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On improper colouring of unit disk graphs

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Submitted for transfer of status to DPhil candidacy

January 2005

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I certify that this is my own work (except where otherwise indicated).

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Abstract

In this paper, we examine the problem of finding the defective chromatic number of unit disk graphs.

In the introduction, we survey the current state of research into improper colouring and into unit disk graphs. This is intended not only to provide background for the following chapter, but also to give a self-contained overview of these two interesting fields of research.

Most of the original work is in the second chapter, where we show that the unit disk improper colourability problem is NP-complete in nearly all cases. This work is joint work with Jean-Sébastien Sereni (Université de Nice and INRIA, France), a fellow doctoral student with whom I worked while on an academic visit to INRIA in Sophia Antipolis.

The concluding chapter gives some open problems to consider, some specific to improper colouring of unit disk graphs but others under the topic of (colouring) unit disk graphs in general.

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Chapter 1

Introduction

In this paper, we examine the problem of finding the defective chromatic number of unit disk graphs. This problem has not been considered before. Most of the original work is in the second chapter, where we show that the unit disk improper colourability problem is NP-complete in nearly all cases. This work is joint work with Jean-Sébastien Sereni (Université de Nice and INRIA, France), a fellow doctoral student with whom I worked while on an academic visit to INRIA in Sophia Antipolis.

In this chapter, we survey the current state of knowledge regarding unit disk graphs and the defective colouring problem. Here, we also give the definitions and background that are required for the next chapter. Note that throughout this paper we will assume rudimentary background in graph theory and in complexity; we refer the reader to West [56] and Garey and Johnson [15], respectively, for basic terminology.

1.1 Unit disk graphs

The study of unit disk graphs stems partly from applications in communication networks. The *frequency assignment problem* (otherwise known as radio channel assignment or frequency allocation) is one of the most prominent and well-studied of these applications. In frequency assignment, we have a fixed configuration of radio transmitter towers. Each tower transmits a radio signal (or set of signals) at a chosen frequency and has a transmission area (usually a disk of given radius centred at the tower) in which receivers can detect the tower's signal. The signals of two towers might interfere if their transmission areas intersect (usually if the distance between the towers is too small). In this case, we must choose the transmitters frequencies that are far enough apart in the radio spectrum; otherwise, receivers in the intersection area will not be able to distinguish between conflicting signals. Since it is expensive to buy bandwidth, our aim is, under these requirements, to allocate frequencies to the transmitter towers so as to minimise the overall amount of frequency bandwidth used. For more thorough and general treatment of the frequency assignment problem, consult [27].

The unit disk colourability problem is one of the simplest models for radio channel assignment and it is also one of the most well-studied [19, 8, 33, 18, 32]. In this model, we assume that transmitter towers are scattered in the plane, all towers have the same transmission range and operating bandwidth, and there are no physical obstacles (such as skyscrapers, clouds or geographical features) to omnidirectional transmission; hence, the transmission areas are uniform disks centred at the towers. We assume that the frequencies used are uniformly spaced, e.g. integers, and we must give two towers different frequencies if their transmission areas intersect. We can derive a graph from the intersection of the transmission areas (i.e. represent each tower by a vertex and connect two vertices if the disks of the respective towers intersect), and it is clear that, under this model, the channel assignment problem is simply to find a proper colouring of the resulting graph that uses the least number of colours.

The graph we derive above is just a unit disk graph, but we now give an alternative, more precise definition. We are given an arbitrary set of n points fixed in the plane and a fixed positive quantity d. At each point, we centre a disk of diameter d. We connect two points if their disk's interiors intersect; that is, we connect two points if they are less than distance d apart. Note that the value of d is arbitrary and we may assume d to be 1 without loss of generality. Any graph that is isomorphic to a graph constructed in such a manner is called a *unit disk graph*. For any unit disk graph G, the configuration of points in the plane together with an appropriate value of d that gives rise to G is called a *representation* (also called model, embedding or realisation) of G.

1.1.1 Classes of graphs related to unit disk graphs

In this part, we describe some graph classes related to the class of unit disk graphs that are also important to the frequency allocation problem.

The class of unit disk graphs is part of an important family of graph classes. Let Σ be a set of sets and $\mathcal{G}$ be a class of graphs. For each family $\mathcal{F} \subseteq \Sigma$, the *intersection graph of* $\mathcal{F}$, denoted $\Omega(\mathcal{F})$, is the graph with vertex set $\mathcal{F}$ and U adjacent to V, $U \neq V$, if their intersection is nonempty. The *intersection class of* Σ, denoted $\Omega(\Sigma)$ is the class of graphs $\{\Omega(\mathcal{F})|\mathcal{F} \subseteq \Sigma\}$. We say $\mathcal{G}$ is an intersection class if $\mathcal{G}$ is isomorphic to $\Omega(\Sigma)$ for some Σ. There is extensive theory on intersection graphs: read [38] for more background.

Clearly, by setting Σ to be the set of open unit-diameter disks in the plane, we see that the class of unit disk graphs is an intersection class; we denote this class by $\mathcal{UD}$. If instead we set Σ to be the set of open disks (of arbitrary radii) in the plane, then we obtain the class of *(general) disk graphs*; we denote this class by $\mathcal{D}$. The class of disk graphs is also important to radio channel assignment [19], since it models radio towers with varying transmission strength.

Another generalisation of the unit disk graph class is obtained by examining higher dimensions. For example, let Σ be the set of open (unit) balls in 3-dimensional space; this graph class could be important to frequency assignment with extensive space travel(!) but, for the present, it has applications in the analysis of molecules (consult [20]).

We can also restrict to one dimension. If we restrict $\mathcal{UD}$ to a line (i.e. let Σ be the set of open unit intervals on the line), then we obtain the class $\mathcal{UI}$ of *unit interval graphs* (also known as *indifference graphs*). Analogously, if we restrict $\mathcal{D}$ to a line, then we obtain the class $\mathcal{I}$ of *(general) interval graphs*. These classes model radio channel assignment on a long, narrow stretch of land such as a highway, for instance; however, there also applications in genetics [55] and in scheduling [17].

Consider intersection graphs of sets of closed disks (of arbitrary radii) whose interiors do not intersect; such graphs are called *disk contact graphs*. Long ago, the following beautiful result was known [26, 47]: the class of disk contact graphs is precisely the class of planar graphs. We will denote the class of planar graphs by $\mathcal{P}$. Clearly, $\mathcal{P} \subseteq \mathcal{D}$. (Note that $\mathcal{P}$ is not an intersection class.)

We note that the star $K_{1,6}$ is a planar interval graph, but not a unit disk graph; the cycle C_4 is a planar unit disk graph, but not an interval graph; also, the complete graph K_5 is a unit disk graph as well as an interval graph, but not planar. Therefore, the classes $\mathcal{I}$, $\mathcal{UD}$ and $\mathcal{P}$ are incomparable under the inclusion relation.

The class $\mathcal{I}$ is well known to be a perfect class – a graph G is *perfect* if for every induced subgraph H of G, the chromatic number $\chi(H)$ is equal to the clique number $\omega(H)$ – however, the cycle C_5 is a planar unit disk graph that is not perfect.

Figure 1.1 illustrates the inclusion relationship between the graph classes we have defined in this section. An arrow from one graph class to another means that the first is included in the second.

For each of these graph classes, it is important to note the complexity of the recognition problem, i.e. given a graph G, determine whether or not G belongs to the class. Unit interval graph recognition is polynomial [45], as is interval graph recognition [28], and contact disk graph recognition is the same as planarity testing and hence polynomial. However, for unit disk graphs [7] and general disk graphs [22], the recognition problem is NP-hard. Surprisingly, it is not yet known whether these NP-hard problems are in NP.

1.1.2 Complexity on restriction to unit disk graphs

In this part, we survey complexity results for the restrictions to the class of unit disk graphs and to related classes.

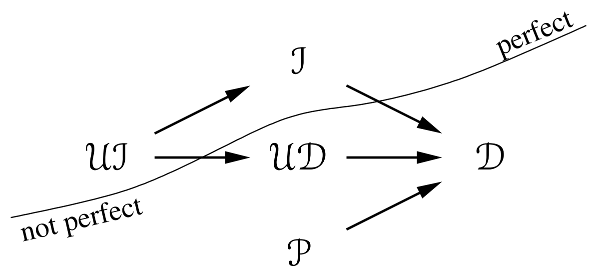


Figure 1.1: A diagram of the relationship between graph classes $\mathcal{UI}$, $\mathcal{I}$, $\mathcal{UD}$, $\mathcal{D}$ and$\mathcal{P}$.

For channel assignment as well as other applications (mostly related to communications networks), it is important to consider various computational problems for the class of unit disk graphs. The classical problem of finding a minimum dominating set (a set of vertices A is a dominating set if every other vertex in the graph is adjacent to some member of A) restricted to $\mathcal{UD}$ is related to the problem of optimally placing emergency senders in a communication network [51]. Given n points in the plane, the problem of finding a maximum subset of points no two of which are at distance d or greater is the same as the maximum clique problem for the corresponding unit disk graph [8]. We will examine the unit disk colourability problem in more detail in the next part. See Table 1.1 for a summary of what is known about these problems, comparing the restrictions to interval graphs, to unit disk graphs, to planar graphs and to disk graphs.

Table 1.1: Relative complexity for certain problems restricted to the graph classes $\mathcal{I}$, $\mathcal{UD}$ and $\mathcal{P}$.

Problem	$\mathcal{I}$	$\mathcal{UD}$	$\mathcal{P}$	$\mathcal{D}$
CHROMATIC NUMBER	P [40]	NPC [8]	NPC	NPC
CLIQUE	P [40]	P [8]	P	Open
INDEPENDENT SET	P [16]	NPC [8]	NPC	NPC
DOMINATING SET	P [4]	NPC [34]	NPC	NPC
CONNECTED DOMINATING SET	P [44]	NPC [29]	NPC	NPC

Clark, Colbourn and Johnson [8] give a systematic analysis of these problems restricted to unit disk graphs. For UD 3-COLOURABILITY, UD INDEPENDENT SET and UD DOMINATING SET, they provide NP-completeness proofs which all rely on the same embedding (by Valiant [52]) and reduction from the corresponding problems restricted to $\mathcal{P}$ (well known to be NP-complete [15]). This indicates a strong complexity relationship between the classes $\mathcal{P}$ and $\mathcal{UD}$.

For UD CLIQUE, the authors exhibit a polynomial-time algorithm which assumes a unit disk embedding is known. Here follows a (very) brief sketch of this algorithm. Let G be a unit disk graph and assume we have an arbitrary fixed unit disk embedding of G. For each pair of points x and y with $xy \in V(G)$, let $D_{x,y}$ be the set of points z such that both $d(x,z)$ and $d(y,z)$ are at most $d(x,y)$ (where $d(x',y')$ is the distance between x' and y' in the embedding). It is shown that any clique must be a subset of one of these "lozenges" $D_{x,y}$. Also, it shown that each such set $D_{x,y}$ is the complement of a bipartite graph (and hence it is polynomial to determine a maximum size clique in $D_{x,y}$). Hence, to find the size of a maximum clique in G, it suffices to run a polynomial-time algorithm on the sets $D_{x,y}$ for each pair of vertices (x,y) and clearly this algorithm runs in polynomial time overall. There is also a polynomial algorithm that does not require a representation [43]. Note that PLANAR CLIQUE is clearly polynomial since K_5 is non-planar (due to the Kuratowski's theorem). The complexity of DISK CLIQUE is still open.

