Acyclic dominating partitions*

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Abstract

Given a graph $G = (V, E)$, let $\mathcal{P}$ be a partition of V. We say that $\mathcal{P}$ is *dominating* if, for each part P of $\mathcal{P}$, the set $V \setminus P$ is a dominating set in G (equivalently, if every vertex has a neighbour of a different part from its own). We say that $\mathcal{P}$ is acyclic if for any parts P, P' of $\mathcal{P}$, the bipartite subgraph $G[P, P']$ consisting of the edges between P and P' in $\mathcal{P}$ contains no cycles. The acyclic dominating number $\mathrm{ad}(G)$ of G is the least number of parts in any partition of V that is both acyclic and dominating; and we shall denote by $\mathrm{ad}(d)$ the maximum over all graphs G of maximum degree at most d of $\mathrm{ad}(G)$. In this paper, we prove that $\mathrm{ad}(3) = 2$, which establishes a conjecture

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of Boiron, Sopena and Vignal [4]. For general d, we prove the upper bound $\mathrm{ad}(d) = O(d \ln d)$ and a lower bound of $\mathrm{ad}(d) = \Omega(d)$.

1 Introduction

Given a graph $G = (V, E)$, let $\mathcal{P}$ be a partition (or colouring) of V. We say that $\mathcal{P}$ is dominating if, for each part (or colour) P of $\mathcal{P}$, the set $V \setminus P$ is a dominating set in G. (Recall that a set $S \subset V$ is a *dominating set* if every vertex of $V \setminus S$ has a neighbour belonging to S.) We may equivalently define the partition $\mathcal{P}$ to be dominating if every vertex has a neighbour of a different colour from its own. We say that $\mathcal{P}$ is *acyclic* if for any parts P, P' of $\mathcal{P}$, the bipartite subgraph $G[P, P']$ consisting of the edges between P and P' in $\mathcal{P}$ contains no cycles. We call any cycle in $G[P, P']$ an *alternating cycle*; thus, $\mathcal{P}$ is acyclic if it contains no alternating cycle. The *acyclic dominating number* $\mathrm{ad}(G)$ of G is the least number of parts in any partition of V that is both acyclic and dominating.

By the definition of a dominating partition, the parameter $\mathrm{ad}(\cdot)$ is only well-defined on graphs with no isolated vertices, and we hereafter assume this to be true for any graph under consideration, unless specified otherwise. Note that $\mathrm{ad}(G) \geq 2$ for any graph G, since any dominating partition has at least two parts.

The quantity $\mathrm{ad}(G)$ is closely related to the *acyclic t-improper chromatic number* $\chi_a^t(G)$ of the graph G. In this graph colouring variant, first introduced by Boiron *et al.* [4, 5] and further investigated in Addario *et al.* [1] one seeks to colour G with the minimum number of colours subject to the constraints that each colour class has maximum degree at most t and that the colouring is acyclic in the sense described above. Clearly, the acyclic 0-improper chromatic number is just the acyclic (proper) chromatic number $\chi_a(G)$ — the subject of many works: *inter alia*, [2, 3, 6, 8]. Observe that $\mathrm{ad}(G) \leq \chi_a(G)$ for any graph G, as any acyclic colouring is also an acyclic dominating partition.

It is easily seen that if G is a regular graph of degree $\Delta(G)$ then $\mathrm{ad}(G)$ is precisely the acyclic $(\Delta(G)-1)$-improper chromatic number $\chi_a^{\Delta(G)-1}(G)$ of G. If G is a graph of *maximum degree* $\Delta(G)$, then $\mathrm{ad}(G)$ is *at least* $\chi_a^{\Delta(G)-1}(G)$; however, these two quantities do not necessarily coincide as the latter allows partitions in which vertices of degree strictly less than the maximum degree may receive the same colour as all of their neighbours.

Given a positive integer d, we let $\mathrm{ad}(d)$ be the maximum possible value of $\mathrm{ad}(G)$ over all graphs with maximum degree at most d. In this paper, we settle the case $d = 3$. In Boiron *et al.* [4] it was conjectured that for any graph G of maximum degree at most three, the acyclic 2-improper chromatic number of G is at most two. We prove this conjecture by showing the following.

Theorem 1. $\mathrm{ad}(3) = 2$.

In other words, *any graph G of maximum degree three may be partitioned into two dominating sets D_1, D_2 such that $G[D_1, D_2]$ is a forest.* The latter formulation of Theorem 1 suggests another question: given a graph $G = (V, E)$, does there always exist an integer $k > 1$ and a partition of V into *dominating sets* $V_1, \ldots, V_k$ such that for distinct $i, j \in \{1, \ldots, k\}$, $G[V_i, V_j]$ is a forest? It turns out that such a partition does not necessarily exist, as the following example shows.

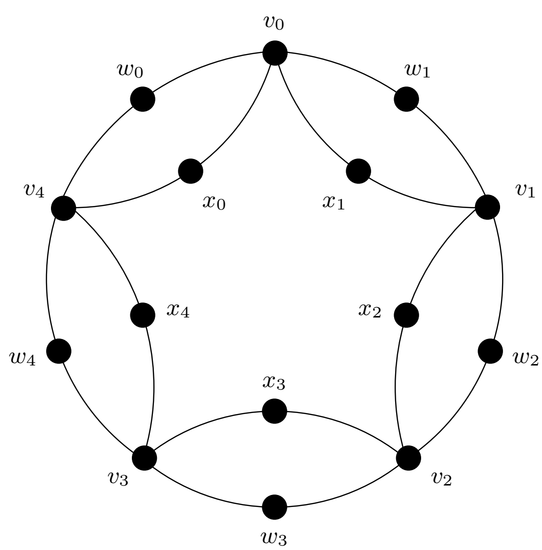


Figure 1: An example of a graph which does not admit an acyclic partition into dominating sets.

Let G have vertex set $V = \bigcup_{i=0}^{4}\{v_i, w_i, x_i\}$, and for each $i \in \{0, \ldots, 4\}$ let v_i be joined to each of $w_i, x_i, w_{i+1}, x_{i+1}$ (where the subscripts are interpreted modulo 5). The graph G is illustrated in Figure 1. Given any 2-colouring of V, there must be $i \in \{0, \ldots, 4\}$ such that v_i and v_{i+1} receive the same colour. In this case, for the colouring to be dominating it must be the case that both w_{i+1} and x_{i+1} receive the opposite colour from v_i; but then $v_i w_{i+1} v_{i+1} x_{i+1}$ forms an alternating cycle. In any colouring with four or more colours, some colour class is not a dominating set as G contains vertices of degree two. Finally, consider any 3-colouring of G. If, for some i, both v_i and v_{i+1} receive the same colour, say, 1, then either $v_i w_{i+1} v_{i+1} x_{i+1}$ forms an alternating cycle or both w_{i+1} and v_{i+1} are not dominated by one of the colours other than 1. But then it must be the case that, for some i, v_i and v_{i+2} receive the same colour, say, v_i and v_{i+2} are coloured 1 and v_{i+1} is coloured 2. In this case, all of $w_{i+1}, x_{i+1}, w_{i+2}, x_{i+2}$ must have colour 3, for otherwise they are not dominated by colour 3; however, now v_{i+1} is not dominated by colour 1. This shows that there is no acyclic colouring of G such that each colour class is a dominating set. We remark that since the graph G has maximum degree four, this example also shows that $\mathrm{ad}(4) \geq 3$.

The fact that acyclic partitions into dominating sets do not always exist lends credence to the idea that the acyclic dominating number and $\mathrm{ad}(d)$ are natural objects of study. Given that a partition of V into two dominating sets is extremely easy to find (any bipartition that maximises the number of edges in the cut is such a partition), it seems *prima facie* plausible that $\mathrm{ad}(d)$ can be bounded independently of d. However, this turns out not to be the case. It was shown in Addario *et al.* [1] that $\chi_a^{d-1}(d) = \Omega(d^{2/3})$. In particular, this shows that $\mathrm{ad}(d) \geq \chi_a^{d-1}(d)$ tends to infinity as $d \to \infty$. We improve upon this result by showing the following.

Theorem 2. *$\chi_a^{d-1}(d) = \Omega(d)$.*

It immediately follows that $\mathrm{ad}(d) = \Omega(d)$. Our lower bound is within a logarithmic factor of optimal as we also give the following upper bound on $\mathrm{ad}(d)$.

Theorem 3. *$\mathrm{ad}(d) = O(d \ln d)$.*

This extends one case of a result in Addario *et al.* [1], which stated that $\chi_a^{d-1}(d) = O(d \ln d)$. It seems plausible that $\mathrm{ad}(d) = \Theta(d)$, but proving the requisite upper bound seems to require a more refined analysis.

