Rational Homotopy Theory

Utrecht University

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PREFACE

These are the notes of a four week crash course that I taught to the master students at Utrecht University in March 2017. I have presented some classical algebraic topology of rational spaces in Chapter 2, which forms a natural continuation of what the students had learnt in their basic algebraic topology course, as well as the equivalence with commutative differential graded algebras over the rationals in Chapter 3. I have tried to make the material self-contained (relative to what the students already knew), and have almost succeeded with the exception of a result based on the Eilenberg-Moore spectral sequence which I had to quote from Bousfield-Gugenheim [1]. I hope to repair this last gap in a next installment of the notes. The supervision of an earlier master's thesis on the same material by Joshua Moerman has turned out to be very helpful in preparing the course.

I would like to thank Peter James and Kay Werndli for turning my handwritten notes into this nice looking text, and for catching several inaccuracies and mistakes of mine in the process of doing so.

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Chapter 1

INTRODUCTION AND OVERVIEW

These are the notes from a mini-course in rational homotopy theory given at Utrecht University in spring 2017 by Ieke Moerdijk. Additional general references for the course are [3], [1], [13], [14]. In the course, a familiarity with some basic concepts in algebraic topology is assumed.

All spaces we consider are assumed to be pointed and simply connected (although the whole theory works a bit more generally, for nilpotent spaces, for example). We usually delete the base point from the notation. Whenever convenient, we restrict ourselves to CW-complexes. Recall that every space is of the weak homotopy type of a CW-complex, and that if the space is pointed and simply connected, the corresponding CW-complex can be taken to have one 0-cell and no 1-cells.

Sometimes, it is more convenient to work with simplicial sets rather than spaces. Recall the definition of a simplicial set, and the adjoint pair

$$|-| : \mathbf{sSets} \rightleftarrows \mathbf{Top} : \mathrm{Sing}$$

given by the geometric realisation and the singular simplicial set functor. A map $X_\bullet \to Y_\bullet$ of simplicial sets is called a *weak equivalence* iff the induced map between geometric realisations $|X_\bullet| \to |Y_\bullet|$ is a homotopy equivalence. For any CW-complex Z, the map $|\mathrm{Sing}(Z)| \to Z$ is a homotopy equivalence. It follows that for any simplicial set $X_\bullet$, the map $X_\bullet \to \mathrm{Sing}|X_\bullet|$ is a weak equivalence. So, every topological space is weakly equivalent to the realisation of a simplicial set, and this simplicial set is unique up to weak equivalence. For simply connected pointed spaces, one can take the corresponding simplicial sets to be 1-reduced, i.e. having exactly one 0-simplex and exactly one (degenerate) 1-simplex.

1. Rational Spaces and Rationalisation

The "homotopy theory of spaces" studies spaces up to weak equivalence. We recall that a weak equivalence of spaces is a continuous map $f\colon X \to Y$ such that the induced map between path components $f_*\colon \pi_0(X) \to \pi_0(Y)$ as well as all induced maps between homotopy groups

$$f_*\colon \pi_n(X,x) \to \pi_n(Y, f(x)) \qquad \text{with } n \geqslant 1$$

are isomorphisms. We can equivalently consider CW-complexes up to homotopy equivalence, simplicial sets up to weak equivalence, or Kan complexes up to homotopy equivalence, but note that CW-approximation takes place from the left (via covers) whereas replacing a simplicial set with a Kan-complex takes place from the right (via embeddings). *Rational homotopy theory* instead studies spaces up to rational equivalence.

(1.1) **Definition.** A continuous map $f\colon X \to Y$ is a *rational equivalence* iff for all $n > 1$, the induced maps

$$f_* \otimes \mathbb{Q}\colon \pi_n(X) \otimes \mathbb{Q} \to \pi_n(Y) \otimes \mathbb{Q}$$

are isomorphisms. Note that, because we only consider simply connected spaces, the conditions for $n = 1$ and $n = 0$ become trivial.

(1.2) **Proposition (Basic Fact 1).** A continuous map $f\colon X \to Y$ is a rational equivalence iff it induces isomorphisms in rational homology

$$H_i(X;\mathbb{Q}) \to H_i(Y;\mathbb{Q})$$

for all $i > 1$.

(1.3) **Definition.** A space X (where we recall that all spaces in this course are assumed to be pointed and simply connected) is *rational* if $\pi_n(X)$ is a $\mathbb{Q}$-vector space for all $n > 1$. A *rationalisation* of a space X is a rational equivalence $X \to \widehat{X}$, where $\widehat{X}$ is rational. We will later show that for a CW-complex X, such a rationalisation exists and is unique up to homotopy equivalence.

(1.4) **Remark.** To be a little more precise, every $\pi_n(X)$ is an abelian group and for one such to be a $\mathbb{Q}$-vector space just means that all multiplication maps

$$\mu_t\colon \pi_n(X) \to \pi_n(X),\ x \mapsto t \cdot x$$

with $t \in \mathbb{Z}^\times$ a non-zero integer are invertible.

(1.5) **Proposition (Basic Fact 2).** A space X is rational iff every integral homology group $H_i(X;\mathbb{Z})$ with $i > 1$ is a $\mathbb{Q}$-vector space.

(1.6) **Remark.** A rational equivalence between rational spaces is of course the same thing as a weak equivalence, or as a homotopy equivalence in case the spaces involved are CW-complexes.

(1.7) **Example (Rationalisation of Spheres).** We begin by describing the rationalisation of the circle S^1. Note that this is not a simply connected space and so this does not technically constitute an example in this course. Recall that $\pi_1(S^1) \cong \mathbb{Z}$, and that S^1 has no other interesting homotopy. Recall that

$$\operatorname{colim}\left(\mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \cdots\right) \cong \mathbb{Z}[\tfrac{1}{p}],$$

where each of the maps in the colimit is given by multiplication by p. Similarly,

$$\operatorname{colim}\left(\mathbb{Z} \xrightarrow{p_0} \mathbb{Z} \xrightarrow{p_1} \mathbb{Z} \xrightarrow{p_2} \ldots\right) \cong \mathbb{Q},$$

where the i^{th} map of the colimit is given by multiplication by p_i, where $(p_i)_{i\in\mathbb{N}}$ is a sequence of primes, where each prime occurs infinitely many times.

We mimic these constructions topologically: recall that any map $f\colon X \to Y$ can be turned into a cofibration by using the mapping cylinder $M_f = (X \times [0,1]) \cup_{X\times\{1\}} Y$, which fits into a diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook, "i_f"'] & & Y \arrow[dl, bend right, hook, "s_f"'] \\ & M_f \arrow[ur, "r_f"'] & \end{tikzcd}$$

where $r_f \circ i_f = f$, $r_f \circ s_f = \mathrm{id}_Y$ and $s_f \circ r_f \simeq \mathrm{id}_{M_f}$ (so that Y is a deformation retract of M_f).

For a sequence of spaces (CW-complexes)

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \dots,$$

one can us this construction to compute its so-called *homotopy colimit* by successively replacing the maps by cofibrations and then taking the usual colimit. So, defining recursively $M_0 := X_0$, $r_0 := \mathrm{id}_{X_0}$ and $M_{n+1} := M_{f_{n+1}\circ r_n}$, $r_{n+1} := r_{f_{n+1}\circ r_n}$, we get a replacement sequence

$$\begin{tikzcd}
X_0 \arrow[r, "f_1"] \arrow[d, equal] & X_1 \arrow[r, "f_2"] \arrow[d, hook, bend right, "s_1"'] & X_2 \arrow[r, "f_3"] \arrow[d, hook, bend right, "s_2"'] & \dots \\
M_0 \arrow[r, hook, "i_1"'] & M_1 \arrow[r, hook, "i_2"'] \arrow[u, two heads, "r_1"'] & M_2 \arrow[r, hook, "i_3"'] \arrow[u, two heads, "r_2"'] & \dots \, .
\end{tikzcd}$$

Then $M_\infty := \operatorname{hocolim} X_n := \operatorname{colim} M_n$ and $\pi_i(M_\infty) = \operatorname{colim} \pi_i(X_n)$. In the case of the sequence

$$S^1 \to S^1 \to S^1 \to \cdots,$$

where the i^{th} map is a degree p_i map, with the p_i as before, we find that

$$\pi_1(M_\infty) = \operatorname{colim} \pi_1(S^1) = \mathbb{Q}.$$

Therefore, the map $S^1 \to M_\infty$ is a rationalisation (up to our earlier comment that S^1 is not in fact simply connected).

More generally, for the n-sphere S^n where $n > 1$, we can do exactly the same. Recall that $\pi_i(S^n) = 0$ for $0 < i < n$, $\pi_i(S^n) = \mathbb{Z}$ for $i = n$, and that the higher homotopy groups of S^n are complicated (and impossible to describe in general). To see that $S^n \to M_\infty$, with M_∞ defined via the analagous sequence

$$S^n \to S^n \to S^n \to \cdots,$$

is again a rationalisation, we can now use the homological criterion, and the fact that $\widetilde{H}_i(S^n) \cong \mathbb{Z}$ for $i = n$ and 0 otherwise. Basic facts 1 and 2 then give us the result.

(1.8) **Exercise.** Prove that the two colimits of the form $\operatorname{colim}(\mathbb{Z} \to \mathbb{Z} \to \dots)$ above are indeed what we claim them to be.

(1.9) **Exercise.** Prove that if X is a rational space, any map $S^n \to X$ extends to a map $M_\infty \to X$, and prove that this extension is unique up to homotopy.

(1.10) **Remark.** The rationalisation of arbitrary CW-complexes also heavily depends on basic facts 1 and 2.

2. CDGAs over $\mathbb{Q}$

Our goal will be to compare the homotopy theory of rational spaces with that of commutative differential graded algebras (cdgas) over $\mathbb{Q}$. First off, let us fix the nomenclature for the more and more highly structured graded objects we are going to encounter.

(2.1) **Definition.** As for the entire section, all the following is over $\mathbb{Q}$.

(a) A *graded vector space* is simply a vector space A, together with a direct sum decomposition $A = \bigoplus_{n\in\mathbb{N}} A^n$. The elements of the $A^n \subseteq A$ are called *homogeneous* or *pure* of degree $\deg a := |a| := n$. Note that a general element in A is a finite sum of pure elements $a_0 + \ldots + a_n$ with $a_k \in A^{i_k}$ for some $i_k \in \mathbb{N}$. To stress that we have a graded object, we sometimes write A^* instead of just A.

(b) A *(non-negatively graded) cochain complex* is such a graded vector space A together with a linear differential $d\colon A \to A$ of degree 1 (i.e. $d(A^n) \subseteq A^{n+1}$ for all $n \in \mathbb{N}$) such that $d \circ d = 0$.

(c) A *graded algebra* is a graded vector space A together with a bilinear, associative multiplication $\mu\colon A \times A \to A$, $(a, b) \mapsto ab$ such that $\mu(A^m, A^n) \subseteq A^{m+n}$ and for which there is a unit (necessarily in A_0).

(d) Such a graded algebra is *(graded) commutative* iff

$$ab = (-1)^{|a||b|} ba \qquad \text{for all homogeneous } a,\, b \in A.$$

(e) A *differential graded algebra* (or *dga*) A is a graded vector space carrying both the structure of a grade algebra and a cochain complex that are compatible in the sense that d is a *derivation*, meaning

$$d(ab) = (da)b + (-1)^{|a|} a(db) \qquad \text{for all homogeneous } a,\, b \in A.$$

(f) Finally, a *(graded) commutative differential graded algebra* (or a *cdga*) is simply a dga that is also commutative (as a graded algebra).

As one might expect, a morphism $\varphi\colon A \to B$ between such objects are just graded linear maps (i.e. $\varphi(A_n) \subseteq B_n$) preserving all the structure present. In this way, we have categories $\mathrm{gr}\,\mathbf{Vect}$, $\mathbf{Ch}^{\geqslant 0}$, $\mathrm{gr}\,\mathbf{Alg}$, $\mathrm{cgr}\,\mathbf{Alg}$, $\mathbf{dga}$ and $\mathbf{cdga}$ of graded vector spaces, cochain complexes, graded algebras, commutative graded algebras, dgas and cdgas, respectively.

(2.2) **Example.** If M is a smooth manifold, then the differential forms $\Omega^*(M)$, together with the usual differential d and the wedge product $\wedge$ form a cdga.

(2.3) **Examples.** As with all algebraic structures, we have free constructions:

$$\begin{tikzcd} \mathrm{gr}\,\mathbf{Vect} \arrow[r, shift left, "T"] \arrow[r, phantom, "\perp"] & \mathrm{gr}\,\mathbf{Alg} \arrow[l, shift left] \arrow[r, shift left, "(-)_{\mathrm{ab}}"] \arrow[r, phantom, "\perp"] & \mathrm{cgr}\,\mathbf{Alg} \arrow[l, shift left] \end{tikzcd} \qquad \text{and}$$

$$\begin{tikzcd} \mathrm{gr}\,\mathbf{Vect} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{Ch}^{\geqslant 0} \arrow[l, shift left] \arrow[r, shift left, "T"] \arrow[r, phantom, "\perp"] & \mathbf{dga} \arrow[l, shift left] \arrow[r, shift left, "(-)_{\mathrm{ab}}"] \arrow[r, phantom, "\perp"] & \mathbf{cdga} \arrow[l, shift left] \end{tikzcd} ,$$

where all the right adjoint functors are forgetful ones. The free functors are given as follows:

(a) For a graded vector space A^*, we put $F^n A := A^n \oplus A^{n-1}$ with $A^{-1} := 0$ and the differential on $F^* A$ given by $d(a, b) := (0, a)$.

(b) For two graded vector spaces or cochain complexes A^*, B^*, recall that their tensor product is given by

$$(A \otimes B)^n := \bigoplus_{i+j=n} A^i \otimes B^j, \qquad \text{with differential (in the cochain case)}$$

$$d(a \otimes b) := (da) \otimes b + (-1)^{|a|} a \otimes (db) \qquad \text{for homogeneous } a,\, b.$$

With this, we define T^*A to be the *tensor algebra*

$$TA := \bigoplus_{n \in \mathbb{N}} T^n A, \qquad \text{where } T^0 A := \mathbb{Q} \text{ and } T^{n+1}A := T^n A \otimes A.$$

Note that the degree of $v = v_1 \otimes \ldots \otimes v_n \in T^n A$ is not n but $|v| := \sum_{i=1}^n |v_i|$. In the cochain case, the differential is defined recursively from the above formula, yielding

$$d(v_1 \otimes \ldots \otimes v_n) = \sum_{i=1}^{n} (-1)^{|v_1| + \ldots + |v_{i-1}|} v_1 \otimes \ldots \otimes v_{i-1} \otimes dv_i \otimes v_{i+1} \otimes \ldots \otimes v_n$$

for $n \in \mathbb{N}$ (in particular $d(\lambda) = 0$ for all $\lambda \in T^0 A = \mathbb{Q}$).

(c) Finally, the functor $(-)_{\mathrm{ab}}$ is the *(graded) abelianisation functor* given by

$$A_{\mathrm{ab}} := A/\mathfrak{a}, \quad \text{where} \quad \mathfrak{a} := \left\langle ab - (-1)^{|a||b|} ba \;\middle|\; a,\, b \text{ homogeneous} \right\rangle.$$

This quotient inherits a grading from A because we only identify elements of the same degree ($\mathfrak{a}$ is a *homogeneous ideal*). It also inherits a multiplication and a differential in the obvious way

$$[a][b] := [ab] \qquad \text{and} \quad d[a] := [da].$$

If A is a graded vector space or a cochain complex, we usually write $\Lambda(A) := (TA)_{\mathrm{ab}}$, which is called the *exterior algebra* on A. In analogy to differential forms, the equivalence class of $v_1 \otimes \ldots \otimes v_n \in T^n A$ in $\Lambda^n(A)$ is usually denoted by $v_1 \wedge \ldots \wedge v_n$.

(2.4) **Exercise.** Check that all these functors are well-defined and left adjoint to the corresponding forgetful functors. In particular check that T and $(-)_{\mathrm{ab}}$ (and thus Λ) are compatible with differentials.

(2.5) **Definition.** A morphism of dgas is called a *quasi-isomorphism* iff it induces isomorphisms in cohomology.

We will consider the homotopy category of these cdgas, which is to say, we will study cdgas up to quasi-isomorphism. One way to define the homotopy category of cdgas is to formally invert all of the quasi-isomorphisms, i.e.

$$\mathrm{Ho}(\mathbf{cdga}) = \mathbf{cdga}[q.i.^{-1}]$$

However, this category is difficult to handle: in principle, morphisms are arbitrarily long zig-zags of morphisms of cdgas. Instead, we want a subclass of objects, which will be called *minimal models*, such that every cdga is quasi-isomorphic to a minimial model, and any quasi-isomorphism between minimal models has an inverse up to (chain) homotopy. This will give an alternative description of the homotopy category. This approach is due to Sullivan; there is an alternative approach due to Quillen, involving model categories, but as there are finiteness concerns, we prefer the former approach in this short course.

3. Polynomial Forms

A particularly relevant example for us is the cdga $A^*(\Delta^n)$ of polynomial differential forms on the n-simplex

$$\Delta^n := \{(t_0, \dots, t_n) \mid t_i \in \mathbb{R}, t_i \geqslant 0, \Sigma t_i = 1\}.$$

Its elements of degree q are of the form $p(t_0, \dots, t_n)dt_{i_1} \cdots dt_{i_q}$, where p is a polynomial with rational coefficients, and one calculates with these using the relation $t_0 + \cdots + t_n = 1$ (and hence $dt_0 + \cdots + dt_n = 0$), together with the usual relations for calculating with differential forms. In other words,

$$A^*(\Delta^n) = \Lambda(t_0, \dots, t_n, dt_0, \dots, dt_n)/(\Sigma t_i = 1, \Sigma dt_i = 0),$$

with $\deg(t_i) = 0$. Notice that by writing $t_0 = 1 - (t_1 + \cdots + t_n)$, we also have that $A^*(\Delta^n)$ is the free cdga on the vector space with basis $t_1, \dots, t_n$, viewed as a graded vector space concentrated in degree 0.

Recall that the category $\mathbf{\Delta}$ has objects the finite linear orders $[n] := \{0, \dots, n\}$ and morphisms the order-preserving maps. In any category $\mathcal{C}$, a *simplicial object* is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ and a *cosimplicial object* is a functor $\mathbf{\Delta} \to \mathcal{C}$. The Δ^n assemble into a cosimplicial space, since each map α of $\mathbf{\Delta}$ gives rise to a map $\Delta^n \to \Delta^m$ by its action on the vertices. We can then pull back forms to get a map $\alpha^* : A^p(\Delta^m) \to A^p(\Delta^n)$. This makes each $A^p(\Delta^\bullet)$ into a simplicial $\mathbb{Q}$-vector space, and hence $A^*(\Delta^\bullet)$ into a simplicial cdga. For an arbitrary simplicial set X, we obtain

$$A^p(X) := \mathbf{sSets}(X, A^p(\Delta^\bullet))$$

for each p, and these assemble into the *cdga of polynomial forms on X*. Similarly, for an arbitrary cdga B, then we may define

$$(KB)_n := \mathbf{cdga}(B, A^*(\Delta^n))$$

for each n, and these assemble into a simplicial set $(KB)_\bullet$. The functors $A^*(-)$ and $(K(-))_\bullet$ are adjoint contravariant.

(3.1) **Exercise.** Check that the functor $A^*(-)\colon \mathbf{sSets} \to \mathbf{cdga}^{\mathrm{op}}$ is indeed a left adjoint to the functor $(K(-))_\bullet\colon \mathbf{cdga}^{\mathrm{op}} \to \mathbf{sSets}$.

(3.2) **Remark.** The method of obtaining an adjunction in this way is called the 'method of schizophrenic objects', and many adjunctions arise in this fashion. In the case at hand, the schizophrenic object is $A^*(\Delta^\bullet)$, which alternately takes the roles of both a simplicial set and a cdga.

Our goal will be to show that the unit map $\eta : X \to K(A^*(X))_\bullet$ is a rational equivalence for X pointed simply connected, and that the counit map $\varepsilon : B \to A^*((KB)_\bullet)$ is a quasi-isomorphism when B satisfies certain conditions. (Note that the counit map appears in an unexpected direction due to the contravariance of the functors: a priori, the counit is a map of $(\mathbf{cdga})^{\mathrm{op}}$.) This will mean that $A^*(-)$ behaves well on homotopy: if $X \to Y$ is a weak equivalence of simplicial sets, then the pullback of forms $A^*(Y) \to A^*(X)$ is a quasi-isomorphism. Conversely, for X and Y simply connected, if $A^*(Y) \to A^*(X)$ is a quasi-isomorphism, then $X \to Y$ induces isomorphisms in rational cohomology, and hence is a rational equivalence. However, for good behaviour of $(K(-))_\bullet$ with respect to quasi-isomorphisms, we will need to restrict ourselves to the minimal models.

The conclusion of all this is that we can faithfully represent the rational homotopy theory of pointed simply connected spaces by the study of cdgas up to quasi-isomorphism, or of minimal models up to chain homotopy. More precise statements will be given later.

Chapter 2

RATIONALISATION OF CW-COMPLEXES

1. The Construction

Recall that in (1.1.7), we saw how to construct a map $S^n \to S^n_{\mathbb{Q}}$ with $S^n_{\mathbb{Q}}$ rational and such that any other map $S^n \to X$ into a rational space factors uniquely through $S^n_{\mathbb{Q}}$ up to homotopy. We can use these rationalisations of spheres to construct rationalisations of arbitrary CW-complexes.