As the table indicates, the problems restricted to interval graphs (hence unit interval graphs) are all polynomial. We note, in particular, a simple algorithm that solves both INTERVAL CLIQUE and INTERVAL CHROMATIC NUMBER simultaneously:

Proposition 1 *If G is an interval graph, then $\chi(G) = \omega(G)$, and $\chi(G) = \omega(G)$ can be found in polynomial time.*

Proof. We assume we have an interval representation for G and order the vertices by the left endpoints of the respective intervals. We apply greedy colouring, i.e. properly colour the vertices in order and always assign the smallest possible colour. Clearly, this is a polynomial-time algorithm, so it remains to show that it gives the best colouring. Suppose we are at the point in the greedy process when vertex x receives c, the largest colour used overall. The interval (a,b) associated with x intersects with $c-1$ intervals whose vertices were coloured 1 through $c-1$. These intervals all have left endpoints that are smaller than a and hence all share the point a. Hence, we have a clique of size c (namely, x and its $c-1$ neighbours coloured 1 through $c-1$). We have that $\omega(G) \geq c \geq \chi(G)$ and, since $\omega(G) \leq \chi(G)$ in general, equality holds. □

Except possibly for DISK (CONNECTED) DOMINATING SET, each of the NP-hard problems in Table 1.1 has a "good" heuristic [3, 33, 23, 13]: for each such problem, there is a polynomial-time approximation algorithm with constant performance guarantee and some even have polynomial-time approximation schemes (a PTAS is a polynomial-time approximation algorithm that, given an instance of the problem and $\varepsilon > 0$, returns a solution within a factor $1+\varepsilon$ of the optimal solution. On the other hand, for general graphs, none of these problems has a PTAS unless P = NP [31, 2, 24, 25].

1.1.3 The unit disk colourability problem

For the frequency assignment problem, our focus is upon UD CHROMATIC NUMBER. We give, in this part, an in-depth discussion of this problem.

In particular, to aid understanding of the proofs in the second chapter, we now give brief descriptions of the NP-completeness proof of UD 3-COLOURABILITY given in [8] and the proof of the following more general result.

Theorem 1 (Gräf, Stumpf and Weißenfels [18]) *For each fixed $l \geq 3$, the problem UD l-COLOURABILITY is NP-complete.*

The proof for NP-completeness of UD 3-COLOURABILITY given in [8] is a reduction from PLANAR 3-COLOURABILITY for graphs with maximum degree at most 4. The most important section of this proof is the (polynomial-time) generation of a restricted planar embedding such that the edges are only drawn on lines of the integer grid (hence the need for graphs with maximum degree at most 4). Subsequently, the embedded edges are replaced by sequences of unit disk graphs that "communicate" the 3-colourability of the original graph. This is a relatively simple NP-completeness technique known as local replacement (cf. [15]).

For Theorem 1, Gräf, Stumpf and Weißenfels attempt to generalize this approach; however, the first main difficulty is that reduction from PLANAR l-COLOURABILITY for $l > 3$ is impossible due to the four-colour theorem. Instead, the reduction is from general l-COLOURABILITY, but this introduces two new difficulties: higher degrees and crossing edges. To handle the first difficulty, they use a new (non-planar) graph embedding in which the edges are still drawn on lines of the integer grid, but each high-degree vertex v is replaced by a set of points $M(v)$ of size $\deg(v)$ spread apart in the integer grid (and such that, for $u \neq v$, the convex hulls of $M(u)$ and $M(v)$ do not intersect). For the second difficulty, it is necessary to find an auxiliary unit disk graph for the crossing of two edges that communicates l-colourability information (see Figure 2.4 in the next chapter).

Our proofs in the next chapter will borrow heavily on these ideas.

We have already noted that $\mathcal{I}$ is a perfect class and, in the last part, showed that there is a polynomial algorithm to find the clique and chromatic numbers. In contrast, $\mathcal{UD}$ is not a perfect class and we just noted that there is no polynomial algorithm to find the chromatic number of unit disk graphs (unless P = NP). However, UD CLIQUE is polynomial and this raises the question of whether there is a bound on the chromatic number in terms of the clique number for unit disk graphs. The following straightforward result answers this question in the affirmative.

Proposition 2 (Peeters [42]) *There is a polynomial algorithm that, for any unit disk graph G, generates a proper colouring of G with at most $3\omega(G)-2$*

colours. This is a polynomial approximation algorithm for UD CHROMATIC NUMBER with performance guarantee of 3.

Corollary 1 *For any unit disk graph G, $\chi(G) \leq 3\omega(G) - 2$.*

Proof of Proposition 2. We assume we have the unit disk representation for G and order the vertices lexicographically, i.e. we order the points first by x-coordinate, then by y-coordinate. Like in the proof of Proposition 1, we apply greedy colouring according to this ordering. Clearly, this is a polynomial-time algorithm, so it remains to show that it uses at most $3\omega - 2$ colours. Suppose we are at the point in the algorithm when vertex x receives c, the largest colour used overall. Let D be the open unit disk centred at x. Because of the ordering we chose, all of the previously coloured neighbours of x lie within a sector of D with angle less than π radians. Note that any two points in the interior of a sector with angle $\pi/3$ radians are less than distance 1 apart (and each less than distance 1 from x). Hence the points in the interior of such a sector form a clique together with x. It follows that the number of previously coloured neighbours of x is at most $3(\omega(G) - 1)$ and this implies that $c \leq 3(\omega(G) - 1) + 1$. □

By similar arguments, it can be shown that

- For any unit disk graph G, the degeneracy $\delta^*(G)$ – the supremum of the minimum degree over all induced subgraphs of G – is at most $3\omega(G) - 3$. (On the other hand, however, there is an example of a unit disk graph whose degeneracy meets this bound [32])
- For any unit disk graph G, the maximum degree $\Delta(G)$ is at most $6\omega(G) - 6$. In particular, the greedy algorithm on *any* ordering is a 6-approximation.
- For any disk graph G, $\chi(G) \leq 6\omega(G) - 5$.

Proposition 2 was given in 1991; however, there has been no significant improvement since. Gräf, Stumpf, and Weißenfels [18] provide a more sophisticated heuristic called the STRIPE algorithm, but it also has performance guarantee of 3. Because of results for colouring of random unit disk graphs (see [35]), it is plausible that there could be a better bound on $\chi(G)$ in terms of $\omega(G)$ for unit disk graphs. Note that there are classes of unit disk graphs with $\chi(G) \geq \frac{3}{2}\omega(G)$ [32]; however, it is still open whether the bound on the ratio $\chi(G)/\omega(G)$ for unit disk graphs should be closer to 3/2 or 3.

1.2 Improper colouring

A *colouring* of a graph G is a labelling $c : V(G) \to S$. The elements of S are called *colours* and the vertices of one colour form a *colour class*. We say that c is an *l-colouring* if $|S| \leq l$. Given a graph G, a colouring c of G, and a subset S of $V(G)$, the *impropriety* (or *defect*) of a vertex v restricted to S under c, denoted $\mathrm{im}_S^c(v)$, is the number of neighbours of v in S in the same colour class. We say that a vertex is *proper* in S under c if $\mathrm{im}_S^c(v) = 0$ and *improper* otherwise. The *impropriety* (or *defect*) of c in S is $\mathrm{im}_S(c) = \max_{v \in S}\{\mathrm{im}_S(v)\}$. For the abovementioned notions, we will often drop the superscript or subscript if the context is clear. A colouring is *k-improper* if its impropriety in G is at most k. We say a graph is *k-improper l-colourable* if it has a k-improper l-colouring. The *k-improper chromatic number* $\chi^k(G)$ is the least l such that G is k-improper l-colourable. This notion was introduced by Cowen, Cowen and Woodward [10] (however, they use the notation *(l, k)-colourable*). Note that 0-improper colouring is exactly proper colouring; hence, the 0-improper chromatic number is precisely the chromatic number $\chi(G)$.

We will also consider a more restrictive version of colouring and its improper analogue. An *l-list assignment* of G is a function L which assigns to each vertex $v \in V(G)$ a set of size at most l. An *L-list colouring* is a colouring c such that $c(v) \in L(v)$ for all $v \in V(G)$. We say a graph G is *k-improper l-choosable* (or *k-improper l-list colourable*) if given any l-list assignment L, G has a k-improper L-list colouring. The *k-improper choosability* $\mathrm{ch}^k(G)$ is the least l such that G is k-improper l-choosable. This notion was first considered by Borowiecki, Drgas-Burchardt and Mihók [5]. The notion of (0-improper) choosability was introduced independently by Erdös, Rubin and Taylor [12] and Vizing [53] in the 1970's.

Note that every k-improper l-choosable graph is k-improper l-colourable and hence $\chi^k(G) \leq \mathrm{ch}^k(G)$, since the assignment $L(v) = \{1, \ldots, l\}$ for all v is an l-list assignment. Also note that if a graph is k-improper l-colourable, then it is k_1-improper l_1-colourable for any $(k_1, l_1) \geq (k, l)$. (By $(x_1, y_1) \geq (x_2, y_2)$, we mean $x_1 \geq x_2$ and $y_2 \geq y_2$.)

1.2.1 General results on improper (list) colouring

A k-improper l-colouring is a partition of the vertices into induced subgraphs each with maximum degree at most k. In particular, this means that, if G is k-improper l-colourable, then it is also (proper) $((k + 1)l)$-colourable.

It is intuitively plausible that finding an optimum k-improper colouring for fixed $k \geq 1$ is more “difficult” than finding an optimum proper colouring because the former requires more information. In particular, suppose that we are colouring the vertices sequentially. Perhaps we are performing an *online* colouring – vertices are presented one at a time and each vertex must be affixed a colour before the next vertex is presented. For proper colouring, to know what colours are available for a vertex, we need only know the colours assigned to the previously coloured neighbours; however, for improper colouring, we also require the defect of each such vertex in the colouring so far.

This intuition bears out when we consider the complexity of the k-improper l-colouring problem. For fixed $l \geq 3$, l-COLOURABILITY is NP-complete, and determining whether a graph is bipartite can be performed in polynomial time. On the other hand, k-IMPROPER l-COLOURABILITY is NP-complete for $(k,l) \geq (1,2)$ and $(k,l) \geq (0,3)$ [9]. Furthermore, as for proper colouring, it is known that, for any k, there exists $\varepsilon > 0$ such that χ^k cannot be approximated to within a factor of n^ε unless P = NP.

This hardness result leads us to consider (weak) bounds. Recall two elementary lower bounds for the (proper) chromatic number:

Proposition 3 *For any graph G, $\chi(G) \geq \omega(G)$ and $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$ (where $\alpha(G)$ is the size of the maximum independent set of G).*

It is straightforward to generalise this result for k-improper colouring. First, define a *k-dependent set* to be a subset of vertices whose induced subgraph has maximum degree at most k and let $\alpha^k(G)$ be the size of the maximum k-dependent set. (Thus, a 0-dependent set is an independent set and $\alpha^0(G) = \alpha(G)$.)