1.1 Notation

For a vertex $v \in V$, we denote the *neighbourhood* $N(v)$ of v to be the set $\{w : vw \in E\}$ and the *degree* $\deg(v)$ of v to be $|N(v)|$; the *closed neighbourhood* $N[v]$ of v is the set $\{v\} \cup N(v)$, the *second neighbourhood* $N^2(v)$ of v is $\bigcup_{u \in N(v)} N(u) \setminus \{v\}$, and the *closed second neighbourhood* $N^2[v]$ of v is $\{v\} \cup N^2(v)$. The *square* of a graph $G = (V, E)$ has vertex set V and edge set $\{uv : u \in N^2(v)\}$. For a given partition $\mathcal{P}$ of V, and $v \in V$, the *colour* $c_{\mathcal{P}}(v)$ of v with respect to $\mathcal{P}$ is the part of $\mathcal{P}$ to which v belongs. We write $c(v)$ in place of $c_{\mathcal{P}}(v)$ when the partition $\mathcal{P}$ is clear from context. Given sets A and S, the *symmetric difference* $A \ominus S$ between A and S is the set $(A \setminus S) \cup (S \setminus A)$.

2 Graphs of maximum degree three

The primary ingredient in proving Theorem 1 is the following lemma.

Lemma 4. *If* $G = (V, E)$ *is* 2*-connected,* $\Delta(G) = 3$ *and* $\mathcal{P} = \{A, B\}$ *is a dominating partition of* V *such that* G *possesses a unique alternating cycle* C_1 *then* $\mathrm{ad}(G) = 2$*.*

We first provide the straightforward proof of Theorem 1 assuming that the lemma holds, then prove the lemma.

Proof of Theorem 1. Consider an arbitrary graph $G = (V, E)$ of maximum degree three. We proceed by induction on $m = |E|$; clearly if $|E| \leq 3$ then $\mathrm{ad}(G) = 2$ as G has no even cycles. We may presume G is connected; if not, we consider each connected component of G separately. We may also assume G has no vertex of degree one, for if $\deg(v) = 1$ then $G \setminus v$ is connected and by induction there is an acyclic dominating partition of $G \setminus v$; such a partition easily extends to an acyclic dominating partition of G.

Now if G contains a cutedge uv and $G \setminus uv$ has connected components G_1, G_2 (each of which contains no isolated vertices), then by induction there is an acyclic dominating partition $\{A_1, B_1\}$ (resp. $\{A_2, B_2\}$) of G_1 (resp. G_2); we may assume, perhaps by switching the names of the parts, that $u \in A_1$, $v \in B_2$. Then $\{A_1 \cup A_2, B_1 \cup B_2\}$ forms an acyclic dominating partition of G.

If G contains no cutedge then G is necessarily 2-connected. Now let uv be any edge of G; by induction $G \setminus uv$ permits an acyclic dominating partition $\mathcal{P} = \{A, B\}$. Since $\mathcal{P}$ is a dominating partition in the entire graph G, either $\mathcal{P}$ is an acyclic dominating partition for G or G possesses a unique alternating cycle C_1 with respect to $\mathcal{P}$. In the latter case it follows by Lemma 4 that $\mathrm{ad}(G) = 2$; thus $\mathrm{ad}(G) = 2$ in both cases and so $\mathrm{ad}(3) = 2$, as claimed. $\square$

We shall prove Lemma 4 by producing a sequence of local alterations that transform $\mathcal{P}$ into an acyclic dominating partition $\mathcal{P}' = \{A', B'\}$. In order to do so, we first introduce a structure that is at the heart of the proof and some basic conditions that allow us to immediately "fix alternating cycles".

We say C is an *almost alternating cycle* with respect to partition $\mathcal{P} = \{A, B\}$ if there exists a vertex $u \in C$ such that C is an alternating cycle with respect to the partition $\{A \ominus \{u\}, B \ominus \{u\}\}$; in other words, if switching u from A to B (or from B to A) yields that C is an alternating cycle. Given an almost alternating cycle C, the unique $u \in C$ such that C is an alternating cycle with respect to the partition $\{A \ominus \{u\}, B \ominus \{u\}\}$ is called the *crucial vertex* of C.

We now define three basic local conditions to check for (almost) alternating cycles. These conditions are illustrated in Figure 2, with the convention that black and white represent the two parts of the partition. Suppose we are given an alternating or almost alternating cycle C and non-crucial vertices v, w of C adjacent along C; if $\deg(v) = 3$ (resp. $\deg(w) = 3$) then denote the neighbour of v (resp w) not along C by x (resp. y). We remark that possibly $x \in C$, in which case vx is a chord of C. We say that v is *flippable* (with respect to C and $\mathcal{P}$) if $\deg(v) = 3$ and $c(v) = c(x)$. We say v and w are *switchable* (with respect to C and $\mathcal{P}$) if neither v nor w is flippable and

(i) either $\deg(v) = 2$ or ($\deg(v) = 3$ and there is $z_1 \in N(x) \setminus \{v\}$ with $c(z_1) = c(v)$); *and*

(ii) either $\deg(w) = 2$ or ($\deg(w) = 3$ and there is $z_2 \in N(y) \setminus \{w\}$ with $c(z_2) = c(w)$).

Finally, v and x are *exchangeable* (with respect to C and $\mathcal{P}$) if $\deg(v) = 3$, v is not flippable and

(i) x is not the crucial vertex of an almost alternating cycle, and

(ii) for any $z_1 \in N(x) \setminus \{v\}$ with $c(z_1) = c(v)$, there exists $z' \in N(z_1) \setminus \{x\}$ such that $c(z') \neq c(v)$.

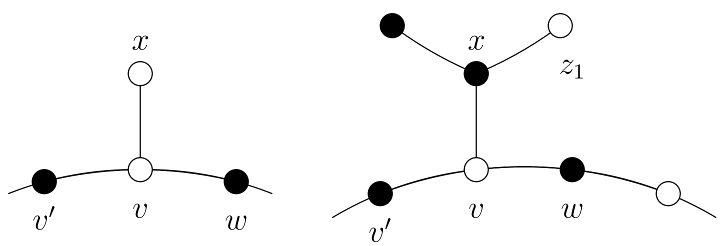
We define exchangeability for w and y symmetrically.

The key properties of flippable vertices and of switchable and exchangeable pairs are the following.

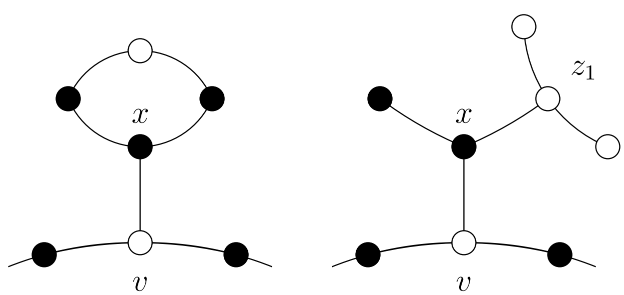
Fact 5. *If $\mathcal{P}_1 = \{A_1, B_1\}$ is a dominating partition for G, $G[A_1, B_1]$ contains a unique alternating cycle C, and v is a flippable vertex with respect to C and $\mathcal{P}_1$, then, letting $A_2 = A_1 \ominus \{v\}$, $B_2 = B_1 \ominus \{v\}$, $\mathcal{P}_2 = \{A_2, B_2\}$ is an acyclic dominating partition for G.*

Fact 6. *If $\mathcal{P}_1 = \{A_1, B_1\}$ is a dominating partition for G, $G[A_1, B_1]$ contains a unique alternating cycle C, and v and w are switchable with respect to C and $\mathcal{P}_1$, then, letting $A_2 = A_1 \ominus \{v, w\}$, $B_2 = B_1 \ominus \{v, w\}$, $\mathcal{P}_2 = \{A_2, B_2\}$ is an acyclic dominating partition for G.*

Fact 7. *If $\mathcal{P}_1 = \{A_1, B_1\}$ is a dominating partition for G, $G[A_1, B_1]$ contains a unique alternating cycle C, and there are vertices $v \in C$ and x such that v and x are exchangeable with respect to C and $\mathcal{P}_1$, then, letting $A_2 = A_1 \ominus \{v, x\}$, $B_2 = B_1 \ominus \{v, x\}$, $\mathcal{P}_2 = \{A_2, B_2\}$ is an acyclic dominating partition for G.*



(a) Examples of a flippable vertex (v, left) and of a switchable pair of vertices (v and w, right).



(b) Two situations where v and x are not exchangeable. On the left, v and x are not exchangeable because x is the crucial vertex of an almost alternating cycle. On the right, v and x are not exchangeable because all of z_1's neighbours (aside from x) have the same colour as v.