Since we are only working with simply connected CW-complexes X, we can assume that $X^{(0)} = X^{(1)} = \{*\}$ and write X as the (homotopy) colimit of its skeleta

$$\{*\} = X^{(0)} = X^{(1)} \subseteq X^{(2)} \subseteq X^{(3)} \subseteq \ldots \subseteq X = \bigcup_{n\in\mathbb{N}} X^{(n)},$$

where each $X^{(n+1)}$ is obtained from $X^{(n)}$ by attaching $(n+1)$-cells, which means that there is a (homotopy) pushout square of the form.

$$\begin{tikzcd}
\coprod_{\alpha\in A} S^n \arrow[r] \arrow[d, hook] & X^{(n)} \arrow[d, hook] \\
\coprod_{\alpha\in A} e^{n+1} \arrow[r] & X^{(n+1)} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

for some index set A (and $e^{n+1} := D^{n+1}$ discs). The idea is to copy this inductive construction of X and adding $\mathbb{Q}$s everywhere. Then

$$\{*\} = X^{(0)}_{\mathbb{Q}} = X^{(1)}_{\mathbb{Q}} \subseteq X^{(2)}_{\mathbb{Q}} \subseteq X^{(3)}_{\mathbb{Q}} \subseteq \ldots \subseteq X_{\mathbb{Q}} := \bigcup_{n\in\mathbb{N}} X^{(n)}_{\mathbb{Q}},$$

where $X^{(n+1)}_{\mathbb{Q}}$ is obtained from $X^{(n)}_{\mathbb{Q}}$ by a (homotopy) pushout

$$\begin{tikzcd}
\coprod_{\alpha\in A} S^n_{\mathbb{Q}} \arrow[r] \arrow[d, hook] & X^{(n)}_{\mathbb{Q}} \arrow[d, hook] \\
\coprod_{\alpha\in A} e^{n+1}_{\mathbb{Q}} \arrow[r] & X^{(n+1)}_{\mathbb{Q}} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd},$$

where the $e^{n+1}_{\mathbb{Q}}$ are contractible spaces such that $S^n_{\mathbb{Q}} \to e^{n+1}_{\mathbb{Q}}$ is a cofibration (e.g. the cone above $S^n_{\mathbb{Q}}$). We claim that this construction is sensible and defines a rationalisation of X.

The first problem, assuming we have $X_{\mathbb{Q}}^{(n)}$, is how to define the maps $S_{\mathbb{Q}}^n \to X_{\mathbb{Q}}^{(n)}$. As part of our construction, we require the two squares above to fit into a cube

$$\begin{tikzcd}
& \coprod_\alpha S^n \arrow[rr] \arrow[dl] \arrow[dd] & & X^{(n)} \arrow[dl] \arrow[dd] \\
\coprod_\alpha S_{\mathbb{Q}}^n \arrow[rr, crossing over] \arrow[dd] & & X_{\mathbb{Q}}^{(n)} & \\
& \coprod_\alpha e^{n+1} \arrow[rr] \arrow[dl] & & X^{(n+1)} \arrow[dl] \\
\coprod_\alpha e_{\mathbb{Q}}^{n+1} \arrow[rr] & & X_{\mathbb{Q}}^{(n+1)} \arrow[from=uu, crossing over] &
\end{tikzcd}.$$

If $X_{\mathbb{Q}}^{(n)}$ is rational, we can easily find maps $S_{\mathbb{Q}}^n \to X_{\mathbb{Q}}^{(n)}$ making the top face commute using the universal property of such a rationalisation. To wit, if $X_{\mathbb{Q}}^{(n)}$ is rational, then every map $S^n \to X^{(n)} \to X_{\mathbb{Q}}^{(n)}$ factors through $S_{\mathbb{Q}}^n$.

With this, the map $X^{(n+1)} \to X_{\mathbb{Q}}^{(n+1)}$ is induced by the universal property of pushouts (since both the front and back face are such) and we only need to verify that $X_{\mathbb{Q}}^{(n+1)}$ is again rational to continue by induction.

Note that for this to work, we don't need $X_{\mathbb{Q}}^{(n)}$ to be the rationalisation of $X^{(n)}$. However we would like $X_{\mathbb{Q}} := \bigcup_n X_{\mathbb{Q}}^{(n)}$ to be the rationalisation of $X = \bigcup_n X^{(n)}$ and the way to do this is to make sure that every $X^{(n)} \to X_{\mathbb{Q}}^{(n)}$ is one and then pass to the (homotopy) colimit $X \to X_{\mathbb{Q}}$. All in all, starting from $X_{\mathbb{Q}}^{(0)} := X_{\mathbb{Q}}^{(1)} := \{*\}$, we need to do two things for the inductive step:

(a) show that $X_{\mathbb{Q}}^{(n+1)}$ is rational and

(b) show that $X^{(n+1)} \to X_{\mathbb{Q}}^{(n+1)}$ is a rationalisation.

If we assume the two basic facts from the first chapter, which allow us to use homology, this is easy. Indeed, letting R be any abelian group (we will only need the cases $R = \mathbb{Z}$ and $R = \mathbb{Q}$), the (homotopy) pushouts

$$X^{(n+1)} = X^{(n)} \cup_{\amalg_\alpha S^n} \coprod_\alpha e^{n+1} \qquad \text{and} \qquad X_{\mathbb{Q}}^{(n+1)} = X_{\mathbb{Q}}^{(n)} \cup_{\amalg_\alpha S_{\mathbb{Q}}^n} \coprod_\alpha e_{\mathbb{Q}}^{n+1}$$

give us a Mayer-Vietoris sequences

$$\begin{tikzcd}
\ldots \arrow[r] & \bigoplus_\alpha H_i(S^n; R) \arrow[r] \arrow[d, "a"'] & H_i(X^{(n)}; R) \arrow[r] \arrow[d, "b"'] & H_i(X^{(n+1)}; R) \arrow[r] \arrow[d, "c"'] & \bigoplus_\alpha H_{i-1}(S^n; R) \arrow[r] \arrow[d, "d"'] & \ldots \\
\ldots \arrow[r] & \bigoplus_\alpha H_i(S_{\mathbb{Q}}^n; R) \arrow[r] & H_i(X_{\mathbb{Q}}^{(n)}; R) \arrow[r] & H_i(X_{\mathbb{Q}}^{(n+1)}; R) \arrow[r] & \bigoplus_\alpha H_{i-1}(S_{\mathbb{Q}}^n; R) \arrow[r] & \ldots
\end{tikzcd}$$

We take $R := \mathbb{Z}$ and consider the homology groups in the lower row that are the codomains of a, b and d (and the next vertical morphism after d, which is just a shifted down one dimension). These groups are $\mathbb{Q}$-vector spaces by hypothesis and hence so is the codomain of c by the 5-lemma (see the lemma below). This shows that $X_{\mathbb{Q}}^{(n+1)}$ is rational. To see that $X^{(n+1)} \to X_{\mathbb{Q}}^{(n+1)}$ is a rationalisation, we take $R := \mathbb{Q}$ above and it again follows from the 5-lemma, that c is an isomorphism.

(1.1) **Lemma.** Given an exact sequence of abelian groups

$$A_1 \to A_2 \to A_3 \to A_4 \to A_5$$

such that A_1, A_2, A_4 and A_5 are $\mathbb{Q}$-vector spaces, then so is A_3.

Proof. Letting $n \in \mathbb{Z}^\times$ be any non-zero integer, we form

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "n"', "\cong"] & A_2 \arrow[r] \arrow[d, "n"', "\cong"] & A_3 \arrow[r] \arrow[d, "n"'] & A_4 \arrow[r] \arrow[d, "n"', "\cong"] & A_5 \arrow[d, "n"', "\cong"] \\
A_1 \arrow[r] & A_2 \arrow[r] & A_3 \arrow[r] & A_4 \arrow[r] & A_5
\end{tikzcd},$$

and apply the 5-lemma. □

(1.2) **Exercise.** Show the stronger version of the above lemma, where we only have a four-term exact sequence $A_1 \to A_2 \to A_3 \to A_4$ with A_1, A_2 and A_4 $\mathbb{Q}$-vector spaces and conclude that so is A_3.
Hint: An abelian group is a $\mathbb{Q}$-vector space iff it is divisible and torsion-free.

(1.3) **Exercise.** Let $i\colon A \rightarrowtail B$ be a closed cofibration (or a relative CW-complex) and consider a pushout square

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, tail, "i"'] & X \arrow[d] \\
B \arrow[r] & Y \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

with f arbitrary. Factoring f as a cofibration, followed by a homotopy equivalence (e.g. using mapping cylinders) yields $A \rightarrowtail M_f \to X$ and we get an intermediate pushout

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, tail, "i"'] & M_f \arrow[r, "\sim"] \arrow[d] & X \arrow[d] \\
B \arrow[r] & Z \arrow[r, dotted] \arrow[ul, phantom, "\ulcorner", very near start] & Y \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}.$$

Prove that $Z \to Y$ is again a homotopy equivalence. Prove the existence of a Mayer-Vietoris long exact sequence for a pushout of two cofibrations (such as the left one above. Conclude the existence of a Mayer-Vietoris long exact sequence for the original square (with only i being a cofibration but not necessarily f).

(1.4) **Exercise.** Consider the construction $X \to X_{\mathbb{Q}}$ from above. Using (1.1.9), show that every continuous map $X \to Y$ with Y rational extends to $X_{\mathbb{Q}} \to Y$ and that this extension is unique up to homotopy. Conclude that $-_{\mathbb{Q}}$ is a functor on the homotopy category of simply connected CW-complexes.

2. Tools to Prove the Basic Facts

In this section, we are going to recall the basic tools from algebraic topology that we are going to use to prove the two basic facts mentioned in the first chapter. All of these can be found in the lecture notes [12] to a previous course in algebraic topology or other textbooks on algebraic topology (such as [6]).

(2.1) **Theorem. (Long Exact Sequence of a Serre Fibration)** Every Serre fibration $p\colon E \to B$ with fibre F (above a fixed base-point coming from E), induces a long exact sequence in homotopy

$$\ldots \to \pi_{n+1}(B) \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \ldots \to \pi_0(F) \to \pi_0(E).$$

(2.2) **Theorem. (Homotopy Groups of Pairs)** Recall that for $x_0 \in A \subseteq X$, one defines $\pi_n(X,A)$ (or really $\pi_n(X,A,x_0)$) with $n \geqslant 1$ to be the set of homotopy classes of maps $I^n \to X$ that map the top face $I^{n-1} \times \{1\}$ to A and all other faces to x_0. These fit into a long exact sequence of homotopy groups

$$\ldots \to \pi_{n+1}(X,A) \to \pi_n(A) \to \pi_n(X) \to \pi_n(X,A) \to \ldots \pi_0(A) \to \pi_0(X).$$

There is an irrefutable similarity between the last two theorems and the reason behind it is that the homotopy groups of a pair are really a special case of the homotopy groups of a fibre in a Serre fibration. To wit, $\pi_{n+1}(X,A) \cong \pi_n\big(P(X,A)\big)$, where $P(X,A)$ is the so-called *homotopy fibre* of the inclusion map $A \hookrightarrow X$. More explicitly, we let $\mathrm{ev}_1\colon PX \to X$ be the usual end-point-evaluation for the path space on X and form the pullback

(2.3)
$$\begin{tikzcd} P(X,A) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & PX \arrow[d, two heads, "\mathrm{ev}_1"] \\ A \arrow[r, hook] & X \end{tikzcd} .$$

The map $P(X,A) \twoheadrightarrow A$ is a Serre fibration (in fact, even a Hurewicz fibration) since ev_1 is one and they are stable under pullback. Moreover, the fibres of ev_1 and $P(X,A) \twoheadrightarrow A$ are isomorphic (as always in a pullback), which, in this case means that they are isomorphic to the loop space ΩX. With this, the long exact sequence of the pair (X,A) is just the long exact sequence of the Serre fibration $P(X,A) \twoheadrightarrow A$.

(2.4) **Remark.** The attentive reader will have noticed that the two long exact sequence don't end with the same term. The reason for this is that we should really have replaced $A \hookrightarrow X$ by a Serre fibration (which one can always do), instead of only getting one into A. This is only a minor point however, especially since all our spaces are simply connected.

(2.5) **Theorem. (Whitehead)** Given a continuous map $f\colon X \to Y$ between simply connected spaces, the following are equivalent:

(a) f is a weak equivalence (i.e. every induced map $\pi_n(X) \to \pi_n(Y)$ with $n \in \mathbb{N}$ is an isomorphism)

(b) every induced map $H_n(X;\mathbb{Z}) \to H_n(Y;\mathbb{Z})$ with $n \in \mathbb{N}$ is an isomorphism.

(2.6) **Remark.** For the direction "(a) $\Rightarrow$ (b)", the assumption that the two spaces be simply connected is not necessary.

For every space X, there are canonical maps $h_n\colon \pi_n(X) \to H_n(X)$ called the *Hurewicz maps* (one for every n). An element $[f\colon S^n \to X]$ of $\pi_n(X)$ is mapped to the image $f_*(\iota)$ of a chosen (and fixed) generator ι of $H_n(S^n) \cong \mathbb{Z}$ via $f_*\colon H_n(S^n) \to H_n(X)$.

(2.7) **Theorem. (Hurewicz)** If X is n-connected for some $n \in \mathbb{N}$, then

$$h\colon \pi_{n+1}(X) \to H_{n+1}(X)$$

is the abelianisation map. For $n = 0$, this means that $H_1(X) \cong \pi_1(X)/\big[\pi_1(X), \pi_1(X)\big]$ with h the quotient map, while for $n > 0$, this means that h is an isomorphism $\pi_{n+1}(X) \cong H_{n+1}(X)$.

There is also a relative version of this result. To define the necessary Hurewicz-map, we identify

$$(I^n, \partial I^n, I^{n-1} \times \{1\}) \cong (D^n, S^{n-1}, *),$$

so that an element of $\pi_n(X, A)$ can be represented by some $f\colon (D^n, S^n) \to (X, A)$. Just as in the absolute case above, we have

$$H_n(D^n, S^{n-1}) \cong \tilde{H}_n(D^n/S^{n-1}) \cong \tilde{H}_n(S^n) \cong \mathbb{Z}.$$

The relative Hurewicz-map $h\colon \pi_n(X, A) \to H_n(X, A)$ then sends $[f\colon (D^n, S^{n-1}) \to (X, A)]$ to $f_*(\iota)$, where ι is a fixed generator of $H_n(D^n, S^{n-1})$ and $f_*\colon H_n(D^n, S^{n-1}) \to H_n(X, A)$.

(2.8) **Theorem. (Hurewicz)** If (X, A) is an $(n-1)$-connected pair (i.e. $\pi_i(X, A) = 0$ for all $i < n$) with $n > 1$, X path-connected and A simply connected, then also $H_i(X, A) = 0$ for all $i < n$ and the Hurewicz map

$$\pi_n(X, A) \to H_n(X, A) \qquad \text{is an isomorphism.}$$

Finally, as a last ingredient, we recall the constructions of the Whitehead- and Postnikov-tower, which, respectively, kill off the lower and higher homotopy groups of a space. To wit, given a path-connected space X, we can form its so-called *Whitehead tower*

$$X =: W_0(X) \xleftarrow{p_1} W_1(X) \xleftarrow{p_2} W_2(X) \xleftarrow{p_3} \dots$$

with $W_n(X)$ being n-connected (i.e. $\pi_i W_n(X) = 0$ for $i \leqslant n$), every $W_{n+1}(X) \to W_n(X)$ a fibration with fibre a $K(\pi_{n+1}(X), n)$ and such that $W_n(X) \to X$ induces an isomorphism $\pi_i W_n(X) \cong \pi_i(X)$ for all $i > n$.

(2.9) **Example.** Taking $n = 1$, the term $W_1(X)$ in the Whitehead tower is simply the universal cover of X.

On the other hand, we can also kill off the higher homotopy groups by forming the so-called *Postnikov tower* of X, which is a commutative diagram

$$\begin{tikzcd}
X \arrow[d] \arrow[dr] \arrow[drr] & & \\
P_1(X) & P_2(X) \arrow[l] & P_3(X) \arrow[l] & \dots \arrow[l]
\end{tikzcd}$$

with $\pi_i P_n(X) = 0$ for all $i > n$, $P_{n+1}(X) \to P_n(X)$ a fibration with fibre $K(\pi_{n+1}(X), n+1)$ and such that $X \to P_n(X)$ induces an isomorphism $\pi_i(X) \cong \pi_i P_n(X)$ for all $i \leqslant n$.

Finally, let us recall a few basic facts about homology and cohomology that we are going to use time and again.

(2.10) **Theorem. (Universal Coefficient Theorem)** For every space X, every abelian group A and $i > 0$, there is a short exact sequence

$$0 \to H_i(X; \mathbb{Z}) \otimes A \to H_i(X; A) \to \operatorname{Tor}_1(H_{i-1}(X; \mathbb{Z}), A) \to 0.$$

Similarly, there is a short exact sequence

$$0 \to \operatorname{Ext}^1(H_{i-1}(X; \mathbb{Z}), A) \to H^i(X; A) \to \operatorname{Hom}_{\mathbb{Z}}(H_i(X; \mathbb{Z}), A) \to 0.$$

In particular, $H_i(X; \mathbb{Q}) \cong H_i(X; \mathbb{Z}) \otimes \mathbb{Q}$ and $H^i(X; \mathbb{Q}) \cong \operatorname{Hom}_{\mathbb{Z}}(H_i(X; \mathbb{Z}), \mathbb{Q})$.

While the universal coefficient theorem is algebraic in nature, the following is more topological and tells us that taking (integral) cohomology is a representable functor, represented by Eilenberg-MacLane spaces. If G is an abelian group and $n > 0$, we fix an isomorphism $G \cong \pi_n K(G,n)$, and thus, by Hurewicz, get a fixed isomorphism

$$G \cong \pi_n K(G,n) \cong H_n(K(G,n);\mathbb{Z}).$$

By the universal coefficient theorem, we then get the so-called *fundamental class*

$$\iota_n \in H^n(K(G,n);G) \cong \operatorname{Hom}_{\mathbb{Z}}\Big(H_n(K(G,n);\mathbb{Z}),G\Big) \cong \operatorname{Hom}_{\mathbb{Z}}(G,G),$$

corresponding to id_G. Now every continuous map $f\colon X \to K(G,n)$ induces a morphism in cohomology $f^*\colon H^n(K(G,n);G) \to H^n(X;G)$ and we can consider the image $f^*\iota_n$ of the fundamental class under it. Since f^* only depends on the homotopy class of f, this means that we get a map

$$[X, K(G,n)] \to H^n(X;G),\ [f] \mapsto f^*\iota_n.$$

Note that there is a group structure on the left-hand set, induced by the concatenation of loops in $K(G,n) \cong \Omega K(G,n+1)$.

(2.11) **Theorem. (Representability of Cohomology)** If X is a CW-complex, G an abelian group and $n > 0$, then

$$[X, K(G,n)] \to H^n(X;G),\ [f] \mapsto f^*\iota_n$$

is an isomorphism of abelian groups.

3. Homology of Fibrations

For this entire section, we are going to be working over a field K that is either $\mathbb{Q}$ or a prime-field $\mathbb{F}_p$. The goal is to analyse the homology of a fibration without resorting to using the Serre spectral sequence. We follow Klaus and Kreck [9] in doing so.

(3.1) **Remark.** Why should we be interested in $H_n(X;\mathbb{F}_p)$? Because the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{F}_p \to 0$$

induces a long exact sequence in homology

$$\ldots \to H_{i+1}(X;\mathbb{F}_p) \to H_i(X;\mathbb{Z}) \xrightarrow{p} H_i(X;\mathbb{Z}) \to H_i(X;\mathbb{F}_p) \to \ldots$$

(this is just the long exact sequence induced by the short exact sequence of singular chain complexes $0 \to C_*(X;\mathbb{Z}) \to C_*(X;\mathbb{Z}) \to C_*(X;\mathbb{F}_p) = C_*(X) \otimes \mathbb{F}_p \to 0$). But $H_i(X;\mathbb{Z})$ is a $\mathbb{Q}$-vector space iff multiplication by p is invertible for all primes p, which, by the above sequence, happens precisely when $H_i(X;\mathbb{F}_p) = H_{i+1}(X;\mathbb{F}_p) = 0$ for all p (as always under the hypothesis that X be simply connected).

(3.2) **Notation.** In this section, we write $h_i(-) := H_i(-;K)$ and $\tilde{h}_i(-) := \tilde{H}_i(-;K)$.

(3.3) **Proposition.** Let $p\colon E \twoheadrightarrow B$ be a Serre fibration between path-connected spaces with fibre F (above some fixed base-point $b_0 \in B$). If B is m-connected and $\tilde{h}_i(F) = 0$ for all $i < n$, then

(a) $h_i(E,F) \to h_i(B,b_0)$ is an isomorphism for all $i \leqslant n+m$;

(b) $h_i(E) \to h_i(B)$ is an isomorphism for all $i < n$ and surjective for $i = n$.

(3.4) **Example.** Two fibrations to which we will later apply this proposition are:

$$K(\pi_n(X), n-1) \to W_n(X) \twoheadrightarrow W_{n-1}(X); \qquad K(G,n) \to E \twoheadrightarrow K(G,n+1)$$

(for E contractible).