Proposition 4 *For any graph G, $\chi^k(G) \geq \frac{\omega(G)}{k+1}$ and $\chi^k(G) \geq \frac{|V(G)|}{\alpha^k(G)}$.*

Similarly, the following is a familiar upper bound:

Proposition 5 *For any graph G, $\chi(G) \leq \Delta(G) + 1$ (where $\Delta(G)$ is the maximum degree of G).*

Consider the following approximation result for graphs of bounded degree.

Proposition 6 (Lovász [30]) *Let G be a graph with maximum degree $\Delta \leq l(k+1) - 1$. There is an algorithm to k-improper l-colour G in $O(\Delta|E(G)|)$ time.*

Now the k-improper analogue to Proposition 5 is a corollary.

Corollary 2 *For any graph G, $\chi^k(G) \leq \left\lceil \frac{\Delta(G)+1}{k+1} \right\rceil$.*

Proof of Proposition 6. Begin with an arbitrary l-colouring c of G. Suppose some vertex v has more than k neighbours in the same colour class. By the bound on the maximum degree, there must be some colour class with at most k members in the neighbourhood of v. We simply change v to this colour. This reduces the total number of monochromatic edges of c by at least one; hence, this process terminates in at most $|E(G)|$ steps. □

The bound in Corollary 2 still holds when χ^k is replaced by ch^k.

1.2.2 Improper (list) colouring planar graphs

Improper colouring for planar graphs was first considered by Cowen, Cowen and Woodward [10]. Improper choosability for planar graphs first considered by Eaton and Hull [11] and Škrekovski [48]. The motivation for this study stems from the four-colour theorem. A generalisation of this much-celebrated result asks the following question: for fixed k, what is the smallest p_k (p^*_k) such that every planar graph is k-improper p_k-colourable (p^*_k-choosable)? Here is a summary of the progress on this question.

The question for k-improper colourability is fully answered. Every planar graph is 4-colourable [1] and K_4 is a planar graph that is not 3-colourable. For the other values of k, the answer is given by [10]. They give simple examples to show that, for each k, there is a planar graph that is not k-improper 2-colourable and there is a planar graph that is not 1-improper 3-colourable. They prove that every planar graph is 2-improper 3-colourable. In summary, $p_0 = p_1 = 4$ and $p_k = 3$ for all $k \geq 2$.

The question for k-improper choosability has one lingering gap. The elegant proof of Thomassen [50] shows that every planar graph is 5-choosable. There are examples of planar graphs that are not 4-choosable [54, 39]. Eaton and Hull [11] and Škrekovski [48] independently showed that every planar graph is 2-improper 3-choosable. These results (and the examples from [10]) imply that $p^*_0 = 5$, $p^*_1 \in \{4, 5\}$, and $p^*_k = 3$ for all $k \geq 2$. It is an open question to determine whether p^*_1 is 4 or 5.

Related questions for graphs on surfaces of higher genus [9], planar graphs with prescribed girth [48, 49] and graphs with prescribed maximum average degree [21] have been studied. The most natural further question, though, is to determine the complexity of improper colouring for planar graphs.

The proof of the four-colour theorem naturally extends to a quartic algorithm for 4-colouring a planar graph [1]; indeed, a quadratic algorithm has been developed [46]. The 2-improper 3-colouring proof of [10] naturally extends to a linear algorithm. It is well-known that it is NP-complete to determine if a planar graph is 3-colourable and that 2-colouring is polynomial for general graphs. The remaining complexity questions are answered as follows.

Theorem 2 (Cowen, Goddard, Jesurum [9])

1. *It is NP-complete to determine if a planar graph is 1-improper 3-colourable.*
2. *For fixed $k \geq 1$, it is NP-complete to determine if a planar graph is k-improper 2-colourable.*

Part 2 of Theorem 2 is important to us since we will be using this result in the next chapter. The proof relies on a reduction from 1-improper

2-colourability, which in turn is reduced from 3-SAT. For an idea of the methods used in proving hardness for improper colouring, let us briefly examine the former reduction.

Let G be a planar graph. For each vertex v in G, we introduce the following (planar) structure D_v. The vertex set of D_v consists of the $k-1$ independent sets of size $2k+1$ $B_1, \ldots, B_{k-1}$ and the $k-1$ vertices $c_1, \ldots, c_{k-1}$. We set $c_0 = v$. Connect all possible edges between B_i and the vertices c_{i-1} and c_i, for $1 \leq i \leq k-1$. See Figure 1.2.

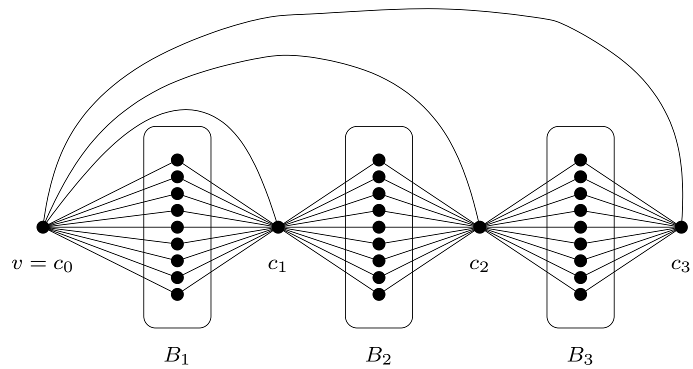


Figure 1.2: The structure D_v for $k = 4$.

In any k-improper 2-colouring of D_v, the vertices c_i must all have the same colour. (If c_{i-1} and c_i have different colours, then one of these vertices has impropriety greater than k, since one of the colours appears at least $k+1$ times in B_i.) Thus, v is adjacent to $k-1$ vertices with the same colour (namely the c_i) and has impropriety at least $k-1$. However, in the 2-colouring where all the c_i have the same colour as v and all members of the B_i have the other colour, v has impropriety exactly $k-1$. Therefore, the resulting planar graph G' is k-improper 2-colourable if and only if G is 1-improper 2-colourable. Clearly, this construction is polynomial.

Chapter 2

Improper colouring of unit disk graphs

In this chapter, we present some analysis for the unit disk improper colourability problem.

Our motivation for this analysis is the following problem proposed by Alcatel Space. Satellites send information to stationary receivers on earth. Each receiver is listening on a chosen frequency (or set of frequencies) and can receive information in a certain area (usually modelled by a disk centred at the receiver). The signals of two receivers might interfere if their areas intersect and, in this case, they should normally be listening on different frequencies. However, the intensity of the signal sent by the satellite to a receiver u is supposed to be large near u, and to decrease quickly with distance. Hence, even if the reception areas of receivers u and v intersect, the intensity of the signal sent to v is assumed to be low near u. Furthermore, if u and v have close reception frequencies, then the signal for v contributes interference (otherwise called noise) to the signal received by u, and vice versa. If the total noise does not exceed a certain threshold, then u can still distinguish its signal (perhaps by use of error-correcting coding conventions). We wish to allocate frequencies to the receivers so as to minimise the total bandwidth used.

Like in the case of traditional radio channel assignment, unit disk graphs can be used in a simple model of this problem. We assume that the receivers are scattered in the plane, all receivers have the same reception range and operating bandwidth, there are no physical obstacles to transmission: the reception areas are uniform disks centred at the receivers. We define the same intersection graph as before: the vertices are the receivers and place an edge between two vertices if their corresponding disks intersect. We furthermore assume that nearby towers contribute noise only if they are on the same channel and that they contribute exactly one unit of noise. We must colour the vertices (i.e. assign a frequency to each receiver) to satisfy the threshold constraint. If we let k be the threshold value (i.e. a receiver can still distinguish its signal if the sum of noises is at most k), then it is clear that, under this model, our problem is to find an optimal k-improper colouring of the resulting unit disk graph.

Before we begin, let us first remark that every clique is a unit disk graph, indeed, a unit interval graph; hence, for each fixed k and l, there exists a unit disk (interval) graph that is not k-improper l-colourable.

2.1 Improper colouring interval graphs

Proposition 7 *For any $k \geq 0$, there exists a unit interval graph I_k with maximum degree and clique number equal to $2k+2$ which is not k-improper 2-colourable.*

Proof. To construct I_k, just start with a $(2k+2)$-clique $K = K_{2k+2}$ and add a vertex u linked to exactly $(k+2)$ vertices of K. Suppose I_k has a k-improper 2-colouring: K must have exactly $(k+1)$ vertices of each colour. Thus any vertex of K has impropriety k in K. As u has $(k+2)$ neighbours in K it must have at least one neighbour of each colour and hence cannot be coloured, a contradiction. I_k is clearly a unit interval graph. □

Note that if we compute Lovász' bounds (from Proposition 4 and Corollary 2) for the graph I_k, we find that its k-improper chromatic number is either 2 or 3, so it is 3. In particular, this shows that there is little hope of improving Lovász' upper bound for general unit interval graphs. On the other hand, for any even k there exist k-improper 2-colourable graphs which have Lovász' upper bound of 3.

Proposition 7 raises the question of the complexity of k-improper l-colouring unit interval graphs for fixed k and l. We prove now that this problem is polynomial for general interval graphs, and we provide a dynamic programming algorithm.

Note that any unit interval graph is bi-simplicial, i.e. the neighbourhood of any vertex induces at most two cliques. As orientation, we make the following observation:

Proposition 8 *Given k and l, any bi-simplicial graph G with maximum degree at least $(2l-1)(k+1)$ is not k-improper l-colourable.*

Proof. Let v be a vertex of degree at least $(2l-1)(k+1)$ and suppose that c is a k-improper l-colouring of G with $c(v) = 1$. Let H be the subgraph of G induced by the neighbours of v not coloured with colour 1. Clearly, H is k-improper $(l-1)$-colourable. Note also that, since v has impropriety at most k, H has at least $(2l-1)(k+1) - k = 2(l-1)(k+1) + 1$ vertices. As G is bi-simplicial, H can be partitioned into two cliques. But then, one of these cliques must be of size at least $(l-1)(k+1)+1$, contradicting the fact that H is k-improper $(l-1)$-colourable. □

Theorem 3 *The k-improper l-colourability problem restricted to interval graphs is in P for any fixed k and l.*

Proof. Let G be an interval graph. We preprocess the graph by computing $\omega(G)$ (and this can be done in polynomial time). We may assume that $\omega(G) \leq l(k+1)$; otherwise, G is not k-improper l-colourable by Proposition 4. Now assume we have an interval representation for G. Let $v_1, \ldots, v_n$ be the vertices of G ordered by the left endpoints of the respective intervals. We will consider the vertices one-by-one according to this order and assign v_1 colour 1.

For this algorithm, we maintain all valid partial k-improper l-colourings of the induced subgraph processed so far; however, we discard vertices that are not required. More precisely, suppose v records the next vertex to be processed and we wish to extend all of the partial colourings (and discard ones that are impossible to extend). We need only maintain a list of all valid partial k-improper l-colourings (together with accumulated improprieties) of a set S, where S contains all previously coloured neighbours of v.

