The distance-t chromatic index of graphs

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Centrum Wiskunde & Informatica / Utrecht

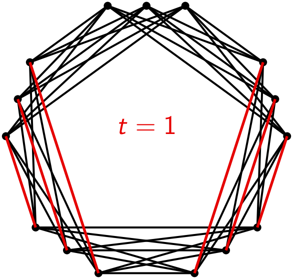
Maastricht, 8/2012
Workshop on Graphs and Matroids

Problem definition

Let $G = (V, E)$ be a (simple) graph.

The distance between two edges in G is the number of *vertices* in a shortest path between them, i.e. distance in the line graph $L(G)$ of G. (So adjacent edges have distance 1.)

A *distance-t matching* of G is a set of edges no two of which are within distance t in G.



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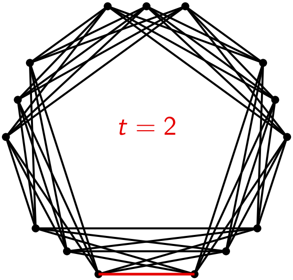


Problem definition

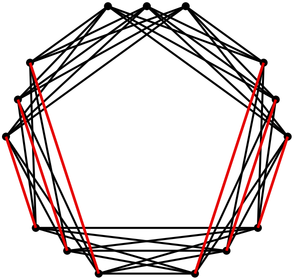
A *distance-t edge-colouring* is an assignment of colours to edges of G such that each colour class induces a distance-t matching.

The *distance-t chromatic index* $\chi'_t(G)$ of G is the least integer k such that there exists a distance-t edge-colouring of G using k colours.

Remarks:

- $\chi'_1(G)$ is the chromatic index $\chi'(G)$ of G.
- A distance-2 matching is an induced matching and so $\chi'_2(G)$ is the strong chromatic index $s\chi'(G)$ of G.
- $\chi'_t(G) = \chi((L(G))^t)$ where $(L(G))^t$ is the t^{th} power of the line graph.

Problem definition



A proposed practical motivation for χ'_t:

Timeslot assignment (TDMA) for wireless sensor networks.

- Each matching in the colouring corresponds to a set of simultaneous pairwise transmissions among sensors in a particular timeslot.
- The distance requirement models the range of network interference that results from transmission between two sensors.

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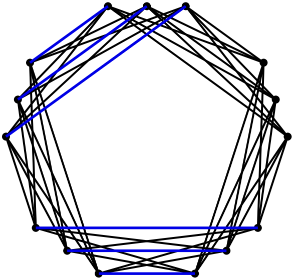

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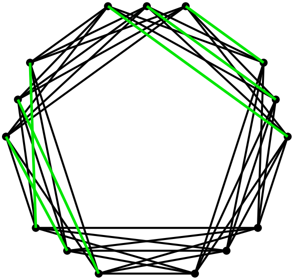

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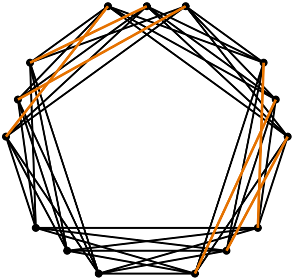

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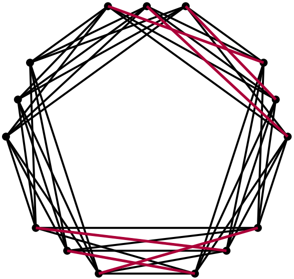

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Scope of current work

Two main settings (with Δ large):

1. $\chi'_t(G)$ for graphs G of maximum degree Δ:

$$\chi'_t(\Delta) := \max\{\chi'_t(G) : \Delta(G) \le \Delta\}.$$

Scope of current work

Two main settings (with Δ large):

1. $\chi'_t(G)$ for graphs G of maximum degree Δ:

$$\chi'_t(\Delta) := \max\{\chi'_t(G) : \Delta(G) \le \Delta\}.$$

2. $\chi'_t(G)$ when G is also prescribed to have girth at least g:

$$\chi'_t(\Delta, g) := \max\{\chi'_t(G) : \Delta(G) \le \Delta, \mathrm{girth}(G) \ge g\};$$

particularly, when does $\chi'_t(\Delta, g)$ becomes $o(\chi'_t(\Delta))$ in terms of g?

