mathbb{Z}$, by Lemma 11. Fix $i < j$, $i, j \in \mathbb{Z}$. By Lemma 17 there are infinitely many m such that $B_i \subset B_m$. Take the least such m for which $m \ge i$—then also $B_j \subseteq B_m$, and so by Lemma 10 there exist paths from i to m and from j to m. The theorem follows. □

2.3. *Random walks and the forward maximal process.* Let P be a point set satisfying condition 1. Given $h > 0$ and $k \in \mathcal{I}$, we define random walks $S^{k,h} = S^{k,h}(P)$ and $L^{k,h} = L^{k,h}(P)$ as follows. We set $S_0^{k,h} = L_0^{k,h} = 0$ and, for $i \ge 1$, set $S_i^{k,h} = |[k, k+i] \times [0, h]|_P - i$, and set $L_i^{k,h} = |[k, k-i] \times [0, h]|_P - i$. We also define random walks S^{k,h^-} and L^{k,h^-}, by replacing $[0, h]$ by $[0, h)$ in the above definitions. In other words, the random walks S^{k,h^-} and L^{k,h^-} ignore points on the line $y = h$. (For fixed h, for any of the random point sets P we will consider, it will be the case that with probability 1, $S_i^{k,h} = S_i^{k,h^-}$ for all i, but we will at times work in conditional settings in which these two random walks are not identical.)

We say that $S^{k,h}$ *survives* if for all $i \ge 0$, $S_i^{k,h} \ge 0$, and otherwise say that $S^{k,h}$ *dies*. Also, we say that $S^{k,h}$ *has a chance* if $S_i^{k,h} \ge 0$ for some $i > 0$, and otherwise that $S^{k,h}$ *has no chance*. We extend these definitions to $L^{k,h}$ by symmetry.

We now establish two more basic properties of $\mathrm{IPC}(\mathcal{P}, \mathcal{I})$, under the following additional assumptions.

7. If $\lambda > 1$, then for any $m \in \mathbb{Z}$, $S_n^{m,\lambda} \leq n$ for at most finitely many $n \in \mathbb{N}$.
8. $S^{k,1}$ dies for all $k \in \mathcal{I}$.

Roughly speaking, condition 7 is a "rightward version" of condition 4. If P is an $\mathcal{I}$-reasonable point set that satisfies conditions 6, 7 and 8, we say that P is *distinguished* (or *$\mathcal{I}$-distinguished*, if $\mathcal{I}$ is not clear from context). The first two examples given in the introduction to this section are almost surely distinguished point sets.

We will see that for distinguished point sets P, when $n_{\mathcal{I}} > -\infty$ and $m_{\mathcal{I}} = \infty$, $\mathrm{BG}(P, \mathcal{I})$ is *not* connected—in this case we call the connected components the *ponds* of $\mathrm{BG}(P, \mathcal{I})$. We will see later that this agrees with the normal use of this term in the invasion percolation literature.

LEMMA 20. *If $n_{\mathcal{I}} > -\infty$, $m_{\mathcal{I}} = \infty$ and P is $\mathcal{I}$-distinguished, then for any $m \in \mathcal{I} \setminus \{n_{\mathcal{I}}, +\infty\}$, if $h_m \geq 1$, then there are at most finitely many $n \in \mathbb{Z}$, $n > m$, such that $B_m \subseteq B_n$.*

PROOF. Suppose otherwise. Without loss of generality, we may assume that $h_m > 1$. Consider the integer sequence $\{n_i\}_{i=0}^{\infty}$, which is defined as follows. Let $n_0 = m$. For $i \in \mathbb{N}$, let n_{i+1} be the smallest integer greater than n_i such that $B_{n_i} \subseteq B_{n_{i+1}}$. Then for any $i \in \mathbb{N}$ and all $n_i < n < n_{i+1}$, we have $B_n \subseteq [n_i, n_{i+1}) \times [0, h_{n_i}]$ for all $n_i < n < n_{i+1}$ by Lemmas 8 and 9. Furthermore, it follows from the definition of B_n and Lemma 9 that $|([n_i, n_{i+1}) \times [0, h_{n_{i+1}})) \setminus (\bigcup_{n_i < n < n_{i+1}} B_n)|_P = 0$ (or otherwise there would be a smaller choice for $h_{n_{i+1}}$). Thus, $|[n_i, n_{i+1}) \times [0, h_{n_i}]|_P = |\bigcup_{n_i < n < n_{i+1}} B_n|_P = n_{i+1} - n_i - 1$. Since $h_{n_0} < h_{n_i}$ for all $i > 0$, it follows that $|[n_0, n_{i+1}) \times [0, h_{n_0}]|_P < n_{i+1} - n_0$ for all $i > 0$. Since $h_{n_0} > 1$, this is a contradiction to condition 7. $\square$

THEOREM 21. *If $n_{\mathcal{I}} > -\infty$, $m_{\mathcal{I}} = \infty$ and P is $\mathcal{I}$-distinguished, then $\mathrm{BG}(P, \mathcal{I})$ contains infinitely many components, all of which are finite. Furthermore, for any given component, if n is the rightmost integer belonging to the component, then $h_n > 1$ and the set of vertices of the component is $\{\ell_n + 1, \ldots, n\}$.*

PROOF. Let P satisfy the hypothesis of the theorem. We construct a sequence of integers $\{n_i\}_{i=0}^{\infty}$ as follows. Let $n_0 = n_{\mathcal{I}}$. For $i \in \mathbb{N}$, let n_{i+1} be the largest integer greater than n_i such that $B_{n_{i+1}}$ contains the point $(n_i, 0)$. We must now show this sequence is well defined. Suppose not, and let i be minimum such that there is no valid choice for n_{i+1}. Then there are infinitely many integers $n > n_i$ such that $B_{n_i} \subseteq B_n$. Since i was chosen minimum, for each such n we have $\ell_n = n_i$. By Lemma 20, it must be that for each such n, $h_n < 1$. But this implies that $|[n_i, n) \times [0, 1]|_P \geq n - n_i$ for all $n \in \mathbb{Z} \cap [n_i, \infty)$, a contradiction to condition 8. Thus the sequence $\{n_i\}_{i=0}^{\infty}$ is well defined.

For all $i \in \mathbb{N}$, it follows from the definition of n_{i+1} that $\ell_{n_{i+1}} = n_i$, and there is no integer $n > n_{i+1}$ for which $B_{n_{i+1}} \subseteq B_n$; thus $B_{n_{i+1}}$ and $B_{n_{i+2}}$ are in separate components of $\mathrm{BG}(P, \mathcal{I})$. By Lemma 9, all B_n such that $n_i < n \leq n_{i+1}$ are contained in $B_{n_{i+1}}$ and hence in the same component of $\mathrm{BG}(P, \mathcal{I})$.

If $h_{n_{i+1}} < 1$ for some $i \in \mathbb{N}$, then by Lemma 17, there are infinitely many $n \in \mathbb{Z} \cap (n_{i+1}, \infty)$ such that $B_{n_{i+1}} \subseteq B_n$, but this is a contradiction to the choice of n_{i+1}. By Lemma 8, we have for all $i \in \mathbb{N}$ that $h_{n_{i+1}} > h_{n_{i+2}}$. We conclude that $h_{n_{i+1}} > 1$ for all $i \in \mathbb{N}$. □

Given an interval $\mathcal{I}$, with $n_{\mathcal{I}} > -\infty$ and $m_{\mathcal{I}} = \infty$, and an $\mathcal{I}$-distinguished point set P, define $\{n_i\}_{i=0}^{\infty} = \{n_i(P, \mathcal{I})\}_{i=0}^{\infty}$ as in the proof of Theorem 21.

COROLLARY 22. *If $n_{\mathcal{I}} > -\infty$, $m_{\mathcal{I}} = \infty$ and P is $\mathcal{I}$-distinguished, then* $\mathrm{IPC}(P, \mathcal{I})$ *is a tree that consists of a unique infinite backbone (i.e., a unique, infinite, self-avoiding path originating from the root) from which emerge finite branches. Furthermore, the backbone contains the points $\{n_i\}_{i=0}^{\infty}$.*

PROOF. Clearly, $\mathrm{IPC}(P, \mathcal{I})$ is acyclic, and is connected by Lemma 10, so is a tree. For each integer $i \geq 1$, let Φ_i be the unique path from n_i to $n_0 = n_{\mathcal{I}}$ in $\mathrm{IPC}(P, \mathcal{I})$. By Lemma 10, it follows that for all $i \geq 1$, Φ_i is a sub-path of Φ_{i+1}, and so the limit $\Phi = \lim_{i \to \infty} \Phi_i$ is a well-defined infinite path starting from $n_{\mathcal{I}}$. Furthermore, for any integer k, any path Φ' starting from v_k, that is, edge-disjoint from Φ must have all its elements among $n_i, \dots, n_{i+1} - 1$, where $n_i \leq v_k < n_{i+1}$. Thus, all branches leaving Φ are finite. □

REMARK. In general, relaxing any of the conditions in the definition of distinguished point sets may cause the conclusions of Corollary 22 to fail. To provide just one example, the following point set satisfies conditions 1–7, but not the conclusion of Corollary 22. The point set P contains no points except the following. Place k points inside $[0, 1) \times [0, 1/3)$. Place each of the points $(i+1, 1-1/(i+2))$ for $i \in \mathbb{N}$. Then $\mathrm{IPC}(P, \mathbb{N})$ has k infinite backbones.

This completes our study of deterministic properties of the invasion percolation procedure. In the next section we begin our study of what happens when the underlying point set is random.

3. Invasion percolation on the PWIT. Throughout Section 3, $\mathcal{P}$ denotes a Poisson process of constant intensity 1 in $[0, \infty) \times [0, \infty)$, so $\mathcal{P}$ is almost surely $\mathbb{N}$-distinguished. By Corollary 5, $\mathrm{IPC}(\mathcal{P}, \mathbb{N})$ is distributed as the invasion percolation cluster $\mathcal{T}_0$ of the PWIT, so results for $\mathrm{IPC}(\mathcal{P}, \mathbb{N})$ apply to $\mathcal{T}_0$ mutatis mutandis. Below, we will derive more precise statements about the structure of $\mathrm{IPC}(\mathcal{P}, \mathbb{N})$ than can be made under the assumptions of Section 2. First, however, we state two "ballot-style" theorems for stochastic processes that we will use repeatedly.

3.1. *Two ballot-style theorems*. The following result was proved independently by Tanner [22] and Dwass [8].