Proof. By cellular approximation, we can assume that B is a CW-complex and since B is m-connected, we can even assume that

$$\{b_0\} = B^{(0)} = B^{(1)} = \ldots = B^{(m)} \subseteq B^{(m+1)} \subseteq B^{(m+2)} \subseteq \ldots.$$

Better still, since every Kan fibration of simplicial sets is a strong deformation retract of a minimal fibration, whose realisation is a fibre bundle, we can assume that $p\colon E \twoheadrightarrow B$ is one such. Explicitly, this means that p is locally trivial, so that for every cell e^k of B, we find a commutative triangle

$$\begin{tikzcd} p^{-1}(e^k) \arrow[rr, "\cong"] \arrow[dr, "p"'] & & e^k \times F \arrow[dl, "\mathrm{pr}"] \\ & e^k & \end{tikzcd}.$$

We now define $E^{(k)} := p^{-1}(B^{(k)})$ and consider $h_i(E^{(k)}, F) \to h_i(B^{(k)}, b_0)$. Our aim is to prove that this map is an isomorphism for all k, as long as i is in an appropriate range. Using the long exact sequences of $F \subseteq E^{(k)} \subseteq E^{(k+1)}$ and $b_0 \in B^{(k)} \subseteq B^{(k+1)}$, we get

$$\begin{tikzcd} \ldots\ h_{i+1}(E^{(k+1)}, E^{(k)}) \arrow[r] \arrow[d, "(**)"'] & h_i(E^{(k)}, F) \arrow[r] \arrow[d] & h_i(E^{(k+1)}, F) \arrow[r] \arrow[d] & h_i(E^{(k+1)}, E^{(k)}) \arrow[r] \arrow[d, "(*)"'] & h_{i-1}(E^{(k)}, F)\ \ldots \arrow[d] \\ \ldots\ h_{i+1}(B^{(k+1)}, B^{(k)}) \arrow[r] & h_i(B^{(k)}, b_0) \arrow[r] & h_i(B^{(k+1)}, b_0) \arrow[r] & h_i(B^{(k+1)}, B^{(k)}) \arrow[r] & h_{i-1}(B^{(k)}, b_0)\ \ldots. \end{tikzcd}$$

To be able to apply the 5-lemma, we need to know when the map $(*)$ is an isomorphism and $(**)$ is surjective. This is clearly the case $k+1 \leqslant m$ because then $B^{(k+1)} = B^{(k)}$ and hence also $E^{(k+1)} = E^{(k)}$, meaning that the two relative homology groups vanish.

So let's assume that $k+1 > m$. By excision, the map $(E^{(k+1)}, E^{(k)}) \to (B^{(k+1)}, B^{(k)})$ looks in homology like

$$\coprod_\alpha (e^{k+1} \times F, \partial e^{k+1} \times F) \longrightarrow \coprod_\alpha (e^{k+1}, \partial e^{k+1})$$

with α indexing the $(k+1)$-cells of B. Using the relative Künneth theorem (and the fact that we are working over a field), in dimension i, we therefore get

$$\bigoplus_\alpha \bigoplus_{p+q=i} h_p(e^{k+1}, \partial e^{k+1}) \otimes_K h_q(F) \longrightarrow \bigoplus_\alpha h_i(e^{k+1}, \partial e^{k+1}).$$

Note that $h_0(F) \cong K$, and the summands with $q = 0$ above correspond precisely to the right-hand side. So the map is always surjective, which means $(**)$ is already done. To have $(*)$ even an isomorphism, the rest of the summands (those with $q > 0$) need to be 0.

For this, we recall that $h_p(e^{k+1}, \partial e^{k+1}) \cong \tilde{h}_p(S^{k+1})$ is K for $p = k+1$ and 0 otherwise, meaning that the summands with $q > 0$ are just $\bigoplus_\alpha h_{i-k-1}(F)$ with $i-k-1 = q > 0$. By hypothesis, this is non-zero only when $i-k-1 \geqslant n$; i.e. $i \geqslant n+k+1 > n+m$.

All in all, we have shown that the map $(**)$ above is always surjective, while $(*)$ is an isomorphism for $i \leqslant n+m$. By the 5-lemma, it follows by induction on k, that $h_i(E^{(k)}, F) \to h_i(B^{(k)}, b_0)$ is an isomorphism in that case. Passing to the colimit, (a) follows.

For the second claim (b), we consider the long exact sequences in homology induced by (E,F) and (B,b_0) and again apply the 5-lemma:

$$\begin{tikzcd} h_{i+1}(E,F) \arrow[r] \arrow[d, "\cong"'] & h_i(F) \arrow[r] \arrow[d, "\cong"'] & h_i(E) \arrow[r] \arrow[d] & h_i(E,F) \arrow[r] \arrow[d, "\cong"'] & h_{i-1}(F) \arrow[d, "\cong"'] \\ h_{i+1}(B,b_0) \arrow[r] & h_i(b_0) \arrow[r] & h_i(B) \arrow[r] & h_i(B,b_0) \arrow[r] & h_{i-1}(b_0) \end{tikzcd} .$$

The two maps induced by $(E,F) \to (B,b_0)$ are isomorphisms by point (a) and the maps induced by $F \to \{b_0\}$ are isomorphisms by hypothesis, as long as $i < n$. For $i = n$, $h_n(F) \to h_n(b_0)$ is only surjective, which means we can still apply the 4-lemma to conclude that $h_n(E) \to h_n(B)$ is surjective, too. □

(3.5) **Corollary.** Let π be an abelian group, and consider the Eilenberg-MacLane space $K(\pi,n)$ for $n > 0$.

(a) If $\pi \otimes \mathbb{Q} = 0$, then $H_i\big(K(\pi,n);\mathbb{Q}\big) = 0$ for all $i > 0$.

(b) If, π is a $\mathbb{Q}$-vector space, then so is $H_i\big(K(\pi,n);\mathbb{Z}\big)$ for all $i > 0$.

Proof. For part (a), when $n = 1$, the homology of $K(\pi,1)$ is the group homology of π and the result follows by looking at the explicit definition in terms of the bar resolution (cf. notes on Homological Algebra [11]). For $n > 1$, we proceed by induction, using the fibration

$$K(\pi,n) \to E \twoheadrightarrow K(\pi,n+1)$$

with a contractible total space E. Then, if we know that $H_i\big(K(\pi,n);\mathbb{Q}\big) = 0$ for all $i > 0$, proposition (3.3) gives the same for $H_i\big(K(\pi,n+1);\mathbb{Q}\big)$.

For (b), exactly the same argument applies to $H_i\big(K(\pi,n);\mathbb{F}_p\big)$ for each prime p, and then the result follows from remark (3.1). □

(3.6) **Corollary.** For a fibre sequence $F \to E \to B$, with B simply connected and F connected, suppose that $h_i(E) \to h_i(B)$ is an isomorphism for $i < k$ and an epimorphism for $i = k$. Then $h_i(F) = 0$ for $0 < i < k$.

Proof. We proceed by induction on k, where the case $k = 1$ is vacuously true. For the inductive step, assuming the claim is true for k, we consider the diagram

$$\begin{tikzcd} h_{k+1}(E) \arrow[r] \arrow[d, two heads] & h_{k+1}(E,F) \arrow[r] \arrow[d, "\cong"'] & h_k(F) \arrow[r] \arrow[d] & h_k(E) \arrow[r] \arrow[d, "\cong"'] & h_k(E,F) \arrow[d, "\cong"'] \\ h_{k+1}(B) \arrow[r] & h_{k+1}(B,b_0) \arrow[r] & h_k(b_0) \arrow[r] & h_k(B) \arrow[r] & h_k(B,b_0) \end{tikzcd} .$$

Using (3.3) with $m = 1$ and $n = k$, we see that the maps of the form $h_i(E,F) \to h_i(B,b_0)$ are isomorphisms for $i \in \{k, k+1\}$. Our assumption tells us that $h_{k+1}(E) \to h_{k+1}(B)$ is surjective, and $h_k(E) \to h_k(B)$ is an isomorphism. Again, this is enough to apply the 5-lemma, so we may conclude that $h_k(F) \cong h_{k+1}(b_0) = 0$. □

(3.7) **Remark.** This corollary gives a converse to part (b) of proposition (3.3).

(3.8) **Corollary.** Suppose we have a commutative diagram as below with the two columns being fibre sequences, F, F' connected and B, B' even simply connected.

$$\begin{tikzcd} F \arrow[r] \arrow[d] & F' \arrow[d] \\ E \arrow[r] \arrow[d] & E' \arrow[d] \\ B \arrow[r] & B' \end{tikzcd}$$

If $B \to B'$ induces an isomorphism in h_* and a surjection on π_2, while the homotopy fibre W of the map $F \to F'$ is connected and has $h_i(W) = 0$ for al $i > 0$, then $h_i(E) \to h_i(E')$ is also an isomorphism for all i.

Proof. Taking horizontal homotopy fibres

$$\begin{tikzcd} W \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & F' \arrow[d] \\ V \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & E' \arrow[d] \\ U \arrow[r] & B \arrow[r] & B' \end{tikzcd},$$

the space W is also the homotopy fibre of the map $V \to U$. Using the long exact sequence of the homotopy fibre, U is simply connected because the map $B \to B'$ induces a surjection $\pi_2(B) \to \pi_2(B')$. By part (b) of Proposition (3.3), $h_i(V) \to h_i(U)$ is an isomorphism for all i (since $\tilde{h}_*(W) = 0$). But by Corollary (3.6) applied to the fibre sequence $U \to B \twoheadrightarrow B'$, we see that $h_i(U) = 0$ for all $i > 0$. Hence $h_i(V) = 0$ for all $i > 0$ and by the proposition as applied to the fibre sequence $V \to E \to E'$, we have the result. □

(3.9) **Exercise.** Formulate and prove a formulation of the above corollary for the partial cases $i < k$.

4. Rational Hurewicz Theorem

We begin with the rational analogue of the classical Hurewicz theorem.

(4.1) **Theorem (The Rational Hurewicz Theorem).** Let X be simply connected. If $\pi_i(X) \otimes \mathbb{Q} = 0$ for all $0 < i < n$ then also $H_i(X;\mathbb{Q}) = 0$ for $0 < i < n$, and

$$\pi_i(X) \otimes \mathbb{Q} \to H_i(X;\mathbb{Q})$$

is an isomorphism for $i = n$.

Proof. The theorem is proved by induction on n. The case $n = 2$ follows from the classical Hurewicz theorem, since X was assumed to be simply connected. Suppose now that for all $0 < i < n$, $\pi_i(X) \otimes \mathbb{Q} = 0$ and that the theorem has been proven for smaller n. Consider the Whitehead tower of X,

$$X = W_1X \xleftarrow{p_2} W_2X \xleftarrow{p_3} W_3X \leftarrow \ldots \xleftarrow{p_n} W_nX \xleftarrow{p_{n+1}} \ldots,$$

in which p_i has fibre $K(\pi_i(X), i-1)$. It follows by proposition (3.3) that p_i induces isomorphisms $H_*(W_i(X);\mathbb{Q}) \to H_*(X;\mathbb{Q})$ in all degrees as long as $i < n$. Now consider the square:

$$\begin{tikzcd}
\pi_n(W_{n-1}(X)) \arrow[r] \arrow[d] & H_n(W_{n-1}(X);\mathbb{Z}) \arrow[d] \\
\pi_n(X) \arrow[r] & H_n(X;\mathbb{Z}).
\end{tikzcd}$$

We have just shown that the map on the right is an isomorphism after tensoring with $\mathbb{Q}$. Furthermore, the map on the top of the square is an isomorphism by the classical Hurewicz theorem, while the map on the left is an isomorphism by construction of the Whitehead tower. Thus, the bottom map becomes an isomorphism once tensored with $\mathbb{Q}$. □

(4.2) **Remark.** There is a stronger version of the rational Hurewicz theorem, a more elementary proof of which can also be found in [9]. This stronger version states that in the situation of the theorem, the Hurewicz map $\pi_i(X) \otimes \mathbb{Q} \to H_i(X;\mathbb{Q})$ is an isomorphism even for $0 < i < 2n-1$ and surjective for $i = 2n-1$.

Now we finally prove proposition (1.1.5), which we earlier called "Basic Fact 2". We restate the proposition here for convenience: If X is a simply connected space, $\pi_n(X)$ is a $\mathbb{Q}$-vector space for all $n > 1$ iff $H_n(X;\mathbb{Z})$ is a $\mathbb{Q}$-vector space for all $n > 1$.

Proof (of (1.1.5), i.e. of Basic Fact 2). Suppose $\pi_i(X)$ is a $\mathbb{Q}$-vector space for all $i > 1$. We will show that $H_n(X;\mathbb{Z})$ is also a $\mathbb{Q}$-vector space for all $n \geqslant 1$, by induction on n. For $n = 1$, there is nothing to prove since X is simply connected. Now suppose that $H_m(X;\mathbb{Z})$ is a $\mathbb{Q}$-vector space for all $m < n$. Consider again the square

$$\begin{tikzcd}
\pi_{n+1}(W_n(X)) \arrow[r] \arrow[d] & H_{n+1}(W_n(X);\mathbb{Z}) \arrow[d] \\
\pi_{n+1}(X) \arrow[r] & H_{n+1}(X;\mathbb{Z}),
\end{tikzcd}$$

in which the top and left maps are isomorphisms, and proceed not with $\mathbb{Q}$ but with $\mathbb{F}_p$. Analogously to before, we find that $H_{n+1}(W_n(X);\mathbb{Z}) \to H_{n+1}(X;\mathbb{Z})$ is an isomorphism after tensoring with each $\mathbb{F}_p$, and so the map $\pi_{n+1}(X) \to H_{n+1}(X;\mathbb{Z})$ is also an isomorphism after tensoring with each $\mathbb{F}_p$. Since $\pi_{n+1}(X)$ is a $\mathbb{Q}$-vector space, this implies that $H_{n+1}(X;\mathbb{F}_p) = 0$ for each p and by remark (3.1), we have that $H_n(X;\mathbb{Z})$ is a $\mathbb{Q}$-vector space.

Conversely, assume each $H_i(X;\mathbb{Z})$ is a $\mathbb{Q}$-vector space. We again proceed by induction and so, given $n \geqslant 1$, we assume that we have already proven that $\pi_m(X)$ is a $\mathbb{Q}$-vector space for all $m \leqslant n$. To prove that $\pi_{n+1}(X)$ is also, it suffices, by the square above, to show that $H_{n+1}(W_n(X);\mathbb{Z})$ is. Again, we have isomorphisms $H_k(W_n(X);\mathbb{F}_p) \cong H_k(X;\mathbb{F}_p) = 0$ for all p and all k; in particular for $k \in \{n+1, n+2\}$. By (3.1), it follows that $H_{n+1}(W_n(X);\mathbb{Z})$ is indeed a $\mathbb{Q}$-vector space. □

(4.3) **Remark.** We have actually shown a stronger result. For a fixed n, in the direction "$\Rightarrow$", we showed that if $\pi_i(X)$ is a $\mathbb{Q}$-vector space for all $i \leqslant n+1$, then so is every $H_i(X;\mathbb{Z})$ with $i \leqslant n$. Conversely, in "$\Leftarrow$", we showed that if $H_i(X;\mathbb{Z})$ is a $\mathbb{Q}$-vector space for all $i \leqslant n+1$, then so is every $\pi_i(X)$ with $i \leqslant n$.

5. Rational Whitehead Theorem

The goal of this section is to prove the rational analogue of the Whitehead theorem. More precisely, we are going to show that a continuous map $f\colon X \to Y$ between simply connected spaces induces an isomorphism in $\pi_*(-) \otimes \mathbb{Q}$ iff it does in $H_*(-;\mathbb{Q})$.

As always, we can replace f by a fibration and consider the long exact sequence of the corresponding homotopy fibre F:

$$\pi_{n+1}(Y) \to \pi_n(F) \to \pi_n(X) \to \pi_n(Y) \to \ldots \to \pi_2(X) \to \pi_2(Y) \to \pi_1(F) \to 0.$$

Our plan to show the direction "$\Rightarrow$" is to simply tensor this long exact sequence with $\mathbb{Q}$ and use the rational Hurewicz theorem. However, for this to even make sense and for Hurewicz to apply, we need $\pi_1(F) = 0$, or equivalently (by exactness), that $\pi_2(X) \to \pi_2(Y)$ is surjective. So let us prove first that we can reduce everything to this case.

(5.1) **Lemma.** Let $f\colon X \to Y$ be a continuous map between simply connected spaces that induces an isomorphism $\pi_2(X)\otimes\mathbb{Q} \cong \pi_2(Y)\otimes\mathbb{Q}$ (or equivalently $H_2(X;\mathbb{Q}) \cong H_2(Y;\mathbb{Q})$ by classical Hurewicz). Then we can factor f as $f = g \circ j\colon X \to W \to Y$ such that

(a) W is again simply connected;

(b) j induces a surjection $\pi_2(X) \twoheadrightarrow \pi_2(W)$;

(c) g induces isomorphisms $H_*(W;\mathbb{Q}) \cong H_*(Y;\mathbb{Q})$ and $\pi_*(W)\otimes\mathbb{Q} \cong \pi_*(Y)\otimes\mathbb{Q}$.

In particular, to prove the Whitehead theorem, we can work with $X \to W$ instead of $X \to Y$.

Proof. Write $\pi_2(X) \to \pi_2(Y) \twoheadrightarrow \Gamma \to 0$ for the cokernel. Since $-\otimes\mathbb{Q}$ is right exact (even exact since $\mathbb{Q}$ is flat) and since $\pi_2(X)\otimes\mathbb{Q} \cong \pi_2(Y)\otimes\mathbb{Q}$, we have $\Gamma\otimes\mathbb{Q} = 0$. Now, writing $K := K(\Gamma, 2)$, the quotient map $\pi_2(Y) \twoheadrightarrow \Gamma$ corresponds to a map $q\colon Y \to K$. Indeed, by the representability of cohomology, the universal coefficient theorem and Hurewicz

$$[Y, K(\Gamma,2)] \cong H^2(Y;\Gamma) \cong \mathrm{Hom}_{\mathbb{Z}}(H_2(Y;\mathbb{Z}),\Gamma) \cong \mathrm{Hom}_{\mathbb{Z}}(\pi_2(Y),\Gamma).$$

Clearly, $q \circ f$ is null-homotopic (it is 0 on π_2 by definition of Γ) and thus factors through the homotopy fibre $g\colon W \to Y$ of $q\colon Y \to K$, say as $j\colon X \to W$:

$$\begin{tikzcd} & W \arrow[d, "g"] & \\ X \arrow[r, "f"'] \arrow[ur, dotted, "j"] & Y \arrow[r, "q"'] & K \end{tikzcd} .$$

By the long exact sequence in homotopy of $W \to Y \to K$, g induces an isomorphism $\pi_n(W) \to \pi_n(Y)$ for $n \geqslant 3$ and $\pi_1(W) = 0$ (which proves (a)). Still part of the same long exact sequence, we have

$$0 \to \pi_2(W) \rightarrowtail \pi_2(Y) \to \pi_2(K) = \Gamma \to 0,$$

meaning that $\pi_2(W) \cong \mathrm{Im}(\pi_2(X))$ is the kernel of $\pi_2(Y) \to \Gamma = \pi_2(Y)/\mathrm{Im}(\pi_2(X))$ and the induced map $\pi_2(X) \to \pi_2(W)$ is thus surjective, which proves (b).

As for the last claim, if we apply $-\otimes\mathbb{Q}$ to the above short exact sequence and note that $\Gamma\otimes\mathbb{Q} = 0$, we get that $\pi_2(W)\otimes\mathbb{Q} \to \pi_2(Y)\otimes\mathbb{Q}$ is an isomorphism and, as already noted, $\pi_n(W) \to \pi_n(Y)$ is an isomorphism for all other n, even before tensoring with $\mathbb{Q}$. Finally, for the homological part of (c), we consider the homotopy fibre of $g\colon W \to Y$, which is $\Omega K \simeq K(\Gamma, 1)$, as observed in (2.3) (because $g\colon W \to Y$ is already the homotopy fibre of the map $Y \to X$). By (3.5), we have

$$H_i(\Omega K; \mathbb{Q}) \cong H_i(K(\Gamma, 1); \mathbb{Q}) = 0 \qquad \text{for all } i > 0$$

and hence, by (3.3), g induces an isomorphism in rational homology. □

(5.2) **Theorem. (Rational Whitehead)** A continuous map $f\colon X \to Y$ between simply connected spaces induces an isomorphism $\pi_i(X) \otimes \mathbb{Q} \cong \pi_i(Y) \otimes \mathbb{Q}$ for all $i \in> 1$ iff it induces an isomorphism $H_i(X; \mathbb{Q}) \cong H_i(X; \mathbb{Q})$ for all $i \in> 1$.

Proof. Ad "⇒": By the above lemma (replacing f by the map j constructed there), we may assume that $\pi_2(X) \to \pi_2(Y)$ is surjective. Replacing f by a fibration and taking the homotopy fibre F, we obtain a long exact sequence

$$\pi_{n+1}(Y) \to \pi_n(F) \to \pi_n(X) \to \pi_n(Y) \to \ldots \to \pi_2(F) \to \pi_2(X) \to \pi_2(Y) \to \pi_1(F).$$

The surjectivity of $\pi_2(X) \to \pi_2(Y)$ implies that $\pi_1(F) = 0$ and when tensoring the long exact sequence with $\mathbb{Q}$, we see that $\pi_i(F) \otimes \mathbb{Q}$ for all $i > 0$. By (3.3), it now follows that f induces an isomorphism $H_*(X; \mathbb{Q}) \cong H_*(Y; \mathbb{Q})$ as claimed.
Ad "⇐": Assuming that f induces an isomorphism $H_*(X; \mathbb{Q}) \cong H_*(Y; \mathbb{Q})$, we show a little more than what's claimed. For every i, we shall show that f induces isomorphisms

(a) $H_*(W_{i-1}X; \mathbb{Q}) \cong H_*(W_{i-1}Y; \mathbb{Q})$ as well as

(b) $\pi_i(X) \otimes \mathbb{Q} \cong \pi_i(Y) \otimes \mathbb{Q}$.