Background

$t = 1$.

Vizing's Theorem implies that $\chi'_1(\Delta) = \Delta + 1$ and $\chi'_1(\Delta, g) \geq \Delta$ for all g.

Background

$t = 2$.

Erdős and Nešetřil proposed the problem of determining $\chi'_2(\Delta)$ in 1985.
They suggested as extremal the multiplied 5-cycle $\implies \chi'_2(\Delta) \geq 1.25\Delta^2$.
Molloy and Reed (1997) showed $\chi'_2(\Delta) \leq 1.998\Delta^2$ for large enough Δ.

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The complete bipartite graphs $K_{\Delta,\Delta} \implies \chi'_2(\Delta, 4) \geq \Delta^2$.
NB: Faudree, Gyárfás, Schelp, Tuza (1990) conjectured $\chi'_2(\Delta, 4) = \Delta^2$.

Mahdian (2000) showed $\chi'_2(\Delta, 5) = O(\Delta^2/\log\Delta)$ (and in fact the stronger result for all C_4-free graphs).
A probabilistic construction shows $\chi'_2(\Delta, g) = \Omega(\Delta^2/\log\Delta)$ for all $g \geq 5$.

A table for $\chi'_t(\Delta)$ and $\chi'_t(\Delta, g)$ (Δ large)

$t \setminus g$	3 (lower/upper)		4	5	$\cdots$
1	$\Delta + 1$		$\Theta(\Delta)$		
2	$1.25\Delta^2$	$1.998\Delta^2$	$\Theta(\Delta^2)$	$\Theta(\Delta^2/\log\Delta)$	
3	?	?	?	?	?
$\vdots$	?	?	?	?	?

A distance-t version of the Erdős-Nešetřil problem

Consider the following upper bound:

$$\chi'_t(\Delta) \le 1 + \Delta((L(G))^t) \le 1 + 2\sum_{j=1}^{t}(\Delta-1)^j < 2\Delta^t.$$

Problem

For each $t \ge 3$, is $\limsup_{\Delta\to\infty} \chi'_t(\Delta)/\Delta^t$ less than $2-\varepsilon$ for some $\varepsilon > 0$?

NB: Molloy and Reed solved the $t = 2$ case with $\varepsilon > 0.002$.

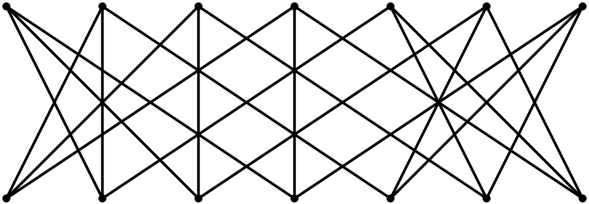
We next show $\limsup_{\Delta\to\infty} \chi'_t(\Delta)/\Delta^t$ is positive for every fixed $t \ge 3$.

Two constructive lower bounds

Proposition (K and Manggala)

For arbitrarily large Δ, there exists a bipartite, Δ-regular graph of girth 6 such that $\chi'_3(G) = \Delta^3 - \Delta^2 + \Delta$.

$t = 3$, $\Delta = 3$: point-line incidence graph of the Fano plane.

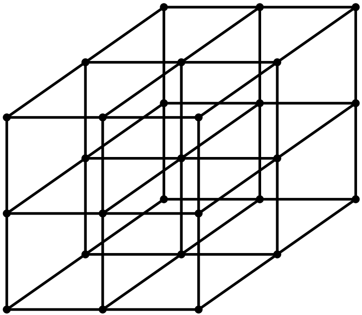


Two constructive lower bounds

Proposition (K and Manggala)

Fix $t \geq 2$. For arbitrarily large Δ, there exists a Δ-regular graph such that $\chi'_t(G) > \Delta^t / \left(2(t-1)^{t-1}\right)$.