LEMMA 23 (Cycle lemma). *Suppose that* $X_1, \dots, X_n$ *are integer-valued, cyclically interchangeable random variables with maximum value* 1. *Then for any integer* $0 \le k \le n$,

$$\mathbf{P}\{S_i > 0\ \forall 1 \le i \le n \mid S_n = k\} = \frac{k}{n}.$$

The next result was proved by Tákacs [21], page 12.

LEMMA 24 (Stationary ballot theorem). *Let* $X_1, X_2, \dots$ *be an infinite sequence of i.i.d. integer random variables with mean* μ *and maximum value* 1, *and for any* $i \ge 1$, *let* $S_i = X_1 + \cdots + X_i$. *Then*

$$\mathbf{P}\{S_n > 0\ \forall n \in \{1, 2, \dots\}\} = \begin{cases} \mu, & \text{if } \mu > 0, \\ 0, & \text{if } \mu \le 0. \end{cases}$$

Now let P be a random point set in, say, $[m, n] \times [0, \infty)$. We recall the definition of the condition $F(m, n, y)$ from Lemma 14, and will abuse notation by also writing $F(m, n, y, P)$ for the *event* that the *condition* $F(m, n, y, P)$ holds. [At times we write $F(m, n, y)$ in place of $F(m, n, y, P)$, when P is clear from context.] Notice that if P is a uniform set of $n-1$ points in $[0, n] \times [0, \lambda)$, for some $\lambda > 0$, then applying the cycle lemma with $X_i = 1 - |[i-1, i) \times [0, \lambda)|_P$ (and so $S_i = i - |[0, i) \times [0, \lambda)|_P$) for $i = 1, \dots, n$, it follows that the probability that $F(0, n, \lambda, P)$ occurs is precisely $1/n$. By an argument of a similar nature, we can straightforwardly derive the following lemma (which can also be deduced from an existing result ([17], Theorem 4) for invasion percolation on K_n and a limiting argument).

LEMMA 25. *Fix an integer* $n \ge 1$, *and list the* n *elements of* $\mathcal{P} \cap ([0, n] \times [0, \infty))$ *of lowest height as* $q_1, \dots, q_n$, *in increasing order of height. Then for each* $i = 1, \dots, n$, $\mathbf{P}\{p_n = q_i\} = 1/n$.

We emphasize that the n elements of $\mathcal{P} \cap ([0, n] \times [0, \infty))$ of lowest height may not all be elements of the set $\{p_1, \dots, p_n\}$, or indeed of the set $\{p_i\}_{i=1}^{\infty}$.

PROOF OF LEMMA 25. Fix n, and let $\lambda = y(q_n)$. Clearly, p_n will be among $q_1, \dots, q_n$. Also, let $\mathcal{I} = \{0, \dots, n\}$, let $P = \{q_1, \dots, q_n\}$ and for each $i \in \{0, \dots, n-1\}$, let $P^i = \{q_1^i, \dots, q_n^i\}$ be the cyclic shift of P to the right by distance i. Then for all $i \in \{1, \dots, n-1\}$, P^i is distributed as $n-1$ uniform points in $[0, n] \times [0, \lambda)$, together with a single uniform point of height λ. We claim that with probability 1, for each $j = 1, \dots, n$, there is exactly one

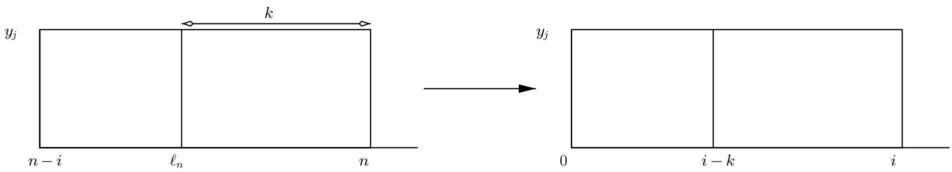

FIG. 3. *The region on the left maps onto the region on the right when P is sent to P^i.*

$i = i(j, P) \in \{0, \dots, n-1\}$ for which $p_n(P^i, \mathcal{I}) = q^i_j$. Since the P^i are identically distributed it follows from this claim that

$$\mathbf{P}\{p_n(P, \mathcal{I}) = q_j\} = \frac{1}{n}\sum_{i=0}^{n-1}\mathbf{P}\{p_n(P^i, \mathcal{I}) = q^i_j\} = \frac{1}{n}\mathbf{P}\left\{\bigcup_{i=0}^{n-1}\{p_n(P^i, \mathcal{I}) = q^i_j\}\right\} = \frac{1}{n},$$

which proves the theorem. It thus remains to prove the above claim, which we do by contradiction. Thus, suppose that for some $j \in \{1, \dots, n\}$, there are distinct $i, i' \in \{0, \dots, n-1\}$ for which $p_n(P^i, \mathcal{I}) = q^i_j$ and $p_n(P^{i'}, \mathcal{I}) = q^{i'}_j$. By replacing P by either P^{n-i} or $P^{n-i'}$ if necessary, we may assume that $i' = 0$. Let $q_j = q^0_j = (x_j, y_j)$. We must have $|[n-i, n] \times [0, y_j)|_P < i$ [or else $p_n(P, \mathcal{I}) \neq q_j$]; on the other hand, $|[\ell_n(P, \mathcal{I}), n] \times [0, y_j]|_P = n - \ell_n(P, \mathcal{I})$.

Let $k = n - \ell_n(P, \mathcal{I})$, the length of $B_n(P, \mathcal{I})$. If $k \geq i$, then we also have $|[\ell_n, n-i] \times [0, y_j]|_P \geq n - \ell_n - i + 1$, so $|[\ell_n + i, n] \times [0, y_j]|_{P^i} \geq n - \ell_n - i + 1$ and $h_n(P^i, \mathcal{I}) < y_j$, contradicting the fact that $p_n(P^i, \mathcal{I}) = q^i_j$. It follows that $k < i$, that is, that $i - (n - \ell_n) \geq 1$. In this case, we have that for each $m \in \{1, \dots, i - (n - \ell_n)\}$, $[\ell_n - m, \ell_n] \times [0, y_j] < m$ (or else we would have either chosen h_n lower or ℓ_n smaller). Translating the above information to P^i, we see that $|[i-k, i] \times [0, y_j]|_{P^i} = k$, that $|[i-k, i] \times [0, y_j)|_{P^i} < k$, and that $|[i-k', i] \times [0, y_j]|_{P^i} < k'$ for each $k' \in \{k+1, \dots, i\}$ (see Figure 3). Thus, by Definition 6 and Lemma 7, $p_i(P^i, \mathcal{I}) = q^i_j$, contradicting the assumption that $p_n(P^i, \mathcal{I}) = q^i_j$. □

We next elaborate on a connection between Poisson Galton–Watson trees and queueing theory that will be useful for many subsequent calculations.

3.2. *A fact from queuing theory and an aside on Poisson Galton–Watson duality.* The following basic result was first noted by Borel [4]. Consider a queue with Poisson rate λ arrivals and constant, unit service time, started at time zero with a single customer in the queue, and with any arbitrary servicing rule (i.e., not necessarily first-in first-out). We may form a rooted tree associated with the queueing process run until the first time τ that there are no customers in the queue (or forever, if the queue is never empty), in the following manner. If a new customer joins the queue at time t, he is joined to the customer being served at time t. We denote the resulting rooted tree by $\mathcal{T}$. Then $\mathcal{T}$ is distributed as a Poisson(λ) Galton–Watson tree [we write PGW(λ), for short] [4].

If the arrival times are given by the x-coordinates of the points of Poisson process $\mathcal{Q} = \mathcal{P} \cap ([0,\infty) \times [0,\lambda))$, we may also associate an interpolated random walk to the process, by setting $S_t = |[0,t) \times [0,\lambda)|_{\mathcal{Q}} - t$ for $t \in \mathbb{R}^+$. Then $|\mathcal{T}|$ is simply the first time t that $S_t = -1$, that is, that $|[0,t) \times [0,\lambda)|_{\mathcal{Q}} = t - 1$. Note that given that $|T| = m < \infty$, $\mathcal{Q}$ is distributed as $m - 1$ independent uniform points in $[0,t) \times [0,\lambda]$, conditioned on $F(0,n,\lambda,\mathcal{Q})$ occurring. We also observe that for all $i \le |\mathcal{T}|$, $S_i = S_i^{0,\lambda^-}$, where S^{0,λ^-} is the random walk defined in Section 2.3. This has immediate implications for the events defined in Section 2.3. In particular, the tree $\mathcal{T}$ is infinite if and only if S^{0,λ^-} survives. It follows that for all $0 \le h \le 1$ we have $\mathbf{P}\{S^{k,\lambda^-} \text{ survives}\} = 0$, and for all $\lambda > 1$ we have $\mathbf{P}\{S^{k,\lambda^-} \text{ survives}\} = \theta(\lambda)$, the probability of survival of a PGW(λ) branching process. Similarly, by Lemma 24, we have that the probability that there is ever a time t at which the total number of arrivals is at least t, is $\min(\lambda, 1)$. Thus, if $0 < \lambda \le 1$, then $\mathbf{P}\{S^{k,\lambda^-} \text{ has a chance}\} = \lambda$, and if $\lambda > 1$ then $\mathbf{P}\{S^{k,\lambda^-} \text{ has a chance}\} = 1$. Of course, the exact same identities hold with S^{k,λ^-} replaced by $S^{k,\lambda}$, L^{k,λ^-} or $L^{k,\lambda}$.

We continue to think of arrival times as given by points of $\mathcal{Q}$. It will be useful for us to view the above queuing procedure as creating a tree whose nodes are labeled by integers rather than by elements of the queue. We do so by re-labeling each node of $\mathcal{T}$ (i.e., each customer c) with the (integer) time at which c begins being served. Furthermore, suppose that we take as our servicing rule the invasion percolation rule—that is, the rule that prioritizes customers (points of $\mathcal{Q}$) with lower y-coordinate over those with higher y-coordinate—and call the resulting tree $\mathcal{T}_\lambda$. Then $\mathcal{T}_\lambda$ is precisely the subtree of IPC($\mathcal{P}, \mathbb{N}$) containing the root and all nodes joined to the root by paths all of whose edges have weight less than λ. Of course, everything still holds if we take $\mathcal{Q} = \mathcal{P} \cap ([0,\infty) \times [0,\lambda])$—that is, if we include points at height precisely λ—as long as we replace $S^{0,\lambda-}$ by $S^{0,\lambda}$ and replace the phrase "less than λ" by "at most λ."

As a consequence of the above discussion we have the following important fact.