It is clear that, if the vertex v_j is not adjacent to $v = v_s$, where $j < s$, then v_j is not adjacent to v_i with $i \geq s$ (and hence we can safely remove v_j from S). Furthermore, the maximum number of vertices in S at any given point in time is $\omega - 1$, since S together with v induces a clique. Thus, a list of size $(lk)^{\omega(G)} \leq (lk)^{l(k+1)}$ is sufficient. Furthermore, the step of colouring a vertex and updating the list is clearly polynomial in time. □

Unfortunately, this result does not fully answer the complexity question for improper colouring of interval graphs. It is unknown whether, for $k > 0$ fixed, there is a polynomial-time algorithm to find $\chi^k(G)$ given an interval graph G.

2.2 Unit disk k-improper l-colourability, $l \geq 3$

Since the l-colourability problem for unit disk graphs is NP-complete for any fixed $l \geq 3$ (cf. Theorem 1 on page 8), we expect that, for any fixed $k \geq 1$, the corresponding k-improper l-colourability problem is also NP-complete. We show now that our expectation is indeed correct, by using a reduction similar to that of [18].

Theorem 4 *Unit disk k-improper l-colourability is NP-complete for fixed $k \geq 0$ and $l \geq 3$.*

Before beginning the outline for the proof of this result, we give the following

Lemma 1 *Suppose K_1 is a $(k+1)$-clique, K_2 is a $((l-1)(k+1))$-clique, and K_3 is a j-clique, $1 \leq j \leq k+1$. Let H be the graph formed by connecting all edges between K_1 and K_2 and between K_2 and K_3. Then H is k-improper l-colourable, and in any k-improper l-colouring of H, any vertex of K_1 and any vertex of K_3 must receive the same colour.*

Proof. Suppose we have a k-improper l-colouring of H, let $u \in K_3$ and assume without loss of generality that u has colour 1. The subgraph induced by $K_1 \cup K_2$ is an $(l(k+1))$-clique, so every colour appears exactly $(k+1)$ times in this clique and any vertex v in $K_1 \cup K_2$ has impropriety k in $K_1 \cup K_2$. Hence, the colour 1 may not appear on the vertices of K_2. As K_2 is an $((l-1)(k+1))$-clique, each colour other than 1 must appear exactly $(k+1)$ times within K_2 and thus has impropriety k in K_2. Clearly, the vertices of K_1 and K_3 must be coloured 1. It is easy to check that this is indeed a valid k-improper l-colouring. □

Our approach will generalise that of Gräf, Stumpf and Weißenfels and we want to show how, given any graph G, to construct a corresponding unit disk graph $\hat{G} = (\hat{V}, \hat{E})$ which is k-improper l-colourable if and only if G is l-colourable. The key to our approach is to generalise the auxiliary graphs. We will describe k-improper l-colourable analogues for each of the four auxiliary graphs that they employ. We shall use the same embedding for the given graph G, and the unit disk graph embedding needs only a slight technical modification to accommodate a larger auxiliary graph for crossings.

2.2.1 Construction of the auxiliary graphs

First, we introduce the graphs that will replace the edges in an embedding of G. All of these graphs are unit disk graphs and, except for the last one, use the same embeddings as in [18]. The remaining properties are given without proof since they generally follow immediately from the construction or a simple application of Lemma 1. Like in the cited reference, our construction makes frequent use of cliques. In figures, these cliques will be represented by circles using the following convention:

- a small circle with a + represents a $(k+1)$-clique;
- a large circle with a $\star$ represents a $(l-2)(k+1)$-clique; and
- a large circle with a $\times$ represents a $(l-1)(k+1)$-clique.

If cliques of other size are needed, they will be represented by a large circle with the number of vertices of the clique written in it. An edge between two cliques means that all possible edges between the two cliques are present.

Definition 1 *A (k,l)-wire of order m, denoted $W^m_{k,l}$, consists of $m+1$ $(k+1)$-cliques $WV_0, \ldots, WV_m$ and m $((l-1)(k+1))$-cliques $WC_1, \ldots, WC_m$. For each $1 \leq i \leq m$, all members of the clique WC_i are connected to the members of both WV_{i-1} and WV_i. The cliques WV_0 and WV_m are called output cliques.*

A (k,l)-wire of order 3 is shown in Figure 2.1.

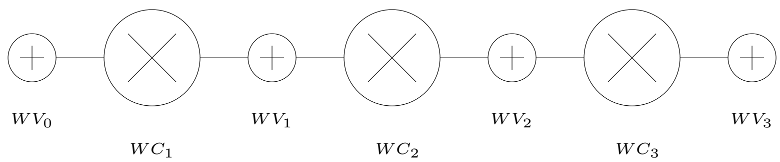


Figure 2.1: The (k,l)-wire $W^3_{k,l}$.

Proposition 9 *A (k,l)-wire of order m has the following properties:*

1. *$W^m_{k,l}$ has $m(l-1)(k+1)+(m+1)(k+1) = (ml+1)(k+1)$ vertices;*
2. *a (k,l)-wire is k-improper l-colourable, but not k-improper $(l-1)$-colourable;*
3. *each k-improper l-colouring assigns the same colour to all members of $WV_0, \ldots, WV_m$, and, in particular, the output cliques receive the same colour; and*
4. *a (k,l)-wire is a unit disk graph.*

Definition 2 *A (k,l)-chain of order m, denoted $K^m_{k,l}$, consists of a $W^m_{k,l}$ together with an additional j-clique WF connected with WV_m, for some $1 \leq j \leq (l-1)(k+1)$. The clique WV_0 is called the fixed output clique while WF is called the forced output clique.*

A (k,l)-chain of order 3 is shown in Figure 2.2.

Proposition 10 *A (k,l)-chain of order m has the following properties:*

1. *$K^m_{k,l}$ has $(ml+1)(k+1)+j$ vertices, where j is the size of WF;*
2. *a (k,l)-chain is k-improper l-colourable, but not k-improper $(l-1)$-colourable;*

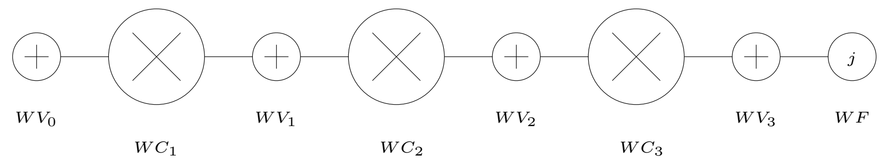


Figure 2.2: A (k, l)-chain of order 3 $K^3_{k,l}$.

3. *each k-improper l-colouring assigns the same colour i to all members of any of the cliques $WV_x, 1 \leq x \leq m$, and each member of the clique WF must receive a colour that is different from i;*

4. *for each pair of different colours (i_1, i_2) from the set $\{1, 2, \ldots, l\}$ there exists a k-improper l-colouring in which the forced and fixed output cliques receive colours i_1 and i_2, respectively; and*

5. *a (k, l)-chain is a unit disk graph.*

We now introduce the graphs that will replace the high degree vertices of G.

Definition 3 *A (k, l)-clone of size $m \geq 2$, denoted $C^m_{k,l}$, consists of the $7m - 7$ $(k + 1)$-cliques $CV_1, \ldots, CV_{7m-7}$, the $7m - 6$ $((l - 1)(k + 1))$-cliques $CC_0, \ldots, CC_{7m-7}$, and the m $(k + 1)$-cliques $O_0, \ldots, O_{m-1}$. For $1 \leq i \leq 7m - 7$, all members of the clique CV_i are connected to the members of both CC_{i-1} and CC_i. For each $0 \leq i \leq m - 1$, all members of O_i are connected to the members of CC_{7i}. The cliques $O_0, \ldots, O_{m-1}$ are called output cliques.*

A (k, l)-clone of size 2 is shown in Figure 2.3.

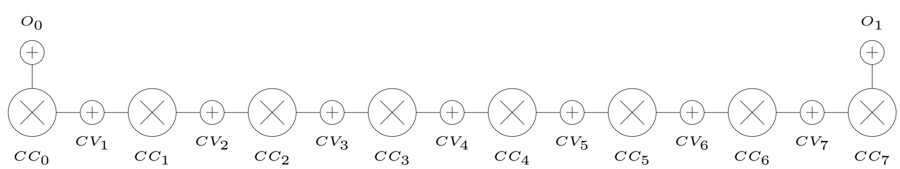


Figure 2.3: The (k, l)-clone $C^2_{k,l}$.

Note that, in the corresponding auxiliary graph described in [18], every third clique was connected to an output vertex for technical reasons. For similar reasons, every seventh $((l-1)(k+1))$-clique is connected to an output clique in our construction.

Proposition 11 *A (k,l)-clone of size m has the following properties:*

1. *$C^m_{k,l}$ has $(7m-6)(l-1)(k+1)+(7m-7)(k+1)+m(k+1) = ((7m-6)l+m-1)(k+1)$ vertices;*

2. *a (k,l)-clone is k-improper l-colourable, but not k-improper $(l-1)$-colourable;*

3. *each k-improper l-colouring assigns the same colour to all members of the output cliques; and*

4. *a (k,l)-chain is a unit disk graph.*

Finally, we introduce the graphs $H_{k,l}$ that will replace the edge crossings in an embedding of G. This construction is based on the graph H_l used in [18] (in the cited paper, it appears as H_k). We replace all vertices of H_l by $(k+1)$-cliques and we replace all edges of H_l by (k,l)-chains of the appropriate order (either 1 or 2) so that the resulting graph has a unit disk representation.

When replacing edges in H_l, we have taken care to orient the (k,l)-chains so that we do not introduce cliques of size greater than $l(k+1)$; in particular, only the forced output cliques of the (k,l)-chains may be incident with the $((l-2)(k+1))$-cliques C_i of $H_{k,l}$. Note then that each $(k+1)$-clique representing a former vertex of H_l is incident to a $((l-1)(k+1))$-clique of some (k,l)-chain, and this ensures that, in a k-improper l-colouring of $H_{k,l}$, each $(k+1)$-clique is assigned a single colour.

See Figure 2.4 for a description of how $H_{k,l}$ is derived.