Figure 2: Illustrations of fixable vertices.

In the proofs of all three facts, we denote the neighbours of v along C by w and v'. If v has a neighbour not along C we denote this neighbour x.

Proof of Fact 5. We show that (a) $\mathcal{P}_2$ contains no alternating cycles, and (b) $\mathcal{P}_2$ is dominating. Any cycle C' that does not pass through v has $C' \cap A_2 = C' \cap A_1$ and $C' \cap B_2 = C' \cap B_1$; therefore, to prove (a) it suffices to show that no cycle containing v is alternating with respect to $\mathcal{P}_2$. Similarly, to prove (b) we need only check that each vertex $u \in N[v]$ is dominated under $\mathcal{P}_2$.

To prove (a), observe that under $\mathcal{P}_2$, v has the same colour as both w and v'; thus no alternating cycle passes through v under $\mathcal{P}_2$. To prove (b), note that since v' is in C, the neighbour of v' along C that is not v dominates v' under $\mathcal{P}_2$; symmetrically, w is dominated under $\mathcal{P}_2$. Finally, under $\mathcal{P}_2$, x is dominated by v, which establishes (b). □

Proof of Fact 6. As in the proof of Fact 5, it suffices to prove that (a) no cycle containing either v or w is alternating with respect to $\mathcal{P}_2$, and (b) each vertex $u \in N[v] \cup N[w]$ is dominated under $\mathcal{P}_2$. Let w' be the neighbour of w along C that is not v; if $\deg(w) = 3$ then denote by y the neighbour of w not along C.

Under $\mathcal{P}_2$, v has the same colour as v' and w has the same colour as w'; thus, no new alternating cycles pass through the edges vv' or ww'. Furthermore, if x and y both exist, then under $\mathcal{P}_2$, v and x have the same colour so no cycle through $xvwy$ is alternating. This establishes (a). To prove (b), first note that v' and w' are dominated by their neighbours along C (other than v and w), and v and w dominate each other under $\mathcal{P}_2$. If x exists, then, by condition (i) in the definition of switchable pairs, x must be dominated under $\mathcal{P}_2$. Symmetrically, if y exists it is dominated under $\mathcal{P}_2$. Thus (b) holds. □

Proof of Fact 7. As in the proof of Fact 5, it suffices to prove that (a) no cycle containing either v or x is alternating with respect to $\mathcal{P}_2$, and (b) each vertex $u \in N[v] \cup N[x]$ is dominated under $\mathcal{P}_2$.

Under $\mathcal{P}_2$, v has the same colour as both v' and w; thus, no new alternating cycles pass through v. Since x is not the crucial vertex of an almost alternating cycle, no new alternating cycles pass through x and (a) holds. To prove (b), note that v' and w are dominated by their neighbours along C (other than v), and v and x are dominated by each other under $\mathcal{P}_2$. Let $z_1 \in N(x) \setminus \{v\}$. If z_1 and x were in the same part of $\mathcal{P}_1$, then they are in different parts of $\mathcal{P}_2$, in which case z_1 is dominated under $\mathcal{P}_2$; otherwise, we know from condition (ii) of exchangeability that z_1 is dominated under $\mathcal{P}_2$ by some $z' \in N(z_1) \setminus \{x\}$, which establishes (b). □

Motivated by these facts, we say that an alternating cycle C is *fixable* (with respect to $\mathcal{P}$), if it has either a flippable vertex or an exchangeable or switchable pair. The proof of Lemma 4 proceeds by first finding a sequence of local alterations to $\mathcal{P}$ resulting in a dominating partition $\{A_1, B_1\}$ such that $G[A_1, B_1]$ contains a unique alternating cycle C that is fixable, then applying one of Facts 5, 6 and 7. We now turn to the details.

Proof of Lemma 4. Let $C_1, \ldots, C_k$ be a sequence of cycles with C_1 the alternating cycle in the statement of the lemma, and such that for $i \in \{2, \ldots, k\}$

(a) C_i is an almost alternating cycle and

(b) denoting the crucial vertex of C_i by u_i and its neighbours along C_i by x_i, y_i, we have $\{u_i, x_i, y_i\} \cap \bigcup_{j=1}^{i-1} C_j = \emptyset$.

For $i \in \{1, \ldots, k\}$, let C_i^* be the largest vertex subset of C_i such that $C_i^* \cap \bigcup_{j=1}^{i-1} C_j = \emptyset$ (so, trivially, $C_1^* = C_1$ and, by (b), C_i^* contains $\{u_i, x_i, y_i\}$); we additionally require that for $i \in \{1, \ldots, k-1\}$

(c) $C_i^* \setminus \{u_i\}$ contains no flippable or exchangeable vertices and no switchable pairs and

(d) u_{i+1} has a neighbour v_i in $C_i^* \setminus \{u_i\}$, and $v_i \neq u_i$ if $i > 1$.

Note that u_{i+1} has the same colour as both its neighbours in C_{i+1} and so it has the opposite colour from v_i. See Figure 3. Choose $C_1, \ldots, C_k$ to maximize k subject to the constraints (a)–(d). It is of course possible that $k = 1$. Let $S = \bigcup_{i=1}^{k-1} \{v_i, u_{i+1}\}$ — if $k = 1$ then $S = \emptyset$ — and let $\mathcal{P}' = \{A', B'\} = \{A \ominus S, B \ominus S\}$. In what follows, we write $c(\cdot)$ in place of $c_{\mathcal{P}}(\cdot)$ and $c'(\cdot)$ in place of $c_{\mathcal{P}'}(\cdot)$.

The lemma follows immediately from Facts 5, 6 and 7 together with the following claims.

Claim 8. *$\mathcal{P}'$ is a dominating partition for G. Furthermore, either* $\mathrm{ad}(G) = 2$ *or C_k is the unique alternating cycle in $G[A', B']$.*

Claim 9. *C_k is fixable with respect to $\mathcal{P}'$.*

Proof of Claim 8. Let us consider the sequence of partitions defined by $\mathcal{P}_1 = \mathcal{P}$, and, for $j \in \{2, \ldots, k\}$, $S_j = \bigcup_{i=1}^{j-1} \{v_i, u_{i+1}\}$ and $\mathcal{P}_j = \{A_j, B_j\} = \{A \ominus S_j, B \ominus S_j\}$. We write $c_j(\cdot)$ in place of $c_{\mathcal{P}_j}(\cdot)$. We show by induction that $\mathcal{P}_j$ is a dominating partition for G and, furthermore, either $\mathrm{ad}(G) = 2$ or C_j is the unique alternating cycle in $G[A_j, B_j]$; this proves the claim since $\mathcal{P}_k = \mathcal{P}'$. The case $j = 1$ holds by assumption, so let $j \in \{2, \ldots, k\}$. Note that $A_j = A_{j-1} \ominus \{v_{j-1}, u_j\}$ and $B_j = B_{j-1} \ominus \{v_{j-1}, u_j\}$.

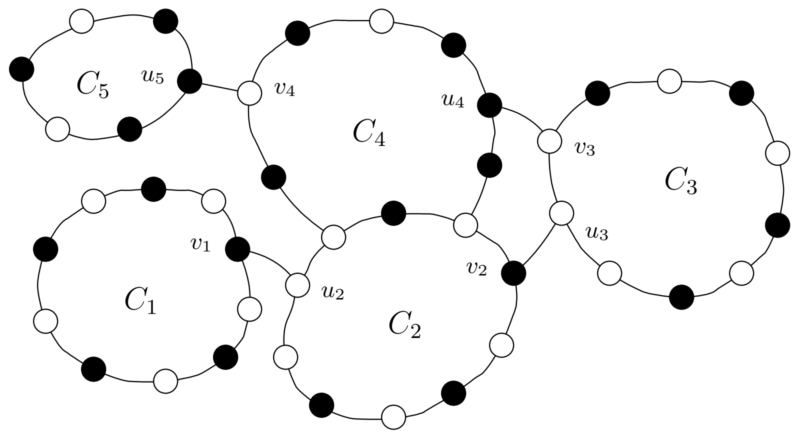


Figure 3: A diagram of an example set of cycles $\{C_1, C_2, C_3, C_4, C_5\}$.