LEMMA 26. *Fix any integer $n \ge 1$, any $\lambda > 0$, and let P be a set of $n - 1$ independent uniform points in $[0,n] \times [0,\lambda)$. Given that $F(0,n,\lambda,P)$ occurs, the tree* IPC($P, \{0,\dots,n-1\}$) *is distributed as* PGW(λ) *conditioned to have n nodes. Furthermore, suppose that p is a uniformly random point on the line segment $[0,n] \times \{\lambda\}$. Then under the same conditioning,* IPC($P \cup \{p\}, \{0,\dots,n\}$) *is distributed as* PGW(λ) *conditioned to have n nodes, together with an additional vertex (vertex n), joined to a uniformly random element of* $0,\dots,n-1$.

We remark that for any fixed n, the distribution of PGW(λ) conditioned to have n nodes does not depend on λ and is precisely that of a uniformly random labelled rooted tree (or Cayley tree) on n nodes, after the labels but not the orders of children have been discarded. It turns out that a version of Lemma 26 also holds for the box tree; see Lemma 37, below. As noted just after Lemma 24, the probability that $F(0, n, \lambda, P)$ occurs is precisely $1/n$. Thus, the distribution of $|\mathrm{PGW}(\lambda)|$ is given by

$$\mathbf{P}\{|\mathrm{PGW}(\lambda)| = n\} = \frac{1}{n}\mathbf{P}\{\mathrm{Poisson}(\lambda n) = n-1\} = \frac{e^{-\lambda n}(\lambda n)^{n-1}}{n!},$$

for all positive integers n (a well-known fact which we record for later reference). When $\lambda = 1$ this is called the *Borel* distribution.

We briefly explain a further basic fact about the function $\theta(\lambda) = \mathbf{P}\{|\mathrm{PGW}(\lambda)| = \infty\}$ and about Poisson Galton–Watson duality. By considering the number of children in the first generation of $\mathrm{PGW}(\lambda)$, we see that $1 - \theta(\lambda) = e^{-\lambda\theta(\lambda)}$, and by differentiating this identity, we see that

$$\theta'(\lambda)\bigl(1 - \lambda\bigl(1 - \theta(\lambda)\bigr)\bigr) = \theta(\lambda)\bigl(1 - \theta(\lambda)\bigr), \tag{1}$$

an equation we will have use of later. Next, given $\lambda > 1$, let $m = m(\lambda) < 1$ be such that $\lambda e^{-\lambda} = me^{-m}$ (we call m the *dual parameter* for λ). Then $m = \lambda(1 - \theta(\lambda))$, from which it is easily seen that conditional on being finite, $\mathrm{PGW}(\lambda)$ is distributed precisely as $\mathrm{PGW}(m)$.

3.3. $\mathrm{IPC}(\mathcal{P}, \mathbb{N})$ *and the forward maximal process.* By Corollary 22, $\mathrm{IPC}(\mathcal{P}, \mathbb{N})$ consists of a unique infinite backbone which in particular passes through the nodes $\{n_i\}_{i=0}^{\infty}$, and from all nodes of which emerge finite branches. Let the edges of the backbone be $e_1, e_2, \dots,$ and for each integer $i \geq 1$ let $W_i = \sup_{j \geq i} W_{e_j}$, so $\{W_i\}_{i=1}^{\infty}$ is the PWIT forward maximal process. From the perspective of the PWIT, the nodes n_i are the nodes at which the forward maximal weight along the backbone decreases.

Lemma 26 allows us to provide another picture of the structure of $\mathrm{IPC}(\mathcal{P}, \mathbb{N})$. First, for each integer $i \geq 0$, let $T_i = T_i(\mathcal{P}, \mathbb{N})$ be the subtree of $\mathrm{IPC}(\mathcal{P}, \mathbb{N})$ on nodes $n_i, \dots, n_{i+1} - 1$ (these nodes induce a tree by Lemma 10). The set $P = \mathcal{P} \cap ([n_i, n_{i+1}) \times [0, h_{n_{i+1}}))$ is distributed as $(n_{i+1} - n_i - 1)$ independent uniform points, conditional on $F(n_i, n_{i+1}, h_{n_{i+1}}, P)$ occurring. Furthermore, $\mathcal{P} \cap ([n_i, n_{i+1}) \times \{h_{n_{i+1}}\})$ contains a single uniform point. Thus, by Lemma 26, we obtain that T_i is distributed as $\mathrm{PGW}(h_{n_{i+1}})$ conditioned to have $n_{i+1} - n_i$ nodes, and that n_{i+1} is joined to a uniformly random element of T_i. Applying this for all i, we obtain the following theorem.

THEOREM 27. *Given $\{n_i\}_{i=0}^{\infty}$, $\mathrm{IPC}(\mathcal{P}, \mathbb{N})$, viewed as an unlabeled tree, can be built as follows. For each integer $i \geq 0$ let T_i be a uniformly random labeled tree on $n_i - n_{i-1}$ vertices. For each integer $i \geq 1$, join the root of T_i to a uniformly random vertex of T_{i-1}. Finally, discard all labels.*

REMARK. It also follows straightforwardly from Lemma 26 that given $\{n_i\}_{i=0}^{\infty}$ and $\{h_{n_{i+1}}\}_{i=0}^{\infty}$, $\mathrm{IPC}(\mathcal{P}, \mathbb{N})$ viewed as a *weighted* unlabeled tree can be built from the tree described in Theorem 27 as follows. Independently for each integer $i \geq 0$ and each edge e of T_i, assign e a random weight with Uniform$[0, h_{n_{i+1}}]$ distribution. Also, for each integer $i \geq 0$, give the unique edge from T_{i+1} to T_i the weight $h_{n_{i+1}}$. We omit the details.

We next show that $\{(|T_{i-1}|, h_{n_i})\}_{i=1}^{\infty} = \{(n_i - n_{i-1}, h_{n_i})\}_{i=1}^{\infty}$ is a Markov process and specify the transition probabilities. First, for any $i \geq 1$, given h_{n_i}, the set $\mathcal{Q} = \mathcal{P} \cap ([n_i, \infty) \times [0, h_{n_i}))$ is precisely a Poisson point process of intensity 1 conditioned on the event that $S^{n_i, h_{n_i}^-}(\mathcal{Q})$ survives (which is precisely the event that $\mathcal{Q}$ is $\{n_i, n_{i+1}, \ldots, \}$-reasonable). Furthermore, given h_{n_i}, the condition $G(n_i, h_{n_i}^-)$ holds for $\mathcal{P}$. Thus, by Lemma 15, we can determine the structure of IPC$(\mathcal{P}, \mathbb{N})$ restricted to $\{n_i, \ldots, \infty\}$ by considering only the points in $\mathcal{Q}$. It follows that $\{(n_i - n_{i-1}, h_{n_i})\}_{i=1}^{\infty}$ is a Markov process, as claimed (and also that $\{h_{n_i}\}_{i=1}^{\infty}$ is a Markov process). Next, for $1 < y < h$, let

$$f_h(y) = \lim_{dy \to 0} \frac{\mathbf{P}\{h_{n_{i+1}} \in dy \mid h_{n_i} = h\}}{dy},$$

and for $n > 0$ let

$$f_h(n, y) = \lim_{dy \to 0} \frac{\mathbf{P}\{h_{n_{i+1}} \in dy, (n_{i+1} - n_i) = n \mid h_{n_i} = h\}}{dy},$$

so $f_h(y) = \sum_n f_h(n, y)$. By the above comments, $f_h(y)$ and $f_h(n, y)$ do not depend on i.

LEMMA 28. *For all $i \geq 1$ and $1 < y < h$,*

$$f_h(y) = \frac{\theta'(y)}{\theta(h)} \quad \text{and} \quad f_h(n, y) = \frac{\theta(y)}{\theta(h)} \frac{e^{-yn}(yn)^{n-1}}{(n-1)!}.$$

Combining the two results in Lemma 28, the following corollary is immediate.

COROLLARY 29. *For all integers $n, i \geq 1$ and $y > 1$,*

$$\mathbf{P}\{n_{i+1} - n_i = n \mid h_{n_{i+1}} = y\} = \frac{\theta(y)}{\theta'(y)} \frac{e^{-ny}(ny)^{n-1}}{(n-1)!}. \tag{2}$$

PROOF OF LEMMA 28. We have

$$\begin{aligned} \mathbf{P}\{h_{n_{i+1}} \leq y \mid h_{n_i} = h\} &= \mathbf{P}\{S^{n_i, y}(\mathcal{P}) \text{ survives} \mid S^{n_i, h^-}(\mathcal{P}) \text{ survives}\} \\ &= \frac{\mathbf{P}\{S^{n_i, y}(\mathcal{P}) \text{ survives}\}}{\mathbf{P}\{S^{n_i, h^-}(\mathcal{P}) \text{ survives}\}} \\ &= \frac{\theta(y)}{\theta(h)}, \end{aligned}$$

and the first claim of the lemma follows by differentiation.

As mentioned, $f_h(y)$ and $f_h(n, y)$ do not depend on i, so we take $i = 0$ (and thus $n_i = 0$). In order to have $n_1 - n_0 = n$ and $h_{n_1} \in dy$, we need that $|[0, n] \times [0, y)|_{\mathcal{P}} = n - 1$, that $|[0, n] \times [y, y + dy)|_{\mathcal{P}} = 1$, that $F(0, n, y, \mathcal{P})$ occurs and that $S^{n,y^-}(\mathcal{P})$ survives. The probabilities of the first two events are easily bounded. The probability of $F(0, n, y, \mathcal{P})$ given that $|[0, n] \times [0, y)|_{\mathcal{P}} = n - 1$ is $1/n$ by the cycle lemma. Finally, the event that $S^{n,y^-}(\mathcal{P})$ survives is independent of the first three events, and has probability $\theta(y)$. Thus,

$$\begin{aligned}\mathbf{P}\{h_{n_{i+1}} \in dy, (n_{i+1} - n_i) = n \mid h_{n_i} = h\} \\ = \frac{e^{-yn}(yn)^{n-1}}{(n-1)!} \cdot \bigl(1 + o(dy)\bigr) n\, dy \cdot \frac{1}{n} \cdot \theta(y) \cdot \frac{1}{\theta(h)},\end{aligned}$$

from which the second claim of the lemma follows. □

We next derive the distribution of the distance along the backbone between n_{i-1} and n_i. For $i \geq 0$ let $d_i = d_i(\mathcal{P}, \mathcal{I}) = d_{\mathrm{IPC}}(n_i, n_{i+1})$. As with the quantities studies above, we have that given $h_{n_{i+1}}$, d_i is independent of the past. For $0 < x < 1$ we say $X \stackrel{d}{=} \mathrm{Geometric}(x)$ if $\mathbf{P}\{X = k\} = x^k(1 - x)$.

THEOREM 30. *For all $i \geq 0$ and all $y > 1$, given that $h_{n_{i+1}} = y$, $d_i \stackrel{d}{=} 1 +$* Geometric$(m(y))$.