The case $i = 1$ is trivial, while $i = 2$ is implied by Hurewicz (as well as the theorem's hypothesis because $W_1X = X$ and $W_1Y = Y$). Now suppose $i > 2$ and that we have the result for all $j \leqslant i$. There is a map between fibrations

$$\begin{tikzcd} W_iX \arrow[r] \arrow[d] & W_iY \arrow[d] \\ W_{i-1}X \arrow[r] & W_{i-1}Y \end{tikzcd} ,$$

where the fibres are $K(\pi_iX, i-1)$ and $K(\pi_iY, i-1)$ respectively. Since $\pi_i(X)\otimes\mathbb{Q} \cong \pi_i(Y)\otimes\mathbb{Q}$ by the inductive hypothesis, the direction we already showed tells us that the map between homotopy fibres

$$K(\pi_iX, i-1) \to K(\pi_iY, i-1)$$

induces an isomorphism in $H_i(-; \mathbb{Q})$. Writing W for the homotopy fibre of this map, the long exact sequence looks like

$$\ldots 0 \to \pi_{i-1}W \to \pi_iX \to \pi_iY \to \pi_{i-2}W \to 0 \to \ldots .$$

In particular, W is connected and since $\pi_i(X) \otimes \mathbb{Q} \cong \pi_i(Y) \otimes \mathbb{Q}$ induced by f, the direction that we already showed tells us that f also induces an isomorphism

$$H_*(K(\pi_iX, i-1); \mathbb{Q}) \cong H_*(K(\pi_iY, i-1); \mathbb{Q}).$$

With this, we can apply (3.6) and conclude that that $H_*(W;\mathbb{Q}) = 0$. Consequently, because $H_*(W_{i-1}X;\mathbb{Q}) \cong H_*(W_{i-1}Y;\mathbb{Q})$ by assumtion, (3.8) then gives us that indeed also $H_*(W_iX;\mathbb{Q}) \cong H_*(W_iY;\mathbb{Q})$, as claimed. For point (b), we form the cubical diagram

$$
\begin{tikzcd}
& \pi_{i+1}(W_iX) \arrow[rr, "\cong"] \arrow[dl] \arrow[dd, "\cong"'] & & H_{i+1}(W_iX;\mathbb{Z}) \arrow[dl] \arrow[dd] \\
\pi_{i+1}(W_iY) \arrow[rr, "\cong"] \arrow[dd, "\cong"'] & & H_{i+1}(W_iY;\mathbb{Z}) \arrow[dd] & \\
& \pi_{i+1}(X) \arrow[rr] \arrow[dl] & & H_{i+1}(X;\mathbb{Z}) \arrow[dl] \\
\pi_{i+1}(Y) \arrow[rr] & & H_{i+1}(Y;\mathbb{Z}) &
\end{tikzcd} .
$$

The two vertical maps in the left face are isomorphisms by definition of the Whitehead tower, while the two rightwards pointing maps in the top face are isomorphisms by the classical Hurewicz theorem.

We want to check that the map $\pi_{i+1}(X) \to \pi_{i+1}(Y)$ becomes an isomorphism after tensoring with $\mathbb{Q}$, which is visibly equivalent to proving that $H_{i+1}(W_iX;\mathbb{Z}) \to H_{i+1}(W_iY;\mathbb{Z})$ becomes an isomorphism after tensoring with $\mathbb{Q}$, which we already showed. $\square$

Chapter 3

MINIMAL MODELS OF CDGAS

1. Free Algebras

The main objective of this section is to introduce our notations and conventions for the exterior algebra on a graded vector space. In particular, we want to quickly address the difference between degree and word length.

Recall from the introduction that for V^* a graded vector space, the *exterior algebra* $\Lambda(V)$ is the free commutative algebra on V, which means that

$$\Lambda(V) = \bigoplus_{n\in\mathbb{N}} \Lambda^n(V), \qquad \text{where } \Lambda^n(V) = \langle v_1 \wedge \ldots \wedge v_n \mid v_1, \ldots, v_n \in V\rangle$$

is obtained from the n-fold tensor product $T^nV = V^{\otimes n}$ (and $v_1 \wedge \ldots \wedge v_n$ denotes the equivalence class of $v_1 \otimes \ldots \otimes v_n$) by imposing a commutativity rule $v \wedge w = (-1)^{|v||w|} w \wedge v$ (for homogeneous v, $w \in V$). Let us explicitly mention that by definition $T^0(V) = \Lambda^0(V) = \mathbb{Q}$, which form the units of our algebra (with the exception of 0 of course).

Elements of $\Lambda^n(V) \subseteq \Lambda(V)$ are said to have *word-length* n. However, the *degree* of $v = v_1 \wedge \ldots \wedge v_n \in \Lambda^n(V)$ is $|v| := |v_1| + \ldots + |v_n|$, rather than n.

(1.1) **Notation.** Given a graded vector space V, we write $\Lambda^n(V) \subseteq \Lambda(V)$ for the elements of word-length n, while $\Lambda(V)^n \subseteq \Lambda(V)$ denotes the subspace of elements of degree n.

(1.2) **Observation.** Note that if $a \in \Lambda(V)$ is of odd degree, then $a \wedge a = 0$ by commutativity. However, if a is of even degree, it might very well happen that $a \wedge a \neq 0$.

If V happens to be a cochain complex, then the differential d on V induces a differential on $\Lambda(V)$, which is explicitly given by

$$d(v_1 \wedge \ldots \wedge v_n) = \sum_{i=1}^{n} (-1)^{|v_1|+\ldots+|v_{i-1}|} v_1 \wedge \ldots \wedge v_{i-1} \wedge dv_i \wedge v_{i+1} \wedge \ldots \wedge v_n.$$

In this case, the differential increases the degree of a homogeneous element by 1, while the word-length is kept the same.

Of particular importance to us are the *polynomial forms* $A^*(\Delta^n)$ on the standard simplices Δ^n. By definition $A^*(\Delta^n) := \Lambda(F\mathbb{Q}^{\oplus n})$, where $\mathbb{Q}^{\oplus n}$ is viewed as a graded vector space concentrated in degree 0 and $F\mathbb{Q}^{\oplus n}$ is the free cochain complex on it, which is just

$$\mathbb{Q}^{\oplus n} \xrightarrow{d=\mathrm{id}} \mathbb{Q}^{\oplus n} \to 0 \to 0 \to \ldots.$$

(1.3) **Example.** Clearly, for $n = 0$, we have $\mathbb{Q}^{\oplus 0} = 0$, whence $F\mathbb{Q}^{\oplus 0} = 0$ and so $A^*(\Delta^0)$ consists simply of the units $\mathbb{Q} = \Lambda^0(0)$ in degree 0.

(1.4) **Exercise.** Show that $A^*(\Delta^1)$ is of the form $\mathbb{Q}[t] \xrightarrow{d} \mathbb{Q}[t]dt \to 0 \to 0 \to \ldots$ with the differential $dp(t)$ of a polynomial given by

$$dp = \frac{\partial p}{\partial t} dt.$$

(1.5) **Exercise.** Convince yourself that $A^*(\Delta^2)$ is of the form

$$\mathbb{Q}[x,y] \xrightarrow{d} \mathbb{Q}[x,y]dx \oplus \mathbb{Q}[x,y]dy \xrightarrow{d} \mathbb{Q}[x,y]dxdy \to 0 \to 0 \to \ldots$$

with the differentials given by

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy \qquad \text{and} \qquad d(pdx + qdy) = \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dxdy.$$

2. Basic Properties of Minimal Models

Recall from section 1.2 the definition of a cdga (A, d) over $\mathbb{Q}$.

(2.1) **Definition.** A cdga (A, d) over $\mathbb{Q}$ is called *reduced* if $A^0 = \mathbb{Q}$. It is *1-reduced* if in addition $A^1 = 0$.

(2.2) **Remark.** As indicated in the introduction, the motivation for this definition comes from the fact that a simply connected space can be represented by a CW-complex with one 0-cell and no 1-cells (up to weak equivalence), or by a simplicial set with only one vertex (0-simplex) and only one degenerate 1-simplex. Such simplicial sets are sometimes also called 1-reduced. If X is a 1-reduced simplicial set, the cdga $A^*(X)$ defined in section 1.2 is quasi-isomorphic to a 1-reduced one.

(2.3) **Definition.** A 1-reduced cdga is called *minimal* iff it is (isomorphic to one) of the form $(\Lambda(V^*), d)$ where:

(a) V^* is a graded vector space with $V^0 = V^1 = 0$ (so that $\Lambda(V)$ is indeed 1-reduced; recall in particular that we add a unit in degree zero as part of the construction);

(b) the differential d has the property that $d(V) \subseteq \Lambda^{\geqslant 2}(V)$.

In other words, the differential of a generator $v \in V^k$ is never a generator, but rather always a sum of wedge-products, each of the form $v_1 \wedge \ldots \wedge v_p$, with $p \geqslant 2$, total degree $k+1$, and each v_i of degree $2 \leqslant |v_i| \leqslant k-1$. We see that for each k,

$$d(V^k) \subseteq \Lambda(V)^{k+1} \cap \Lambda^{\geqslant 2}(V) = \Lambda(V^{<k})^{k+1},$$

and that this condition is equivalent to condition (b) in the definition above. Note that since taking the differential strictly increases the word-length for all elements of V, it cannot possibly be induced by a differential coming from a cochain complex structure on V.

(2.4) **Definition.** For an arbitrary cdga A, a minimal model for A is a minimal cdga M together with a quasi-isomorphism $M \xrightarrow{\sim} A$.

We will prove the following three important properties of cdgas. Keeping our end-goal in mind that the model category of 1-reduced finite cdgas is homotopically dual to that of 1-reduced finite simplicial sets (all rationally), they tell us that minimal cdgas are dual to minimal Kan complexes.

(2.5) **Proposition.** Minimal cdgas have the following properties:

(a) *(Existence)* Every 1-reduced cdga A admits a minimal model $M \xrightarrow{\sim} A$.

(b) *(Cofibrancy)* If $p\colon B \to A$ is a surjective quasi-isomorphism between cdgas, any map $f\colon M \to A$ from a minimal cdga lifts to a $g\colon M \to B$, with $pg = f$.

(c) *(Rigidity)* Any quasi-isomorphism between minimal cdgas is an isomorphism.

For the corollary (2.11) below, which shows the uniqueness of minimal models, we will need a notion of homotopy for cdgas.

(2.6) **Definition.** For every cdga A, we can form the *path object* $P(A) := A \otimes A^*(\Delta^1)$, which comes equipped with maps

$$A \xrightarrow{c} P(A) \underset{\mathrm{ev}_1}{\overset{\mathrm{ev}_0}{\rightrightarrows}} A \qquad \text{(explicitly } c(a) := a \otimes 1,\ \mathrm{ev}_i(a \otimes (p + q dt)) := ap(i))$$

satisfying $\mathrm{ev}_0 \circ c = \mathrm{id}_A = \mathrm{ev}_1 \circ c$. A *homotopy* between two maps $f,\ g\colon B \rightrightarrows A$ is a map $h\colon B \to P(A)$ such that $f = \mathrm{ev}_0 \circ h$ and $g = \mathrm{ev}_1 \circ h$. If such a homotopy exists then we say that the two maps are *homotopic.*

(2.7) **Remark.** The elements of $P(A) = A \otimes A^*(\Delta^1) = A \otimes \Lambda(t, dt)$ can be identified with all $p(t) + q(t)dt$, where p and q are polynomials with coefficients in A, and t has degree 0. Under this identification, ev_0 and ev_1 are the evaluation maps sending each $p(t) + q(t)dt$ to $p(0)$ and $p(1)$ respectively, and c is the constant polynomial map $c(a) = a$.

(2.8) **Remark.** As in topology, every map $f\colon B \to A$ of cdgas can be replaced by a fibration (by which, in this context, we simply mean a surjection that is also a quasi-isomorphsim):

$$\begin{tikzcd} B \arrow[r, tail, "i"'] \arrow[dr, "f"'] & B \times_A P(A) \arrow[l, bend right, "r"'] \arrow[d, two heads, "\pi"] \\ & A \end{tikzcd} ,$$

where $B \times_A P(A)$ is obtained as the pullback

$$\begin{tikzcd} B \times_A P(A) \arrow[r, "\mathrm{pr}_2"] \arrow[d, two heads, "r"'] \arrow[dr, phantom, "\lrcorner", very near start] & P(A) \arrow[d, two heads, "\mathrm{ev}_0"] \\ B \arrow[r, "f"'] & A \end{tikzcd} ,$$

$\pi := \mathrm{ev}_1 \circ \mathrm{pr}_2$ and $i\colon b \mapsto (b, f(b))$ is the obvious section of r induced by the section c of ev_0.

(2.9) **Exercise.** Show that for B a cofibrant cdga (see (A.1.3)), our notion of homotopy for cdgas is an equivalence relation on the set $\mathrm{Hom}_{\mathbf{cdga}}(B, A)$. For this, use the subdivided interval simplicial set $X := \{\bullet \to \bullet \to \bullet\} = \Delta[1] +_{\Delta[0]} \Delta[1]$ (obtained by gluing the vertex 1 of a copy of $\Delta[1]$ to the vertex 0 of another copy) and note that we have inclusions

$$X \overset{\sim}{\hookrightarrow} \Delta[2] \overset{\sim}{\hookleftarrow} \Delta[1] \qquad \rightsquigarrow \qquad A^*(X) \overset{\sim}{\twoheadleftarrow} A^*(\Delta^2) \overset{\sim}{\twoheadrightarrow} A^*(\Delta^1).$$

Two compatible homotopies $B \rightrightarrows P(A)$ then give rise to $B \to A \otimes A^*(X)$ and we can use the cofibrancy of B to get back to $P(A)$.

(2.10) **Exercise.** Show that π is surjective by constructing a section, that $\pi \circ i = f$ and that r is a deformation retraction. *Hint: Indefinite integration $p(t) + q(t)dt \mapsto \int q(t)dt$ is a homotopy $c \circ \mathrm{ev}_0 \sim \mathrm{id}_A$.*

(2.11) **Corollary.** Let $p\colon M \xrightarrow{\sim} A \xleftarrow{\sim} N\colon q$ be two minimal models for a cdga A. Then there exists an isomorphism $\varphi\colon M \cong N$ with $q \circ \varphi$ homotopic to p.

Proof. Replace $q\colon N \to A$ with a surjective quasi-isomorphism π, as in remark (2.8). By cofibrancy (see (2.5)(b)), p lifts along π to give a map $\psi\colon M \to N \times_A P(A)$. Composing with the projection $\mathrm{pr}_1\colon N \times_A P(A) \to N$ gives the map $\varphi = \mathrm{pr}_1 \circ \psi\colon M \to N$. Since pr_1 is a deformation retract, $q \circ \varphi = q \circ \mathrm{pr}_1 \circ \psi \sim \pi \circ \psi = p$. The map is an isomorphism by rigidity (see (2.5)(c)). □

Proof (of (2.5)(a)). Let A be a 1-reduced cdga. We recursively build up a graded vector space V^*, a differential d on $\Lambda(V)$ and a quasi-isomorphism $\Lambda(V) \to A$. More precisely, we will recursively define $V^0, \dots, V^k$ and $d\colon V^{\leqslant k} \to \Lambda(V^{<k})$ together with a map of graded vector spaces $f_k\colon V^{\leqslant k} \to A$ such that the unique extension to an algebra map $\widetilde{f}_k\colon \Lambda(V^{\leqslant k}) \to A$ makes the following diagram commute:

$$\begin{tikzcd} V^{\leqslant k} \arrow[r, "f_k"] \arrow[d, "d"'] & A \arrow[d, "d"] \\ \Lambda(V^{<k}) \arrow[r, "\widetilde{f}_{k-1}"] & A. \end{tikzcd}$$

Moreover, to ensure that on the colimit $\Lambda(V) = \bigcup_k \Lambda(V^{\leqslant} k)$, the $\widetilde{f}_k$ together define a quasi-isomorphism $\tilde{f}$, we arrange that:

(a) $H^i(\Lambda(V^{\leqslant k})) \to H^i(A)$ is surjective in degrees $i \leqslant k$;

(b) $H^i(\Lambda(V^{\leqslant k})) \to H^i(A)$ is injective in degrees $i \leqslant k+1$.

For a start, we take $V^0 := V^1 := 0$, $V^2 := Z^2(A) \subseteq A^2$, and we define $d = 0$ on V^2. Then all desired properties are satisfied because in fact $V^2 \cong H^2(A)$ (since A is 1-reduced) and $\Lambda(V^{\leqslant 2})^3 = 0$, whence $H^3(\Lambda(V^{\leqslant 2}) = 0$.

Suppose we have defined everything up to but not including level k. Since V is 1-reduced, we have $\Lambda(V^{\leqslant k})^k = \Lambda(V^{<k})^k \oplus V^k$ and $\Lambda(V^{\leqslant k})^{k+1} = \Lambda(V^{<k})^{k+1}$. By minimality, we thus get the following diagram depicting the situation in degrees $k-1$, k and $k+1$:

$$\begin{tikzcd} V^{<k}: & \dots \arrow[r, "d"] & \Lambda(V^{<k})^{k-1} \arrow[r, "d"] \arrow[d, equal] & \Lambda(V^{<k})^k \arrow[r, "d"] \arrow[d, hook] & \Lambda(V^{<k})^{k+1} \arrow[r, "d"] \arrow[d, equal] & \dots \\ V^{\leqslant k}: & \dots \arrow[r, "d"'] & \Lambda(V^{<k})^{k-1} \arrow[r, "d"'] \arrow[ur, "d"] & \Lambda(V^{<k})^k \oplus V^k \arrow[r, "d"'] & \Lambda(V^{<k})^{k+1} \arrow[r, "d"'] & \dots \end{tikzcd}$$

In order to satisfy (b) for $i = k+1$, we need to make V^k sufficiently large and ensure that we obtain enough coboundaries in degree $k+1$ from it. To wit, we let

$$U := \operatorname{Ker}\left(Z^{k+1}(\Lambda(V^{<k})) \xrightarrow{\widetilde{f}_{k-1}} Z^{k+1}(A) \twoheadrightarrow H^{k+1}(A) \right).$$

Writing $B^{k+1}(A) := \operatorname{Im}(d\colon A^k \to Z^{k+1}(A)) = \operatorname{Ker}(Z^{k+1}(A) \twoheadrightarrow H^{k+1}(A))$ for the coboundaries in A. We note that U is mapped to $B^{k+1}(A)$ and so, we may choose a lift

$$\begin{tikzcd} & A^k \arrow[d, two heads] & \\ U \arrow[r] \arrow[ur, dotted] & B^{k+1}(A) \arrow[r, hook] & Z^{k+1}(A). \end{tikzcd}$$

This takes care of point (b) above. To also have (a) for $i = k$, we let $W \subseteq H^k(A)$ be a vector space complementary to the image of the composite

$$Z^k(\Lambda(V^{<k})) \xrightarrow{\widetilde{f}_{k-1}} Z^k(A) \twoheadrightarrow H^k(A)$$

(which is also the image of $H^k(\Lambda(V^{<k}))$ in $H^k(A)$) and choose a lift

$$\begin{tikzcd} & Z^k(A) \arrow[r, hook] \arrow[d] & A^k \\ W \arrow[ur, dotted] \arrow[r, hook] & H^k(A) & \end{tikzcd} .$$

With this, we put $V^k := U \oplus W$, with the differential being zero on W and the inclusion $U \hookrightarrow Z^{k+1}(\Lambda(V^<k)) \subseteq \Lambda(V^{<k})$ on U. The two chosen lifts $U \to A^k$ and $W \to A^k$ then assemble to the required map f_k extending f_{k-1} to V^k. With this, the desired properties (a) and (b) are readily verified. □

The proof of the cofibrancy property (b) in the above proposition (2.5) is based on the following simple observation.

(2.12) **Lemma.** Let $p\colon B \to A$ be a surjective quasi-isomorphism between cdgas (in fact, it is enough to have such a map of cochain complexes). Then, for each k, the map

$$(p, d)\colon B^k \to A^k \times_{Z^{k+1}(A)} Z^{k+1}(B)$$

is surjective.

Proof. An element of $A^k \times_{Z^{k+1}(A)} Z^{k+1}(B)$ consist of $a \in A^k$ and $b \in B^{k+1}$ with $pb = da$ and $db = 0$. Given such a pair, we are looking for some $c \in B^k$ with $pc = a$ and $dc = b$. First, b defines a cohomology class $[b] \in H^{k+1}(B)$ with $p_*[b] = [pb] = [da] = [0] \in H^{k+1}(A)$. Since p is a quasi-isomorphism, $[b] = 0$, meaning that there is an $x \in B^k$ with $dx = b$. Then $d(px - a) = dpx - da = pdx - pb = 0$. So $[px - a]$ defines a cohomology class, which must be in the image of $p_*\colon H^k(B) \cong H^k(A)$. Thus, we find $y \in B^k$ with $dy = 0$ and $p_*[y] = [py] = [px - a]$, i.e. $py = px - a - da'$ for some $a' \in A^{k-1}$. Since p is surjective, we can write $a' = pz$ for a suitable $z \in B^{k-1}$. Then $a = p(x - y - dz)$ and $d(x - y - dz) = b$. So $c := x - y - dz$ does the job. □

Proof (of (2.5)(b)). Write $M = (\Lambda(V^*), d)$ and consider a map $f\colon M \to A$. Then f is uniquely determined by $f_0\colon V \to A$ satisfying that for each k, the diagram

$$\begin{tikzcd} V^k \arrow[r, "f_0"] \arrow[d, "d"'] & A \arrow[d, "d"] \\ \Lambda(V^{<k})^{k+1} \arrow[r, "f"] & A^{k+1} \end{tikzcd}$$

commutes. We define a lift $g_0\colon V \to B$ with the same properties by defining $g_0\colon V^{\leqslant k} \to B$ recursively on k. If g_0 has been defined on $V^{<k}$ (along with its extension to an algebra map $g\colon \Lambda(V^{<k}) \to B$), define $g_0\colon V^k \to B^k$ to be any map of vector spaces making the following diagram commute.

$$\begin{tikzcd} V^k \arrow[r, dotted, "g_0"] \arrow[dr, "(f_0, g\circ d)"'] & B^k \arrow[d, two heads, "(p,d)"] \\ & A^k \times_{Z(A^{k+1})} Z(B^{k+1}). \end{tikzcd}$$

Such a map exists, by the lemma. □

The proof of the rigidity property (c) in the proposition (2.5) requires some further preparations.

