For almost all graphs H, almost all H-free graphs have a linear homogeneous set.

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Background: homogeneous sets

A *homogeneous set* in a graph G is a stable or complete subset of $V(G)$. We shall be interested in

$$h(G) \equiv \max\{\alpha(G), \omega(G)\},$$

the order of a largest homogeneous set in G.

Background: homogeneous sets

Lower bounds on h (a.k.a. upper bounds on Ramsey numbers) are fundamental in extremal combinatorics.

Erdős & Szekeres ('35) and Erdős ('47) showed

$$h(G) \geq \frac{1}{2}\log_2|V(G)| \text{ for all } G, \text{ while}$$
$$h(G) \leq 2\log_2|V(G)| \text{ for some } G \text{ with } |V(G)| \text{ large enough.}$$

The factors $1/2$ and 2 are best known even after more than six decades[1].

[1] Lower-order improvements by Spencer ('75), Thomason ('88) and Conlon ('09).

Background: the Erdős–Hajnal conjecture

We will focus on the following question.

How is $h(G)$ affected by the exclusion from G of a fixed graph H as an induced subgraph?

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Erdős & Hajnal proved that $h(G)$ is significantly larger than in general.

Theorem (Erdős and Hajnal, 1989)

For any H, there exists $\varepsilon' = \varepsilon'(H) > 0$ such that

$$G \not\supseteq_i H \implies h(G) > e^{\varepsilon' \sqrt{\log |V(G)|}}.$$

They also conjectured something stronger.

Background: the Erdős–Hajnal conjecture

Conjecture (Erdős and Hajnal, 1989)

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Notation: If there exists $\varepsilon > 0$ such that $G \not\supseteq_i H \implies h(G) > |V(G)|^{\varepsilon}$, then we say H has the *Erdős–Hajnal property*.

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The state of the art.

Erdős & Hajnal, Alon, Pach & Solymosi ('01), Chudnovsky & Safra ('08):

1. K_1, the path P_4 and the bull graph have the E–H property;
2. the E–H property is closed under complementation and substitution.

The cycle C_5 and the path P_5 are at present open.

Background: forbidden induced subgraphs

Notation:[2]

Forb(H) $\equiv$ induced H-free graphs.
Forb(H)^n $\equiv$ induced H-free graphs of order n.

Our approach towards the E–H conjecture has roots going back at least to Erdős, Kleitman & Rothschild ('76) on the asymptotic enumeration of K_k-free graphs, which sparked a rich and active line of research.

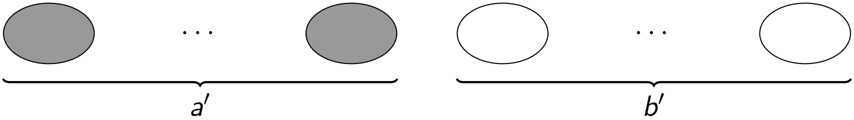
A basic method behind this programme is, e.g. for Forb(H)^n, *to find some well-structured family* $Q \subseteq$ Forb(H)^n *and then show* $|$Forb(H)$^n|$ *is close to* $|Q|$.

[2] Sometimes Forb*(H) is used instead of Forb(H).

Background: forbidden induced subgraphs

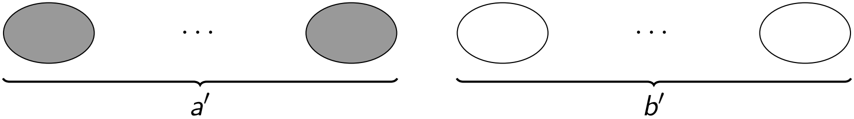
Prömel & Steger ('92/3) implemented this method for $\mathrm{Forb}(H)^n$:
$|\mathrm{Forb}(H)^n|$ is governed by the *colouring number* $\tau(H)$ of H,
defined as the least t such that, for all a, b with $a + b = t$,
$V(H)$ can be partitioned into a cliques and b stable sets.

There exist a', b' with $a' + b' = \tau(H) - 1$ so that $V(H)$ does not admit a partition into a' cliques and b' stable sets. If $V(G)$ can be partitioned into a' cliques and b' stable sets, then $G \in \mathrm{Forb}(H)$. Partition $[n]$ as follows:



with near equal-sized parts, edges arbitrary between parts.