Definition 4 *Let a (k,l)-crossing, denoted $H_{k,l}$, $l \geq 3$ be the graph in Figure 2.5. The cliques $V_0, \ldots, V_3$ are called output cliques.*

Proposition 12 *A (k,l)-crossing has the following properties:*

1. *$H_{l,k}$ has $(37l-2)(k+1)$ vertices;*

2. *a (k,l)-crossing is k-improper l-colourable, not k-improper $(l-1)$-colourable;*

3. *each k-improper l-colouring c satisfies*
   $c(V_0) = c(V_2)$ and $c(V_1) = c(V_3)$;

4. *there exist two k-improper l-colourings c_1 and c_2 which satisfy*
   $c_1(V_0) = c_1(V_2) = c_1(V_1) = c_1(V_3)$ and
   $c_2(V_0) = c_2(V_2) \neq c_2(V_1) = c_2(V_3)$; and

5. *a (k,l)-crossing is a unit disk graph.*

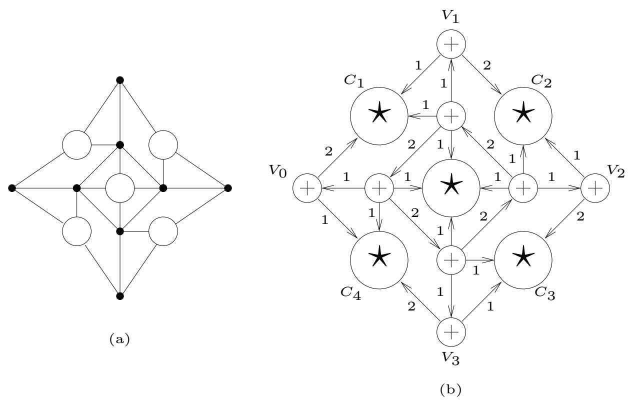


Figure 2.4: The derivation of the (k, l)-crossing $H_{k,l}$ from the l-crossing H_l: (a) H_l, where the circles represent $(l-2)$-cliques and (b) a schematic figure of $H_{k,l}$, where each (k, l)-chain is represented by a directed edge (the edge is directed from the fixed output vertex to the forced output vertex of the chain) together with an integer (the order of the chain).

2.2.2 Embedding of the unit disk graph

As mentioned earlier, we shall use the same embedding of the given graph G, or rather, the embedding of the graph G'. However, for our auxiliary graphs, we must accommodate for the necessity of a larger unit disk representation for $H_{k,l}$.

As in [18], we will make use of so-called representatives and hence we shall generate proper distance models. Recall from their paper that a) because of certain properties, it suffices to represent each clique by one vertex called a *representative*, and b) a unit disk model with distance value d is *proper* if the distance between any two vertices or representatives in the model is unequal to d. With the use of representatives, our (k, l)-wires and (k, l)-chains are no different from l-wires and l-chains, respectively. We can thus use exactly the same embeddings. Similarly, our (k, l)-clone embedding is identical to the l-clone embedding, except that output vertices are placed on the x-axis at distance 56 (instead of 24). The reason we have chosen this larger distance is to give room for the (k, l)-crossing embedding. Figure 2.6 shows how a distance model with distance 6 can be constructed for the (k, l)-crossing. The output vertices are placed at distance 24 from the centre.

The final construction of the unit disk graph $\hat{G}$ is now straightforward and follows [18].

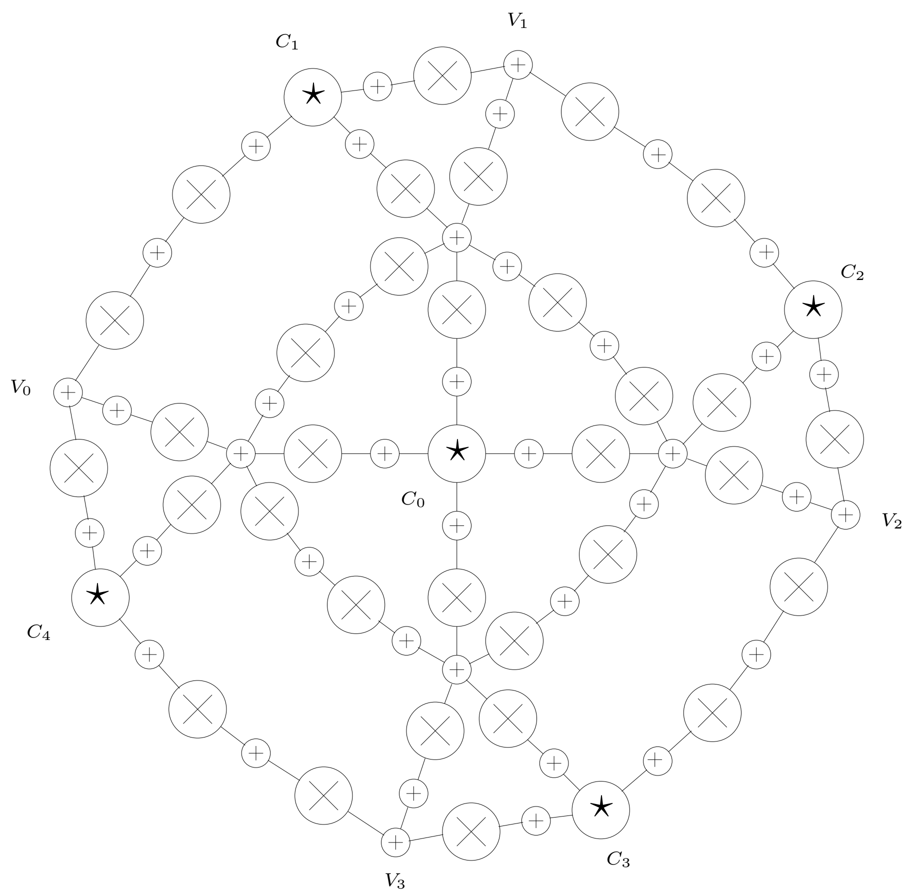


Figure 2.5: The (k, l)-crossing $H_{k,l}$.

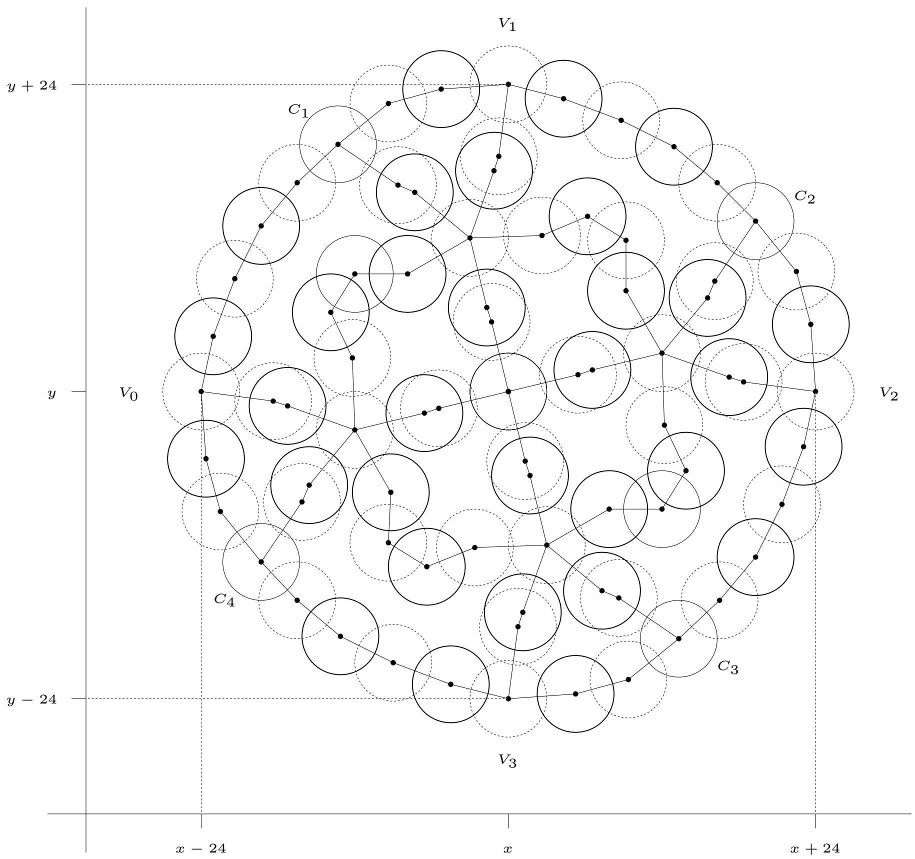


Figure 2.6: An embedding of the (k, l)-crossing: a bold-lined disk represents $(l - 1)(k + 1)$ copies of the same disk, a dash-lined disk represents $k + 1$ copies of the same disk while each of the five remaining disks represents $(l - 2)(k + 1)$ copies of the same disk.

2.2.3 Proof of theorem

Proof of Theorem 4. To prove that $\hat{G}$ is k-improper l-colourable if and only if G is l-colourable, we again follow [18]. The only difficulty lies in the possibility of multiple colours being assigned to a forced output clique of a (k, l)-chain. This difficulty can only arise for vertices in G with degree 1, and we can choose $\hat{G}$ so that a fixed output clique is incident to each such vertex. □

2.3 Unit disk k-improper 2-colourability, $k \geq 1$

It is not clear if we should expect the k-improper 2-colourability problem for unit disk graphs to be NP-complete, as 2-colourability is polynomial in general, while the planar k-improper 2-colourability problem, $k \geq 1$, is NP-complete (cf. Theorem 2(2) on page 12).

Theorem 5 *Unit disk k-improper 2-colourability is NP-complete for any fixed $k \geq 1$.*

Our reduction is from k-improper 2-colourability of planar graphs. Given any planar graph G, we will show how to construct, in polynomial time, a unit disk graph $\hat{G}$ which is k-improper 2-colourable if and only if G is. Our construction is based on [18], but, for the embedding, we have added the condition of planarity. Hence, we do not require a crossing auxiliary graph. On the other hand, since we are dealing entirely with k-improper 2-colouring, we must take care to handle impropriety appropriately.

2.3.1 Construction of the auxiliary graphs

These graphs are unit disk graphs. We will give the corresponding unit disk representations later. First, we introduce the graphs that will replace the edges in an embedding of G.

Definition 5 *A $(k, 2)$-bond, denoted $B_{k,2}$, has vertex set $\{v_0, \ldots, v_{2k+2}\}$. For the edge set, the vertices $\{v_1, \ldots, v_{2k+1}\}$ induce a clique, v_0 is adjacent to any $v_i, i \leq k+1$, and v_{2k+2} is adjacent to any $v_i, i \geq k+1$. The vertices v_0 and v_{2k+2} are called output vertices.*

A $(k, 2)$-bond is shown in Figure 2.7.

Proposition 13 *A $(k, 2)$-bond has the following properties:*

1. *$B_{k,2}$ has $2k+3$ vertices;*
2. *a $(k, 2)$-bond is k-improper 2-colourable, not k-improper 1-colourable;*

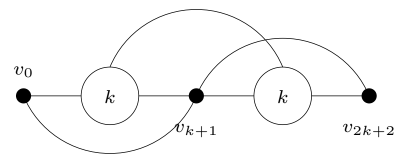


Figure 2.7: The $(k, 2)$-bond $B_{k,2}$.

3. *each k-improper 2-colouring of $B_{k,2}$ assigns the same colour to v_0 and v_{2k+2};*

4. *suppose v_0 is adjacent to $j \in \{0, \dots, k\}$ additional vertices $u_1, \dots, u_j$ and furthermore suppose that $v_0, u_1, \dots, u_j$ are precoloured with the same colour: then v_{2k+2} has impropriety at least j in any k-improper 2-colouring of $B_{k,2}$;*

5. *under the same conditions as the previous property, there exists a k-improper 2-colouring of $B_{k,2}$ such that v_{2k+2} has impropriety j; and*

6. *a $(k, 2)$-bond is a unit disk graph.*

Note that, in the case of the third and fourth properties, we say that v_0 is coloured with external impropriety j.

Proof. The first two properties are easy to establish. For the remainder of the proof we will assume that c is a k-improper 2-colouring of $B_{k,2}$.