$t = 4$, $\Delta = 6$.



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$t \setminus g$	3 (lower/upper)		4	5	6	$\cdots$
1	$\Delta + 1$		$\Theta(\Delta)$			
2	$1.25\Delta^2$	$1.998\Delta^2$	$\Theta(\Delta^2)$	$\Theta(\Delta^2 / \log \Delta)$		
3	Δ^3	$2\Delta^3$	$\Theta(\Delta^3)$			?
4	$0.0185\Delta^4$	$2\Delta^4$	?	?	?	?
5	$0.00195\Delta^5$	$2\Delta^5$	?	?	?	?
$\vdots$	$\vdots$	$\vdots$	?	?	?	?

Main theorem I

Theorem (Kaiser and K)

For each $t \geq 2$, $2 - \limsup_{\Delta\to\infty} \chi'_t(\Delta)/\Delta^t \geq 0.00008$.

I.e. the t-E-N problem affirmed with a *uniform* choice of ε for all t.

Main theorem I: proof idea

Theorem (Kaiser and K)

For each $t \geq 2$, $2 - \limsup_{\Delta \to \infty} \chi'_t(\Delta)/\Delta^t \geq 0.00008$.

This relies on colouring graphs with sparse neighbourhood counts.

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Theorem (Kaiser and K)

For each $t \geq 2$, $2 - \limsup_{\Delta\to\infty} \chi'_t(\Delta)/\Delta^t \geq 0.00008$.

This relies on colouring graphs with sparse neighbourhood counts.

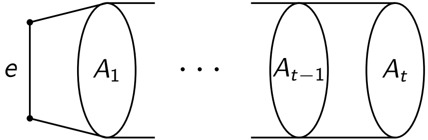
Lemma (Molloy and Reed (1997))

Let $\delta, \varepsilon > 0$ *be such that* $\varepsilon < \frac{\delta}{2(1-\varepsilon)} e^{-\frac{3}{1-\varepsilon}}$ *and let* $\hat{\Delta}_0$ *be large enough. If* $\hat{G} = (\hat{V}, \hat{E})$ *is a graph with maximum degree at most* $\hat{\Delta} \geq \hat{\Delta}_0$ *such that at most* $(1-\delta)\binom{\hat{\Delta}}{2}$ *edges span each* $N(\hat{v})$, $\hat{v} \in \hat{V}$, *then* $\chi(\hat{G}) \leq (1-\varepsilon)\hat{\Delta}$.

Thus the t-E-N problem can be resolved by showing neighbourhood counts in $(L(G))^t$ with $\Delta(G) \leq \Delta$ are at most $(1-\delta) \cdot 2\Delta^{2t}$.

Main theorem I: proof idea

Assume $G = (V, E)$ is Δ-regular. Let $e \in E$ be arbitrary.
Set $\hat{N} := N_{L(G)^t}(e)$.



Set $\hat{S} := E(L(G)^t[\hat{N}])$ and, for contradiction, assume $|\hat{S}| > (1 - \delta) \cdot 2\Delta^{2t}$.

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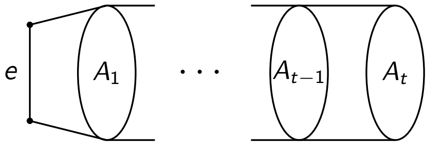

Set $\hat{S} := E(L(G)^t[\hat{N}])$ and, for contradiction, assume $|\hat{S}| > (1 - \delta) \cdot 2\Delta^{2t}$.

Consider

$\tau(e, f) := \max\{0, (\#ef\text{-walks with } \leq t + 1 \text{ edges}) - 1\}$.

$\text{Esc} := \#$ walks with $\leq t + 1$ edges, first edge in $\hat{N}$, last edge in $E - \hat{N}$.