Background: forbidden induced subgraphs



certifies that

$$|\operatorname{Forb}(H)^n| \geq 2^{\left(1-\frac{1}{\tau(H)-1}+o(1)\right)\binom{n}{2}}.$$

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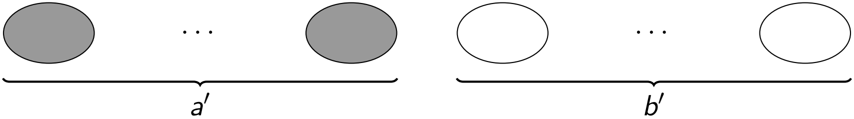

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Using Szemerédi's regularity lemma, Prömel & Steger showed

$$|\operatorname{Forb}(H)^n| \leq 2^{\left(1-\frac{1}{\tau(H)-1}+o(1)\right)\binom{n}{2}}.$$

An extension of this to all hereditary graph properties was obtained by, independently, Alekseev ('92) and Bollobás & Thomason ('95).

Asymptotic E–H: a strengthening of Prömel–Steger

There is a form of the E–H conjecture, with a flavour of the above asymptotic enumeration. If there exists $\varepsilon = \varepsilon(H) > 0$ such that

$$\frac{|\{G \in \mathrm{Forb}(H)^n : h(G) \geq n^{\varepsilon}\}|}{|\,\mathrm{Forb}(H)^n|} \to 1 \text{ as } n \to \infty,$$

then we say H has the *asymptotic Erdős–Hajnal property*.

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Theorem (Loebl, Reed, Scott, Thomason, Thomassé, 2010)

Every graph has the asymptotic E–H property.

They proved this by combining SRL with Chudnovsky & Safra's bull result.

Asymptotic E–H: further strengthening

P_3: $\mathrm{Forb}(P_3)$ is the class of disjoint unions of cliques
$\implies$ for almost all $G \in \mathrm{Forb}(P_3)^n$, $h(G) = \Theta(n/\log n)$.

Asymptotic E–H: further strengthening

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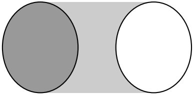
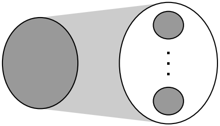
$\Longrightarrow$ for almost all $G \in \text{Forb}(P_3)^n$, $h(G) = \Theta(n/\log n)$.

C_4: The class of split graphs forms almost all of Forb(C_4)
$\Longrightarrow$ for almost all $G \in \text{Forb}(C_4)^n$, $h(G) = \Theta(n)$.

C_5: The class of generalised split graphs forms almost all of Forb(C_5)
$\Longrightarrow$ for almost all $G \in \text{Forb}(C_5)^n$, $h(G) = \Theta(n)$.





or complement

Asymptotic E–H: further strengthening

A stronger asymptotic property: if there exists $\hat{\varepsilon} = \hat{\varepsilon}(H) > 0$ such that

$$\frac{|\{G \in \mathrm{Forb}(H)^n : h(G) \geq \hat{\varepsilon} n\}|}{|\mathrm{Forb}(H)^n|} \to 1 \text{ as } n \to \infty,$$

then we say H has the *asymptotic linear Erdős–Hajnal property*.

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NB: C_4 and C_5 have asymptotic linear E–H property, while P_3 does not.

What graphs have the asymptotic linear E–H property?

Apart from P_3 and P_4?

Asymptotic linear E–H: main theorem

We "almost" answer this question.

Theorem (K, McDiarmid, Reed, Scott, 2012+)

Almost every graph has the asymptotic linear E–H property.

"Almost every graph" here should read "A.a.s. $G^n_{1/2}$".

Asymptotic linear E–H: proof outline

Theorem (K, McDiarmid, Reed, Scott, 2012+)

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The proof depends on a variant colouring number $\tau_1(H)$ described later. It naturally breaks into two parts.

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Lemma

For almost every H, $\tau_1(H) < \tau(H)$, i.e. $\tau_1(G^n_{1/2}) < \tau(G^n_{1/2})$ a.a.s.

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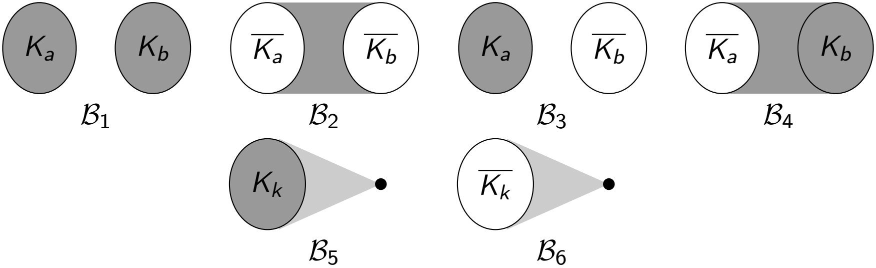
Lemma

$\tau_1(H) < \tau(H) \implies$ *H has the asymptotic linear E–H property.*

Asymptotic linear E–H: a variant colouring number

Recall $\tau(H)$ is the least t such that, for all t_1, t_2 with $t_1 + t_2 = t$, $V(H)$ can be partitioned into t_1 cliques and t_2 stable sets.