(2.13) **Definition.** An *augmentation* of a cdga A is a map $\varepsilon_A\colon A \to \mathbb{Q}$ of graded algebras, where $\mathbb{Q}$ is viewed as a graded algebra concentrated in degree 0. An *augmented cdga* is such a pair (A, ε_A), although we usually just write A. Of course, every 0-reduced cdga has a unique augmentation given by the identity in degree 0.

Again thinking of cdgas as homotopically dual to simplicial sets when working rationally, augmented cdgas correspond to pointed spaces. And just like for spaces, we have a notion of pointed homotopies for augmented cdgas, using a reduced path object $P_\varepsilon(A)$ for an augmented cdga A. It is explicitly given by the following pullback:

$$\begin{tikzcd} P_\varepsilon(A) \arrow[r, tail] \arrow[d, "\varepsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \otimes \Lambda(t,dt) =: P(A) \arrow[d, "\varepsilon\otimes\mathrm{id}"] \\ \mathbb{Q} \arrow[r, tail] & \Lambda(t,dt). \end{tikzcd}$$

Thus, elements of $P_\varepsilon(A)$ are again of the form $p(t) + q(t)dt$ where $p(t)$ and $q(t)$ are polynomials with coefficients in A, but where, moreover, the induced polynomials $\varepsilon p(t)$ and $\varepsilon q(t)$ in $\mathbb{Q}[t]$ are respectively constant and identically zero. In other words, if $p(t) + q(t)dt$ is $a_0 + a_1 t + \cdots a_n t^n + b_0 dt + b_1 t dt + \cdots b_m t^m dt$, then $\varepsilon(a_i) = 0 = \varepsilon(b_j)$ for all i, j except possibly $i = 0$.

(2.14) **Exercise.** Check that $\mathbb{Q} \rightarrowtail \Lambda(t,dt)$ is a quasi-isomorphism and infer that so is the inclusion of the pullback $P_\varepsilon(A) \rightarrowtail P(A)$.

Just like in the unaugmented case, we have maps

$$\begin{tikzcd} A \arrow[r, tail, "c"'] & P_\varepsilon(A) \arrow[l, bend right, "r"'] \arrow[r, shift left, "\mathrm{ev}_0"] \arrow[r, shift right, "\mathrm{ev}_1"'] & A \end{tikzcd}$$

defined analogously but now, these are all maps of augmented algebras (i.e. algebra maps compatible with the respective augmentations).

(2.15) **Definition.** Two maps $f, g\colon B \rightrightarrows A$ between augmented algebras (i.e. algebra maps such that additionally, $\varepsilon_A f = \varepsilon_B = \varepsilon_A g$) are said to be *homotopic relative to augmentation*, briefly “rel ε”, iff there exists an $h\colon B \to P_\varepsilon(A)$ with $\mathrm{ev}_0 \circ h = f$ and $\mathrm{ev}_1 \circ h = g$ (and hence $\varepsilon_A \circ h = \varepsilon_B$). We write $f \sim g$ (rel ε) in this case.

(2.16) **Exercise.** Repeat the exercise (2.9) for augmented homotopies.

(2.17) **Lemma.** Let f, $g\colon (\Lambda(V), d) \rightrightarrows (\Lambda(W), d)$ be two maps between minimal cdgas. If h is a homotopy rel ε between f and g, then h restricts to a homotopy

$$h\colon \Lambda^{\geqslant 2}(V) \to \Lambda^{\geqslant 2}(W) \otimes \Lambda(t,dt)$$

of maps between non-unital cdgas $(\Lambda^{\geqslant 2}(V), d) \to (\Lambda^{\geqslant 2}(W), d)$.

Proof. Note first that f and g map the kernel of $\varepsilon\colon \Lambda(V) \to \mathbb{Q}$ (which is just $\Lambda^{\geqslant 1}(V)$) to that of $\Lambda(W) \to \mathbb{Q}$. In particular, $\Lambda^{\geqslant 2}(V)$ is mapped to $\Lambda^{\geqslant 2}(W)$. For a $v \in \Lambda^{\geqslant 1}(V)$, writing $h(v) = p_v(t) + q_v(t)dt$, we have $0 = \varepsilon(v) = \varepsilon(p_v(t) + q_v(t)dt) \in \mathbb{Q}(t, dt)$. In other words, $p_v(t)$ and $q_v(t)$ are polynomials with coefficients in $\Lambda^{\geqslant 1}(W)$. Thus h maps $\Lambda^{\geqslant 1}(V)$ into $\Lambda^{\geqslant 1}(W) \otimes \Lambda(t, dt)$, hence $\Lambda^{\geqslant 2}(V)$ into $\Lambda^{\geqslant 2}(W) \otimes \Lambda(t, dt)$, since h is an algebra map. □

Again as in the unaugmented case, any map $f\colon B \to A$ between two augmented algebras factors as

$$B \underset{\sim}{\overset{i}{\rightarrowtail}} B \times_A P_\varepsilon(A) \overset{\pi}{\twoheadrightarrow} A,$$

where $B \times_A P_\varepsilon(A)$ is the pullback of f along ev_0 and $i = (f, c)$ while $\pi = \mathrm{ev}_1 \circ \mathrm{pr}_2$.

(2.18) **Exercise.** Repeat the exercise (2.10) for maps between augmented algebras and homotopies relative to augmentation.

Moreover, if f is a quasi-isomorphism, so is its pullback $\mathrm{pr}_2\colon B \times_A P_\varepsilon(A) \to P_\varepsilon(A)$ and hence, so is π. We will use this construction in the following lemma.

(2.19) **Lemma.** Let $f\colon B \to M$ be a quasi-isomorphism into a minimal cdga. Then there exists a map $g\colon M \to B$ with $f \circ g \sim \mathrm{id}_M$ rel ε; i.e. f has a section up to homotopy rel ε.

Proof. We use the factorization above with M for A, and the cofibrancy of minimal cdgas (see (2.5)(b)) to find a lift as in

$$\begin{tikzcd} & B \times_M P_\varepsilon(M) \arrow[d, two heads, "\pi"] \\ M \arrow[ur, dotted] \arrow[r, "\mathrm{id}_M"'] & M \end{tikzcd} .$$

This lift is a pair (g, h), where $h\colon M \to P_\varepsilon(M)$ is a homotopy rel ε from the map

$$\mathrm{ev}_0 \circ h = \mathrm{ev}_0 \circ \mathrm{pr}_2 \circ (g, h) = f \circ \mathrm{pr}_1 \circ (g, h) = f \circ g$$

to the map $\mathrm{ev}_1 \circ \mathrm{pr}_2 \circ (g, h) = \pi \circ (g, h) = \mathrm{id}_M$. □

Proof (of (2.5)(c)). Let $f\colon M \to N$ be a quasi-isomorphism between minimal cdgas and write $M = (\Lambda(V), d)$, $N = (\Lambda(W), d)$. Also write $\overline{M} = \operatorname{Ker} \varepsilon$ and $QM = \overline{M}/(\overline{M} \cdot \overline{M})$; this vector space is called the space of indecomposable elements. Of course, if $M = \Lambda(V)$, then $\overline{M} = \Lambda^{\geqslant 1}(V)$ and QM is canonically isomorphic to V. With the same notations for N, we have a diagram

$$\begin{tikzcd} V \arrow[r, hook, "i"] & \overline{M} \arrow[r, two heads, "p"] \arrow[d, "f"] & QM \arrow[d, dotted, "\widetilde{f}"] \\ W \arrow[r, hook, "j"] & \overline{N} \arrow[r, two heads, "q"] & QN, \end{tikzcd}$$

where i and j are inclusions and f factors as $\widetilde{f}$ through the quotient maps p and q. Now consider the short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker} p \arrow[r] \arrow[d, "f^{\geqslant 2}"] & \overline{M} \arrow[r, "p"] \arrow[d, "f"] & QM \arrow[r] \arrow[d, "\widetilde{f}"] & 0 \\ 0 \arrow[r] & \operatorname{Ker} q \arrow[r] & \overline{N} \arrow[r, "q"] & QN \arrow[r] & 0. \end{tikzcd}$$

By lemma (2.19), f has a right inverse g up to homotopy rel ε. This inverse maps $\overline{M} = \Lambda^{\geqslant 1}(V)$ into $\overline{N}$ and $\operatorname{Ker} p = \Lambda^{\geqslant 2}(V)$ into $\operatorname{Ker} q = \Lambda^{\geqslant 2}(W)$, as any map of augmented algebras does. Moreover, by lemma (2.17) the homotopy restricts as well. Thus, the map $f_*\colon H^*(\operatorname{Ker} p) \to H^*(\operatorname{Ker} q)$ has a right inverse g_*, and hence f_* is surjective. The same argument gives that g_* is also surjective, so both are mutually inverse isomorphisms. Thus, in the diagram above, the map f is a quasi-isomorphism by assumption and $f^{\geqslant 2}$ is a quasi-isomorphism as well. By the 5-lemma, $\widetilde{f}$ must also be a quasi-isomorphism. But QM and QN are cochain complexes with zero differential, so $\widetilde{f}$ is itself an isomorphism. Notice that since canonically $V \cong QM$ and $W \cong QN$, this is an isomorphism $V \cong W$ but it need not be the restriction of f.

We now prove that f itself is an isomorphism $M \to N$. Let us first remark that it in fact suffices to prove that f is a surjection. Because in that case, the cofibrancy property (2.5)(b) provides us with a section $s\colon N \to M$ as a lift of id_N, and the same argument applied to s would then show that s is surjective as well. Hence s and f are both isomorphisms.

So, to conclude the proof, let us show that f is surjective. We proceed by induction, and prove that $\Lambda(W^{<k})$ is contained in the image of f for each $k > 0$. The first relevant case is $k = 3$, for which this is clear since $f\colon V^2 \to \Lambda(W)^2 = W^2$ is $H^2(V) \to H^2(W)$ and hence an isomorphism. Assume that $\Lambda(W^{<k})$ is contained in the image of f, and consider an arbitrary $w \in W^k$. Since $\widetilde{f}$ is an isomorphism, we can write $w = \widetilde{f}(v)$ for some $v \in V^k$, or $f(v) = w + x$ for some $x \in \Lambda(W^{<k})^k$. Since x is in the image of f, so is w. This proves that $W^k \subseteq \operatorname{Im}(f)$. Since f is an algebra map and $\Lambda(W^{<k}) \subseteq \operatorname{Im}(f)$, we conclude that $\Lambda(W^{\leqslant k}) \subseteq \operatorname{Im}(f)$. This completes the proof. □

3. Polynomial Differential Forms on Simplicial Sets

In the introduction, we sketched how one can mimic de Rham cohomology of manifolds, replacing smooth differential forms by polynomial ones with rational coefficients, and manifolds by simplicial sets. For the standard n-simplex

$$\Delta^n = \{t_0, \dots, t_n \mid t_i \geqslant 0, \Sigma t_i = 1\}$$

we defined the cdga $A^*(\Delta^n)$ of rational polynomial forms on Δ^n,

$$A^*(\Delta^n) = \Lambda(F\mathbb{Q}^{\oplus n}) = \Lambda(t_0, \dots, t_n, dt_0, \dots, dt_n)/(\Sigma t_i = 1, \Sigma dt_i = 0)$$

(cf. also section 1 of this chapter). Any non-decreasing function $\alpha\colon [n] \to [m]$ induces a map $\alpha\colon \Delta^n \to \Delta^m$ (denoted by the same letter). It is the unique affine map sending the i^{th} vertex $(0, \dots, 0, 1, 0, \dots, 0)$ of Δ^n (with a 1 in the i^{th} position) to the $\alpha(i)^{\text{th}}$ vertex of Δ^m. Explicitly,

$$\alpha(t_0, \dots, t_n)_j = \sum_{\alpha(i)=j} t_i \qquad (t = 0, \dots, m).$$

In calculations, one often works with faces $\delta_i\colon [n-1] \rightarrowtail [n]$, which skip i (for $i \in \{0, \dots, n\}$) and degeneracies $\sigma_j\colon [n] \twoheadrightarrow [n-1]$ which hit j twice (for $j \in \{0, \dots, n-1\}$). For these, one has:

$$\begin{aligned} &\delta_i\colon \Delta^{n-1} \rightarrowtail \Delta^n, && \delta_i(t_0, \dots, t_{n-1}) = (t_0, \dots, t_{t-1}, 0, t_i, \dots, t_{n-1});\\ &\sigma_j\colon \Delta^n \rightarrowtail \Delta^{n-1}, && \sigma_j(t_0, \dots, t_n) = (t_0, \dots, t_j + t_{j+1}, \dots, t_n). \end{aligned}$$

Each such affine map $\alpha\colon \Delta^n \to \Delta^m$ induces a map of cdgas

$$\alpha^*\colon A^*(\Delta^m) \to A^*(\Delta^n)$$

simply by pulling back forms (i.e. precomposing with α). This map α^* is uniquely determined by what it does on the coordinates $t_0, \ldots, t_m$ of Δ^m, by

$$\alpha^*(t_j) = \sum_{\alpha(i)=j} t_i$$

(where, on the right, the t_i are the coordinates on Δ^n). For a fixed degree p, this gives $A^p(\Delta^\bullet)$ the structure of a simplicial $\mathbb{Q}$-vector space, and $A^*(\Delta^\bullet)$ that of a simplicial cdga. We record in particular the simplicial face and degeneracy maps $\delta_i^*\colon A^p(\Delta^n) \to A^p(\Delta^{n-1})$ and $\sigma_j^*\colon A^p(\Delta^{n-1}) \to A^p(\Delta^n)$:

$$\delta_i^*(t_j) = \begin{cases} t_j & j < i \\ 0 & j = i \\ t_{j-1} & j < i \end{cases} \qquad (j \in \{0, \ldots, n\});$$

$$\sigma_i^*(t_j) = \begin{cases} t_j & j < i \\ t_i + t_{i+1} & j = i \\ t_{j+1} & j < i \end{cases} \qquad (j \in \{0, \ldots, n-1\}).$$

It is common to write

$$\begin{aligned} d_i &= \delta_i^*\colon A^p(\Delta^n) \to A^p(\Delta^{n-1}) \\ s_i &= \sigma_i^*\colon A^p(\Delta^{n-1}) \to A^p(\Delta^n). \end{aligned}$$

For an arbitrary simplicial set X, we defined the cdga $A^*(X)$ of polynomial forms on X as

$$A^*(X) := \mathbf{sSets}(X, A^*(\Delta^\bullet)).$$

Explicitly, an element $\omega \in A^p(X)$ (i.e. a *p-form* on X) is a family of forms $\omega_x \in A^p(\Delta^m)$, one for every simplex $x \in X_m$ with $m \geqslant 0$, that are compatible in the sense that

$$\omega_{\alpha^*(x)} = \alpha^*(\omega_x)$$

for each $\alpha\colon [n] \to [m]$ (where the α^* on the left refers to the induced map $X_m \to X_n$ and that on the right refers to the map $A^*(\Delta^m) \to A^*(\Delta^n)$). For a map $f\colon X \to Y$ of simplicial sets, there is a pullback operation on differential forms defined by

$$f^*(\omega)_x = \omega_{f(x)} \qquad \text{for } \omega \in A^*Y \text{ and } x \in X_m.$$

This is in fact a map of cdgas $f^*\colon A^*(Y) \to A^*(X)$.

(3.1) **Exercise.** Show that there is a quasi-isomorphism $A^*(X) \otimes A^*(Y) \to A^*(X \times Y)$.

Recall also that for each simplicial set X one has the classical simplicial cochain complex $C^\bullet(X; \mathbb{Q})$ defined by

$$C^p(X; \mathbb{Q}) = \operatorname{Hom}(X_p, \mathbb{Q})$$

and with differential $d\colon C^{p-1}(X; \mathbb{Q}) \to C^p(X; \mathbb{Q})$ defined by

$$d(x) = \sum_{i=0}^{n} (-1)^i d_i(x).$$

Its cohomology $H^*(X; \mathbb{Q})$ agrees with that of the geometric realization $H^*\big(|X|; \mathbb{Q}\big)$. There is a standard integration map

$$\int\colon A^*(X) \to C^*(X; \mathbb{Q}), \qquad \left(\int \omega\right)(x) := \int_{\Delta^p} \omega_x \quad \text{for } \omega \in A^p(X) \text{ and } x \in X_p.$$

(3.2) **Exercise.** Write an explicit formula for $\int$.

This is indeed a map of cochain complexes, because

$$\int_{\Delta^p} d\omega_x = \int_{\partial\Delta^p} \omega_x$$

by Stokes's theorem. In addition, the map is natural in X; i.e. for $f\colon X \to Y$, the diagram

$$\begin{tikzcd} A^*(Y) \arrow[r, "\int"] \arrow[d, "f^*"'] & C^*(Y;\mathbb{Q}) \arrow[d, "f^*"] \\ A^*(X) \arrow[r, "\int"] & C^*(X;\mathbb{Q}) \end{tikzcd}$$

commutes, as is immediate from the definitions. Our aim in this section is to prove the following.

(3.3) **Theorem.** For each simplicial set X, the map

$$\int : A^*(X) \to C^*(X;\mathbb{Q})$$

is a quasi-isomorphism.

For this we will need some preliminary results, which require some familiarity with the basics of the theory of simplicial sets and with the general method of *cellular induction*. General references include [10] (in German) and [4].

(3.4) **Proposition.** For a fixed degree p, the simplicial set $A^p(\Delta^\bullet)$ is a trivial Kan complex. That is to say, the unique map $A^p(\Delta^\bullet) \to \Delta[0]$ is a trivial fibration.

Proof. $A^p(\Delta^\bullet)$ is not just a simplicial set, of course, but a simplicial vector space. Any simplicial group – so in particular any simplicial vector space – is a Kan complex, and for any simplicial group, its homotopy groups are the homology groups of the associated chain complex. It thus suffices to prove that the associated chain complex

$$A^p(\Delta^0) \xleftarrow{\partial} A^p(\Delta^1) \xleftarrow{\partial} A^p(\Delta^2) \leftarrow \dots,$$

with $\partial\colon A^p(\Delta^n) \to A^p(\Delta^{n-1})$ defined by $\delta\omega = \sum_{i=0}^n (-1)^i \delta_i^*\omega$, has trivial homology. This follows from the existence of a contracting homotopy

$$h\colon A^p(\Delta^n) \to A^p(\Delta^{n+1}) \qquad \text{satisfying} \qquad h\circ\partial + \partial\circ h = \mathrm{id}.$$

This map h is defined as a map of non-unital cdgas as follows:

$$h(1) := (1-t_0)^2, \qquad h(t_i) := (1-t_0)t_{i+1} \qquad (\text{for } i \in \{0,\dots,n\}).$$

One checks that $\delta_0^* h = \mathrm{id}$ and $\delta_{i+1}^* h = h\delta_i^*$, from which it follows that $h\circ\partial + \partial\circ h = \mathrm{id}$. $\square$

(3.5) **Corollary.** For any injective map $f\colon X \rightarrowtail Y$ of simplicial sets, the pullback of polynomial forms $f^*\colon A^*(Y) \to A^*(X)$ is surjective.

Proof. A set of generating cofibrations $\mathcal{I}$ in **sSets** consists of the inclusions $\partial\Delta[n] \hookrightarrow \Delta[n]$ and that the class of relative cell complexes $\mathrm{Cell}(\mathcal{I})$ consists of all monomorphisms of simplicial sets. In other words, every monomorphism can be obtained as a transfinite composition of pushouts of inclusions $\partial\Delta[n] \hookrightarrow \Delta[n]$.

Therefore, it is enough to prove the result for such boundary inclusions and that it is stable under the dual of the two colimit constructions (since A^* sends pushouts to pullbacks and sequential colimits to sequential limits). This is clear sinc in the category of cdgas, the surjections are fibrations for the transferred model structure (cf. (A.5.2)), which are stable under pullback and sequential limits, so the second requirement is met.

To prove that $F_n^*\colon A^*(\Delta[n]) \to A^*(\partial\Delta[n])$ is surjective, it is enough, by the Yoneda Lemma, to show that every map $\eta\colon \partial\Delta[n] \to A^p(\Delta^\bullet)$ extends to a map $\omega\colon \Delta[n] \to A^p(\Delta^\bullet)$, such that $\omega \circ F_n = \eta$. But this is an immediate consequence of the fact that $A^p(\Delta^\bullet)$ is a trivial Kan complex, as we showed in (3.4). □

(3.6) **Proposition.** For a fixed simplicial degree n, the cdga $A^*(\Delta^n)$ has trivial cohomology, i.e.

$$H^p(A^*(\Delta^n);\mathbb{Q}) = \begin{cases} \mathbb{Q} & p = 0 \\ 0 & p > 0. \end{cases}$$

Proof. Write $A^*(\Delta^n)$ as the free cdga on generators $t_1, \dots, t_n$ in degree 0 by throwing away t_0 and dt_0 using the given relations:

$$\begin{aligned} A^*(\Delta^n) &= \Lambda(t_0, t_1, \dots, t_n, dt_0, dt_1, \dots, dt_n)/(\Sigma t_i = 1, \Sigma dt_i = 0) \\ &= \Lambda(t_1, \dots, t_n, dt_1, \dots, dt_n). \end{aligned}$$

Now notice that this is the coproduct of n copies of $\Lambda(t, dt)$, and that in the category of cdgas the coproduct is the same as the tensor product:

$$A^*(\Delta^n) = \bigotimes_{i=1}^{n} \Lambda(t, dt) = \bigotimes_{i=1}^{n} A^*(\Delta^1).$$

Clearly $A^*(\Delta^1)$ has the desired property, and the proposition follows from Künneth. □

(3.7) **Exercise.** A more geometric reason for the previous result is that Δ^n is contractible. More generally, mimic the proof of the Poincaré lemma and show that if $h\colon \Delta[1] \times X \to Y$ is a simplicial homotopy between two maps $f = h_0$ and $g = h_1$, then

$$f^* = g^*\colon H^p(A^*(Y)) \to H^p(A^*(X)).$$

Proof (of theorem (3.3)). We use induction on the skeleta of X (this is another example of cellular induction, which in this case gives a result for all X as every simplicial set is cellular in the sense described below). We write $X^{(n)}$ for the n-skeleton of X. Then

$$X^{(0)} \subseteq X^{(1)} \subseteq X^{(2)} \subseteq \ldots \subseteq X = \bigcup_{n \geqslant 0} X^{(n)}$$

and we have a ladder

$$\begin{tikzcd} A^*(X^{(0)}) \arrow[d] & A^*(X^{(1)}) \arrow[l] \arrow[d] & A^*(X^{(2)}) \arrow[l] \arrow[d] & \ldots \arrow[l] \\ C^*(X^{(0)};\mathbb{Q}) & C^*(X^{(1)};\mathbb{Q}) \arrow[l] & C^*(X^{(2)};\mathbb{Q}) \arrow[l] & \ldots \arrow[l] \end{tikzcd}$$

with $A^*(X) = \lim_n A^*(X^{(n)})$ and $C^*(X;\mathbb{Q}) = \lim_n C^*(X^{(n)};\mathbb{Q})$. All the upper horizontal maps are surjections by corollary (3.5), and one easily checks that the same is true for the lower horizontal maps.