The following theorem derives the distributions of the trees hanging off the backbone and within a given pond.

THEOREM 31. *Fix $i \geq 0$ and $y > 1$. Given that $h_{n_{i+1}} = y$, for all k with $n_i \leq k < n_{i+1}$ and for which k is on the backbone, the subtree of* IPC$(\mathcal{P}, \mathbb{N})$ *containing k and containing no other vertices of the backbone, is distributed as* PGW$(m(y))$.

Together, Theorems 30 and 31 provide another Markovian characterization of IPC$(\mathcal{P}, \mathbb{N})$: we may construct a tree with the distribution of IPC$(\mathcal{P}, \mathbb{N})$ by growing the trees hanging off the backbone one-by-one, where the branching distribution of the trees depends on the current forward maximal weight. (This characterization is exactly that which is claimed in Theorem 3, which is proved below.) The forward maximal weight process evolves according to the dynamics implied by Lemma 28 and Theorem 30: first stay constant for a geometric amount of time depending on the current forward maximal weight, then decrease the maximal weight according to Lemma 28. This characterization is essentially the $\sigma \to \infty$ limit of results of Angel et al. [3] described in the Introduction, for invasion percolation on the regular σ-ary tree. However, it does not seem trivial to derive these results from theirs by a limiting argument and local weak convergence, since they depend not only on the graph structure of the tree but also on the weights.

If one wishes, at this point one can apply all the methodology of [3] to see that corresponding results hold for invasion percolation on the PWIT: notably, convergence to the Poisson lower envelope, mutual singularity of IPC and IIC measures and spectral asymptotics all hold for invasion percolation on the PWIT. We have not included the details as the development is essentially technical, requiring no significant ideas not already found in [3].

Since $m(\lambda) = \lambda(1 - \theta(\lambda))$, we also obtain the following corollary of Theorems 27, 30 and 31 and Corollary 29, which is new as far as we know. For any $0 < p < 1$, let $D = \text{Geometric}(p)$, and let $r = v_0, v_1, \dots, v_D = v$ be a path of length D. For each $k \in \{0, \dots, D\}e$, starting from v_k grow a PGW(p) tree with root v_k. This yields a triple $(T, r, v)_p$.

COROLLARY 32. *Fix any $0 < p < 1$ and let $(T, r, v)_p$ have the distribution described above. Then conditional on $|T|$, T is distributed as* PGW(p) *conditioned to have size $|T|$, and v is distributed as a uniformly random node of T. Furthermore, let $y = m^{-1}(p)$. Then for all $n \geq 1$, $\mathbf{P}\{|T| = n\}$ is given by the right-hand side of* (2).

We now turn to the proof of Theorem 30. Recall that for $y > 1$, $m(y)$ is the dual parameter for y. We will make use of the following easy (and known) lemma.

LEMMA 33. *Let T be a Cayley tree of order n with root r, and let v be a uniformly random node in T. Then*

$$\mathbf{P}\{d_T(r, v) = k - 1\} = \frac{k(n)_k}{n^{k+1}}$$

where $(n)_k = n(n-1)\cdots(n-k+1)$.

PROOF. We view T as a doubly-rooted tree with roots r and v. By removing the edges on the path from r to v, we obtain a forest of $d_T(r, v)$ rooted trees, whose roots are ordered (as $r = v_1, \dots, v_k = v$, say). The number of such forests is $\binom{n}{k}k! \cdot (kn^{n-k-1})$; see, for example, [18], Theorem 3.2. The result follows by dividing by the total number of doubly-rooted trees on n labeled vertices, which is n^n by Cayley's formula. □

PROOF OF THEOREM 30. As mentioned at the start of Section 3.3, for all i, n_{i+1} is joined to a uniformly random element of T_i—say v_i—and $d_{\text{IPC}}(n_i, n_{i+1}) = 1 + d_{T_i}(n_i, v_i)$. By Corollary 29 and Lemma 33, it follows that

$$\begin{aligned}
&\mathbf{P}\{d_{\text{IPC}}(n_i, n_{i+1}) = k \mid h_{n_{i+1}} = y\} \\
&\qquad = \sum_{n=k}^{\infty} \mathbf{P}\{d_{T_i}(n_i, v_i) = k - 1 \mid |T_i| = n\}\mathbf{P}\{|T_i| = n \mid h_{n_{i+1}} = y\}
\end{aligned}$$

$$= \sum_{n=k}^{\infty} \frac{k(n)_k}{n^{k+1}} \cdot \frac{\theta(y)}{\theta'(y)} \frac{e^{-ny}(ny)^{n-1}}{(n-1)!}$$

$$= y^{k-1} \frac{\theta(y)}{\theta'(y)} \sum_{n=k}^{\infty} \frac{k}{n} \cdot \frac{e^{-ny}(ny)^{n-k}}{(n-k)!}$$

$$= y^{k-1} \frac{\theta(y)}{\theta'(y)} \sum_{n=k}^{\infty} \frac{k}{n} \mathbf{P}\{\text{Poisson}(ny) = n-k\}.$$

By the cycle lemma,

$$\frac{k}{n} \mathbf{P}\{\text{Poisson}(ny) = n-k\} = \mathbf{P}\{S_n^{0,y} = -k, S_i^{0,y} > -k \ \forall 0 \le i < n\},$$

so

$$\mathbf{P}\{d_{\text{IPC}}(n_i, n_{i+1}) = k \mid h_{n_{i+1}} = y\} = y^{k-1} \frac{\theta(y)}{\theta'(y)} \mathbf{P}\{S_n^{0,y} = -k \text{ for some } n\}.$$

But by the connection with queueing theory explained above, $\mathbf{P}\{S_n^{0,y} = -k$ for some $n\}$ is precisely the probability that k independent PGW(y) all fail to survive, which is $(1-\theta(y))^k$. We complete the proof by applying the identity (1) from page 951. □

PROOF OF THEOREM 31. For this theorem we revert to viewing IPC$(\mathcal{P}, \mathbb{N})$ as a subtree of the PWIT $\mathcal{T}$; our proof is based on the proof of Proposition 2.3 of [3]. Given a node $v \in \mathcal{T}$, write $\mathcal{T}_v$ [resp. $\mathcal{T}_v(\lambda)$] for the subtree of $\mathcal{T}$ rooted at v (resp. rooted at v and containing all edges of weight at most λ to descendants of v)—so $\mathcal{T}_v(\lambda) \stackrel{d}{=} \text{PGW}(\lambda)$—and write $\lambda^*(v) = \inf\{\lambda : \mathcal{T}_v(\lambda) \text{ is infinite}\}$.

Let r be the root of $\mathcal{T}$, and fix any node $v \in \mathcal{T}$. Fix $y > 1$ and integers $1 \le j \le k$. Let $E_{v,j,k,dy}$ be the event that v is on the backbone, has k children of whom the backbone passes through the jth (in the left-to-right ordering of the PWIT) and $\lambda^*(v) \in dy$. We split $E_{v,j,k,dy}$ into four events depending on distinct edge sets of the PWIT:

- F_1 $\mathcal{T}_r(y) - \mathcal{T}_v(y)$ is finite. (This event depends only on edges of $\mathcal{T} - \mathcal{T}_v$.)
- F_2 v has precisely k children of weight at most $y + dy$—say $v_1, \dots, v_k$. (This depends only on the weights of edges from v to its children.)
- F_3 For $i \in \{1, \dots, k\} \setminus \{j\}$, $\mathcal{T}_{v_i}(y + dy)$ is finite. (This depends only on the weights edges in the subtrees $\mathcal{T}_{v_i}$ for $i \ne j$.)
- F_4 $\mathcal{T}_{v_j}(y)$ is finite, but $\mathcal{T}_{v_j}(y + dy)$ is infinite. (This depends only on the weights of edges in $\mathcal{T}_{v_j}$.)

Since the edge sets determining the events $F_1, \dots, F_4$ are disjoint, if E occurs, then the conditioning on subtrees $\mathcal{T}_{v_i}(y)$ for $i \in \{1, \dots, k\} \setminus \{j\}$ is precisely that they are finite. Thus, given that E occurs, for each $i \in \{1, \dots, k\} \setminus \{j\}$, $\mathcal{T}_{v_i}(y)$ is distributed as $\mathrm{PGW}(m(y))$.

Now let $E_{v,dy} = \bigcup_{i,j} E_{v,i,j,dy}$—so $E_{v,dy}$ is the event that v is on the backbone and that $\lambda^*(v) \in dy$. Given the observation at the end of the previous paragraph, to prove the theorem it suffices to show that as $dy \to 0$, given $E_{v,dy}$, the number $N_v(y+dy)$ of children of v in $\mathcal{T}_v(y+dy)$ approaches $\mathrm{Poisson}(m(y))+1$ in distribution [so that the number of children off the backbone approaches $\mathrm{Poisson}(m)$]. To see this is an easy calculation. First,

$$\begin{aligned}\mathbf{P}\{E_{v,dy}\} &= \mathbf{P}\{F_1\} \cdot \mathbf{P}\{\mathcal{T}_v(y) \text{ is finite but } \mathcal{T}_v(y+dy) \text{ is infinite}\} \\ &= \mathbf{P}\{F_1\} \cdot \big(1+o(dy)\big)\theta'(y)\,dy.\end{aligned}$$

Next, fixing $k \geq 1$, by symmetry,

$$\begin{aligned}&\mathbf{P}\{N_v(y+dy) = k, E_{v,dy}\} \\ &\qquad = \big(1+o(dy)\big)\mathbf{P}\{F_1\} \cdot \mathbf{P}\{\mathrm{Poisson}(y) = k\} \cdot k \cdot \big(1-\theta(y)\big)^{k-1} \cdot \theta'(y)\,dy.\end{aligned}$$

The factor k above selects which of the k children of v is on the backbone. Since $m = y(1-\theta(y))$, it follows that

$$\begin{aligned}\lim_{dy\to 0} \mathbf{P}\{N_v(y+dy) = k \mid E_{v,dy}\} &= \mathbf{P}\{\mathrm{Poisson}(y) = k\} \cdot k \cdot \big(1-\theta(y)\big)^{k-1} \\ &= \mathbf{P}\{\mathrm{Poisson}(m) = k-1\},\end{aligned}$$

which completes the proof. □

4. The stationary graph and box processes. Throughout this section, $\mathcal{P}$ denotes a Poisson point process of intensity 1 in the upper half-plane, so $\mathcal{P}$ is almost surely exemplary and $\mathbb{Z}$-distinguished.