To prove the third property, suppose that $c(v_0) = 1$ and $c(v_{2k+2}) = 2$. Remark that, as $\{v_1, \dots, v_{2k+1}\}$ induces a $(2k+1)$-clique, then one colour, say 2, must appear exactly $k+1$ times. Hence, any such vertex coloured 2 has impropriety k in the clique, and so cannot be a neighbour of v_{2k+2}. However, among $v_1, \dots, v_{2k+1}$, there are only k non-neighbours of v_{2k+2}. This is a contradiction.

To prove the fourth property, suppose that $c(v_0) = c(u_1) = \cdots = c(u_j) = 1$. Since v_0 has impropriety j, colour 1 appears at most $k - j$ times among $v_1, \dots, v_{k+1}$. As $v_1, \dots, v_{2k+1}$ is a $(2k+1)$-clique, there are at least k vertices of colour 1. We deduce that there are at least j vertices among $\{v_{k+1}, \dots, v_{2k+1}\}$ coloured 1. Since $c(v_{2k+2}) = 1$ by the above, v_{2k+2} has impropriety at least j.

For the fifth property, again suppose that $c(v_0) = c(u_1) = \cdots = c(u_j) = 1$. Set $c(v_1) = c(v_2) = \cdots = c(v_{k-j}) = 1$. Set $c(v_{k-j+1}) = c(v_{k-j+2}) = \cdots = c(v_{2k-j+1}) = 2$. Set $c(v_{2k-j+2}) = c(v_{2k-j+3}) = \cdots = c(v_{2k+2}) = 1$. It is routine to check that this colouring satisfies our requirement.

We will describe the embedding of $B_{k,2}$ in the next section. □

Definition 6 *A $(k,2)$-wire of order m, denoted $W^m_{k,2}$, is the left-to-right concatenation of m $(k,2)$-bonds $B_1,\ldots,B_m$. The extreme vertices, v_0 of B_1 and v_{2k+2} of B_m, are called output vertices.*

A $(k,2)$-wire of order 3 is shown in Figure 2.8. The following properties follow from Proposition 13.

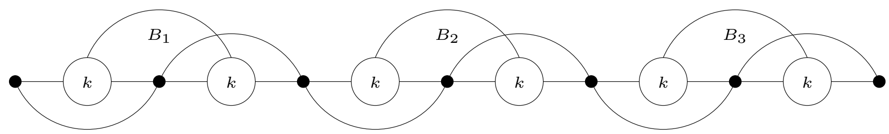


Figure 2.8: The $(k,2)$-wire of order 3 $W^3_{k,2}$.

Proposition 14 *A $(k,2)$-wire of order m has the following properties:*

1. *$W^m_{k,2}$ has $m(2k+2)+1$ vertices;*
2. *a $(k,2)$-wire is k-improper 2-colourable, not k-improper 1-colourable;*
3. *each k-improper 2-colouring of $W^m_{k,2}$ assigns the same colour to the output vertices;*
4. *if an output vertex v of B_i has external impropriety $j \in \{0,\ldots,k\}$, then, in any k-improper 2-colouring of $W^m_{k,2}$, the other output vertex of B_i has impropriety at least j;*
5. *if an output vertex v of B_i has external impropriety $j \in \{0,\ldots,k\}$, there exists a k-improper 2-colouring of $W^m_{k,2}$ such that the other output vertex of B_i has impropriety j; and*
6. *a $(k,2)$-wire is a unit disk graph.*

Definition 7 *A $(k,2)$-clone of size $m \geq 2$, denoted $C^m_{k,2}$, consists of m output vertices $o_1,\ldots,o_m$, such that there is an $(k,2)$-wire W_i between o_i and o_{i+1} for each $1 \leq i < m$.*

A $(k,2)$-clone of size 3 is shown in Figure 2.9. Note that we have defined the $(k,2)$-clone to have arbitrarily order, but we will apply $(k,2)$-clones of bounded order to our embedding.

Proposition 15 *A $(k,2)$-clone has the following properties:*

1. *$C^m_{k,2}$ has $l(2k+2)+1$ vertices, for some $l \geq m$;*
2. *a $(k,2)$-clone is k-improper 2-colourable, not k-improper 1-colourable;*

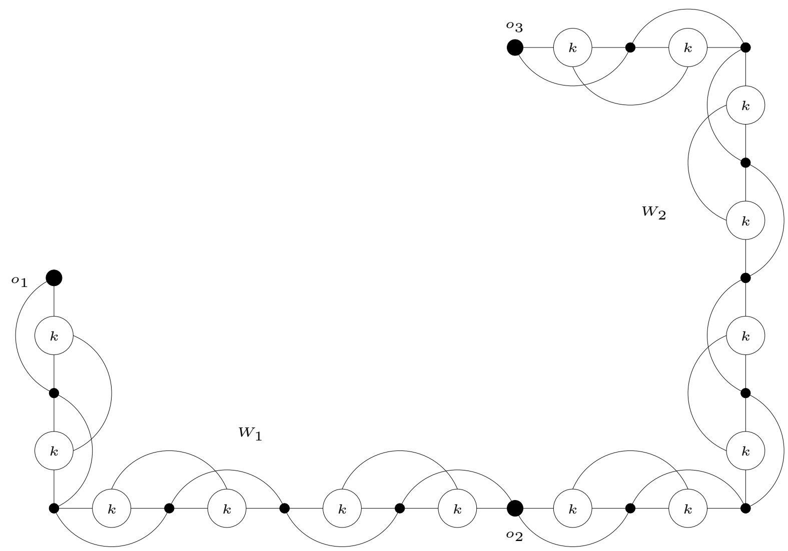


Figure 2.9: A $(2, 2)$-clone of size 3 $C^3_{2,2}$.

3. *each k-improper 2-colouring of $C^m_{k,2}$ assigns the same colour to all output vertices;*

4. *in any k-improper 2-colouring of $C^m_{k,2}$, the sum of external improprieties of the output vertices (cf. the remark following Proposition 13) is at most k;*

5. *given a sequence $s_1, \ldots, s_m$ of non-negative integers whose sum is at most k, there is a k-improper 2 colouring of $C^m_{k,2}$ such that the external impropriety of o_i is s_i, $1 \leq i \leq m$; and*

6. *a $(k, 2)$-clone is a unit disk graph.*

Proof. For the fifth property, we colour the vertices of $C^m_{k,2}$ starting at o_1. Suppose $c(o_1) = 1$. By Proposition 14(5), since o_1 and o_2 are output vertices of W_1, there exists a k-improper 2-colouring of W_1 such that o_2 has impropriety s_1. Now, $c(o_2) = 1$ and, if we set the external impropriety of o_2 in W_2 to $s_1 + s_2$, we will have external impropriety s_2 for o_2 in $C^m_{k,2}$. By Proposition 14(5), there exists a k-improper 2-colouring of W_2 such that o_3 has impropriety $s_1 + s_2$. We can carry on like this until we have coloured all of $C^m_{k,2}$, since $s_1 + s_2 + \cdots + s_m \leq k$.

The other properties use similar applications of Proposition 13. □

Definition 8 *For any odd positive integer m, a $(k, 2)$-link of order m, de-*

noted $K^m_{2,k}$, is defined as follows. The vertex set is $\{v_0, \ldots, v_{x(k,m)+1}\}$, where

$$x(k,m) = \begin{cases} mk(k+1) & \text{if } k \text{ is even} \\ mk(k+1)+k+1 & \text{if } k \text{ is odd} \end{cases}$$

For the edge set, we join v_i and v_j if and only if $|i-j| \leq k+1$. The vertices v_0 and $v_{x(k,m)+1}$ are called output vertices.

A $(2,2)$-link of order 1 is shown in Figure 2.10.

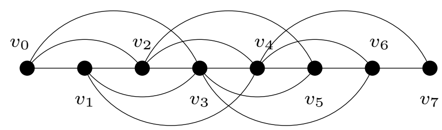


Figure 2.10: The $(2,2)$-link of order 1 $K^1_{2,2}$.

Proposition 16 *A $(k,2)$-link has the following properties:*

1. *$K_{k,2}$ has $x(k,m)+2$ vertices;*
2. *a $(k,2)$-link is k-improper 2-colourable, not k-improper 1-colourable;*
3. *for any k-improper 2-colouring of $K_{k,2}$ in which the output vertices receive the same colour, the output vertices both have non-zero improprieties;*
4. *there exists a k-improper 2-colouring of $K_{k,2}$ such that the output vertices receive different colours and both vertices have impropriety zero;*
5. *there exists a k-improper 2-colouring of $K_{k,2}$ such that the output vertices receive the same colour and both vertices have impropriety one; and*
6. *a $(k,2)$-link is a unit disk graph.*

Proof. The first two properties are easy to establish.

For the third property, suppose c is a k-improper 2-colouring of $K_{k,2}$ such that both output vertices are coloured 1. By symmetry, suppose that v_0 has impropriety 0. Then we must have $c(v_i) = 2$ for all $i \in \{1, \ldots, k+1\}$. In particular, note that $\text{im}_{\{v_1,\ldots,v_k\}}(v_{k+1}) = k$ so any vertex v_i with $i \in \{k+2, \ldots, 2k+2\}$ must be coloured 1. More generally, we see that the only possibility is that $c(v_i) = 1$ if and only if $(m-1)(k+1)+1 \leq i \leq m(k+1)$ for m an even integer. However, since $\frac{x(k,m)}{k+1}$ is even, the $k+1$ vertices with indices between $(\frac{x(k,m)}{k+1} - 1)(k+1)+1$ and $x(k,m)$ are coloured $1 =$

$c(v_{x(k,m)+1})$. Since these $k+1$ vertices are adjacent to $v_{x(k,m)+1}$ we have a contradiction.

For the fourth property, we use the above forced colouring. In other words, set $c(v_0) = 1$, $c(v_{x(k,m)+1}) = 2$ and for $1 \leq i \leq x(k,m)$, set $c(v_i) = 1$ if and only if $(m-1)(k+1)+1 \leq i \leq m(k+1)$ for m an even integer. It is simple to check that the output vertices have impropriety zero.

For the fifth property, we use the following colouring. Set $c(v_0) = c(v_{x(k,m)+1}) = 1$. For any $1 \leq i \leq x(k,m)$, set $c(v_i) = 1$ if and only if the index i is between $(m-1)k+1$ and mk for m an even integer. Clearly, under this colouring, v_0 is adjacent to exactly one vertex with colour 1, namely, v_{k+1}. For the impropriety of $v_{x(k,m)+1}$, we have to check the parity cases for k. If k is even, then $\frac{x(k,m)}{k}$ is odd and the only neighbour of $v_{x(k,m)+1}$ with colour 1 is $v_{x(k,m)-k}$; if k is odd, then $\frac{x(k,m)-1}{k}$ is odd and the only neighbour of $v_{x(k,m)+1}$ with colour 1 is $v_{x(k,m)}$. In either case, $v_{x(k,m)+1}$ has impropriety one.