We first show that $\mathcal{P}_j$ is dominating. Observe that if $v \notin N[v_{j-1}] \cup N[u_j]$ then none of its neighbours change colour in the transition from $\mathcal{P}_{j-1}$ to $\mathcal{P}_j$. Thus, since, by induction, v is dominated under $\mathcal{P}_{j-1}$, it is also dominated under $\mathcal{P}_j$. Also, $c_{j-1}(v_{j-1}) \neq c_{j-1}(u_j)$, so $c_j(v_{j-1}) \neq c_j(u_j)$ and both v_{j-1} and u_j are dominated under $\mathcal{P}_j$. If $v \in N(u_j) \setminus \{v_{j-1}\}$, say $v = x_j$, then $x_j, u_j \notin S_{j-1}$ by condition (b) in the definition of the cycles $C_1, \ldots, C_k$ and so $c_{j-1}(x_j) = c_{j-1}(u_j)$. Then necessarily $c_j(x_j) \neq c_j(u_j)$ and v is dominated under $\mathcal{P}_j$. If $v \in N(v_{j-1}) \setminus \{u_j\}$, then let v' denote the neighbour of v along C_{j-1} that is not v_{j-1}. Since C_{j-1} is an alternating cycle in $G[A_{j-1}, B_{j-1}]$ by induction, it follows that $c_{j-1}(v) \neq c_{j-1}(v')$. Since $v' \in C_{j-1}$, it follows by condition (b) in the definition of the cycles $C_1, \ldots, C_k$ that $v' \neq u_j$. Thus, $c_j(v) \neq c_j(v')$ and v is dominated under $\mathcal{P}_j$. This completes the proof that $\mathcal{P}_j$ is dominating.

Next, we show that C_j is the only possible alternating cycle in $G[A_j, B_j]$. Since $A_j = A_{j-1} \ominus \{v_{j-1}, u_j\}$ and $B_j = B_{j-1} \ominus \{v_{j-1}, u_j\}$, it follows that C_{j-1} is not an alternating cycle under $\mathcal{P}_j$. We shall show that we have created no alternating cycles other than C_j in the transition from $\mathcal{P}_{j-1}$ to $\mathcal{P}_j$. Under $\mathcal{P}_j$, the two members of $N(v_{j-1}) \setminus \{u_j\}$ have the same colour as v_{j-1}; thus, no new alternating cycle passes through v_{j-1}. This means any new alternating cycle must pass through $x_j u_j y_j$. If some $C \neq C_j$ is alternating under $\mathcal{P}_j$, then the subgraph $C \cup C_j \setminus \{u_j\}$ contains an alternating cycle $C' \neq C_{j-1}$. As C' contains neither v_{j-1} nor u_j, C' is alternating under $\mathcal{P}_{j-1}$, but this contradicts the uniqueness of C_{j-1} in $G[A_{j-1}, B_{j-1}]$. Now to complete the inductive step and the proof we note that if C_j is not alternating in $G[A_j, B_j]$ then $\mathcal{P}_j$ is acyclic, so $\mathrm{ad}(G) = 2$. □

To prove that C_k is fixable with respect to $\mathcal{P}'$ we first show that C_k is fixable with respect to $\mathcal{P}$. In fact, we show the following stronger statement.

Claim 10. *One of the following holds:*

(A) $C_k^* \setminus \{u_k\}$ *contains a vertex that is flippable with respect to* C_k *and* $\mathcal{P}$*;*

(B) $C_k^* \setminus \{u_k\}$ *contains a pair of vertices that are switchable with respect to* C_k *and* $\mathcal{P}$*;*

(C) *there are vertices* v *and* x *with* $v \in C_k^* \setminus \{u_k\}$ *such that* v *and* x *are exchangeable with respect to* C_k *and* $\mathcal{P}$ *and such that*

$$\forall\ z_1 \in N(x) \setminus \{v\}, c(z_1) = c(x) \neq c(v); \tag{1}$$

(D) *there are vertices* v *and* x *with* $v \in C_k^* \setminus \{u_k\}$ *such that* v *and* x *are exchangeable with respect to* C_k *and* $\mathcal{P}$*, and such that there exists* $z_1 \in N(x) \setminus \{v\}$ *with* $c(z_1) = c(v)$*.*

Note that ((C) or (D)) is equivalent to the condition that there are vertices v and x with $v \in C_k^* \setminus \{u_k\}$ such that v and x are exchangeable with respect to C_k and $\mathcal{P}$. Thus, Claim 10 is equivalent to the condition that C_k is fixable with respect to $\mathcal{P}$ **within** $C_k^* \setminus \{u_k\}$. We have separated (C) and (D) because they require different treatments not only in this proof but also later in the proof of Claim 9.

Proof of Claim 10. As a first step, we prove the following.

($\star$) If none of (A), (B), (C) holds, then there exists a vertex $\hat{v} \in C_k^*$ with a neighbour $\hat{x} \notin C_k$ that is crucial for some almost alternating cycle $\hat{C}$.

We break the proof of ($\star$) into two main cases.

In the first case, suppose there are two adjacent vertices $\hat{v}, \hat{w} \in C_k^* \setminus \{u_k\}$. If it exists, denote by $\hat{x}$ (resp. $\hat{y}$) the neighbour not along C_k of $\hat{v}$ (resp. $\hat{w}$). If $\hat{v}$ or $\hat{w}$ is flippable with respect to C_k and $\mathcal{P}$, then (A) holds; otherwise, it follows that either $\deg(\hat{v}) = 2$ or ($\deg(\hat{v}) = 3$ and $c(\hat{x}) \neq c(\hat{v})$), and either $\deg(\hat{w}) = 2$ or ($\deg(\hat{w}) = 3$ and $c(\hat{y}) \neq c(\hat{w})$). If $\hat{v}$ and $\hat{w}$ are switchable with respect to C_k and $\mathcal{P}$, then (B) holds; otherwise, either $\hat{v}$ or $\hat{w}$ has degree three — without loss of generality, we may presume $\hat{v}$ — and by the definition of switchability (1) holds with $v = \hat{v}$, $x = \hat{x}$. Clearly, $\hat{x} \notin C_k$. We note that since (1) holds with $v = \hat{v}$, $x = \hat{x}$, no vertex in $N(\hat{x}) \setminus \{\hat{v}\}$ has the same colour as $\hat{v}$, so that condition (ii) in the definition of exchangeable vertices holds vacuously. Therefore, if (C) does not hold then it must be the case that $\hat{x}$ is the crucial vertex of an almost alternating cycle $\hat{C}$.

In the second case, suppose there are two adjacent vertices $\hat{v}, \hat{w} \in C_k$ with $\hat{v} \in C_k^* \setminus \{u_k\}$ and $\hat{w} \notin C_k^*$. Since $\hat{w} \notin C_k^*$, choose $j < k$ as small as possible such that $\hat{w} \in C_j$; necessarily, $\hat{w} \in C_j^*$. By the definition of C_k^*, $\hat{v} \notin C_j$ and so the other two neighbours of $\hat{w}$ are in C_j. Since C_k is an almost alternating cycle and $\hat{w} \notin \{u_k, x_k, y_k\}$, it follows that $c(\hat{v}) \neq c(\hat{w})$. Therefore, if $\hat{w} = v_j$, then it must be that $\hat{v} = u_{j+1}$, but this contradicts either that $\hat{v} \in C_k^*$ or that $\hat{v} \neq u_k$. Similarly, if $\hat{w} = u_j$, then it must be that $\hat{v} = v_{j-1}$, again a contradiction.

Next, we consider the assumption — by condition (c) in the definition of the cycles $C_1, \ldots, C_k$ — that $\hat{w}$ and $\hat{v}$ are not exchangeable with respect to C_j and $\mathcal{P}$. If condition (i) of exchangeability does not hold, then $\hat{v}$ is the crucial vertex of some almost alternating cycle. This implies that the two neighbours of $\hat{v}$ other than $\hat{w}$ have the same colour as $\hat{v}$. Then it must be the case that $\hat{v}$ is flippable with respect to C_k and $\mathcal{P}$, in which case (A) holds. If (A) does not hold and additionally condition (ii) of exchangeability fails, then $\hat{v}$ has a neighbour $\hat{x} \neq \hat{w}$ for which $c(\hat{x}) = c(\hat{w})$ and such that every element of $N(\hat{x}) \setminus \{\hat{v}\}$ has the same colour as $\hat{w}$. If $\hat{x} \in C_k$, then it must be that $\hat{x} \in \{x_k, y_k\}$ and $\hat{x}$ is flippable with respect to C_k and $\mathcal{P}$, so that (A) holds. Otherwise, (1) holds with $v = \hat{v}$ and $x = \hat{x}$. Furthermore, in this case condition (ii) for the exchangeability of $\hat{v}$ and $\hat{x}$ with respect to C_k and $\mathcal{P}$ is vacuously satisfied. Thus, either (C) holds or $\hat{x}$ is crucial for some almost alternating cycle $\hat{C}$.