4.1. *Rooted subtrees in the box tree.* Recall that $\mathrm{BG} = \mathrm{BG}(\mathcal{P}, \mathbb{Z})$ is the tree with vertex set $\mathbb{Z}$ defined in Section 2. For $n \in \mathbb{Z}$, we let BG^n denote the subtree of BG rooted at n, and write $|\mathrm{BG}^n|$ for its size (number of vertices). By the definition of BG, it is immediate that $|\mathrm{BG}^n| = n - \ell_n$. Our main aim in this section is to prove the following theorem.

THEOREM 34. $\mathrm{BG}(\mathcal{P}, \mathbb{Z})$ *is distributed as the Poisson IIC, in the local weak sense.*

A key step in proving Theorem 34, one that additionally introduces several of the main ideas, is the following theorem.

THEOREM 35. *For all $n \in \mathbb{Z}$, conditional on $\ell^n(\mathcal{P}, \mathbb{Z})$ and $h^n(\mathcal{P}, \mathbb{Z})$, $\mathrm{BG}^n(\mathcal{P}, \mathbb{Z})$ is distributed as $\mathrm{PGW}(h^n)$ conditioned to have $n - \ell^n$ nodes. Furthermore, unconditionally BG^n is distributed as $\mathrm{PGW}(1)$.*

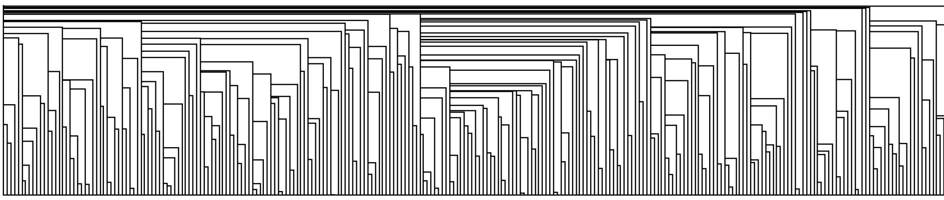

FIG. 4. *The boxes for a random set of* 256 *points. The points themselves are omitted. The code for generating this image was written by Omer Angel.*

COROLLARY 36 ([17], Theorem 1). *We have* $\lfloor x(p_0) \rfloor \stackrel{d}{=} -\lfloor AV \rfloor$, *where* A *is Borel distributed, and* V *is* Uniform[0, 1] *and independent of* A.

PROOF. Given ℓ_0 and h_0, the line segment $[\ell_0, 0] \times \{h_0\}$ contains a single uniformly random point, and this point is p_0. The second assertion of the theorem implies that ℓ_0 is Borel distributed, and the corollary follows. □

For the next several pages, we focus on developing the tools needed for the proof of Theorem 35. By translation invariance, it suffices to prove Theorem 35 with $n = 0$. We prove the theorem by way of the following analog of Lemma 26 that holds for the box tree.

LEMMA 37. *Let* n, λ, P *and* p *be as in Lemma* 26. *Given that* $F(0, n, \lambda, P)$ *occurs*, $\mathrm{BG}(P \cup \{p\}, \{0, \dots, n\})$ *is distributed as* $\mathrm{PGW}(\lambda)$ *conditioned to have* n *nodes*.

We remark that for any $\lambda > 0$, $\mathrm{PGW}(\lambda)$ conditioned to have k nodes and PGW(1) conditioned to have k nodes are identically distributed. Thus, in proving Theorem 35 and Lemma 37 we may and shall at times assume without loss of generality that $\lambda = 1$. Figure 4 contains an example of $\mathrm{BG}(P \cup \{p\}, \{0, \dots, n\})$ for P, p as in Lemma 37, with $n = 256$. (By the preceding comment, the value of λ is not important.) In proving the lemma, it will be important to view PGW(1) both as an ordered (plane) tree and as an unordered tree. The ordered perspective is natural for PGW(1) when viewed as a subtree of the PWIT. Next, fix an unordered, rooted tree U. We will abuse notation by writing $\mathrm{PGW}(1) = U$ if $\mathrm{PGW}(1) = T$ for some ordered tree T with underlying unordered tree U. Fix one such tree T, and let $\mathrm{aut}(U)$ be the number of rooted automorphisms of T. [Note: by this we mean the number of distinct plane trees with underlying unrooted tree U; e.g., for the tree U in Figure 5, interchanging the pair of leaves x and y does not affect the plane tree, and $\mathrm{aut}(U) = 12$.] We then have $\mathbf{P}\{\mathrm{PGW}(1) = U\} = \mathrm{aut}(T) \cdot \mathbf{P}\{\mathrm{PGW}(1) = T\}$.

We next turn our attention to BG^0. There is again a natural ordering of children in BG^0—vertices are integers, and when we refer to a box tree as an ordered tree, we are referring to the ordering inherited from the integers. However, unlike in

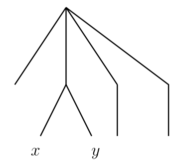

FIG. 5. *Viewed as an unordered tree, the above tree has* $\mathrm{aut}(U) = 12$.

PGW(1), we cannot expect the distributions of distinct subtrees to be identical under this ordering. Given an unordered tree U, we will also abuse notation by writing $\mathbf{P}\{\mathrm{BG}^0 = U\}$ if BG^0 is unlabeled, rooted isomorphic to U.

Our proof of Lemma 37 makes use of the following easy fact.

LEMMA 38. *Let $r \geq 2$ and let $s_1, \dots, s_r$ be natural numbers. Then*

$$\sum_{\pi} \prod_{j=2}^{r} \frac{s_{\pi(j)}}{\sum_{i=1}^{j} s_{\pi(i)}} = 1,$$

where the summation is over all permutations π of $\{1, \dots, r\}$.

PROOF. We proceed by induction on r. For $r = 2$, the sum is over just two permutations, and the result is

$$\frac{s_1}{s_2 + s_1} + \frac{s_2}{s_1 + s_2} = 1,$$

as required. For general r, we partition the set of permutations π of $\{1, \dots, r\}$ depending on the value of $\pi(r)$—for each $k = 1, \dots, r$, let Π_k be the set of permutations π of $\{1, \dots, r\}$ with $\pi(r) = k$. Since our aim is to prove that the sum

$$\sum_{\pi} \prod_{j=2}^{r} \frac{s_{\pi(j)}}{\sum_{i=1}^{j} s_{\pi(i)}} = \sum_{k=1}^{r} \sum_{\pi \in \Pi_k} \prod_{j=2}^{r} \frac{s_{\pi(j)}}{\sum_{i=1}^{j} s_{\pi(i)}}$$

has the value one, it suffices to prove that for each k we have

$$\sum_{\pi \in \Pi_k} \prod_{j=2}^{r} \frac{s_{\pi(j)}}{\sum_{i=1}^{j} s_{\pi(i)}} = \frac{s_k}{\sum_{i=1}^{r} s_i}.$$

Since the expression on the right-hand side here is the $j = r$ term of the product for all $\pi \in \Pi_k$, it suffices to show that

$$\sum_{\pi \in \Pi_k} \prod_{j=2}^{r-1} \frac{s_{\pi(j)}}{\sum_{i=1}^{j} s_{\pi(i)}} = 1.$$

By re-labeling if necessary, this may be deduced from the induction hypothesis. □

PROOF OF LEMMA 37. Fix an unordered tree U with n vertices and root r. We will show that

$$n \cdot \mathbf{P}\{\mathrm{BG}(P, [n]) = U\} = \mathbf{P}\{\mathrm{PGW}(1) = U \mid |\mathrm{PGW}(1)| = n\}. \tag{3}$$

Proving this equality will prove the lemma, since $F(0, n, \lambda, P)$ must occur in order to have $\mathrm{BG}(P, [n]) = U$, and $\mathbf{P}\{F(0, n, \lambda, P)\} = 1/n$ as noted just after Lemma 24. The case $n = 1$ of (3) is trivial, so suppose that $n > 1$ and that the proposition holds for all n' with $1 \le n' < n$.

Order the children of the root r of U arbitrarily, and suppose that the subtrees $U_1, \dots, U_k$ of U rooted at the children of the root have sizes $n_1, \dots, n_k$ with respect to this order. Let $\mathrm{aut}(r)$ be the number of permutations of the children of r which induce automorphisms of U. [For example, the tree in Figure 5 has $\mathrm{aut}(r) = 2$.]

We note that

$$\begin{aligned}
&\mathbf{P}\{\mathrm{PGW}(1) = U \mid |\mathrm{PGW}(1)| = n\} \\
&\quad = \frac{\mathbf{P}\{\mathrm{PGW}(1) = U\}}{\mathbf{P}\{|\,\mathrm{PGW}\,1| = n\}} \\
&\quad = \frac{n!}{n^{n-1}e^{-n}} \cdot \frac{e^{-1}}{k!} \frac{k!}{\mathrm{aut}(r)} \prod_{i=1}^{k} \mathbf{P}\{\mathrm{PGW}(1) = U_i\} \qquad (4)\\
&\quad = \frac{1}{\mathrm{aut}(r)} \frac{n!}{n^{n-1}} \prod_{i=1}^{k} \frac{n_i^{n_i-1}}{n_i!} \mathbf{P}\{\mathrm{PGW}(1) = U_i \mid |\mathrm{PGW}(1) = n_i\} \\
&\quad = \frac{n}{\mathrm{aut}(r)} \binom{n-1}{n_1, \dots, n_k} \prod_{i=1}^{k} \left(\frac{n_i}{n}\right)^{n_i} \frac{1}{n_i} \mathbf{P}\{\mathrm{PGW}(1) = U_i \mid |\mathrm{PGW}(1) = n_i\}.
\end{aligned}$$

Next, assume the points of P are listed in increasing order of height as $\{p_1, \dots, p_n\}$. We first consider the sizes of the subtrees of B_0.

Let $m_0 = 0$ and for $i = 1, \dots, k$, let $m_i = m_{i-1} + n_i$ (so in particular $m_k = n - 1$). Also for $i = 1, \dots, k$, let Q_i be the set of points $p \in P \setminus \{p_n\}$ satisfying $m_{i-1} \le x(p) < m_i$, and let p^i be the point of Q_i with greatest y-coordinate.

We recall the definitions of the events E, F and G from Lemma 14. In order for n to have children with subtrees $U_1, \dots, U_k$ in that order, it is necessary and sufficient that the following events occur:

(I) $E(m_{i-1}, m_i, y(p^i), P)$ occurs for each $i \in \{1, \dots, k\}$;
(II) We have $y(p^i) > y(p^2) > \cdots > y(p^k)$;
(III) $F(m_{i-1}, m_i, y(p^i), P)$ and $G(m_{i-1}, y(p^i), P)$ occur for each $i = \{1, \dots, k\}$;

(IV) $\mathrm{BG}^{m_i} = U_i$ for each $i = \{1, \ldots, k\}$.