This is important insofar as we need it in order to be able to pass to the limit. To wit, by the exercise (3.8) below, when passing to the limit, we obtain a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \lim\limits_n A^*(X^{(n)}) \arrow[r] \arrow[d] & \prod\limits_n A^*(X^{(n)}) \arrow[r] \arrow[d, "\sim"'] & \prod\limits_n A^*(X^{(n)}) \arrow[r] \arrow[d, "\sim"'] & 0 \\
0 \arrow[r] & \lim\limits_n C^*(X^{(n)};\mathbb{Q}) \arrow[r] & \prod\limits_n C^*(X^{(n)};\mathbb{Q}) \arrow[r] & \prod\limits_n C^*(X^{(n)};\mathbb{Q}) \arrow[r] & 0.
\end{tikzcd}$$

Here, the two vertical maps on the right are quasi-isomorphisms because cohomology commutes with products. Using the 5-lemma, it then follows by comparing the long exact sequences in cohomology that the left-hand vertical map is also a quasi-isomorphism.

Having dealt with the passage to the limit, all that is left to do is prove the result for each $X^{(n)}$, which is obtained from $X^{(n-1)}$ as a pushout of the form

$$\begin{tikzcd}
\coprod \partial\Delta[n] \arrow[r] \arrow[d, tail] & X^{(n-1)} \arrow[d, tail] \\
\coprod \Delta[n] \arrow[r] & X^{(n)}
\end{tikzcd} .$$

For $n = 0$, the simplicial set $X^{(0)}$ is a coproduct of copies of $\Delta[0]$, and the claim trivial. Now suppose that for each X the claim is true for $X^{(n-1)}$. The functors A^* and C^* send monomorphisms to epimorphisms, pushouts to pullbacks, and coproducts to products, so we find ourselves in the following situation:

$$\begin{tikzcd}
& \prod A^*(\partial\Delta[n]) \arrow[dl, "\sim"'] & & A^*(X^{(n-1)}) \arrow[ll] \arrow[dl, "\sim"'] \\
\prod C^*(\partial\Delta[n];\mathbb{Q}) & & C^*(X^{(n-1)};\mathbb{Q}) \arrow[ll] & \\
& \prod A^*(\Delta[n]) \arrow[uu, two heads] \arrow[dl, "\sim"'] & & A^*(X^{(n)}) \arrow[ll] \arrow[uu, two heads] \arrow[dl] \\
\prod C^*(\Delta[n];\mathbb{Q}) \arrow[uu, two heads] & & C^*(X^{(n)};\mathbb{Q}) \arrow[ll] \arrow[uu, two heads] &
\end{tikzcd} ,$$

where all maps from the back to the front face are integration maps. Now the map $\prod A^*(\Delta[n]) \to \prod C^*(\Delta[n];\mathbb{Q})$ is a quasi-isomorphism by proposition (3.6) and the other two maps indicated are quasi-isomorphisms by the inductive hypothesis (note that $\partial\Delta[n] = \Delta[n]^{(n-1)}$ is an $(n-1)$-skeleton). With this, the result follows from Exercise (3.9) (which is in fact a special case of a model categorical statement). $\square$

(3.8) **Exercise.** Show that every tower of surjective morphisms between cochain complexes $V_0 \twoheadleftarrow V_1 \twoheadleftarrow V_2 \twoheadleftarrow \ldots$ gives rise to a short exact sequence

$$0 \to \lim_n V_n \to \prod_n V_n \xrightarrow{\mathrm{id}-i} \prod_n V_n \to 0,$$

where $i\colon \prod_n V_n \to \prod_n V_n$ is the shift map, sending the $(n+1)^{\text{st}}$ factor V_{n+1} to the n^{th} factor V_n via $V_{n+1} \twoheadrightarrow V_n$.

(3.9) **Exercise (Dual Gluing Lemma).** Consider two cospans

$$\begin{tikzcd} A' \arrow[r, two heads] \arrow[d, "\sim"'] & A \arrow[d, "\sim"'] & A'' \arrow[l] \arrow[d, "\sim"'] \\ B' \arrow[r, two heads] & B & B'' \arrow[l] \end{tikzcd}$$

of cochain complexes of $\mathbb{Q}$-vector spaces, in which the two rightwards-pointing maps are surjections and all vertical maps are quasi-isomorphism. Then the map between the pullbacks of the two two cospans is again a quasi-isomorphism.

(3.10) **Corollary.** If $f\colon X \to Y$ is a weak equivalence between simplicial sets then

$$f^*\colon A^*(Y) \to A^*(X)$$

is a quasi-isomorphism.

Proof. This follows from the corresponding fact for $C^*(Y;\mathbb{Q}) \to C^*(X;\mathbb{Q})$, which computes $H^*(Y;\mathbb{Q}) \to H^*(X;\mathbb{Q})$. □

(3.11) **Remark.** Call a simplicial set simply connected if $|X|$ is a simply connected space. Then, clearly, the corollary can be strengthened to one direction of the following corollary. Indeed, by (1.1.2), a map between simply connected spaces is a rational equivalence iff it induces isomorphisms in rational cohomology.

(3.12) **Corollary.** For a map $X \to Y$ between simply connected simplicial sets, the induced map $A^*(Y) \to A^*(X)$ is a quasi-isomorphism iff $|X| \to |Y|$ is a rational equivalence.

(3.13) **Remark.** For a topological space X, the singular complex $\mathrm{Sing}(X)$ is a simplicial set for which $|\mathrm{Sing}(X)|$ is weakly homotopy equivalent to X. Let us define the algebra of polynomial forms on X as $A^*(X) := A^*(\mathrm{Sing}(X))$. Then a map $X \to Y$ of simply connected topological spaces is a rational equivalence iff $A^*(Y) \to A^*(X)$ is a quasi-isomorphism.

4. The Space Associated to a Minimal CDGA

Recall from earlier sections, that we have a pair of adjoint functors $A^*\colon \mathbf{sSets} \rightleftarrows \mathbf{cdga}^{\mathrm{op}} : K$ (left adjoint on the left). Moreover, there are model structures (cf. the appendix) on both categories. Explicitly, the cofibrations in **sSets** are the monomorphisms, while the weak equivalences are defined via geometric realisation. For **cdga**, the fibrations are the surjections (and hence these are the cofibrations in $\mathbf{cdga}^{\mathrm{op}}$), while the weak equivalences are the quasi-isomorphisms. We also showed in (2.5) that minimal cdgas are cofibrant in **cdga** (whence fibrant in $\mathbf{cdga}^{\mathrm{op}}$). As seen in the last section, A^* maps monomorphisms to surjections and weak equivalences to quasi-isomorphisms, meaning that the above is in fact a Quillen adjunction.

Now a pointed simplicial set is simply an object in $\mathbf{sSets}_* := \Delta[0] \downarrow \mathbf{sSets}$ (the point $\Delta[0]$ is a terminal object in **sSets**), meaning a simplicial set together with a chosen base-vertex. On the other hand, since the terminal object in $\mathbf{cdga}^{\mathrm{op}}$ is the initial object $\mathbb{Q}$ in **cdga** (viewed as concentrated in degree 0), a pointed object in $\mathbf{cdga}^{\mathrm{op}}$ is simply an *augmented cdga*, i.e. a cdga A, together with an algebra map $\varepsilon\colon A \to \mathbb{Q}$ (cf. Definition (2.13)). We define $\mathbf{cdga}_* := \mathbf{cdga} \downarrow \mathbb{Q}$ to be the category of augmented cdgas. Since $A^*(\Delta[0]) \cong \mathbb{Q}$

and $K(\mathbb{Q})_\bullet = \mathbf{cdga}(\mathbb{Q}, A^*(\Delta^\bullet)) \cong \Delta[0]$, every pointed simplicial set $\Delta[0] \to X$ is mapped to an augmented cdga $A^*(X) \to \mathbb{Q}$ and vice versa, meaning we get an induced adjunction

$$A^* \colon \mathbf{sSets}_* \rightleftarrows (\mathbf{cdga}_*)^{\mathrm{op}} : K.$$

This is again a Quillen adjunction with respect to the induced model structures on $\mathbf{sSets}_*$ and $\mathbf{cdga}_*$ (cf. (A.1.5)). Consequently, we have a derived adjunction

(4.1) $$\mathbb{L}A^* = \mathrm{Ho}\, A^* \colon \mathrm{Ho}(\mathbf{sSets}_*) \rightleftarrows \mathrm{Ho}(\mathbf{cdga}_*)^{\mathrm{op}} : \mathbb{R}K.$$

For the following theorem, recall that a simply connected space X is called rational iff its homotopy groups (or equivalently reduced integral homology groups) are rational vector spaces. We will apply the same terminology to simplicial sets. That is, a simplicial set is called *rational* iff its geometric realisation is.

(4.2) **Theorem.** The derived adjunction (4.1) above restricts to an equivalence on the following two full subcategories (of $\mathrm{Ho}(\mathbf{sSets}_*)$ and $\mathrm{Ho}(\mathbf{cdga}_*)^{\mathrm{op}}$ respectively):

(a) on the left, simply connected rational pointed simplicial sets X such that $H^p(X;\mathbb{Q})$ is finite-dimensional in each degree p;

(b) on the right, augmented cdgas B that are quasi-isomorphic to a 1-reduced one and with the same property that $H^p(B)$ is finite-dimensional for all p.

(4.3) **Remark.** Observe that the functor $A^* \colon \mathbf{sSets}_* \to (\mathbf{cdga}_*)^{\mathrm{op}}$ does indeed restrict to these subcategories since $H^p(A^*(X)) \cong H^p(X;\mathbb{Q})$ as seen in the last section. We shall show the same for K later on. For the time being, we take it for granted and proceed with the proof of the theorem.

Proof. We need to show that for X a simplicial set and B a cdga, both from the respective full subcategories indicated above, the derived unit and counit

$$\bar{\eta} \colon X \to (\mathbb{R}K)(A^*(X)) \qquad \text{and} \qquad \bar{\varepsilon} \colon B \to A^*((\mathbb{R}K)(B))$$

are isomorphisms in $\mathrm{Ho}(\mathbf{sSets}_*)$ and $\mathrm{Ho}(\mathbf{cdga}_*)$, respectively. Since we can pick any cofibrant replacement for B in $\mathbf{cdga}$, we might just as well take a minimal model $M \xrightarrow{\sim} B$. To see that such a minimal model exists for B an object in the above subcategory of $\mathbf{cdga}$, let B_{red} be the *1-reduction* of B given by $B^0_{\mathrm{red}} := \mathbb{Q}$, $B^1_{\mathrm{red}} := 0$, $B^2_{\mathrm{red}} := B^2/\operatorname{Im}(d \colon B^1 \to B^2)$ and $B^n_{\mathrm{red}} := B^n$ for $n > 2$. We have an obvious map $B \to B_{\mathrm{red}}$, which is a quasi-isomorphism because B is quasi-isomorphic to a 1-reduced cdga (in particular $H^0(B) \cong \mathbb{Q}$ and $H^1(B) \cong 0$). Now B_{red} admits a minimal model by (2.5) and hence, so does B.

All in all, for X a simplicial set of the subcategory in question, $e \colon M \xrightarrow{\sim} A^*(X)$ a minimal model and N any minimal cdga, we need to show that the derived unit $\bar{\eta}$ and the derived counit $\bar{\varepsilon}$ given by

$$\bar{\eta} \colon X \xrightarrow{\eta} KA^*X \xrightarrow{Ke} KM, \qquad \bar{\varepsilon} = \varepsilon \colon N \to A^*KN$$

are, respectively, a weak equivalence and a quasi-isomorphism. For this, we apply A^* to the derived unit and consider the commutative diagram

$$\begin{tikzcd} A^*X & A^*KA^*X \arrow[l, "A^*\eta"'] & A^*X \arrow[l, "\varepsilon_{A^*X}"'] \\ & A^*KM \arrow[ul, "A^*\bar{\eta}"] \arrow[u, "A^*Ke"'] & M \arrow[l, "\varepsilon_M"'] \arrow[u, "e", "\sim"'] \end{tikzcd}.$$

The horizontal composite at the top is the identity by a triangle identity and hence, $A^*\bar{\eta}$ is a quasi-isomorphism iff ε_M is one. Using (3.12), this then implies that $\bar{\eta}$ is a rational equivalence, whence even a weak equivalence because X and KM are rational.

From the construction of minimal models on page 23, we see that if a 1-reduced cdga B has $H^*(B)$ finite-dimensional in each degree, then it admits a minimal model of the form $M = (\Lambda(V), d)$, where $V^0 = V^1 = 0$ and V^n is finite-dimensional for each $n > 1$. Together with what we observed above, we have reduced the proof of the theorem to the following proposition. □

(4.4) **Proposition.** For every minimal cdga $M = \big(\Lambda(V), d\big)$ with V^n finite-dimensional for every $n > 1$, the simplicial set KM is simply connected and rational, and the counit $\varepsilon_M \colon M \to A^*KM$ is a quasi-isomorphism.

As a preparation for the proof of this proposition, let us examine the space KM for $M = \big(\Lambda(M), d\big)$ a little more closely. Writing $M_{(n)} := \big(\Lambda(V^{\leqslant n}), d\big)$, we obtain a filtration

$$\mathbb{Q} = M_{(0)} = M_{(1)} \subseteq M_{(2)} \subseteq M_{(3)} \subseteq \dots \qquad \text{with colimit} \qquad M = \bigcup_n M_{(n)}$$

and where every inclusion $M_{(n)} \hookrightarrow M_{(n+1)}$ is a cofibration of cdgas, as seen in the proof of (2.5)(b) on page 24. Moreover, for every $n > 1$, there is a pushout of cdgas

$$\begin{tikzcd} M_{(n-1)} \arrow[r, tail] \arrow[d] & M_{(n)} \arrow[d] \\ \mathbb{Q} \arrow[r, tail] & \big(\Lambda(V^n), 0\big) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} .$$

Since K is a right Quillen functor, it turns this into a tower of fibrations

$$\Delta[0] = KM_{(0)} = KM_{(1)} \twoheadleftarrow KM_{(2)} \twoheadleftarrow KM_{(3)} \twoheadleftarrow \dots \qquad \text{with limit } KM$$

and pullbacks

$$\begin{tikzcd} K\big(\Lambda(V^n), 0\big) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & KM_{(n)} \arrow[d, two heads] \\ \Delta[0] \arrow[r] & KM_{(n-1)} \end{tikzcd} .$$

The observant reader with notice that this looks suspiciously like the Postnikov tower of KM, which is indeed the case as we shall prove now.

(4.5) **Lemma.** The Kan complex $K\big(\Lambda(V^n), 0\big)$ is an Eilenberg-MacLane space of type $K(\pi, n)$ for $\pi = (V^n)^\vee$, the dual of V^n. Consequently, for the tower of fibrations

$$KM_{(0)} = KM_{(1)} \twoheadleftarrow KM_{(2)} \twoheadleftarrow KM_{(3)} \twoheadleftarrow \dots \twoheadleftarrow KM,$$

we have

$$\pi_k(KM_{(n)}) = \pi_k(KM) \cong (V^k)^\vee \text{ for } k \leqslant n \qquad \text{and} \qquad \pi_k(KM_{(n)}) = 0 \text{ for } k > n.$$

In particular, KM is a simply connected and rational simplicial set.

Proof. The second part of the lemma follows from the long exact sequence of a fibration and the exercise below. As for the actual calculation of the homotopy groups, the case $n = 1$ is trivial, because there, $V^1 = 0$ and hence $K(\Lambda(V^1), 0) = K\mathbb{Q} = \Delta[0]$. For $n > 1$, we pick simplicial sets S^k, whose geometric realisations $|S^k|$ are equivalent to the k-spheres (e.g. $S^k = \Delta[k]/\partial\Delta[k]$) and have

$$\pi_k K(\Lambda(V^n), 0) = [S^k, K(\Lambda V^n, 0)] \cong [(\Lambda V^n, 0), A^*(S^k)] \cong [V^n, A^*(S^k)],$$

where V^n as a cochain complex concentrated in degree n and where the brackets denote Hom-sets in $\mathrm{Ho}(\mathbf{sSets}_*)$, $\mathrm{Ho}(\mathbf{cdga}_*)$ and $\mathrm{Ho}(\mathbf{Ch}^{\geqslant 0})$, respectively. We can do this by taking derived adjunctions (as we did above) and noting that the $(\Lambda(V^n), 0)$ are cofibrant, while in $\mathbf{Ch}^{\geqslant 0}$, all objects are cofibrant (same as in $\mathbf{sSets}$) and fibrant.

Since these are just Hom-sets in a category, we may replace $A^*(S^k)$ by any isomorphic object (in $\mathrm{Ho}(\mathbf{Ch}^{\geqslant 0})$, which is to say quasi-isomorphic). Let us pick $\mathbb{S}^k$, defined to be $\mathbb{Q}$ in degrees 0 and k, and 0 everywhere else. This is quasi-isomorphic to $A^*(S^k)$ by (3.3). Now

$$\pi_k K(\Lambda(V^n), 0) \cong [V^n, \mathbb{S}^k]$$

and we note that for $k \neq n$, all maps $V^n \to \mathbb{S}^k$ are 0, while for $k = n$, there are maps $V^n \to \mathbb{S}^k$ but no chain-homotopies between such. Therefore, we conclude that

$$\pi_k K(\Lambda(V^n), 0) = 0 \quad \text{for } k \neq n \text{ and} \quad \pi_n K(\Lambda(V^n), 0) \cong \mathrm{Hom}(V^n, \mathbb{Q}) = (V^n)^\vee.$$

However, all we have established is that there is such a bijection and some argument is needed why this is even an isomorphism of groups. For this, we consider the diagonal map $\Delta\colon V^n \to V^n \oplus V^n$, which induces a coalgebra structure

$$\Lambda(V^n) \to \Lambda(V^n \oplus V^n) \cong \Lambda(V^n) \otimes \Lambda(V^n).$$

Applying the contravariant functor K, which maps coproducts to products, we see that $K(\Lambda(V^n), 0)$ is a topological monoid. By the Eckmann-Hilton argument, the usual group structure on $\pi_n K(\Lambda(V^n), 0)$ is the same as the one induced by this monoid structure. On the other side of the bijection, the group structure on $(V^n)^\vee$ induced by Δ is the usual one (i.e. pointwise addition of functions). □

Before tackling the proof of (4.4), we recall the *Gysin sequence* of a vector bundle. For more details on this, see for example [8]. Given a vector bundle

$$\mathbb{R}^r \to V \xrightarrow{p} B,$$

we consider the pair (V, V_0), where $V_0 \subseteq V$ is the subbundle of non-zero vectors. The fibres of this pair of bundles then are

$$(\mathbb{R}^r, \mathbb{R}^r \setminus \{0\}) \simeq (D^r, D^r \setminus \{0\}) \simeq (D^r, S^{r-1}) \simeq (S^r, *),$$

where D^r is the unit disc in $\mathbb{R}^r$. If $V \to B$ is orientable (e.g. if B is simply connected), there is a fundamental class $\xi \in H^r(V, V_0)$ that restricts to a generator of the cohomology in each fibre and induces isomorphisms (called *integration along the fibre*)

$$H^{i+r}(V, V_0) \xrightarrow{\cong} H^i(B) \qquad \text{with inverse } \alpha \mapsto p^*(\alpha) \cdot \xi.$$

This isomorphism, together with the fact that p is a homotopy equivalence allows us to rewrite the long exact sequence of the pair (V, V_0) as follows:

$$\begin{tikzcd}
\dots \arrow[r] & H^{i+r}(V, V_0) \arrow[r] & H^{i+r}(V) \arrow[r] & H^{i+r}(V_0) \arrow[r] & H^{i+r+1}(V, V_0) \arrow[r] & \dots \\
\dots \arrow[r] & H^i(B) \arrow[r] \arrow[u, "\sim"'] & H^{i+r}(B) \arrow[r] \arrow[u, "p^*", "\sim"'] & H^{i+r}(V_0) \arrow[r] \arrow[u, equal] & H^{i+r+1}(B) \arrow[r] \arrow[u, "\sim"'] & \dots .
\end{tikzcd}$$

The lower exact sequence is called the *Gysin sequence* of the vector bundle and the map $H^i(B) \to H^{i+r}(B)$ is given by multiplication by $(p^*)^{-1}(\xi)$.