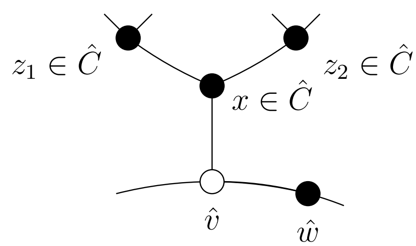


Figure 4: The situation ($\star$) in Claim 10.

In both main cases, we found that ($\star$) holds; we hereafter assume that (A), (B), and (C) do not hold and let $\hat{x}$, $\hat{v}$, and $\hat{C}$ be as in ($\star$). Let $N(\hat{x}) \setminus \{\hat{v}\} = \{z_1, z_2\}$ and note that $c(z_1) = c(z_2) = c(\hat{x})$. See Figure 4. Since $\hat{v} \in C_k^*$, it follows that if $\hat{x} \in C_i$ for some $i < k$, then $\{\hat{x}, z_1, z_2\} \subset C_i$, which then implies that $\hat{x} = u_i$ and $\hat{v} = v_{i-1}$, an impossibility. If $z_1 = u_i$ for some $i \leq k$, then $\hat{x} \in \{x_i, y_i\}$ and $\hat{x}$ is flippable with respect to C_i and $\mathcal{P}$, a contradiction. Also, if z_1 is some non-crucial vertex in C_i (take i minimum so that $z_1 \in C_i^*$), then, since $c(z_1) = c(\hat{x})$ and $\hat{x} \notin C_i$, z_1 is flippable with respect to C_i, which contradicts either the condition (c) in the definition of the cycles $C_1, \ldots, C_k$ (if $i < k$), or the assumption that (A) does not hold (if $i = k$). Symmetrically, we can show that $z_2 \notin C_i$ for any $i \leq k$. If furthermore (D) does not hold, then it follows that conditions (a)–(d) in the definition of the cycles hold for $C_1, \ldots, C_{k+1}$ by letting $C_{k+1} = \hat{C}$, $u_{k+1} = \hat{x}$, $x_{k+1} = z_1$, $y_{k+1} = z_2$. This would contradict the maximality of k. □

Finally, to conclude that C_k is fixable with respect to $\mathcal{P}'$ and complete the proof of Claim 9, we show that in fact C_k is fixable with respect to $\mathcal{P}'$ at "the same place" that it is fixable with respect to $\mathcal{P}$. In doing so, we use the following easy observation, which guarantees that $\mathcal{P}'$ and $\mathcal{P}$ are not very different near to C_k^*.

Fact 11. *If $v \in C_k^* \setminus \{u_k\}$, then $(N(v) \setminus \{u_k\}) \cap S$ is empty.*

Proof. Fix $j \in \{1, \ldots, k-2\}$, and let $w \in N(v) \setminus \{u_k\}$. If $w \in \{v_j, u_{j+1}\}$ then $N(w) \subset C_j \cup C_{j+1}$, so $v \in C_j \cup C_{j+1}$, which contradicts the fact that $v \in C_k^*$. Likewise, if v_{k-1} is adjacent to v then v is in C_{k-1}, a contradiction. □

Proof of Claim 9. We split the proof up according to which of (A), (B), (C), and (D) in Claim 10 holds.

(A): Suppose $v \in C_k^* \setminus \{u_k\}$ is flippable with respect to C_k and $\mathcal{P}$, and denote by x the neighbour not along C_k of v. Since $v \in C_k^* \setminus \{u_k\}$, $v \notin S$. Furthermore, $x \notin S$ by Fact 11, so $c'(x) = c(x) = c(v) = c'(v)$ and v is flippable with respect to $\mathcal{P}'$.

(B): Next, suppose v and w in $C_k^* \setminus \{u_k\}$ are switchable with respect to C_k and $\mathcal{P}$. If v (resp. w) has degree three then denote the neighbour of v (resp. w) not along C_k by x (resp. y). Since neither v nor w is in S, $c'(v) = c(v) \neq c(w) = c'(w)$. If $\deg(v) = 3$ then since v and w are switchable, $c(x) \neq c(v)$, and furthermore there exists $z_1 \in N(x) \setminus \{v\}$ with $c(z_1) = c(v)$. Since x has two neighbours with the opposite colour under $\mathcal{P}$, $x \neq u_k$, and so $x \notin S$ by Fact 11. Thus, $c'(x) = c(x) \neq c(v) = c'(v)$.

Suppose that $z_1 \in S$. Note that if $z_1 = u_{i+1}$ for some $i \in \{1, \ldots, k-1\}$, then x must be v_i (and the other two neighbours of z_1 have the same colour as z_1 under $\mathcal{P}$), but this contradicts that $x \notin S$. So it must be that $z_1 = v_i$ for some $i \in \{1, \ldots, k-1\}$. Then, as $x \neq u_{i+1}$ and C_i does not intersect v, necessarily $x \in C_i$ and also x must have another neighbour $z_2 \notin \{z_1, v\}$ with $z_2 \in C_i$. See Figure 5. If $c(z_2) \neq c(x)$, it follows, as we showed for z_1, that $z_2 \notin \{u_2, \ldots, u_k\}$. If $z_2 = v_j$ for some $j \neq i$, then, as $x \neq u_{j+1}$ and C_j does not intersect v, it must be that x and z_1 are both in C_j. In particular, $v_j = z_2 \in C_i$ and $v_i = z_1 \in C_j$, but this contradicts either that $v_i \in C_i^*$ or

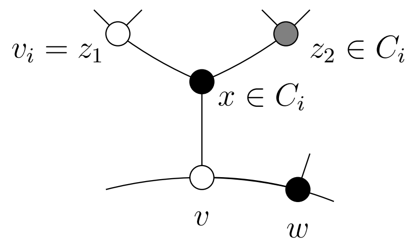


Figure 5: A situation in case (B) in Claim 9. (Half-fill indicates that the colour of the vertex under $\mathcal{P}$ is not yet determined.)

that $v_j \in C_j^*$. So if $c(z_2) \neq c(x)$, then $c'(z_2) = c'(v)$. If $c(z_2) = c(x)$, then necessarily $z_2 = u_i$, so that $c'(z_2) \neq c(z_2) = c(x)$ and $c'(z_2) = c'(v)$.

So we have just shown that if $\deg(v) = 3$ then $c'(x) \neq c'(v)$ and x has a neighbour that is not v with the same colour as v under $\mathcal{P}'$. Symmetrically, if $\deg(w) = 3$ then $c'(y) \neq c'(w)$ and y has a neighbour that is not w with the same colour as w under $\mathcal{P}'$. Therefore, v and w are switchable with respect to C_k and $\mathcal{P}'$.

(C): Next, suppose that there are vertices v and x with $v \in C_k^* \setminus \{u_k\}$ such that v and x are exchangeable with respect to C_k and $\mathcal{P}$ and such that (1) holds. If $x = u_k$, then it must be that $v = v_{k-1}$ which contradicts that $v \in C_k^*$; thus, $x \notin S$ by Fact 11. Let z_1 be an element of $N(x) \setminus \{v\}$ and suppose $z_1 \in S$. If $z_1 = v_i$ for some $i \in \{1, \ldots, k-1\}$, then necessarily $x = u_i$ since $c(z_1) = c(x)$, contradicting that $x \notin S$. Thus, $z_1 = u_{i+1}$ for some $i \in \{1, \ldots, k-2\}$, in which case $x \in C_{i+1}$ (as $c(x) = c(z_1)$). Since $v \notin C_{i+1}$, x has another neighbour z_2 with $z_2 \in C_{i+1} \setminus \{v, z_1\}$. Howevever, it must be that $c(z_2) \neq c(x)$, contradicting (1). Thus, $z_1 \notin S$ so $c'(z_1) = c(z_1) = c(x) = c'(x)$. As z_1 was arbitrary in $N(x) \setminus \{v\}$, it follows that (1) holds with respect to $\mathcal{P}'$.

We now must show that x does not become the crucial vertex of an almost alternating cycle C under $\mathcal{P}'$. If it does, then C contains none of the vertices v_i. (If $v_i \in C$ for some i, then, since no element of $N(x) \setminus \{v\}$ is in $\{v_1, \ldots, v_{k-1}\}$, C would contain two consecutive vertices, one of them v_i, with the same colour in $\mathcal{P}'$ and so would not be alternating under $\mathcal{P}'$.) Also, C contains at least one of the vertices u_i, or else C was already an alternating cycle under $\mathcal{P}$. If $u_i \in C$, then since $v_{i-1} \notin C$, it must be that $x_i \in C$ and $y_i \in C$. Let $I = \{i : u_i \in C\}$ and let

$$H = \bigcup_{i \in I} C_i \cup (C \setminus \{x\}) \setminus \bigcup_{i \in I} \{u_i\}.$$

All of the edges between vertices in H cross the partition $\mathcal{P}$; also, H is connected and there is a path P in H between the two neighbours of x other than v. But then $V(P) \cup \{x\}$ induces an almost alternating cycle under $\mathcal{P}$ for which x is the crucial vertex, contradicting that v and x are an exchangeable pair with respect to C_k and $\mathcal{P}$. Therefore, v and x are exchangeable with respect to C_k and $\mathcal{P}'$.