First, (I) is equivalent to the requirement that $|Q_i| = n_i$ for each $i = \{1, \ldots, k\}$. The x-coordinates of points in P, are uniformly distributed on $[-n, 0]$ so

$$\mathbf{P}\{(\mathrm{I})\} = \binom{n-1}{n_1, \ldots, n_k} \prod_{i=1}^{k} \left(\frac{n_i}{n}\right)^{n_i}. \tag{5}$$

Given (I), for (II) to occur it suffices that for each $i = \{1, \ldots, k\}$, the point of $(P \setminus \{p_n\}) \setminus (Q_1 \cup \cdots \cup Q_{i-1})$ with the largest y-coordinate, is a member of Q_i. Thus,

$$\mathbf{P}\{(\mathrm{II}) \mid (\mathrm{I})\} = \prod_{i=1}^{k} \frac{n_i}{n - 1 - \sum_{j=1}^{i-1} n_j}. \tag{6}$$

Since $y(p^i) < y(p^j)$ for $j < i$, if (I), (II) and $\bigcap_{i=1}^{k-1} F(m_{i-1}, m_i, y(p^i))$ all hold for some $k \geq 1$, then it is immediate that $G(m_{i-1}, y(p^i))$ occurs for each $i = \{1, \ldots, k\}$. Thus,

$$\mathbf{P}\{(\mathrm{III}) \mid (\mathrm{I}), (\mathrm{II})\} = \mathbf{P}\left\{\bigcap_{i=1}^{k} F(m_{i-1}, m_i, y^*(i)) \,\middle|\, (\mathrm{I}), (\mathrm{II})\right\}.$$

Furthermore, given (I) and (II), independently for each $i = \{1, \ldots, k\}$, the points of $Q_i \setminus \{p^i\}$ are independently and uniformly distributed in $[m_{i-1}, m_i) \times [0, y(p^i))$. Thus, by the cycle lemma,

$$\mathbf{P}\{(\mathrm{III}) \mid (\mathrm{I}), (\mathrm{II})\} = \prod_{i=1}^{k} \mathbf{P}\{F(m_{i-1}, m_i, y(p^i)) \mid (\mathrm{I}), (\mathrm{II})\} = \prod_{i=1}^{k} \frac{1}{n_i}.$$

Finally, given (I), (II) and (III) and independently for each $i = \{1, \ldots, k\}$, $Q_i \setminus \{p^i\}$ is precisely a uniform set of $n_i - 1$ points, conditioned on $F(m_{i-1}, m_i, y(p^i), Q_i)$ holding, and p^i is a uniform point on $[m_i - n_i, m_i] \times \{y(p^i)\}$. Furthermore, by Lemma 15, given $E(m_{i-1}, m_i, y(p^i))$, $F(m_{i-1}, m_i, y(p^i))$ and $G(m_{i-1}, y(p^i))$, we have

$$\mathrm{BG}^{m_i}(P, \{0, \ldots, n\}) = \mathrm{BG}(Q_i, \{m_{i-1}, \ldots, m_i\}).$$

Thus, by the induction hypothesis,

$$\begin{aligned} &\mathbf{P}\{(\mathrm{IV}) \mid (\mathrm{I}), (\mathrm{II}), (\mathrm{III})\} \\ &\quad = \prod_{i=1}^{k} \mathbf{P}\{\mathrm{BG}^{m_i}(P, \{0, \ldots, n\}) = U_i \mid E(m_{i-1}, m_i, y(p^i)), \\ &\qquad\qquad F(m_{i-1}, m_i, y(p^i)), G(m_{i-1}, y(p^i))\} \end{aligned} \tag{7}$$

$$= \prod_{i=1}^{k} \mathbf{P}\big\{\mathrm{BG}(Q_i, \{m_{i-1}, \dots, m_i\}) = U_i \mid F(m_{i-1}, m_i, y(p^i))\big\}$$

$$= \prod_{i=1}^{k} \mathbf{P}\{\mathrm{PGW}(1) = U_i \mid |\mathrm{PGW}(1)| = n_i\}.$$

Combining (5)–(7), and rearranging, we obtain that

$$\mathbf{P}\big\{n \text{ has children } U_1, \dots, U_k \text{ in that order in } \mathrm{BG}(P \cup p, \{0, \dots, n\}\big\}$$

$$= \binom{n-1}{n_1, \dots, n_k} \prod_{i=1}^{k} \frac{n_i}{n - 1 - \sum_{j=1}^{i-1} n_j} \times \prod_{i=1}^{k} \left(\frac{n_i}{n}\right)^{n_i} \frac{1}{n_i} \mathbf{P}\{\mathrm{PGW}(1) = U_i \mid |\mathrm{PGW}(1)| = n_i\},$$

$$= \frac{\mathrm{aut}(r)}{n} \prod_{i=1}^{k} \frac{n_i}{n - 1 - \sum_{j=1}^{i-1} n_j} \mathbf{P}\{\mathrm{PGW}(1) = U \mid |\mathrm{PGW}(1)| = n\},$$

the latter equality holding due to (4). To obtain $\mathbf{P}\{\mathrm{BG}(P \cup p, \{0, \dots, n\}) = U\}$, we now must sum this bound over distinct orderings of $U_1, \dots, U_k$. We instead sum over all permutations $\pi : [k] \to [k]$, and note that this counts each distinct ordering $\mathrm{aut}(r)$ times. We thus obtain

$$\mathbf{P}\big\{\mathrm{BG}(P \cup p, \{0, \dots, n\}) = U\big\}$$

$$= \frac{1}{n} \mathbf{P}\{\mathrm{PGW}(1) = U \mid |\mathrm{PGW}(1)| = n\} \cdot \sum_{\pi : [k] \to [k]} \prod_{i=1}^{k} \frac{n_{\pi(i)}}{n - 1 - \sum_{j=1}^{i-1} n_{\pi(j)}}.$$

By Lemma 38, the above sum is 1, which establishes (3) by induction and so completes the proof. □

In proving Theorem 35, we will use the following identity, which we quote in advance.

LEMMA 39. *For integers $a \geq 0$, $b > 0$, let $I_{a,b} := \int_0^1 x^a e^{-bx}\, dx$. Then $I_{b-1,b} - I_{b,b} = e^{-b}/b$ for each $b > 0$.*

PROOF. Integration by parts. □

The final step before proving Theorem 35 is to derive the conditional distribution of $\ell_0 = |\mathrm{BG}^0|$ given h_0. As this will be useful later in the paper, we state it as a separate lemma. Write

$$\varphi_y(n) = \lim_{dy \to 0} \frac{\mathbf{P}\{|\ell_0| = n, h_0 \in [y, y + dy)\}}{dy}.$$

LEMMA 40. *For all* $0 < y < 1$ *and all* $n \geq 1$,

$$\varphi_y(n) = (1-y) \cdot \frac{e^{-ny}(ny)^{n-1}}{(n-1)!}. \tag{8}$$

PROOF. Fix $n \in \mathbb{N}$ and $0 < y < 1$. In order to have $\ell_0 = -n$ and $h_0 = y$, it is necessary and sufficient that $E = E(-n, 0, y, P)$, $F = F(-n, 0, y, P)$ and $G = G(-n, y, P)$, from Lemma 14 all occur. We first calculate the density of the event E.

$$\begin{aligned}
&\mathbf{P}\{|[-n, 0) \times [0, y)|_{\mathcal{P}} = n-1, |[-n, 0) \times [y, y+dy)|_{\mathcal{P}} = 1\} \\
&\quad = \big(1 + o(dy)\big)\mathbf{P}\{\text{Poisson}(ny) = n-1\} \cdot n\, dy \\
&\quad = \big(1 + o(dy)\big)\frac{e^{-ny}(ny)^{n-1}}{(n-1)!} \cdot n\, dy.
\end{aligned}$$

Now let $f_E(y) = f_E(0, n, y) = \frac{e^{-ny}(ny)^{n-1}}{(n-1)!} \cdot n$. Given that $|[-n, 0) \times [0, y)|_{\mathcal{P}} = n-1$ occurs, $\mathcal{P}([-n, 0] \times [0, y))$ consists of $n-1$ uniformly random points. Independently of this, given that $|[-n, 0) \times \{y\}|_{\mathcal{P}} = 1$, the line segment $[-n, 0] \times \{y\}$ contains a single uniformly random point. By the first of the two preceding observations and by the cycle lemma, it follows that $\mathbf{P}\{F \mid E\} = \frac{1}{n}$. Furthermore, G is independent of E, and so by Lemma 24, $\mathbf{P}\{G\} = 1 - y$. We thus have

$$\varphi_y(n) = \mathbf{P}\{F, G \mid E\} f_E(y) = \mathbf{P}\{G\}\mathbf{P}\{F \mid E\} f_E(y),$$

from which the lemma follows. □

PROOF OF THEOREM 35. We assume without loss of generality that $n = 0$. Let $P = ([\ell_0, 0] \times [0, h_0]) \cap \mathcal{P}$, and let $p = ([\ell_0, 0] \times \{h_0\}) \cap \mathcal{P}$. Then P is precisely distributed as a set of $|\ell_0| - 1$ uniform points in $([\ell_0, 0] \times [0, h_0])$, conditional on $F(\ell_0, 0, h_0)$, and p has uniform distribution on $([\ell_0, 0] \times \{h_0\})$. The first claim of the theorem then follows from Lemma 37. Next, by Lemma 40, for any positive integer m we have

$$\begin{aligned}
\mathbf{P}\{|\ell_0| = m\} &= \int_0^1 \varphi_y(m)\, dy \\
&= \frac{m^{m-1}}{(m-1)!}(I_{m-1,m} - I_{m,m}),
\end{aligned} \tag{9}$$

where the notation $I_{a,b}$ is that defined in Lemma 39. Applying that lemma, we obtain that $\mathbf{P}\{|\mathrm{BG}^0| = m\}$ is $m^{m-1}e^{-m}/m!$ [exactly the probability that a PGW(1) has size m]. The second claim of the theorem then follows from the first and the fact that the conditional distribution of PGW(λ) given its size, is independent of λ. □

Before proving Theorem 34, we first state a consequence of the above development.