Claim

$\sum_{e,f \in \hat{N}} \tau(e, f) + \text{Esc} < 4\delta \cdot \Delta^{2t}$.

Main theorem I: proof idea

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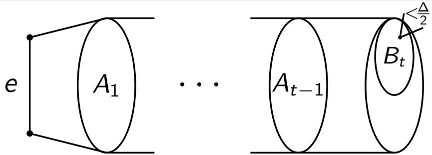
$\sum_{e,f\in\hat{N}} \tau(e,f) + \text{Esc} < 4\delta\cdot\Delta^{2t}$.



Set

$A^* := A_1 \cup \cdots \cup A_{t-1} \cup B_t,$

$\sigma_t(u,v) := \#uv$-walks with $\leq t$ edges and first edge in $\hat{N}$.

Claim

$\sum_{u,v\in A^*} \sigma_t(u,v) > \alpha\cdot\Delta^{2t-1}$.

Main theorem I: $t = 3$

For $t = 3$, we can extend the argument of Molloy and Reed for $t = 2$, which applies Jensen's Inequality twice for a lower bound on the number of C_4s in $N_{(L(G))^3}(e)$, $\forall e \in V$.

Theorem (Kaiser and K)

$2 - \limsup_{\Delta\to\infty} \chi'_3(\Delta)/\Delta^3 \geq 0.0002.$

A table for $\chi'_t(\Delta)$ (Δ large)

t	lower	upper
1	$\Delta + 1$	
2	$1.25\Delta^2$	$1.998\Delta^2$
3	Δ^3	$1.9998\Delta^3$
4	$0.0185\Delta^4$	$1.99992\Delta^4$
5	$0.00195\Delta^5$	$1.99992\Delta^5$
$\vdots$	$\vdots$	$\vdots$

Remarks:

- The general proof gives an alternative solution to the E-N problem, albeit with a much weaker constant.
- It remains possible that $\limsup_{\Delta\to\infty} \chi'_t(\Delta)/\Delta^t = o(1)$ as $t \to \infty$.

Main theorem II

Theorem (Kaiser and K)

For $t \geq 2$, all graphs G of girth at least $2t + 1$ and maximum degree at most Δ have $\chi'_t(G) = O(\Delta^t / \log \Delta)$.

1

Main theorem II

Theorem (Kaiser and K)

For $t \geq 2$, all graphs G of girth at least $2t+1$ and maximum degree at most Δ have $\chi'_t(G) = O(\Delta^t / \log \Delta)$.

By a probabilistic construction, this bound is tight up to a constant factor dependent upon t[1].

Proposition (Kaiser and K)

There is a function $f = f(\Delta, t) = (1+o(1))\Delta^t/(t \log \Delta)$ (as $\Delta \to \infty$) such that, for every $g \geq 3$ and every Δ, there is a graph G of girth at least g and maximum degree at most Δ with $\chi'_t(G) \geq f(\Delta, t)$.

[1] If girth at least $3t-2$, the upper bound can be strengthened to $O(\Delta^t/(t \log \Delta))$.

A table for $\chi'_t(\Delta, g)$ (Δ large)

<table>
<tr><th>$t \setminus g$</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th><th>10</th><th>11</th><th>$\cdots$</th></tr>
<tr><td>1</td><td colspan="10">$\Theta(\Delta)$</td></tr>
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<tr><td>4</td><td>$\Theta(\Delta^4)$</td><td colspan="5">?</td><td colspan="4">$\Theta(\Delta^4/\log\Delta)$</td></tr>
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<tr><td>$\vdots$</td><td>$\vdots$</td><td colspan="9">$\vdots$</td></tr>
</table>

Open problems

1. Is there some $\varepsilon > 0$ such that $\limsup_{\Delta\to\infty} \chi'_t(\Delta)/\Delta^t \geq \varepsilon$ for all t?
2. Is it true that $\limsup_{\Delta\to\infty} \chi'_t(\Delta, 2t)/\Delta^t > 0$ for all $t \geq 4$?

Thank you!