(D): Finally, suppose there are vertices v and x with $v \in C_k^* \setminus \{u_k\}$ such that v and x are exchangeable with respect to C_k and $\mathcal{P}$, and such that there exists $z_1 \in N(x) \setminus \{v\}$ with $c(z_1) = c(v)$. If it exists, denote the vertex in $N(x) \setminus \{v, z_1\}$ by z_2. By condition (ii) for the exchangeability of v and x with respect to C_k and $\mathcal{P}$, there exists $z' \in N(z_1) \setminus \{x\}$ such that $c(z') \neq c(v)$. Denote the vertex in $N(z_1) \setminus \{x, z'\}$ (if it exists) by z''. By condition (i) of exchangeability, $x \neq u_k$; thus, $x \notin S$ by Fact 11.

First, assume that $z_1 \in S$. Now, $z_1 \notin \{u_2, \ldots, u_k\}$ since it has two neighbours, z' and x, with the opposite colour under $\mathcal{P}$. Thus $z_1 = v_i$ for some $i \in \{1, \ldots, k-1\}$. Since $x \notin S$, $x \neq u_{i+1}$ and $x \in C_i$. Pick j to be the smallest such that $x \in C_j$, so that $x \in C_j^*$. Since $v \in C_k^*$, $v \notin C_j$, so z_1 and z_2 are both in C_j. Since $c(z_1) \neq c(x)$, $x \neq u_j$. We now claim that x and v are exchangeable with respect to C_j and $\mathcal{P}$, which contradicts condition (c) in the definition of the cycles $C_1, \ldots, C_k$. Since $v \neq u_k$ and C_k is an almost alternating cycle, one of the neighbours of v along C_k has the opposite colour from v under $\mathcal{P}$. See Figure 6(a). This verifies condition (i) for the exchangeability of x and v with respect to C_j and $\mathcal{P}$. If $w \in N(v) \setminus \{x\}$ and $c(w) = c(x) (\neq c(v))$, then either the neighbour $w' (\neq v)$ of w along C_k satisfies $c(w') \neq c(x)$ or $w' = u_k$. In the latter case, $w \in C_k^* \setminus \{u_k\}$ and, since we are not in case (A) or (B), w is not flippable and w and x are not switchable with respect to C_k and $\mathcal{P}$; thus, we must have $\deg(w) = 3$ and the neighbour y of w not along C_k must satisfy $c(y) \neq c(x)$. As w was arbitrary in $N(v) \setminus \{x\}$, this verifies condition (ii) for the exchangeability of x and v with respect to C_j and $\mathcal{P}$.

We hereafter assume that $z_1 \notin S$. Then $c'(z_1) = c'(v)$, so x is not the crucial vertex of an alternating cycle with respect to $\mathcal{P}'$, which verifies condition (i) for the exchangeability of v and x with respect to C_k and $\mathcal{P}'$. In order to verify condition (ii) with respect to z_1 for the exchangeability of v and x with respect to C_k and $\mathcal{P}'$, we must show that there exists $z \in N(z_1) \setminus \{x\}$ such that $c'(z) \neq c'(v)$. (With respect to condition (ii), the situation for z_2, if it exists, will be handled separately.) Suppose otherwise; in other words, assume both of the following:

(I) if $z \in N(z_1) \setminus \{x\}$ and $c(z) \neq c(v)$, then $z \in S$; and,

(II) if $z \in N(z_1) \setminus \{x\}$ and $c(z) = c(v)$, then $z \notin S$.

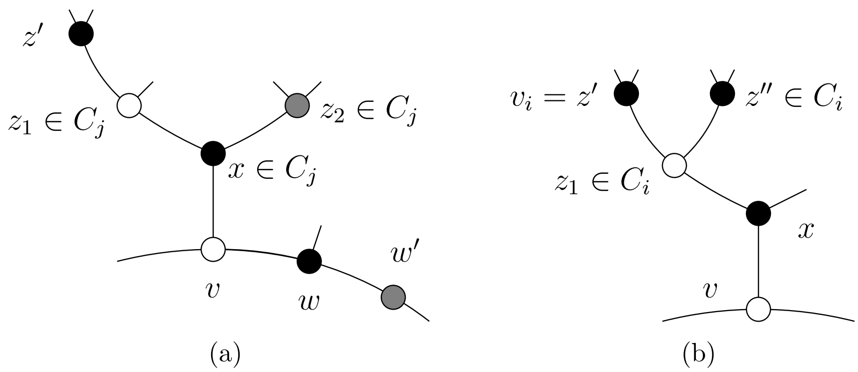


Figure 6: Illustrations of situations in case (D) in Claim 9. (Half-fill indicates that the colour of the vertex under $\mathcal{P}$ is not yet determined.)

In particular, by (I), $z' \in S$. Note that $z' \notin \{u_2 \ldots, u_k\}$ since z_1 is a neighbour of z' with the opposite colour and $z_1 \notin S$, so we may suppose that $z' = v_i$ for some $i \in \{1, \ldots, k-1\}$. Since $z_1 \neq u_{i-1}$, $z_1 \in C_i$. We now show that $x \in C_j$ for some $j < k$. If not, then z'' exists and $z'' \in C_i$. It must be that $c(z'') = c(z) \neq c(v)$, for otherwise, by (II), $z'' \notin S$ and then C_i contains two consecutive vertices, z_1 and z'', that are both not u_i and both the same colour, a contradiction. So $c(z'') \neq c(v)$ and, by (I), $z'' \in S$. See Figure 6(b). Then, as we argued for z', $z'' = v_{i'}$ for some $i' \in \{1, \ldots, k-1\}$, $i \neq i'$. However, it cannot be that both $x \notin C_i$ and $x \notin C_{i'}$, as otherwise there is a contradiction either with $v_i \in C_i^*$ or with $v_{i'} \in C_{i'}^*$. We now have that $x \in C_j$ for some $j < k$, and may pick j smallest, so that $x \in C_j^*$. However, we can argue, as we did in the case $z_1 \in S$, that $v \notin C_j$, $x \neq u_j$ and, ultimately, that x and v are exchangeable with respect to C_j and $\mathcal{P}$, a contradiction.

We have just shown that, under the assumption that $z_1 \notin S$, there exists $z \in N(z_1) \setminus \{x\}$ such that $c'(z) \neq c'(v)$. As we argued above for z_1, if z_2 exists, $c(z_2) = c(v)$ and $z_2 \notin S$, then there exists $z \in N(z_2) \setminus \{x\}$ such that $c'(z) \neq c'(v)$; therefore, in this case condition (ii) with respect to z_2 is satisfied and v and x are exchangeable with respect to C_k and $\mathcal{P}'$. If z_2 does not exist, then we need not check condition (ii) with respect to z_2. We also need not check condition (ii) with respect to z_2 if we assume that z_2 exists, $c(z_2) \neq c(v)$ and $z_2 \notin S$. If z_2 exists, $c(z_2) \neq c(v)$ and $z_2 \in S$, then, since $x \notin S$, there is some element $z'' \in N(z_2) \setminus \{x\}$ such that $c(z'') \neq c(z_2)$ and $z'' \in S$; thus, in this case, $c'(z_2) = c'(v)$ and $c'(z'') \neq c'(v)$ so that condition (ii) with respect to z_2 holds. Note that the case in which z_2 exists, $c(z_2) = c(v)$ and $z_2 \in S$ reduces to the case $z_1 \in S$, by exchanging the labels z_1 and z_2. We conclude that, if $z_1 \notin S$, then v and x are exchangeable with respect to C_k and $\mathcal{P}'$. This also concludes the analysis of case (D).

This establishes that in all cases, C_k is fixable with respect to $\mathcal{P}'$, which completes the proof of Claim 9. $\square$

This also completes the proof of Lemma 4. $\square$

3 Graphs of arbitrary maximum degree

3.1 Proof of Theorem 3

We make use of the following result, which may also be of independent interest.

Theorem 12. *There exists a universal constant $c > 0$ such that every graph $G = (V, E)$ with maximum degree d has a dominating set D satisfying $|N^2[v] \cap D| \le cd \ln d$ for all $v \in V$.*

The proof of Theorem 3 is straightforward given Theorem 12.