COROLLARY 41. *For any positive integer n, any $0 < y < 1$, and any unordered rooted tree U with $|U| = n$,*

$$\lim_{dy \to 0} \frac{\mathbf{P}\{\mathrm{BG}^0 = U, h_0 \in dy\}}{dy} = (1-y)n\mathbf{P}\{\mathrm{PGW}(y) = U\}.$$

PROOF. Immediate from (8) and Theorem 35. □

PROOF OF THEOREM 34. We will in fact prove that for any unordered rooted tree U', conditional on $\mathrm{BG}^0 = U'$, $\mathrm{BG}^{a_0} \setminus \mathrm{BG}^0$ is distributed as PGW(1), which implies the statement of the theorem. Thus, let U and U' be unordered rooted trees with roots r and r', and let U^* be the unordered rooted tree with root r obtained by adding an edge between r and r'. We define

$$\begin{aligned} E &= \{\mathrm{BG}^0 = U', \mathrm{BG}^{a_0} \setminus \mathrm{BG}^0 = U\} \\ &= \{\mathrm{BG}^0 = U', \mathrm{BG}^{a_0} = U^*\}. \end{aligned}$$

Next let $k = k(U^*)$ be the number of children of r in U^*, and let $j = j(U^*, U') \geq 1$ be the number of children of r in U^* whose subtree is isomorphic to U'. Also, let $\mathrm{aut}(r)$ [resp. $\mathrm{aut}^*(r)$] be the number of permutations of the children of r in U (resp. U^*) which induce automorphims of U (resp. U^*). Note that $\mathrm{aut}^*(r) = j \cdot \mathrm{aut}(r)$.

Given an ordering $\mathbf{u} = (U_1, \ldots, U_k)$ of the children of r in U^*, let $i_1 = i_1(\mathbf{u}), \ldots, i_j = i_j(\mathbf{u})$ be the indices i for which U_i is isomorphic to U'. For each $\ell \in \{0, \ldots, k\}$, let $m_\ell = m_\ell(\mathbf{u}) = 1 + \sum_{q=\ell+1}^{k} |U_q|$ (which is 1 when $\ell = k$). Let

$$E_{\mathbf{u}} = E \cap \{a_0 \text{ has children } U_1, \ldots, U_k \text{ in that order}\},$$

and for each $p \in \{1, \ldots, j\}$, let

$$E_{\mathbf{u},p} = E_{\mathbf{u}} \cap \{a_0 = m_{i_p}(\mathbf{u})\} = \{m_{i_p}(\mathbf{u}) \text{ has children } U_1, \ldots, U_k \text{ in that order}\}.$$

(The above definitions are depicted in Figure 6.) Then $E = \bigcup_{\mathbf{u}} \bigcup_{p=1}^{j} E_{\mathbf{u},p}$, where the first union is over all distinct orderings $\mathbf{u}$ [we say $\mathbf{u} = (U_1, \ldots, U_k)$ and $\mathbf{u}' = (U_1', \ldots, U_k')$ are distinct if there is some $i \in \{1, \ldots, k\}$ for which U_i and U_i' are not isomorphic]. The double union is then over disjoint terms, and so

$$\mathbf{P}\{E\} = \sum_{\mathbf{u}} \sum_{p=1}^{j} \mathbf{P}\{E_{\mathbf{u},p}\}.$$

Reversing the order of summation above, by translation invariance and Theorem 35, we obtain

$$\begin{aligned} \mathbf{P}\{E\} &= \sum_{p=1}^{j} \sum_{\mathbf{u}} \mathbf{P}\{E_{\mathbf{u},p}\} \\ &= j \cdot \mathbf{P}\{\mathrm{BG}^0 = U^*\} \\ &= j \cdot \mathbf{P}\{\mathrm{PGW}(1) = U^*\}. \end{aligned}$$

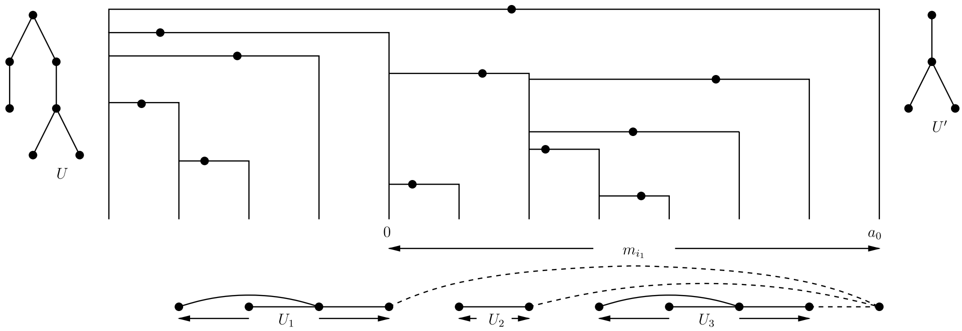

FIG. 6. *In the above figure, an ordering* $\mathbf{u} = (U_1, U_2, U_3)$ *of the children of* U^* *is fixed. Dashed edges lead from* a_0 *to its three children. In this example,* $j(U^*, U') = 2$ *since* U_1 *and* U_3 *are (unordered, rooted) isomorphic to* U' *but* U_2 *is not, and* $i_1(\mathbf{u}) = 1$, $i_2(\mathbf{u}) = 3$ *for the same reason. Finally, this example relates to the event* $E_{\mathbf{u},p}$ *with* $p = 1$, *since* $a_0 = m_{i_1}(\mathbf{u})$.

Now fix some ordering $U_1, \dots, U_k$ of the children of r in U^* with $U_k = U'$. Then by the definition of PGW(1) and the preceding equality, we have

$$\begin{aligned}\mathbf{P}\{E\} &= j \cdot \frac{e^{-1}}{k!} \frac{k!}{\mathrm{aut}^*(r)} \prod_{i=1}^{k} \mathbf{P}\{\mathrm{PGW}(1) = U_i\} \\ &= \mathbf{P}\{\mathrm{PGW}(1) = U'\} \cdot \frac{e^{-1}}{(k-1)!} \frac{(k-1)!}{\mathrm{aut}(r)} \prod_{i=1}^{k-1} \mathbf{P}\{\mathrm{PGW}(1) = U_i\} \\ &= \mathbf{P}\{\mathrm{PGW}(1) = U'\}\mathbf{P}\{\mathrm{PGW}(1) = U\}.\end{aligned}$$

It follows by Theorem 35 that

$$\begin{aligned}\mathbf{P}\{\mathrm{BG}^{a_0} \setminus \mathrm{BG}^0 = U | \mathrm{BG}^0 = U'\} &= \frac{\mathbf{P}\{E\}}{\mathbf{P}\{\mathrm{BG}^0 = U'\}} \\ &= \mathbf{P}\{\mathrm{PGW}(1) = U\},\end{aligned}$$

proving the theorem. □

4.2. *An ancestral process in* $\mathrm{IPC}(\mathcal{P}, \mathbb{Z})$. By the end of this section we will have proved Theorems 1–3 from the Introduction. To warm up, we prove the following theorem.

THEOREM 42. *The subtree of* $\mathrm{IPC}(\mathcal{P}, \mathbb{Z})$ *rooted at zero and containing only nodes with positive label, is distributed as* PGW(1).

PROOF. By Proposition 18, each point of $\mathcal{P}$ in $[0, 1] \times [0, 1]$ yields a child of 0 in T_0, so 0 has Poisson(1) children. Let $r_0 = 0$, and let $r_1 = \min(i > 0 : \lfloor x(p_i) \rfloor =$

0), so r_1 is "the first integer with 0 as a parent." For any $i > 0$, the event that $r_1 = i$ is independent of $\mathcal{P} \cap ([i, \infty) \times [0, \infty))$, so r_1 also has Poisson(1) children. More generally, let $r_k = \min(i > r_{k-1} : \lfloor x(p_i) \rfloor \in \{r_0, \dots, r_{k-1}\})$. Then for all $k \geq 1$ and all $i > 0$, the event that $r_k = i$ is independent of $\mathcal{P} \cap ([i, \infty) \times [0, \infty))$, so r_k has Poisson(1) children. The nodes $r_0, r_1, r_2, \dots$ are precisely the descendants of 0, and we have just seen that each has Poisson(1) children independently of all the others. This proves the theorem. □

Heuristically, the fact that IPC($\mathcal{P}, \mathbb{Z}$) is equal in distribution to the IIC can be seen as follows. By symmetry, from Theorem 42, at each node of IPC($\mathcal{P}, \mathbb{Z}$) is rooted a copy of PGW(1). Also, by exploring the nodes of multiple trees in a left-to-right fashion as in Theorem 42, we see that the offspring distribution for distinct branches of IPC($\mathcal{P}, \mathbb{Z}$) are independent. Furthermore, the parent of 0 in IPC($\mathcal{P}, \mathbb{Z}$) is more likely to be a node with many children than one with few children. This should "size-bias" the number of children of the parent of zero, in such a way as to precisely compensate for the edge from 0 to its parent, so that a Poisson(1) number of children remain. The same argument should also hold for the parent of the parent of zero, and so on ad infinitum. It is possible to make (parts of) this heuristic argument rigorous; however, we obtain the result as a relatively direct byproduct of our argument for Theorem 3, whose proof requires a different approach.

The key to the proof is the definition of a "backward maximum process" which is extremely similar to the forward maximal process. We begin by listing 0 and its ancestors in IPC($\mathcal{P}, \mathbb{Z}$) in decreasing order as n_0, n_1, n_2, et cetera, so in particular $n_0 = 0$ and in general $n_{i+1} = \lfloor x(p_{n_i}) \rfloor$. For $i \geq 0$ let $w_i = h_{n_i}$, and let $m_i = \max_{0 \leq j \leq i} w_j$, the greatest weight of any of the first $(i + 1)$ edges. In particular, $m_0 = w_0 = h_0$. Finally, let $i_0 = 0$ and, for $k \geq 1$, let i_k be the smallest integer $i > i_{k-1}$ for which $m_{i_k} > m_{i_{k-1}}$. Then the following lemma is basic (but important).

LEMMA 43. *For all $k \geq 1$, $n_{i_k} = \ell(n_{i_{k-1}})$, and so $m_{i_k} = h_{n_{i_k}}$.*

PROOF. The fact that $n_{i_k} \leq \ell(n_{i_{k-1}})$ is immediate from Lemma 9. But $\ell = \ell(n_{i_{k-1}})$ is an ancestor of $n_{i_{k-1}}$ by Lemma 10, and $h_\ell > h_{n_{i_{k-1}}}$ by Lemma 8. Thus, $\ell(n_{i_{k-1}}) = n_{i_k}$ as claimed. □

Above, we derived the joint distribution of the height and length of B_0. We next show that the sequence $\{m_{i_k}\}_{k \in \mathbb{N}}$ has a particularly simple and pleasing description.

LEMMA 44. *The sequence $\{m_{i_k}\}_{k \in \mathbb{N}}$ is a homogeneous Markov chain, and for all k, given m_{i_k}, $m_{i_{k+1}}$ has distribution* Uniform$[m_{i_k}, 1]$.