We will describe the embedding of $K_{k,2}$ in the next section. □

Definition 9 *A $(k,2)$-chain of order (m,n), denoted $K_{k,2}^{(m,n)}$, consists of the concatenation of a $(k,2)$-wire of order j $(B_1B_2 \cdots B_j)$ with a single $(k,2)$-link of order n (K_1) then with another $(k,2)$-wire of order $m-j$ $(B_{j+1}B_{j+2} \cdots B_m)$ for some $1 < j < m$. The extreme vertices, v_0 of B_1 and v_{2k+2} of B_m, are called output vertices.*

A $(2,2)$-chain of order $(2,1)$ is shown in Figure 2.11. The following properties follow from Propositions 13 and 16

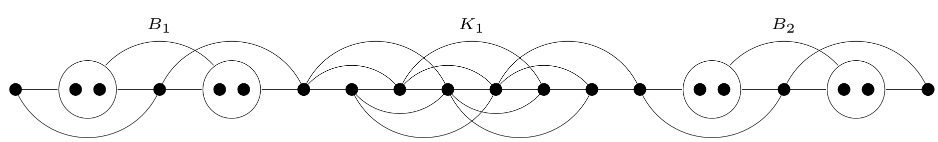


Figure 2.11: The $(2,2)$-chain of order $(2,1)$ $K_{2,2}^{(2,1)}$.

Proposition 17 *A $(k,2)$-chain of order (m,n) has the following properties:*

1. *$K_{k,2}^{(m,n)}$ has $m(2k+2)+x(k,m)+2$ vertices;*
2. *a $(k,2)$-chain is k-improper 2-colourable, not k-improper 1-colourable;*
3. *for any k-improper 2-colouring of $K_{k,2}^{(m,n)}$ in which the output vertices receive the same colour, the output vertices both have non-zero improprieties;*

4. *there exists a k-improper 2-colouring of $K_{k,2}^{(m,n)}$ such that the output vertices receive different colours and both vertices have impropriety zero;*

5. *there exists a k-improper 2-colouring of $K_{k,2}^{(m,n)}$ such that the output vertices receive the same colour and both vertices have impropriety one; and*

6. *a $(k, 2)$-chain is a unit disk graph.*

Now, we introduce the graphs that will replace the high-degree vertices in G.

2.3.2 Embedding of the unit disk graph

Given any graph G, we will now show how to compute a distance model with distance value 1 of a unit disk graph $\hat{G}$ which is k-improper 2-colourable if and only if G is k-improper 2-colourable. First, we embed G in the plane in a suitable way. Then we construct $\hat{G}$ so that the vertices and edges of the original graph are replaced by the auxiliary graphs described above. Because of the definition of the $(k, 2)$-chain, there are naturally two different classes of unit disk embeddings depending on the parity of k. We will only fully describe the case of even k since the other case is similar.

In [8], the authors use an orthogonal embedding of G, i.e. a planar embedding of G such that each edge corresponds to an arc made up of horizontal and vertical line segments. In [18], each edge corresponds to an arc made up of horizontal and vertical line segments in the embedding of G; however, crossing edges are permitted and, to take account of high-degree vertices, each vertex is represented by a (possibly degenerate) line segment. Here, we will use what is called a box-orthogonal embedding. A box-orthogonal embedding of G is a planar embedding of G such that each edge is represented by alternate horizontal and vertical line segments and each vertex is represented by a (possibly degenerate) rectangle, called a box (See Figure 2.12). We assume that all line segments, including those at the perimeter of a box, lie on lines of the integer grid. There is a box-orthogonal embedding for each planar graph and one can be computed in polynomial time [14, 41].

Let $G = (V, E)$ be any planar graph. We generate a box-orthogonal embedding of G. Let us assume that no two edges meet at a point, i.e. no box is degenerate and no two edges meet at the corner of a box. (We can do this by expanding each box by distance 1/2 in each of the four directions then doubling the scale of the grid).

Each vertex $v \in V$ is replaced by a box $Box(v)$, and we denote the $\deg(v)$ points of contact with edges by $M(v)$. We aim to embed a $(k, 2)$-clone in the perimeter of $Box(v)$ such that its output vertices replace the vertices in $M(v)$. We can do this by starting at an arbitrary point of $M(v)$ and proceed

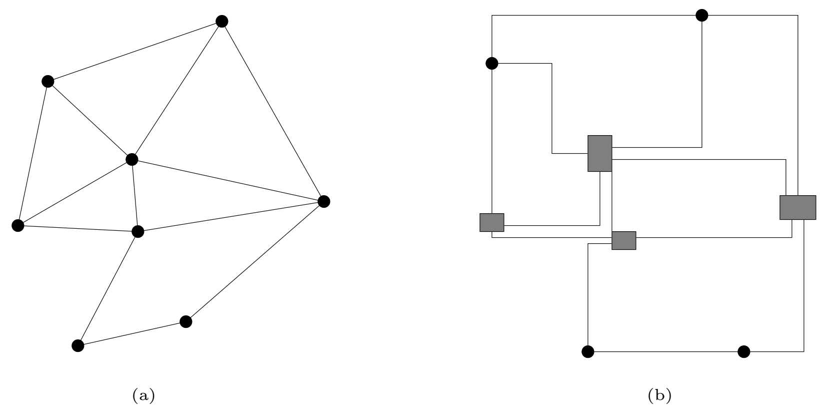


Figure 2.12: (a) An arbitrary planar graph G and (b) a box-orthogonal embedding of G.

in clockwise direction about the perimeter. We extend the $(k,2)$-clone with an embedding of a $(k,2)$-wire to the next grid point in the perimeter and continue until all members of $M(v)$ have been included. It only remains to describe the unit disk embedding of some $(k,2)$-wire between two adjacent grid points.

Each edge $e \in E$ is replaced by a line $A(e)$ consisting of alternate horizontal and vertical line segments of the grid. Clearly, $A(e)$ has integer grid length. We aim to embed a $(k,2)$-chain along $A(e)$. Since we use $(k,2)$-wires to extend a $(k,2)$-chain to arbitrary length, it suffices to describe the unit disk embedding of some $(k,2)$-chain between two adjacent grid points.

We first describe unit disk embeddings for the elementary auxiliary graphs: the $(k,2)$-bonds and $(k,2)$-links.

We will denote the embedding of a $(k,2)$-link of order m by E_K^m. Each centre of the disk replacing a vertex of $K_{k,2}^m$ lies on a line. The points are distributed equidistant from each other. Let the distance between adjacent vertices v_i and v_{i+1} be $d = \frac{mk}{mk(k+1)+1}$. Since $(k+2)^{-1} \leq d < (k+1)^{-1}$, v_i is adjacent to v_j if and only if $|i-j| \leq k+1$. Also, we can easily check that the distance in E_K^m between output vertices is precisely mk. See Figure 2.13.

We will use two different embeddings for the $(k,2)$-bonds. In the first embedding, we will denote it E_B^a, the disks for the output vertices of $B_{k,2}$ are touching (but not intersecting), hence, the distance between the output vertices is 1. The first embedding is illustrated in Figure 2.14(a). Note that the two bold disks represent cliques of size k. In the second embedding, we will denote it E_B^b, all of the disks lie on a line. The output vertices of $B_{k,2}$ are at distance $2-2d'$, where $d' = \frac{1}{k+3}$ and the central vertex v_{k+1} of $B_{k,2}$ is midway between them. The centres of the two k-clique disks are at distance $1 - \frac{d+d'}{2}$ from the nearer respective output vertices. See Figure 2.14(b).

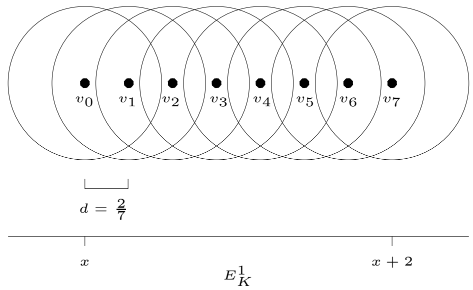


Figure 2.13: An embedding of the $(2, 2)$-link of order 1.

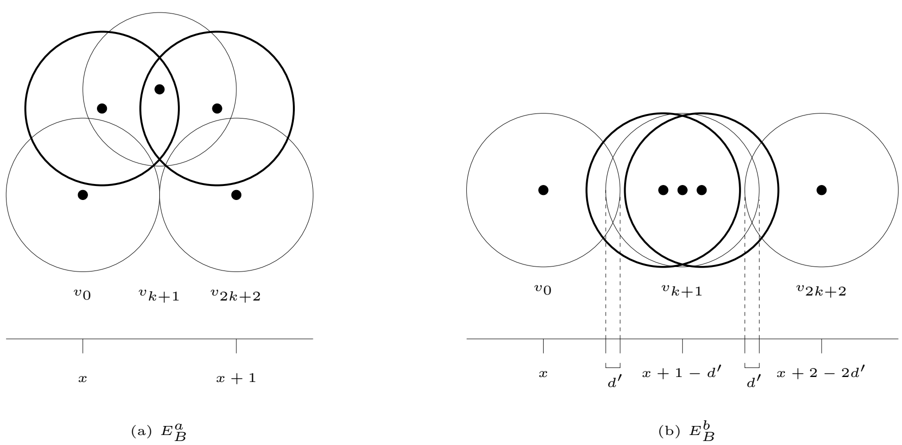
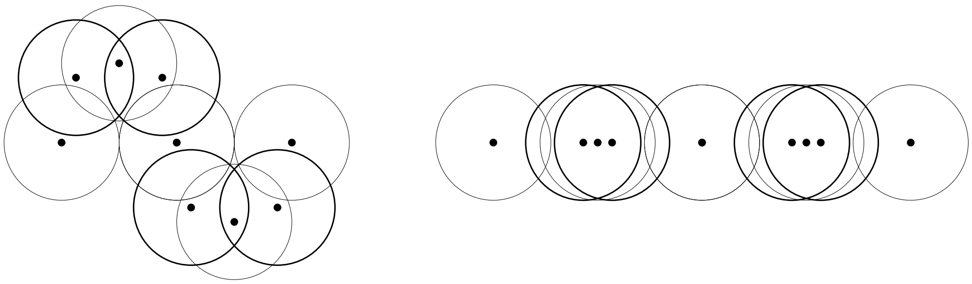


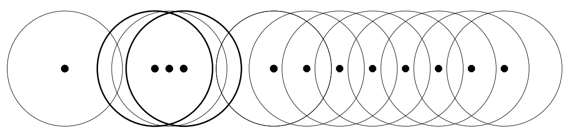
Figure 2.14: Two embeddings of a $(k, 2)$-bond: (a) E_B^a and (b) E_B^b.

Clearly, E_B^a can be concatenated with itself, as can E_B^b. See Figure 2.15. Also, E_B^b can be concatenated with E_K^m and with E_B^a. See Figure 2.16.

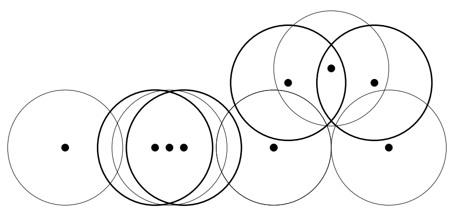


(a) $E_B^a E_B^a$ (b) $E_B^b E_B^b$

Figure 2.15: (a) The concatenation of two copies of E_B^a and (b) of two copies of E_B^b.