Proof of Theorem 3. Given a graph $G = (V, E)$ of maximum degree d, let D be the dominating set that is guaranteed by Theorem 12. We first assign colours to the members of D by greedily colouring the vertices of D in the square of G; this requires at most $k = \lfloor cd \ln d \rfloor$ colours, since vertices of D are adjacent at most $cd \ln d - 1$ other vertices of D in the square of G. To extend this colouring to the entire graph, we use one new colour for members of the set $V \setminus D$. It can be checked that this assignment of colours gives an acyclic dominating partition with $k + 1 = O(d \ln d)$ parts. $\square$

The following lemma is a crucial element in the proof of Theorem 12 and we show it using a linear programming approach.

Lemma 13. *For any graph $G = (V, E)$ with maximum degree d, there exist nonnegative reals $(w_v)_{v \in V}$ such that $\sum_{u \in N[v]} w_u \ge 1$ and $\sum_{u \in N^2[v]} w_u \le d + 1$ for all $v \in V$.*

Proof. Without loss of generality, let us assume $V = \{1, \ldots, n\}$. We shall consider the optimisation problem of minimising

$$\max_{i \in \{1,\ldots,n\}} \sum_{j \in N^2[i]} w_j$$

subject to the constraints $\sum_{j \in N[i]} w_j \geq 1$ for all $i \in \{1, \dots, n\}$, over all $w_1, \dots, w_n \geq 0$. This optimisation problem can be written as a linear program as follows:

$$\begin{aligned} \text{minimise} \quad & z \\ \text{subject to} \quad & \sum_{j \in N^2[i]} w_j \leq z \quad (i \in \{1, \dots, n\}), \\ & \sum_{j \in N[i]} w_j \geq 1 \quad (i \in \{1, \dots, n\}), \\ & w_1, \dots, w_n \geq 0. \end{aligned} \tag{2}$$

Let us write

$$A = \left(\begin{array}{c|c} -N^2[G] & \underline{1} \\ N[G] & \underline{0} \end{array} \right), \quad b = \left(\begin{array}{c} \underline{0} \\ \underline{1} \end{array} \right),$$
$$y = (w_1, \dots, w_n, z)^T, \quad c = (0, \dots, 0, 1)^T.$$

Here, $\underline{1}$ (resp. $\underline{0}$) denotes the all-ones (resp. all-zeros) vector of length n, and $N[G]$ (resp. $N^2[G]$) denotes the $(n \times n)$-matrix whose rows are the incidence vectors of the closed neighbourhoods $N[i]$ (resp. $N^2[i]$). With these definitions we can write (2) in the standard form as follows:

$$\begin{aligned} \text{minimise} \quad & c^T y \\ \text{subject to} \quad & Ay \geq b, \\ & y \geq 0. \end{aligned}$$

The dual linear program is the following:

$$\begin{aligned} \text{maximise} \quad & b^T x \\ \text{subject to} \quad & A^T x \leq c, \\ & x \geq 0. \end{aligned}$$

Equivalently, writing $x = (\theta_1, \dots, \theta_n, \xi_1, \dots, \xi_n)^T$ we see that this dual program can be written as follows:

$$\begin{aligned} \text{maximise} \quad & \xi_1 + \cdots + \xi_n \\ \text{subject to} \quad & \sum_{j \in N[i]} \xi_j \leq \sum_{j \in N^2[i]} \theta_j \quad (i \in \{1, \dots, n\}), \\ & \theta_1 + \cdots + \theta_n \leq 1, \\ & \theta, \xi \geq 0. \end{aligned} \tag{3}$$

We shall now show that the optimum of (3) is bounded above by $d+1$, which proves the result.

First notice that we may add the constraints

$$\xi_i \leq \sum_{j \in N^2[i]} \theta_j \quad (i \in \{1, \ldots, n\}), \tag{4}$$

without altering the value of the optimum, since they are trivially satisfied by any choice of θ, ξ that is feasible for (3).

Given a vector $\theta = (\theta_1, \ldots, \theta_n)$ of nonnegative numbers, let $P(\theta)$ denote the set of all $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ that satisfy $\xi_i \leq \sum_{j \in N^2[i]} \theta_j$ and $\sum_{j \in N[i]} \xi_j \leq \sum_{j \in N^2[i]} \theta_j$ for all $i \in \{1, \ldots, n\}$ — notice we are no longer requiring ξ to be nonnegative — and let $f(\theta)$ denote the supremum over all $\xi \in P(\theta)$ of $\xi_1 + \cdots + \xi_n$. To finish the proof, it suffices to show that $f(\theta) \leq (d+1)(\sum_{i=1}^n \theta_i)$ for all nonnegative θ with $\theta_1 + \cdots + \theta_n \leq 1$. We in fact prove that, letting $k = \max\{i : \theta_i > 0\}$ (which we interpret as 0 if $\theta_i = 0$ for all i), $f(\theta) \leq (d+1)(\sum_{i=1}^k \theta_i)$; we prove this stronger statement by induction on k.

By the constraints in (4), the claim trivially holds when $k = 0$, so consider $0 < k \leq n$ and suppose the claim holds for all $k' < k$. Pick $\xi \in P(\theta)$ arbitrarily and denote $\xi' = \xi - \theta_k 1_{N[k]}$ (where $1_{N[k]}$ denotes the incidence vector of $N[k]$). Note that $\xi' \in P(\theta_1, \ldots, \theta_{k-1}, 0, \ldots, 0)$, because $(\xi')_k = \xi_k - \theta_k$ and any i with $k \in N^2[i]$ satisfies $\sum_{j \in N[i]} (\xi')_j \leq \sum_{j \in N[i]} \xi_j - \theta_k$ as i is incident to at least one $j \in N[k]$. This gives that $f(\theta_1, \ldots, \theta_{k-1}, 0, \ldots, 0) \geq \sum_{j=1}^k \xi'_j = \sum_{j=1}^k \xi_j - \theta_k(\deg(k) + 1)$. Taking the supremum over all $\xi \in P(\theta)$ and applying induction, we thus have

$$f(\theta) - \theta_k(\deg(k) + 1) \leq f(\theta_1, \ldots, \theta_{k-1}, 0, \ldots, 0) \leq (d+1)(\sum_{i=1}^{k-1} \theta_i),$$

which completes the inductive step and the proof. $\square$

Now, for the proof of Theorem 12, we also need two standard probabilistic tools. One is a symmetric version of the Lovász Local Lemma. The other is a Chernoff-Hoeffding type bound for sums of indicator variables.

Lemma 14 (Lovász Local Lemma, [7])**.** *Let $\mathcal{A}$ be a finite set of events and suppose that p, δ satisfy that*

1. $\mathbb{P}(A) \leq p$ *for all* $A \in \mathcal{A}$*, and*

2. *each* $A \in \mathcal{A}$ *is independent of all but at most* δ *of the other events in* $\mathcal{A}$*.*

If $ep(\delta+1) \le 1$, then $\mathbb{P}(\bigcap_{A\in\mathcal{A}} \overline{A}) > 0$.

Lemma 15. *Let $Z = \sum_{i=1}^m I_i$ be a sum of independent $\{0,1\}$-valued random variables, and pick $k > \mu = \mathbb{E}Z$. Then*

$$\mathbb{P}(Z > k) \le e^{-\mu H(k/\mu)},$$

where $H(x) = x\ln x - x + 1$.

Lemma 15 is essentially what is found in Janson, Łuczak and Ruczinski [9], but in a form that we desire. A short proof of this lemma is given in the appendix.

Proof of Theorem 12. Let $w = (w_v)_{v\in V}$ be the vector from Lemma 13, and set $p_v = \min(100 w_v \ln d, 1)$ for all v. Let us now construct the set D at random, by selecting each vertex v with probability p_v independently of all other vertices. We claim that with positive probability, the set D has the required properties. In order to prove our claim we apply the Lovász Local Lemma. For $v \in V$, let A_v denote the event that either $D \cap N[v] = \emptyset$ or $|D \cap N^2[v]| > 200 d\ln d$. If none of the events A_v occur, then the set D will satisfy the conclusion of the theorem.

If $p_u = 1$ for some $u \in N[v]$ then $\mathbb{P}(D \cap N[v] = \emptyset) = 0$. If $p_u < 1$ for all $u \in N[v]$, then

$$\begin{aligned}\mathbb{P}(D\cap N[v] = \emptyset) &\le \prod_{u\in N[v]} (1 - 100 w_u \ln d)\\ &\le \prod_{u\in N[v]} \exp[-100 w_u \ln d] = d^{-100\sum_{u\in N[v]} w_u}\\ &\le d^{-100},\end{aligned}$$

where the last inequality is due to Lemma 13. Next, let us consider the probability that $|D\cap N^2[v]| > 200 d\ln d$. Let us write $\mu = \sum_{u\in N^2[v]} p_u$. Note that $1 \le \mu \le 100 d \ln d$ by Lemma 13. By Lemma 15, we have that

$$\begin{aligned}\mathbb{P}(|D\cap N^2[v]| > 200 d\ln d) &\le \exp[-\mu H(\tfrac{200 d \ln d}{\mu})]\\ &\le \exp[-100\cdot H(2)\cdot d\ln d] \ll d^{-100}.\end{aligned}$$

Thus, $\mathbb{P}(A_v) \le 2d^{-100}$ for d sufficiently large.