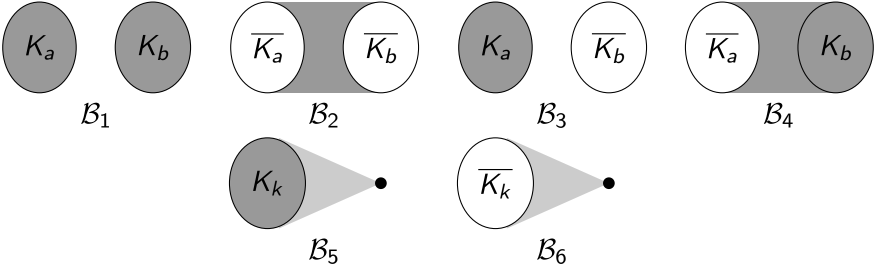
We define a family $\mathcal{F}_1$ of six graph classes:



The $\mathcal{F}_1$-colouring number $\tau_1(H)$ of H is the least t such that, for all $t_1, t_2, t_3, t_4, t_5, t_6$ with $t_1 + \cdots + t_6 = t$, $V(H)$ can be partitioned into t_1 $\mathcal{B}_1$'s, . . . , and t_6 $\mathcal{B}_6$'s.

Asymptotic linear E–H: a variant colouring number

Note the class of stable sets is a subclass of each of $\mathcal{B}_2$, $\mathcal{B}_4$, $\mathcal{B}_6$.
Also the class of cliques is a subclass of each of $\mathcal{B}_1$, $\mathcal{B}_3$, $\mathcal{B}_5$.



$\implies \tau_1(H) \leq \tau(H)$ for any graph H.

Asymptotic linear E–H: the first random part

Lemma

For almost every H, $\tau_1(H) < \tau(H)$, i.e. $\tau_1(G^n_{1/2}) < \tau(G^n_{1/2})$ a.a.s.

To prove this random graphs part, we define yet another variant colouring number τ_2.

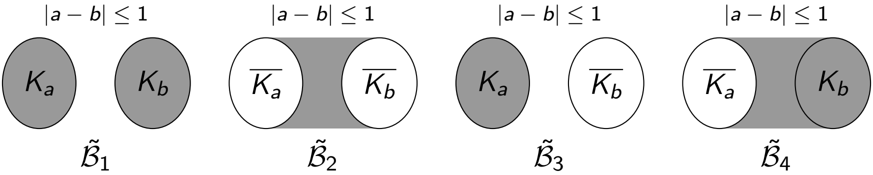
Then, by straightforward reductions, it suffices to prove that

$$\chi(G^n_{1/2}) - \tau_2(G^n_{1/2}) = \Omega\left(\frac{n}{(\log n)^2}\right),$$

with the aid of recent methodology (of Panagiotou & Steger (2009) and Fountoulakis, K & McDiarmid (2010)) to colour random graphs.

Asymptotic linear E–H: another variant colouring number

Let $\mathcal{F}_2$ be four "balanced" versions of the six graph classes:



The $\mathcal{F}_2$-colouring number $\tau_2(H)$ of H is the least t such that, for all t_1, t_2, t_3, t_4 with $t_1 + t_2 + t_3 + t_4 = t$, $V(H)$ can be partitioned into t_1 $\tilde{\mathcal{B}}_1$'s, ..., and t_4 $\tilde{\mathcal{B}}_4$'s.

Asymptotic linear E–H: the second random part

Lemma

$\tau_1(H) < \tau(H) \implies$ *H has the asymptotic linear E–H property.*

This second part of the main result follows from another application of SRL and another structural lemma (that does not require bull).

Lemma

Let $k = |V(H)|$ *and* $c = 1/2R(k)$ *where* $R()$ *denotes the Ramsey number. If* G *is a graph of order* $n \geq \max\{R(k^2+k), 2(R(k)+k^2+k)\}$, *then it contains either*

- *a homogeneous set of size* cn, *or*
- *an induced copy of a "balanced" member of* $\mathcal{F}_1$ *of order* $2k$ *or* $2k+1$.

Concluding remarks

The following is as yet unresolved:

Could it be that every graph except P_3 and P_4 has the asymptotic linear E–H property?

A weaker form of the above question (which does not follow from the Loebl *et al.* result) is also open:

Is there some universal constant $\varepsilon > 0$ such that for all H

$$\frac{|\{G \in \mathrm{Forb}(H)^n : h(G) \geq n^{\varepsilon}\}|}{|\mathrm{Forb}(H)^n|} \to 1 \text{ as } n \to \infty?$$