If $S^{r-1} \to E \to B$ is a sphere bundle (rather than a vector bundle) with simply connected base B (or more generally orientable), then one can mimic the above construction. One way to do this is to first construct a vector bundle V over B by taking the pushout

$$\begin{tikzcd}
E \times \{0\} \arrow[r] \arrow[d, tail] & B \times \{0\} \arrow[d, tail] \\
E \times I \arrow[r] & V \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd} ,$$

where then $E \times \{1\} \simeq V_0 \subseteq V$, so that we get a long exact sequence

$$\ldots \to H^i(B) \to H^{i+r}(B) \to H^{i+r}(E) \to H^{i+1}(B) \to \ldots$$

as above, which is the *Gysin sequence* of the sphere bundle $S^{r-1} \to E \to B$.

Proof (of (4.4)). Consider the tower $\Delta[0] = KM_{(0)} = KM_{(1)} \twoheadleftarrow KM_{(2)} \twoheadleftarrow KM_{(3)}$ from above with limit KM. Since the universal maps $KM \to KM_{(n)} \to KM_{(n-1)}$ induce isomorphisms on π_k for small k (more precisely $k \leqslant n-1$), they also induce isomorphisms in (rational) cohomology for small k by Hurewicz. In other words, for small k, the sequence

$$\ldots \to A^k(KM_{(n)}) \to A^k(KM_{(n+1)}) \to \ldots \to A^k(KM)$$

eventually consists of quasi-isomorphisms and hence, the map $\operatorname{colim}_n A^*(KM_{(n)}) \to A^*(KM)$ is a quasi-isomorphism. Consequently, to prove our claim that $\varepsilon\colon M \to A^*KM$ is a quasi-isomorphism, it suffices to check that $\varepsilon\colon M_{(n)} \to A^*KM_{(n)}$ is one for all n. So let us consider $M_{(n)} = \big(\Lambda(V^{\leqslant n}), d\big)$ and argue by induction on n and the dimension of V^n.

For $n = 0$ and $n = 1$, there is nothing to do because $V^0 = V^1 = 0$. Assuming the claim is shown for all $M_{(m)}$ with $m < n$, then again the case where $V^n = 0$ is trivial because then $M_{(n-1)} = M_{(n)}$. So suppose the claim is true for $V^{<n} \oplus W$, with $W \subseteq V^n$ an arbitrary proper subspace. We write $V^n := W^n \oplus \mathbb{Q}v$ for some non-zero $v \in V^n$, $W := V^{<n} \oplus W^n$ and consider the following pushout:

$$\begin{tikzcd}
\partial B := \Lambda S(n+1) \arrow[r, "dv"] \arrow[d, tail] & (\Lambda W, d) =: N \arrow[d, tail] \\
B := \Lambda D(n+1) \arrow[r, "{v,dv}"'] & \big(\Lambda(V^{\leqslant n}), d\big) =: M \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd} ,$$

where $S(n+1)$ and $D(n)$ are as in (A.4.7) ($S(n+1)$ is $\mathbb{Q}$ in degree $n+1$ and 0 everywhere else, while $D(n+1)$ is $\mathbb{Q}$ in degrees n and $n+1$ with the identity differential between them and 0 everywhere else), while the labels on the arrows denote the images of the corresponding generators. Then the counits fit into a cube

$$\begin{tikzcd}
& \partial B \arrow[rr] \arrow[dl] \arrow[dd] & & N \arrow[dl, "\sim"'] \arrow[dd] \\
A^*K(\partial B) \arrow[rr] \arrow[dd] & & A^*KN \arrow[dd] & \\
& B \arrow[rr] \arrow[dl] & & M \arrow[dl] \\
A^*KB \arrow[rr] & & A^*KM &
\end{tikzcd} ,$$

where the back face is a (homotopy) pushout and the counit $N \to A^*KN$ is a quasi-isomorphism by the inductive hypothesis. It would then follow from a standard cube argument already used (in its dual form) in the proof of (3.3), that the counit $M \to A^*KM$ is a quasi-isomorphism, too, provided

(a) the counit $\varepsilon\colon B \to A^*KB$ is a quasi-isomorphism;

(b) the counit $\varepsilon\colon \partial B \to A^*K(\partial B)$ is a quasi-isomorphism;

(c) the front square of the cube is a homotopy pushout.

Property (a) holds because the unit $\mathbb{Q} \to B$ is a quasi-isomorphism and so KB is a contractible Kan complex. The proofs of (b) and (c) are harder and somewhat beyond the scope of this short course. We will sketch the proof of (b) and quote a result giving the proof of (c).

For the latter, note that $K(\partial B)$ is a $K(\pi, m)$-space for $\pi = \mathbb{Q}^\vee \cong \mathbb{Q}$ as seen in Lemma (4.5) and $m = n+1 \geqslant 3$. So K maps the back of the cube to a pullback of the form

$$\begin{tikzcd} Y \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads] \\ X \arrow[r] & K(\mathbb{Q}, m) \end{tikzcd}$$

with the vertical maps being fibrations, E a contractible Kan complex and $H^*(A^*X)$ finite-dimensional in each degree (by the inductive hypothesis applied to $X = KN$). An Eilenberg-Moore spectral sequence argument shows that A^* maps such a square to a homotopy pushout (see [1, Lemma 10.5] for details).

Assertion (b) on the other hand says that $\Lambda S(m) \to h^*(K(\mathbb{Q}, m))$ is an isomorphism (at least for $m \geqslant 3$), where h^* is rational cohomology. Notice that $\Lambda S(m)$ is a polynomial ring $\mathbb{Q}[\gamma^m]$ on a generator γ in degree m if m is even and a ring of dual numbers $\mathbb{Q}[\varepsilon^m]$ on a generator ε in degree m with $\varepsilon^2 = 0$ if m is odd. We will now sketch why the same is true for $h^*(K(\mathbb{Q}, m))$.

First, we may replace $K(\mathbb{Q}, m)$ by $K(\mathbb{Z}, m)$ because the map $K(\mathbb{Z}, m) \to K(\mathbb{Q}, m)$ induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is a rational equivalence. Indeed, this follows from (2.3.3) because the homotopy fibre of the map is a $K(\mathbb{Q}/\mathbb{Z}, m-1)$-space (as implied by the long exact sequence in homotopy), which has vanishing rational cohomology (in strictly positive degrees) by (2.3.5). All in all, we need to show that

$$h^*(K(\mathbb{Z}, m)) \cong \begin{cases} \mathbb{Q}[\gamma^m] & m \text{ even} \\ \mathbb{Q}[\varepsilon^m] & m \text{ odd.} \end{cases}$$

For $m = 1$, we can take $K(\mathbb{Z}, 1)$ to be S^1 and the result is clear. To proceed from $m-1$ to m, we distinguish two cases: from even to odd and from odd to even. We give a brief sketch and more details can be found in [9].

From even to odd: Let $f\colon S^m \to K(\mathbb{Z}, m)$ represent a generator of $\pi_m K(\mathbb{Z}, m) \cong \mathbb{Z}$. We want to show that f induces an isomorphism $h^*(S^m) \to h^*(K(\mathbb{Z}, m))$, for which it suffices to check that the homotopy fibre of f is 0 in reduced rational cohomology by (2.3.3). To construct the homotopy fibre, we pick a contractible space E together with a fibration $E \twoheadrightarrow K(\mathbb{Z}, m)$ (e.g. the path space). Now the homotopy fibre of f is simply the pullback f^*E of $E \twoheadrightarrow K(\mathbb{Z}, m)$ along f. Moreover, we have a fibre sequence

$$K(\mathbb{Z}, m-1) \to E \twoheadrightarrow K(\mathbb{Z}, m)$$

(by the long exact homotopy sequence) and so the pulled back fibration $f^*E \twoheadrightarrow S^m$ in

$$\begin{tikzcd} F \arrow[r, "\sim"] \arrow[d] & K(\mathbb{Z}, m-1) \arrow[d] \\ f^*E \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads] \\ S^m \arrow[r, "f"'] & K(\mathbb{Z}, m) \end{tikzcd}$$

has a weakly equivalent homotopy fibre $F \simeq K(\mathbb{Z}, m-1)$. To show that $\tilde{h}^*(f^*E) = 0$, we divide S^m into a northern hemisphere D_N and a southern on D_S, both contractible with intersection S^{m-1}. Then f^*E restricts to a trivial bundle over those hemispheres and hence over their intersection. Let's denote the restrictions of f^*E to the two hemispheres by $f^*(E)_N$ and $f^*(E)_S$, respectively. There is a Mayer-Vietoris sequence

$$\ldots \to h^i(f^*E) \to h^i f^*(E)_N \oplus h^i f^*(E)_S \to h^i\big(f^*(E)_N \cap f^*(E)_S\big) \to h^{i+1}(f^*E) \to \ldots$$

and we have $f^*(E)_N \simeq F \times D_N \simeq F$, as well as $f^*(E)_S \simeq F \times D_S \simeq F$, while on the intersection $f^*(E)_N \cap f^*(E)_S \simeq F \times S^{m-1}$. Thus

$$h^i\big(f^*(E)_N\big) \cong h^i(F) \cong h^i\big(f^*(E)_S\big) \qquad \text{and by Künneth}$$

$$h^i\big(f^*(E)_N \cap f^*(E)_S\big) \cong h^i(F) \oplus h^{i-(m-1)}(F).$$

The above Mayer-Vietoris sequence can now be rewritten as

$$\ldots \to h^i(f^*E) \to h^i F \oplus h^i F \to h^i F \oplus h^{i-(m-1)} F \to h^{i+1}(f^*E) \to \ldots.$$

Since the middle map is simply the identity on the first summand, we can again rewrite the sequence as

$$\ldots \to h^i(f^*E) \to h^i F \to h^{i-(m-1)} F \to h^{i+1}(f^*E) \to \ldots.$$

By the inductive hypothesis, $h^*(F) \cong h^*\big(K(\mathbb{Z}, m-1)\big)$ is $(m-1)$-periodic, whence so is $h^*(f^*E)$. In particular, it suffices to prove that $h^i(f^*E) = 0$ for $0 < i \leqslant m-1$. But from the long exact sequence in homotopy of the fibre sequence

$$F \simeq K(\mathbb{Z}, m-1) \to f^*E \twoheadrightarrow S^m,$$

we see that $\pi_i(f^*E) = 0$ for $0 \leqslant i < m-1$. For $i = m-1$, we use the special nature of the map f and the 5-lemma to conclude from the diagram

$$\begin{tikzcd} \pi_m(S^m) \arrow[r] \arrow[d, equal] & \pi_{m-1}(F) \arrow[r] \arrow[d, equal] & \pi_{m-1}(f^*E) \arrow[r] \arrow[d] & \pi_{m-1}(S_m) \arrow[r] \arrow[d, equal] & \pi_{m-2}(F) \arrow[d, equal] \\ \pi_m K(\mathbb{Z}, m) \arrow[r] & \pi_{m-1}(F) \arrow[r] & \pi_{m-1}(E) \arrow[r] & \pi_{m-1} K(\mathbb{Z}, m) \arrow[r] & \pi_{m-2}(F) \end{tikzcd}$$

that $\pi_{m-1}(f^*E) = 0$. It then follows from Hurewicz, that $h_i(f^*E) = 0$ for $0 < i \leqslant m-1$.

From odd to even: We again consider the fibre sequence $K(\mathbb{Z}, m-1) \to E \twoheadrightarrow K(\mathbb{Z}, m)$ as in the previous case. From the inductive hypothesis, we know that $h^*K(\mathbb{Z}, m-1) \cong h^*(S^{m-1})$.

So, from the point of view of cohomology, the above fibre sequence is a sphere bundle (with simply connected base) and we get a Gysin long exact sequence in cohomology

$$\ldots \to h^i K(\mathbb{Z}, m) \to h^{i+r} K(\mathbb{Z}, m) \to h^{i+r}(E) \to h^{i+1} K(\mathbb{Z}, m) \to \ldots .$$

But E is contractible and so

$$h^i K(\mathbb{Z}, m) \cong h^{i+m} K(\mathbb{Z}, m).$$

This isomorphism is given by multiplication with an element in $h^m K(\mathbb{Z}, m)$, which we call γ. Since $h^0 K(\mathbb{Z}, m) \cong \mathbb{Q}$ and $h^i K(\mathbb{Z}, m) = 0$ for $0 < i < m$, we apply Hurewicz to conclude that $h^* K(\mathbb{Z}, m) \cong \mathbb{Q}[\gamma^m]$, as claimed. $\square$

Appendix

MODEL CATEGORIES

In this appendix, we briefly recount some key definitions and results from the theory of Quillen model categories. General references include [2] and [7].

1. Model Structures

(1.1) **Definition.** A *(closed) model structure* on a category $\mathcal{E}$ consists of three classes of maps in $\mathcal{E}$, called *weak equivalences*, *fibrations* and *cofibrations*, with the following properties:

(MC1) $\mathcal{E}$ has finite limits and colimits (in practice, $\mathcal{E}$ will have all small colimits);

(MC2) weak equivalences have the *two-out-of-three property*, i.e. if $fg = h$ and any two of f, g and h are weak equivalences, then so is the third;

(MC3) all three classes of maps are closed under retracts, where a map f is said to be a *retract* of g iff there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & A' \arrow[r, "p"] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r, "j"'] & B' \arrow[r, "q"'] & B \end{tikzcd},$$

such that the two horizontal composites $p \circ i$ and $q \circ j$ are identities.

(MC4) In a commutative square of solid arrows

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "i"'] & X \arrow[d, two heads, "p"] \\ B \arrow[r] \arrow[ur, dotted, "d"] & Y \end{tikzcd}$$

with i a cofibration and p a fibration, if i or p is also a weak equivalence, there is a dotted *diagonal filler* d making everything commute.

(MC5) every map f can be factored in two ways: as $f = p \circ i$ with i a cofibration and p a fibration as well as a weak equivalence, or as $f = q \circ j$ with j a cofibration as well as a weak equivalence and q a fibration.

With this, a *model category* is simply a category together with a model structure on it.

(1.2) **Notation.** A cofibration is usually indicated by the arrow being of the form $\rightarrowtail$, while fibrations are indicated by $\twoheadrightarrow$. This notation conflicts with the notation for mono- and epimorphisms but usually, there is no risk of confusion.

(1.3) **Terminology.** For a model category as above,

(a) a map that is both a cofibration and a weak equivalence is called a *trivial* (or *acyclic*) *cofibration* and similarly, a map that is both a fibration and a weak equivalence is called a *trivial* (or *acyclic*) *fibration*;

(b) an object X is *fibrant* iff the unique map $X \to 1$ into the terminal object is a fibration and dually, it is *cofibrant* iff the unique map $0 \to X$ from the initial object is a cofibration; an object that is both fibrant and cofibrant is sometimes called *bifibrant*.

(c) for an arbitrary object X, factoring $X \to 1$ as $X \overset{\sim}{\rightarrowtail} X_f \twoheadrightarrow 1$ provides a so-called *fibrant replacement* (or *resolution* or *approximation*) of X and dually, factoring $0 \to X$ as $0 \rightarrowtail X_c \overset{\sim}{\twoheadrightarrow} X$ provides a *cofibrant replacement* (or *resolution* or *approximation*);

(d) for two maps f and g, we say that f has the *left lifting property* (or *llp*) with respect to g or that g has the *right lifting property* (or *rlp*) with respect to f iff every square of solid arrows

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

has a dotted diagonal filler making everything commute. More generally, for two classes of maps $\mathcal{F}$, $\mathcal{G}$, we say that $\mathcal{F}$ has the *left lifting property* with respect to $\mathcal{G}$ and that $\mathcal{G}$ has the *right lifting property* with respect to $\mathcal{F}$ iff every $f \in \mathcal{F}$ has the left lifting property with respect to every $g \in \mathcal{G}$.

(1.4) **Exercise.** Convince yourself that the axioms for a model structure are self-dual. Consequently, every model structure on a category $\mathcal{M}$ gives rise to a *dual model structure* on $\mathcal{M}^{\mathrm{op}}$ with the same weak equivalences but the cofibrations and fibrations switched.

(1.5) **Exercise.** Show that if $\mathcal{M}$ is a model category and $A \in \mathcal{M}$, we obtain a model structure on $A \downarrow \mathcal{M}$ by declaring a morphism to be, respectively, a weak equivalence, fibration and cofibration iff it is so in $\mathcal{M}$. In particular, if $\mathcal{M}$ is a model category, the category $\mathcal{M}_* := 1 \downarrow \mathcal{M}$ of pointed objects in $\mathcal{M}$ is again one in a canonical way.

(1.6) **Exercise.** Check that for an object X, both X_{cf} and X_{fc} are bifibrant, so that every object X is weakly equivalent to a bifibrant one in two ways:

$$X_{cf} \overset{\sim}{\leftarrowtail} X_c \overset{\sim}{\twoheadrightarrow} X \qquad \text{and} \qquad X \overset{\sim}{\rightarrowtail} X_f \overset{\sim}{\twoheadleftarrow} X_{fc}.$$

(1.7) **Exercise.** There is a converse to (MC5) above, namely that if a map has the left lifting property with respect to all trivial fibrations (resp. fibrations) then it is a cofibration (resp. trivial cofibration). Dually for a map that has the right lifting property with respect to all trivial cofibrations (resp. cofibrations).

(1.8) **Exercise.** Using the last exercise, show that (trivial) fibrations are stable under base change; i.e. the pullback of an (acyclic) fibration along an arbitrary map is again one. Dually, (trivial) cofibrations are stable under cobase change.

(1.9) **Exercise.** Similarly to the last exercise, using the characterisation of (acyclic) cofibrations by a left lifting property, show that if

$$X_0 \rightarrowtail X_1 \rightarrowtail X_2 \rightarrowtail \dots$$

is a diagram of cofibrations, then the so-called *transfinite composition* $X_0 \to \operatorname{colim}_n X_n$ is again a cofibration.

2. The Homotopy Category

In general, given a category $\mathcal{E}$ and a class of morphisms $\mathcal{W}$ in $\mathcal{E}$ we would like to invert, we can always form the localisation $\mathcal{E}[\mathcal{W}^{-1}]$. However, in general, we do not have any control over it and it might well happen that the localisation has proper Hom-classes instead of Hom-sets, even if this wasn't the case for $\mathcal{E}$.

The usefulness of model structures lies precisely in the fact that the localisation of a model category at its weak equivalences becomes more tangible (including asserting that we still have Hom-sets).

(2.1) **Definition.** If $\mathcal{E}$ is a model category, its *homotopy category* $\operatorname{Ho}\mathcal{E}$ is the localisation of $\mathcal{E}$ at the class of weak equivalences (together with the universal functor $\mathcal{E} \to \operatorname{Ho}\mathcal{E}$).

Using the model structure, we can give a very concrete description of $\operatorname{Ho}\mathcal{E}$ using "actual" homotopies.

(2.2) **Definition.** For an object X in a model category, a *(good) path object* is an object $P(X)$, together with two maps

$$X \xrightarrow[\sim]{c} P(X) \xrightarrow{(\mathrm{ev}_0, \mathrm{ev}_1)} X \times X$$

that factor the diagonal with the first one a weak equivalence and the second one a fibration. With this, two maps $f,\ g\colon Y \to X$ are called *(right) homotopic* (denoted by $f \simeq g$) iff there is a path object $P(X)$ together with a factorisation

$$\begin{tikzcd} & P(X) \arrow[d, two heads, "{(\mathrm{ev}_0,\mathrm{ev}_1)}"] \\ Y \arrow[ur, dotted, "h"] \arrow[r, "{(f,g)}"'] & X \end{tikzcd}$$

of (f, g) through the path object. Such a factorisation h is then called a *(right) homotopy* between f and g.

(2.3) **Remark.** A priori, if $f,\ g\colon Y \to X$ are homotopic and $P(X)$ is any path object for X, we may not find a homotopy $f \simeq g$ through $P(X)$ but might be forced to use another path object. However, one can show that if Y is cofibrant, X is fibrant, and $f \simeq g$ as above are homotopic, we can find a homotopy through an arbitrary path object (cf. Exercise (3.2.9)).

(2.4) **Example.** Prove the last sentence of the remark. *Hint: If f, $g\colon Y \to X$ are homotopic, with Y cofibrant and X fibrant, first construct a homotopy through a path object $P(X)$ such that $c\colon X \to P(X)$ is even a trivial cofibration.*

(2.5) **Remark.**

(a) There is a dual construction using cylinder objects instead of path objects. However, these two approaches are equivalent (at least as long as all objects involved are bifibrant) and in our application of the theory to cdgas, we will be mostly concerned with path objects.

(b) One can show that for X fibrant, $\simeq$ is an equivalence relation on $\mathrm{Hom}(Y, X)$ for all Y and we write $[Y, X] := \mathrm{Hom}(Y, X)/{\simeq}$.

(c) One can also show that for Z cofibrant, the homotopy relation is compatible with composition in the sense that for all other objects X, Y, it induces a map

$$[Z, Y] \times [Y, X] \longrightarrow [Z, X].$$

With these remarks, for $\mathcal{E}$ a model category and $\mathcal{E}_b \subseteq \mathcal{E}$ the full subcategory of bifibrant objects, there is a well-defined category $\mathcal{E}_b/{\sim}$ with objects $\mathrm{Ob}(\mathcal{E}_b/{\sim}) := \mathrm{Ob}(\mathcal{E}_b)$, morphisms $\mathrm{Hom}_{\mathcal{E}_b/\sim}(X, Y) := [X, Y]$ and identities as well as the composition inherited from $\mathcal{E}$. The importance of this category lies in the following fact.

(2.6) **Proposition.** The inclusion $\mathcal{E}_b \hookrightarrow \mathcal{E}$ induces an equivalence $\mathcal{E}_b/{\simeq} \xrightarrow{\simeq} \mathrm{Ho}\,\mathcal{E}$.