Each event A_v is independent of all but at most d^4 others; therefore, for sufficiently large d, it holds that

$$e\cdot \mathbb{P}(A_v)\cdot (d^4+1) < 1.$$

Applying the Lovász Local Lemma, we conclude $\mathbb{P}(\bigcap_{v\in V} \overline{A_v}) > 0$, as required. □

3.2 Proof of Theorem 2

Let n, m be integers and let us define a graph $G_{n,m} = (V, E)$ with $2nm$ vertices as follows. Set $V = \{v_{i,j}^1, v_{i,j}^2 : i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}\}$ and add $v_{i,j}^x v_{i',j'}^{x'}$ to E if and only if $i = i'$ or $j = j'$. The graph $G_{n,m}$ may be envisaged as a $(n \times m)$-matrix with two vertices in each entry, where vertices are adjacent if and only if they share the same row or column. Let us also define $H_{n,m} = G_{n,m} \setminus \{v_{i,m}^2 : i \in \{1, \ldots, n\}\}$, i.e. $H_{n \times m}$ is the same as $G_{n,m}$ except that it has only one vertex in each entry of the last column. Thus, $G_{n,m}$ is a regular graph with degree $2(n+m) - 3$, and $H_{n,m}$ has maximum degree $2(n+m) - 4$.

Lemma 16. *Suppose $n \leq m$. Then*

$$\chi_a^{2(n+m)-4}(G_{n,m}) \geq n/2 \text{ and } \chi_a^{2(n+m)-3}(H_{n,m+1}) \geq n/2.$$

Let us first show how this lemma implies Theorem 2.

Proof of Theorem 2. Let d be an arbitrary positive integer. There is a positive integer n such that we can write either $d = 4n-4$, $d = 4n-3$, $d = 4n-2$ or $d = 4n-1$; thus, d is the maximum degree of one of $H_{n,n}$, $G_{n,n}$, $H_{n,n+1}$, and $G_{n,n+1}$, respectively. By Lemma 16, it follows that $\chi_a^{d-1}(d) \geq n/2 \geq (d+1)/8$, so that $\chi_a^{d-1}(d) = \Omega(d)$ as required. □

Proof of Lemma 16. Our proof improves upon the corresponding analysis in Addario *et al.* [1]. We shall focus on the case of $H_{n,m+1}$, since the case of $G_{n,m}$ is similar. Let d be the maximum degree of $H_{n,m+1}$ and suppose that there exists a $(d-1)$-improper colouring $c : V \to \{1, \ldots, k\}$ for some $k < n/2$.

In any row, there is at most one colour that occurs more than once, because if two distinct colours occur more than once in the same row, there is a 4-cycle alternating between them. Pick an arbitrary row. As the number of colours used is less than $n/2$, there is some colour that appears at least $2m + 2 - n/2 \geq 3(m+1)/2 + 1$ times in this row. We call this colour the "dominant colour" of that row. Moreover, for any $i \in \{1, \ldots, n\}$, there are more than $(m+1)/2$ values $j \in \{1, \ldots, m\}$ for which both vertices $v_{i,j}^1, v_{i,j}^2$ are coloured by the dominant colour.

Now consider rows i, i' for $i \neq i'$. By the above, there must exist $j \in \{1, \ldots, m\}$ such that the pair $v_{i,j}^1, v_{i,j}^2$ both have the dominant colour of row i and the pair $v_{i',j}^1, v_{i',j}^2$ both have the dominant colour of row i'. We conclude that rows i and i' must have the same dominant colour, for otherwise the 4-cycle $v_{i,j}^1 v_{i',j}^2 v_{i,j}^2, v_{i',j}^1$ is alternating. As i and i' were arbitrary, it follows that all rows have the same dominant colour. By similar aguments, there is a single dominant colour for the columns 1 to m; furthermore, the dominant colour for the rows and the dominant colour for the columns must coincide and we may assume this colour is, say, 1.

Because the colouring is $(d-1)$-improper, it must either hold that none of the rows is monochromatic or that none of columns 1 to m is monochromatic, for if both row i and column j (with $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$) are monochromatic then the vertices $v^1_{i,j}, v^2_{i,j}$ and their d neighbours all have colour 1. Let us assume none of columns 1 through m is monochromatic. (The case when no row is monochromatic is similar.) For technical reasons, let us assume by permuting the rows that if column $m+1$ is not monochromatic with colour 1, then a colour different from 1 occurs in the intersection of row 1 and column $m+1$.

Now let $A_1 \subseteq \{2, \ldots, k\}$ be the set of non-dominant colours appearing in the first row, and let $J_1 \subseteq \{1, \ldots, m+1\}$ be the set of columns in which these colours appear (in the first row). Note that either $m+1 \in J_1$ or column $m+1$ is monochromatic with colour 1, by assumption. If a colour from A_1 appears in column $j \in \{1 \ldots, m\} \setminus J_1$ then there is an alternating 4-cycle through the vertices $v^1_{1,j}, v^2_{1,j}$, both of colour 1; thus, colours from A_1 appear only in the columns from J_1. For $i \in \{2, \ldots, n\}$, let $A_i \subseteq \{2, \ldots, k\}$ be the set of colours that appear in row i and columns $\{1, \ldots, m+1\} \setminus \bigcup_{j=1}^{i-1} J_j$; let J_i be the corresponding set of columns in which these colours appear (in row i). By the same logic, the colours from A_i do not appear outside the columns from J_i. Observe that $|A_i| \geq |J_i|$ and the sets $A_1, \ldots, A_n$ are mutually disjoint. Since none of the columns 1 to m is monochromatic, each is a member of exactly one J_i and hence

$$k - 1 \geq |A_1| + \cdots + |A_n| \geq |J_1| + \cdots + |J_m| \geq m \geq n.$$

But this contradicts the assumption that $k < n/2$. □

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References

[1] L. Addario-Berry, L. Esperet, R. J. Kang, C. McDiarmid, and A. Pinlou, *Acyclic t-improper colourings of graphs with bounded maximum degree*, 2009+. To appear in *Discrete Mathematics*.

[2] M. O. Albertson and D. M. Berman, *The acyclic chromatic number*, Proceedings of the Seventh Southeastern Conference on Combinatorics, Graph Theory, and Computing (Louisiana State U., Baton Rouge, La., 1976), 1976, pp. 51–69. Congressus Numerantium, No. XVII.

[3] N. Alon, C. McDiarmid, and B. Reed, *Acyclic coloring of graphs*, Random Structures and Algorithms **2** (1991), no. 3, 277–288.

[4] P. Boiron, É. Sopena, and L. Vignal, *Acyclic improper colorings of graphs with bounded degree*, DIMACS/DIMATIA Conference "Contemporary Trends in Discrete Mathematics", 1997, pp. 1–10.

[5] ———, *Acyclic improper colorings of graphs*, J. Graph Theory **32** (1999), no. 1, 97–107.

[6] O. V. Borodin, *On acyclic colorings of planar graphs*, Discrete Math. **25** (1979), no. 3, 211–236.

[7] P. Erdős and L. Lovász, *Problems and results on* 3*-chromatic hypergraphs and some related questions*, Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II, 1975, pp. 609–627. Colloq. Math. Soc. János Bolyai, Vol. 10.

[8] B. Grünbaum, *Acyclic colorings of planar graphs*, Israel J. Math. **14** (1973), 390–408.

[9] S. Janson, T. Łuczak, and A. Rucinski, *Random graphs*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, 2000.

Appendix

Proof of Lemma 15. Let $p_i = \mathbb{E}I_i$. The moment generating function of Z equals

$$\mathbb{E}e^{tZ} = \prod_i \mathbb{E}e^{tI_i} = \prod_i ((1-p_i) + e^t p_i) \leq e^{-(1-e^t)\sum_i p_i} = e^{-\mu(1-e^t)}.$$

For any $t > 0$, Markov's inequality gives

$$\mathbb{P}(Z > k) = \mathbb{P}(e^{tZ} > e^{tk}) \leq \mathbb{E}e^{tZ}/e^{tk} = e^{-\mu(t(k/\mu)-e^t+1)}.$$

Setting $t = \ln(k/\mu)$ gives the result. ◻