PROOF. For all $0 < y < 1$ and all k, $h_k \leq y$ if and only if the random walk $L^{k,y}$ has a chance. As remarked in Section 3.2, the probability of this is precisely y.

Given $m_{i_{k-1}}$ and n_{i_k}, by Lemmas 8 and 43, we know precisely that $m_{i_k} = h_{n_{i_k}} > h_{n_{i_{k-1}}} = m_{i_{k-1}}$. In other words, we know precisely that the random walk $L^{n_{i_k}, m_{i_{k-1}}}$ has no chance. Thus, for $0 < m < y < 1$,

$$\begin{aligned}\mathbf{P}\{m_{i_k} \le y \mid m_{i_{k-1}} = m\} &= \mathbf{P}\{h_{n_{i_k}} \le y \mid m_{i_{k-1}} = m\} \\ &= \mathbf{P}\{L^{n_{i_k}, y} \text{ has a chance} \mid L^{n_{i_k}, m} \text{ has no chance}\} \\ &= \frac{1 - \mathbf{P}\{L^{n_{i_k}, y} \text{ has no chance}\} - \mathbf{P}\{L^{n_{i_k}, m} \text{ has a chance}\}}{\mathbf{P}\{L^{n_{i_k}, m} \text{ has no chance}\}} \\ &= \frac{y - m}{1 - m},\end{aligned}$$

which proves the lemma. □

Note also, by the first remark in the proof of the lemma, we have the following proposition.

PROPOSITION 45 ([17], Theorem 2). $h_0 \stackrel{d}{=} \text{Uniform}[0, 1]$.

We now prove a more substantial result, about the structure of the portion of $\mathrm{IPC}(\mathcal{P}, \mathbb{Z})$ that lives "under the backward maximum process." It essentially states that, like the forward maximal process, the portion of $\mathrm{IPC}(\mathcal{P}, \mathbb{Z})$ that lives under the backward maximum process looks like a single infinite backbone, to which subcritical Poisson Galton–Watson trees are attached at each point. Also, these subcritical trees become closer and closer to critical the further along the backbone from 0 they are.

THEOREM 46. *Let* $\mathrm{IPC}^-(\mathcal{P}, \mathbb{Z})$ *denote the restriction of* $\mathrm{IPC}(\mathcal{P}, \mathbb{Z})$ *to the nonpositive integers, and let* 0 *be its root. Then* $\mathrm{IPC}^-(\mathcal{P}, \mathbb{Z})$ *is distributed as* $\mathcal{T}^*_{\mathrm{IIC}}$.

We will prove this theorem at the end of the section. For each $k \ge 0$, let $\mathcal{P}_k = \mathcal{P} \cap ([n_{i_{k+1}}, n_{i_k}] \times [0, m_{i_k}))$ and let $\mathcal{I}_k = \{n_{i_{k+1}}, n_{i_{k+1}} + 1, \ldots, n_{i_k}\}$. By Lemma 26, given $(n_{i_{k+1}} - n_{i_k})$, $\mathrm{IPC}(\mathcal{P}_k, \mathcal{I}_k)$ is distributed as $\mathrm{PGW}(m_{i_k})$ conditioned to have $(n_{i_{k+1}} - n_{i_k})$ nodes, together with a single additional node (the node n_{i_k}) attached to a uniform vertex.

THEOREM 47. *For all* $k \ge 0$ *and* $0 < m < 1$, *given that* $m_{i_k} = m$, $\mathrm{IPC}(\mathcal{P}_k, \mathcal{I}_k)$ *is distributed as a path with* $(1 + \text{Geometric}(m))$ *edges, from* $n_{i_{k+1}}$ *to* n_{i_k}, *with an independent* $\mathrm{PGW}(m)$ *tree attached to each node of the path except* n_{i_k}.

PROOF. The proof uses a correspondence between $\mathrm{IPC}(\mathcal{P}_k, \mathcal{I}_k)$ and a pond of $\mathrm{IPC}(\mathcal{P}, \mathbb{N})$ of appropriate height. Let $\lambda > 1$ be such that $me^{-m} = \lambda e^{-\lambda}$, so then

$m = \lambda(1-\theta(\lambda))$. For all $n \geq 1$, by Lemma 40 we have

$$\begin{aligned}\mathbf{P}\{n_{i_{k+1}} - n_{i_k} = n \mid m_{i_k} = m\} &= (1-m) \cdot \frac{(mn)^{n-1} e^{-mn}}{(n-1)!} \\ &= \frac{n^{n-1}}{(n-1)!} (me^{-m})^n \frac{1-m}{m} \\ &= \frac{n^{n-1}}{(n-1)!} (\lambda e^{-\lambda})^n \frac{1-\lambda(1-\theta(\lambda))}{\lambda(1-\theta(\lambda))} \\ &= \frac{n^{n-1}}{(n-1)!} (\lambda e^{-\lambda})^n \frac{\theta(\lambda)}{\lambda\theta'(\lambda)} \\ &= \frac{\theta(\lambda)}{\theta'(\lambda)} \frac{(\lambda n)^{n-1} e^{-\lambda n}}{(n-1)!}.\end{aligned}$$

By Corollary 29, the latter is the probability that a pond of IPC$(\mathcal{P}, \mathbb{N})$, conditioned to have height λ, has size n. Thus, IPC$(\mathcal{P}_k, \mathcal{I}_k)$ is distributed as a pond of IPC$(\mathcal{P}, \mathbb{N})$ conditioned to have height λ. By Theorem 30, it follows that the length of the path from $n_{i_{k+1}}$ to n_{i_k} has distribution $1 + \text{Geometric}(\lambda(1-\theta(\lambda))) \stackrel{d}{=} 1 + \text{Geometric}(m)$. Furthermore, by Theorem 31, to each vertex of the path except n_{i_k} is attached an independent copy of PGW(m). This completes the proof. □

Having proved Theorem 47, we are now prepared for the last ingredient needed for the proofs of Theorems 1 and 46.

THEOREM 48. IPC$(\mathcal{P}, \mathbb{Z})$ *is distributed as the Poisson IIC*, *in the local weak sense.*

PROOF OF THEOREM 48. For each $j \geq 0$, let T_j be the subtree of IPC$(\mathcal{P}, \mathbb{Z})$ rooted at n_j and containing all nodes reachable from n_j without passing through n_{j-1} or n_{j+1} (so T_j contains neither n_{j-1} nor n_{j+1}). We show that independently for each j, T_j is distributed as PGW(1), which proves the theorem.

For each j, let $k = k(j)$ be the largest integer k for which $i_k \leq j$. Let U_j be the subtree of T_j containing only nodes of index less than n_k (in other words, U_j is the subtree of T_j which lives under the backward maximum process). Also, let V_j be the subtree containing n_j and all nodes of T_j not in U_j. Then by Theorem 47, U_j is distributed as PGW$(m_{i_{k(j)}})$, independently of $\{U_{j'}\}_{j' \neq j}$.

Next, for each node $\ell \in U_j$, the number of children of ℓ in IPC$(\mathcal{P}, \mathbb{Z})$ that are not in U_j is precisely the number of points of $\mathcal{P}$ in $[\ell - 1, \ell) \times (m_{i_{k(j)}}, 1)$, and therefore has Poisson$(1 - m_{i_{k(j)}})$ distribution. Furthermore, all such children have strictly positive index by the definition of U_j.

Finally, as in Theorem 42, let $r_1 = \min(i > 0 \colon \lfloor x(p_i) \rfloor \in U_j)$. Then r_1 has a Poisson(1) number of children, independently of U_j. More generally, exposing the descendants of U_j in a left-to-right fashion as in Theorem 42, we see that each descendant of U_j with positive index has a Poisson(1) number of children, independently of all the others.

To sum up: U_j is distributed as $\mathrm{PGW}(m_{i_{k(j)}})$; each node of U_j independently has $\mathrm{Poisson}(1 - m_{i_{k(j)}})$ children in $T_j \setminus U_j$; and each of these children is the root of a PGW(1) tree, independently of each other and of U_j. It follows that T_j is distributed as PGW(1), as claimed. □

PROOF OF THEOREM 1. The two graphs T and T' are $\mathrm{BG}(\mathcal{P}, \mathbb{Z})$ and $\mathrm{IPC}(\mathcal{P}, \mathbb{Z})$. Part (a) follows from Lemma 17. Part (b) is trivial. Part (c) follows from the example given in Figure 2. Part (d) is trivial. Part (e) follows from Theorems 34 and 48. □

PROOF OF THEOREM 2. For each n, $\mathrm{IPC}(\mathcal{P}, \{-n, -n + 1, \ldots\})$, viewed as rooted at 0, is distributed as $\mathrm{IPC}(\mathcal{P}, \mathbb{N})$, viewed as rooted at n. The theorem then follows from the fact that $\mathrm{IPC}(\mathcal{P}, \{-n, -n + 1, \ldots\}) \to \mathrm{IPC}(\mathcal{P}, \mathbb{Z})$ almost surely. □

PROOF OF THEOREM 3. Let $q : [1, \infty) \to (0, 1]$ be the unique map satisfying the implicit equation $\frac{d\theta(q(\lambda))}{d\lambda} = -1$ for all $\lambda \in [0, \infty)$. (It is possible to write down a more detailed—though still implicit—formula for q, but this is unilluminating and we omit it.) If W is a random variable satisfying $\mathbf{P}\{W \le x\} = \theta(x)$ for all $x \ge 1$, then $q(W) \stackrel{d}{=} \mathrm{Uniform}[0, 1]$, from which the first part of the theorem follows immediately. The second part of Theorem 3 then follows from the first, together with Theorems 30, and 31. □

PROOF OF THEOREM 46. Let $U_0, U_1, U_2, \ldots$ be the subtrees of $\mathrm{IPC}(\mathcal{P}, \mathbb{Z})$ introduced in the proof of Theorem 48. $\mathrm{IPC}^-(\mathcal{P}, \mathbb{Z})$ consists exactly of an infinite backbone path through vertices $n_0, n_1, n_2, \ldots,$ with U_j attached at n_j for each $j \ge 0$. Each U_j is distributed as $\mathrm{PGW}(m_{i_{k(j)}})$. So all that remains to prove the theorem is to prove that the sequence $(m_{i_{k(j)}})$ is distributed as the sequence $(M_j)_{j\ge 0}$ defined in the Introduction. This follows from Theorem 47 [which states that the backward maximum process weights $m_{i_{k(j)}}$ stay constant for one plus a geometric number of values of j, with parameter dependent on the current weight $m_{i_{k(j)}}$, and the fact that the sequence $(m_{i_k})_{k\ge 0}$ is as described in Lemma 44]. □

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