(a) $E_B^b E_K^1$



(b) $E_B^b E_B^a$

Figure 2.16: (a) The concatenation of E_B^b with E_K^1 and (b) of E_B^b with E_B^a.

We will use these constructions to show that there are embeddings of some $(k, 2)$-wire and of some $(k, 2)$-chain between two adjacent grid points. We first scale the grid so that two adjacent grid points are distance $u = 5k+8$ apart. We embed a $(k, 2)$-wire W^* of order $3k + 6$ by concatenating $k + 3$ copies of E_B^b with k copies of E_B^a with $k + 3$ more copies of E_B^b. This embedding has length $(k + 3)(2 - 2d') + k + (k + 3)(2 - 2d') = 5k + 8$, as required. We embed a $(k, 2)$-chain K^* of order $2(k+3)+1$ by concatenating $k+3$ copies of E_B^b with E_K^1 with $k+3$ more copies of E_B^b. This embedding has also has length $5k + 8$, as required.

Since, in E_B^b, the distance between an output vertex and any other vertex

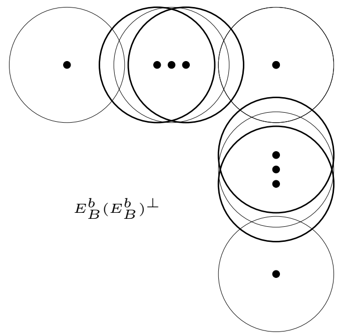


Figure 2.17: The embedding of $W^2_{k,2}$ around a right-angle turn.

is at least $\frac{1}{2}(\frac{2}{3}+\frac{3}{4}) > \frac{1}{\sqrt{2}}$, E^b_B can be concatenated with a perpendicular copy of itself (to bend around corners). See Figure 2.17. Now, for each vertex v, we embed W^* between grid points along the perimeter of $Box(v)$ to obtain a $(k,2)$-clone whose output vertices are precisely $M(v)$. Also, for each edge e we embed W^* between grid points along $A(e)$, except for one pair of grid points between which we embed a K^*, to obtain an embedding of a $(k,2)$-chain along $A(e)$. The resulting graph is $\hat{G}$.

Remark: for the case of odd k, we choose the values $d = \frac{mk+1}{(mk+1)(k+1)+1}$, $d' = \frac{1}{k+3}$ and $u = 5k+9$.

2.3.3 Proof of theorem

Proof of Theorem 5. Let $G = (V,E)$ be any graph. The construction of the corresponding unit disk graph $\hat{G}$ and its embedding can clearly be done in polynomial time. It remains to show that G is k-improper 2-colourable if and only if $\hat{G}$ is. Each vertex $v \in V$ is replaced by a box, and then the points of contact with edges are denoted by $M(v)$. These points are then replaced by the output vertices of a $(k,2)$-clone if $|M(v)| \geq 2$. The set of output vertices is denoted $I(v)$ (where $I(v) = \{v\}$ if $|M(v)| = 1$).

"$\Longrightarrow$": Let c be a k-improper 2-colouring of G. We want to construct a k-improper 2-colouring of $\hat{G}$. First, for any vertex v of G, we colour the vertices of $I(v)$ by $c(v)$.

Second, for any edge $e = xy$ of G, let K_e be the $(k,2)$-chain which connects $I(x)$ to $I(y)$ in $\hat{G}$. If $c(x) \neq c(y)$, then we apply Proposition 17(4) to colour K_e. If $c(x) = c(y)$, then we apply Proposition 17(5).

Last, for any vertex v of G, let C_v be the $(k,2)$-clone whose output vertices are $I(v)$. Since c is a k-improper 2-colouring, we can apply Proposition 15(5) to colour C_v. In this way, we obtain a k-improper 2-colouring of $\hat{G}$.

"$\Longleftarrow$": Let $\hat{c}$ be a k-improper 2-colouring of $\hat{G}$. We want to construct a k-improper 2-colouring of G. By Proposition 15(3), for any vertex v of G, we can assign the colour of the vertices of $I(v)$. By Proposition 15(4) and Proposition 17(3), it is clear that the colouring generated is indeed a k-improper 2-colouring of G. □

Chapter 3

Plan for future work

There is a plethora of promising open problems related to the material in this paper. First, we review some of the established open problems related to sections 1.1 and 1.2 of the introduction. Second, we discuss three major candidates for further work arising from the unit disk improper colourability problem. All of these problems are candidates for future doctoral research. For each problem, the statement is followed by a discussion of background and possible approaches.

3.1 Established open problems

Problem 1 *Is the (unit) disk graph recognition problem in NP?*

As we noted earlier, (unit) disk graph recognition is NP-hard. To show it is in NP, the obvious approach, by guessing a representation then checking it, does not work, since it may be possible that, for some (unit) disk graph, every representation has a disk with irrational coordinates. It was shown in Lemma 8 of [7] that (unit) disk graph recognition is in PSPACE.

Problem 2 *For fixed n, how many unit disk graphs of order n are there?*

The enumeration problem for unit disk graphs has not been well-studied and it would be interesting to obtain good bounds on the number of such graphs.

Problem 3 *Is there a disk graph that is not a containment disk graph?*

We are given an arbitrary set of n points $v_1, \ldots, v_n$ and n positive reals $d_1, \ldots, d_n$. At each point v_i, we centre a disk of diameter d_i. We connect two points if one point is contained in the other's disk. Any graph that is isomorphic to a graph constructed in such a manner is called a *containment disk graph*. We denote the class of containment disk graphs by $\mathcal{CD}$. Note that the term containment here is different from the usual meaning (cf. [38]).

The class of containment disk graphs was introduced in [32]. They showed that $K_{3,3}$ is a containment disk graph but not a disk graph. Thus, $\mathcal{CD} \not\subseteq \mathcal{D}$. The above asks whether the converse is true. It is possible that the following class of graphs may provide a solution to Problem 3. Let $T_1 = K_3$. We obtain T_{k+1} from T_k as follows. For each face of T_k other than the unbounded face, place a vertex at the centre and connect it to all vertices on the boundary of the face. Clearly, since it is planar, T_k is a disk graph for each k.

Conjecture 1 *For some k, T_k is not a containment disk graph.*

Problem 4 *Is DISK CLIQUE NP-hard?*

In [6], the authors introduce the class of bounded-ratio disk graphs. A graph is an *r-bounded disk graph* if it is the intersection graph of a set of disks whose largest disk has radius at most r times the radius of its smallest disk. We denote the class of such graphs by $r\mathcal{BD}$. Clearly,

$$\mathcal{UD} \subseteq r\mathcal{BD} \subseteq \mathcal{D} \subseteq \{\text{general graphs}\}$$

for each r. At one end of this continuum, the problem of finding a maximum clique is in P while, at the other end, the problem is NP-complete. We wish to determine where the change in complexity is along this continuum.

Problem 5 *What is the tight upper bound on the ratio $\chi(G)/\omega(G)$ for unit disk graphs?*

As we mentioned in section 1.1.3, the tight upper bound on χ/ω for unit disk graphs is between 3/2 and 3. The work of McDiarmid [35] on random unit disk colouring suggests that this value should be closer to 3/2. There are similar gaps for disk graphs and double disk graphs (consult [32]).

Problem 6 *Is every planar graph 1-improper 4-choosable?*

In the notation of section 1.2.2, this would answer the question of whether p_1^* is equal to 4 or 5.

3.2 Open problems related to unit disk improper colourability

Since this area of study has not been visited before, there are many new problems to consider. Faced with the establishment of NP-completeness for nearly all cases, there are three natural questions to consider.

The first natural question is to consider bounds and approximation. Analagous to Problem 5,

Problem 7 *Is there a good bound on the ratio* $\frac{(k+1)\chi^k(G)}{\omega(G)}$ *for unit disk graphs?*

The remark in section 1.1.3 that $\Delta(G) \leq 6\omega - 6$ for any unit disk graph G combined with Corollary 2 gives that this ratio is at most 6; however, we would expect that this ratio would be much closer to 3, if not lower.

A related area of interest is to develop good heuristics for the k-improper chromatic number for any fixed $k \geq 1$. One avenue of investigation is the STRIPE algorithm of [18] which is a 3-approximation for the chromatic number of unit disk graphs. In this algorithm, the unit disk representation is divided into strips of small enough width so that they can be coloured efficiently. They prove that strips of width at most $\sqrt{3}/2$ induce cocomparability graphs which, being perfect graphs, can be coloured efficiently with $\omega(\{\text{strip graph}\})$ colours. If, for the improper colouring problem for cocomparability graphs, there is an efficient algorithm or good approximation, this approach could produce a better ratio than 6.

The second natural question is to restrict the UD improper colourability problem and investigate whether the problem remains NP-hard. For (unit) interval graphs, we showed that the problem is in P when we fix both k and l; however, it is open to determine if, for fixed k, finding the k-improper chromatic number of a (unit) interval graph is polynomial. We here give two other natural restrictions worth further study.

Problem 8 *Restricted to weighted induced subgraphs of the triangular lattice, is k-IMPROPER l-COLOURABILITY NP-complete?*

For communication networks, we often work with the triangular lattice T, as this is known to give the most efficient cover (cf. [37]). The points of the triangular lattice are integer linear combinations of the vectors $\vec{p} = (1, 0)$ and $\vec{q} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. To produce T, we join any two points at distance 1 apart. Note that T is a unit disk graph.

For any graph G, a *weight assignment* ω is an association of each vertex $v \in V(G)$ with a non-negative weight ω_v. Given a graph G and a weight assignment ω, a *weighted colouring* c_ω of G is an assignment to each vertex $v \in V(G)$ of a multiset $c_\omega(v)$ of size ω_v. Our definition here differs from [37] to generalise to improper colouring: a vertex may be assigned multiple instances of the same colour. Given that the weights ω_v represent replacing v by a clique of size ω_v, there is a natural analogue for weighted k-improper colouring for fixed k.

The authors in [37] show that the (proper) 3-colourability problem restricted to weighted induced subgraphs of the triangular lattice is NP-complete. It is possible to generalise their approach to show that, for fixed k, the corresponding k-improper 3-colourability problem is NP-complete.

Conjecture 2 *For fixed* $(k, l) \geq (1, 2)$ *or* $(k, l) \geq (0, 3)$*, the* k*-improper* l*-colourability problem restricted to weighted induced subgraphs of the triangular lattice is NP-complete.*

Note that it is not yet known whether, for weighted induced subgraphs of the triangular lattice, l-colourability for fixed $l > 3$ is NP-complete.

Problem 9 *What is the complexity of distinct weighted improper colourability?*

In light of the motivating problem given by Alcatel (cf. page 14), it is sensible to prevent receivers at the same site from having the same frequency. We again consider weighted graphs, and, given a graph G and a weight assignment ω, we define a *distinct weighted colouring* to be a weighted colouring such that the colours at each vertex are all distinct. Thus, a distinct weighted k-improper colouring is a weighted k-improper colouring such that predetermined cliques must have distinct colours. We expect this problem to be NP-complete.

The third natural question is to consider the k-improper chromatic number for random unit disk graphs. It already seems likely that extensions of [36] and [35] for improper colourability are feasible.

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