3. The Small Object Argument

It often happens that there is a set (rather than a proper class) of cofibrations as well as a set of trivial cofibrations, which serve as "test functions" to test (trivial) fibrations against. Having such generating sets allows us to create factorisations by general abstract non-sense without having to create them by hand to establish a model structure. For this entire section, we fix a cocomplete category $\mathcal{E}$, which we work in.

(3.1) **Definition.** Consider a class of morphisms $\mathcal{I}$ in $\mathcal{E}$. An object $A \in \mathcal{E}$ is *finite* (or *compact* or *$\aleph_0$-small* or *finitely presentable*) relative to $\mathcal{I}$ iff for every sequence of maps in $\mathcal{I}$

$$X\colon X_0 \to X_1 \to X_2 \to \dots$$

indexed by the first infinite ordinal $\omega = \{0 < 1 < \dots\}$, every morphism $A \to \mathrm{colim}_n X_n$ factors essentially uniquely through one of the X_n. More explicitly, writing $l_i\colon X_i \to \mathrm{colim}_n X_n$ for the colimiting cocone, we require that

(a) for every $f\colon A \to \mathrm{colim}_n X_n$, there be some $i \in \omega$ together with $g\colon A \to X_i$ such that $f = l_i \circ g$ and that

(b) for any two factorisations $f = l_i \circ g = l_j \circ g'$, one through X_i and one through X_j, there be some $k > i$, j such that $(X_i \to X_k) \circ g = (X_j \to X_k) \circ g'$.

(3.2) **Remark.**

(a) Strictly speaking, the essential uniqueness of the factorisation is not necessary for what follows but is a standard requirement in the literature.

(b) There is really no reason to restrict the above to the countable sequences. More generally, one can define κ-small and κ-presentable objects (which for a general κ is not the same) for arbitrary infinite cardinals κ.

If $\mathcal{I}$ happens to be the class of all morphisms in $\mathcal{E}$, we simply say that A is *finite* and omit the "relative to $\mathcal{I}$"-part.

(3.3) **Exercise.** Show that a vector space is finite (relative to the class of all linear maps) iff it is finite-dimensional. Inspect your proof and conclude that the same method works for basically any algebraic structure (such as groups or modules or cdgas).

(3.4) **Exercise.** Show that a simplicial set is finite iff it only has a finite number of non-degenerate simplices. For this, first show that all $\Delta[n]$ are finite then use the fact that a finite colimit of finite simplicial sets is finite.

(3.5) **Example.** The category **Top** is not algebraic or combinatorial in any reasonable sense. However, it is still accessible to these techniques. While the only small objects (with respect to all continuous maps) are discrete spaces, one can show [7, Proposition 2.4.2], that every compact space is finite relative to T_1-inclusions (which are closed inclusions $i\colon A \hookrightarrow B$ such that every point in $B \setminus A$ is closed).

The usefulness of finite (or small) objects in the context of model categories lies in the fact that we have generalised cellular approximations as known from topology, to produce factorisations.

(3.6) **Definition.** If $\mathcal{A}$ is a class of morphisms, a relative $\mathcal{A}$-*cell complex* is a morphism $X_0 \to X_\infty = \operatorname{colim}_n X_n$ that arises as a transfinite composition of a diagram

$$X_0 \to X_1 \to X_2 \to \dots$$

indexed by ω such that every $X_n \to X_{n+1}$ fits into a pushout square

$$\begin{tikzcd} \coprod\limits_{k\in K_n} A_k \arrow[r] \arrow[d, "\amalg_k i_k"'] & X_n \arrow[d] \\ \coprod\limits_{k\in K_n} B_k \arrow[r] & X_{n+1} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

for some indexing set K_n and some family $(i_k\colon A_k \to B_k)_{k\in K_n}$ of morphisms in $\mathcal{A}$. We write $\mathrm{Cell}(\mathcal{A})$ for the class of all relative $\mathcal{A}$-cell complexes.

(3.7) **Remark.** Again, we could really allow general ordinals as indexing categories for transfinite sequences rather than just ω. However, there is then an issue concerning intermediate limit ordinals and having just ω is enough for our purposes.

(3.8) **Exercise.** As follows from (1.7) and (1.8), if $\mathcal{A}$ is a class of (acyclic) cofibrations in a model category, then so is every relative $\mathcal{A}$-cell complex.

Knowing what generalised cell complexes are, we can now construct a cellular approximation in a "brute force" fashion by just attaching as many cells as needed.

(3.9) **Definition.** Given a morphism $f\colon X \to Y$ and a class of morphisms $\mathcal{A}$, we define a sequence of objects $X_0, X_1, \dots$ together with morphisms $\iota_n(f)\colon X \to X_n$, $\pi_n(f)\colon X_n \to Y$ recursively as follows: We start with $X_0 := X$, $\iota_0 := \mathrm{id}_X$ and $\pi_0 := f$. Assuming we have constructed everything up to stage n, we let S_n be the set of all triples

$$s = (i_s, g_s, h_s) \quad \text{fitting into a commuting square} \quad \begin{tikzcd} A_s \arrow[r, "g_s"] \arrow[d, "i_s"'] & X_n \arrow[d, "\pi_n(f)"] \\ B_s \arrow[r, "h_s"'] & Y \end{tikzcd} \quad \text{with } i_s \in \mathcal{A}.$$

With this, we then define X_{n+1} by the pushout

$$\begin{tikzcd} \coprod\limits_{s \in S_n} A_s \arrow[r, "{[g_s]_s}"] \arrow[d, "\amalg_k i_s"'] & X_n \arrow[d] \\ \coprod\limits_{s \in S_n} B_s \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & X_{n+1} \end{tikzcd} .$$

Now $\iota_{n+1}(f)$ is simply the composite of $X_n \to X_{n+1}$ with $\iota_n(f)$, while $\pi_{n+1}(f)$ is the unique morphism induced by $\pi_n(f)$ and $[h_s]_s$. The *cellular approximation* of f (*of length* ω) is the map $\iota_\infty(f)\colon X = X_0 \to \operatorname{colim}_n X_n =: X_\infty$, which comes with a second morphism $\pi_\infty(f)\colon X_\infty \to Y$ induced by the $\pi_n(f)$.

By construction, the cellular approximation $\iota_\infty(f)$ of a morphism f is always a relative $\mathcal{A}$-cell complex. Now if all domains of morphisms in $\mathcal{A}$ are sufficiently small, then we can also say something about $\pi_\infty(f)$.

(3.10) **Theorem (Small Object Argument).** If $\mathcal{A}$ is a set of morphisms, all of whose domains are finite, and f any map, then $\pi_\infty(f)$ has the right lifting property with respect to all morphisms in $\mathcal{A}$.

Proof. Letting $f\colon X \to Y$ be arbitrary, $i\colon A \to B$ in $\mathcal{A}$ and given a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X_\infty \arrow[d, "\pi_\infty(f)"] \\ B \arrow[r] & Y \end{tikzcd} ,$$

by the finiteness of A, the map $A \to X_\infty$ factors through some X_n in the diagram defining the $\mathcal{A}$-cellular approximation of f. Consequently, the square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X_n \arrow[d, "\pi_n(f)"] \\ B \arrow[r] & Y \end{tikzcd}$$

is an element of the set S_n used for the construction of X_{n+1}. This means that there is a map $B \to X_{n+1}$ and we get a diagonal filler $B \to X_{n+1} \to X_\infty$ for the original square. □

(3.11) **Remark.** The theorem holds more generally for a set of morphisms, all of whose domains are finite relative to $\mathcal{A}$-cobase changes (i.e. pushouts of maps in $\mathcal{A}$ along arbitrary morphisms). This is for example relevant for the case of **Top**, where non-discrete spaces are never finite in our sense. Moreover, there is again a generalisation to arbitrary infinite cardinals.

In the context of model structures, this construction is useful because if $\mathcal{A}$ is a set of (trivial) cofibrations, we know that relative $\mathcal{A}$-cell complexes are still (trivial) cofibrations and so, we can use the above construction to factor every map f into an (trivial) cofibration $\iota_\infty(f)$, followed by a map $\pi_\infty(f)$ that has the right lifting property at least with respect to the (trivial) cofibrations in $\mathcal{A}$. If these happen to be enough to test (trivial) fibrancy against, we get the factorisations required for a model structure for free. Think for example of the set $I^n \hookrightarrow I^{n+1}$ in **Top**, which are enough to recognise Serre fibrations.

(3.12) **Definition.** A model category $\mathcal{E}$ is *(finitely) cofibrantly generated* iff there is a set $\mathcal{I}$ of arrows (called the *generating cofibrations*) and a set of arrows $\mathcal{J}$ (called the *generating trivial cofibrations*) such that

(a) the domains of all arrows in $\mathcal{I}$ (resp. $\mathcal{J}$) are finite relative to $\mathcal{I}$-cobase changes (resp. $\mathcal{J}$ cobase changes), meaning that they are finite relative to all maps that arise as pushouts of maps in $\mathcal{I}$ (resp. $\mathcal{J}$) along arbitrary arrows;

(b) the morphisms having the right lifting property with respect to $\mathcal{I}$ (resp. $\mathcal{J}$) are precisely the acyclic fibrations (resp. fibrations).

(3.13) **Remark.** In algebraic or combinatorial settings (such as chain complexes, cdgas or simplicial sets), a version of the inital smallness condition is automatically satisfied (though possibly for a larger cardinal than $\aleph_0$ as in our case) and can be safely ignored.

4. Examples of Model Structures

The two classical examples of Quillen model categories are the categories **Top** and **sSets**. The theory of model categories allows us to make precise in what sense the two are just different approaches to "the same homotopy theory".

(4.1) **Example.** There is a cofibrantly generated model structure on **Top**, where

(a) the weak equivalences are the usual ones (i.e. the continuous $f\colon X \to Y$ such that $f_*\colon \pi_0(X) \to \pi_0(Y)$ is bijective and $f_*\colon \pi_n(X,x) \to \pi_n(Y, f(x))$ an isomorphism for all $n > 0$ and all $x \in X$);

(b) fibrations are Serre fibrations;

(c) cofibrations are retracts of relative cell complexes.

This is sometimes called the *classical* or *Quillen model structure* on **Top**. By definition of a Serre fibration, a generating set of trivial cofibrations is given by the set of all

$$\{(I^n \times \{0\}) \cup (\partial I^n \times I) \hookrightarrow I^n\}_{n\in\mathbb{N}} \quad \text{or equivalently} \quad \left\{I^{n-1} \hookrightarrow I^n\right\}_{n\in\mathbb{N}},$$

while a set of generating cofibrations is given by

$$\left\{S^{n-1} \hookrightarrow D^n\right\}_{n\in\mathbb{N}}.$$

To define what the weak equivalences between simplicial sets are, one usually uses the *geometric realisation* functor $|{-}| : \mathbf{sSets} \to \mathbf{Top}$, which is uniquely determined by

$$|\Delta[n]| := \Delta^n = \left\{(t_0, \ldots, t_n) \in I^{n+1} \,\middle|\, \sum_{i=0}^n t_i = 1\right\}$$

and the requirement that it preserve colimits (every simplicial set can be built by gluing together copies of standard simplices $\Delta[n]$).

We also recall the notion of *horns* of the standard simplices. For $n \in \mathbb{N}$ and $k \in \{0, \dots, n\}$, the k^{th} horn $\Lambda^k[n]$ consist of those faces of $\Delta[n]$ that contain the vertex k. More explicitly, by definition, $\Delta[n]_m$ consists of all weakly increasing maps $\varphi\colon [m] \to [n]$, while the subset $\Lambda^k[n]_m$ only contains those $\varphi\colon [m] \to [n]$ that have neither $[n]$ nor $[n] \setminus \{k\}$ as their image. In other words

$$\Lambda^k[n]_m := \{\varphi\colon [m] \to [n] \mid [n] \setminus \{k\} \not\subseteq \operatorname{Im}\varphi\}\,.$$

(4.2) **Example.** There is a cofibrantly generated model structure on **sSets**, where

(a) the weak equivalences are those maps $f\colon X \to Y$ such that $|f| : |X| \to |Y|$ is a weak equivalence in **Top**;

(b) fibrations are *Kan fibrations*; i.e. those maps that have the right lifting property with respect to all *horn inclusions* $\Lambda^k[n] \hookrightarrow \Delta[n]$;

(c) cofibrations are monomorphisms (which are just dimensionwise injections).

By definition, a set of generating trivial cofibrations is given by the set of all horn inclusions $\Lambda^k[n] \hookrightarrow \Delta[n]$ and one can show that all boundary inclusions $\partial\Delta[n] \hookrightarrow \Delta[n]$ form a set of generating cofibrations.

The fibrant objects with respect to this model structure on **sSets** are usually called *Kan complexes*. Explicitly, they are those simplicial sets X for which every map $\Lambda^k[n] \to X$ can be extended to a map $\Delta[n] \to X$. It is not too hard to show that every simplicial group (i.e. every simplicial object in **Grp** or equivalently, every group object in **sSets**) is a Kan complex (see for example [10, I.9.6] or [5, I.3.4]).

(4.3) **Definition.** The way to compare model categories are so-called *Quillen adjunctions.* If $\mathcal{M}$ and $\mathcal{N}$ are model categories, then a *Quillen adjunction* is simply a pair of adjoint functors $F\colon \mathcal{M} \rightleftarrows \mathcal{N} : G$ (left adjoint on the left) such that F preserves cofibrations and trivial cofibrations (or equivalently, such that G preserves fibrations and trivial fibrations). This is enough to get a *derived adjunction* $\mathbb{L}F\colon \operatorname{Ho}\mathcal{M} \rightleftarrows \operatorname{Ho}\mathcal{N} : \mathbb{R}G$ and the Quillen adjunction is called a *Quillen equivalence* iff this derived adjunction is an equivalence of categories.

(4.4) **Remark.** The usual way to calculate the derived adjunction of a Quillen adjunction $F\colon \mathcal{M} \rightleftarrows \mathcal{N} : G$ goes as follows. On objects, we simply need to replace them either cofibrantly or fibrantly first:

$$(\mathbb{L}F)(X) := F(X_c) \qquad \text{and} \qquad (\mathbb{R}G)(Y) := G(Y_f)$$

(this is a slight abuse of notation since what we really mean are the images of $F(X_c)$ and $G(Y_f)$ in the respective homotopy categories). For morphisms, one can show that every $f\colon X \to X'$ lifts to $f_c\colon X_c \to X'_c$ and this lift is unique up to homotopy. With this, we can then define

$$(\mathbb{L}F)\left(X \xrightarrow{f} X'\right) := F\left(X_c \xrightarrow{f_c} X'_c\right)$$

using the same abuse of notation as above. Since the lift f_c of f is unique up to homotopy, it is actually unique in the homotopy category and therefore, $(\mathbb{L}F)(f)$ is independent of the chosen lift f_c. Dually for $\mathbb{R}G$. If F happens to preserve weak equivalences, then all these replacements are not necessary and $\mathbb{L}F = \operatorname{Ho}F$ is simply the functor induced between the homotopy categories. Dually for G.

Now, we already know of a pair of adjoint functors between **sSets** and **Top**, namely $|-|: \mathbf{sSets} \rightleftarrows \mathbf{Top} : \mathrm{Sing}$, where $\mathrm{Sing}(X)_\bullet := C(\Delta^\bullet, X)$ is the singular simplicial set associated to a topological space X. One can show that this is a Quillen equivalence and that therefore, the homotopy theory of simplicial sets is "the same" as that of topological spaces.

We finish this section with a few easier examples, which can be checked directly and are left as exercises.

(4.5) **Exercise.** There is a cofibrantly generated model structure on the category **Gpd** of small groupoids (sometimes called the *canonical* or *folk model structure*), where

(a) weak equivalences are equivalences of categories;

(b) cofibrations are functors that are injective on objects;

(c) fibrations are *isofibrations*, which are those functors having the right lifting property with respect to $\{0\} \hookrightarrow \{0 \overset{\sim}{\leftrightarrow} 1\}$. In other words, $p\colon \mathcal{H} \to \mathcal{G}$ is an isofibration iff for every object $X \in \mathcal{H}$, any isomorphism $g\colon p(X) \to Y$ in $\mathcal{G}$ can be lifted to an isomorphism $h\colon X \to Z$ in $\mathcal{H}$ (meaning that $p(h) = g$).

(4.6) **Exercise.** Similarly, there is a cofibrantly generated *folk model structure* on **Cat**, where the three classes of morphisms are the same ones as in **Gpd**; i.e.

(a) weak equivalences are again equivalences of categories;

(b) cofibrations are again functors that are injective on objects;

(c) fibrations are again isofibrations.

(4.7) **Exercise.** There is a cofibrantly generated model structure on the category $\mathbf{Ch}^{\geqslant 0}$ of non-negatively graded cochain complexes over $\mathbb{Q}$, where

(a) weak equivalences are the quasi-isomorphisms;

(b) fibrations are the degreewise surjections;

(c) cofibrations are those maps that are injections in strictly positive degrees.

We define $S(n)$ to be the cochain complex that is 0 everywhere except in degree n, where it is $\mathbb{Q}$; and $D(n+1)$ to be 0 everywhere except in degrees n and $n+1$, where it is $\mathbb{Q}$ and has the identity as the differential between them. With this, a set of generating cofibrations is given by all inclusions $S(n) \hookrightarrow D(n+1)$ for $n \geqslant 0$ as well as $0 \hookrightarrow \mathbb{Q}$, where $\mathbb{Q}$ is viewed as a cochain complex concentrated in degree 0. A set of generating trivial cofibrations is given by all inclusions $0 \hookrightarrow D(n+1)$ for $n \geqslant 0$.

5. Transfer of Model Structures

One useful feature of cofibrantly generated model categories is that there is a *recognition theorem* (which we haven't touched on), which characterises such model structures by a list of axioms that are easier to verify than the usual ones. Another important result is that cofibrantly generated model structures can be transferred through adjunctions under suitable conditions. To wit, starting with a model category $\mathcal{M}$ and an adjunction $F\colon \mathcal{M} \rightleftarrows \mathcal{N} : G$, we can try to get a model structure on $\mathcal{N}$ (and turn the adjunction into a Quillen adjunction) by declaring a map in $\mathcal{N}$ to be a weak equivalence (resp. fibration) iff its image under G is so.

Assuming $\mathcal{M}$ is cofibrantly generated with generating cofibrations $\mathcal{I}$ and generating acyclic cofibrations $\mathcal{J}$, then by adjointness, a map in $\mathcal{N}$ is a fibration (resp. trivial fibration) in the above sense, iff it has the right lifting property with respect to all $Fj\colon FA \to FB$ for $j \in \mathcal{J}$ (resp. $j \in \mathcal{I}$). Assuming suitable finiteness of all objects involved, one can try to go through the small object argument using these new generating (trivial) cofibrations

$$F\mathcal{I} := \{Fi \mid i \in \mathcal{I}\}, \qquad F\mathcal{J} := \{Fj \mid j \in \mathcal{J}\}$$

in $\mathcal{N}$ and sees that this works under the hypotheses of the following theorem.

(5.1) **Theorem (Transfer).** Let $F\colon \mathcal{M} \rightleftarrows \mathcal{N} : G$ be adjoint functors (left adjoint on the left) and $\mathcal{M}$ a cofibrantly generated model category with generating cofibrations $\mathcal{I}$ and generating trivial cofibrations $\mathcal{J}$. If

(a) all domains of morphisms in $F\mathcal{I}$ and $F\mathcal{J}$ are finite in $\mathcal{N}$ and

(b) the images under G of all relative $F\mathcal{J}$-cell complexes in $\mathcal{N}$ are weak equivalences in $\mathcal{M}$,

then there is a cofibrantly generated model structure on $\mathcal{N}$ where a morphism is a weak equivalence or a fibration iff its image under G is and where $F\mathcal{I}$, $F\mathcal{J}$ are sets of generating cofibrations and trivial cofibrations, respectively.

The main example of a transferred model structure for our purposes is the model structure on **cdga**, transferred from that of $\mathbf{Ch}^{\geqslant 0}$ (say over $\mathbb{Q}$ for simplicity).

(5.2) **Example.** We have the usual free forgetful adjunction

$$\begin{tikzcd} \mathbf{Ch}^{\geqslant 0} \arrow[r, shift left, "\Lambda"] \arrow[r, phantom, "\perp"] & \mathbf{cdga} \arrow[l, shift left] \end{tikzcd},$$

which we can use to transfer the model structure from (4.7) above to **cdga**. To see that the condition (b) is satisfied, we consider a generating trivial cofibration $0 \hookrightarrow D(n+1)$ in $\mathbf{Ch}^{\geqslant 0}$ (with $n \geqslant 0$), which is mapped to

$$\mathbb{Q} \hookrightarrow \Lambda(t, dt) \qquad \text{where } t \text{ has degree } n.$$

More explicitly, for $n = 0$, the underlying cochain complex of $\Lambda(t, dt) = A^*(\Delta^1)$ is simply $\mathbb{Q}[t] \to \mathbb{Q}[t]dt \to 0 \to \dots$ (with the usual differentiation of polynomials); for $n > 0$ odd, it is

$$\mathbb{Q} \to 0 \to \dots \to 0 \to \mathbb{Q} \xrightarrow{\mathrm{id}} \mathbb{Q} \to 0 \to \dots \qquad \text{(where the second } \mathbb{Q} \text{ has degree } n);$$

and for $n > 0$ even, it is $\mathbb{Q}$ in degree 0, has $\mathrm{id}\colon \mathbb{Q} \to \mathbb{Q}$ in degrees $(n, n+1)$, $(2n, 2n+1)$, etc. and is 0 everywhere else. In all cases, $\mathbb{Q} \hookrightarrow \Lambda(t, dt)$ is a quasi-isomorphism.

To see that this is even true for all relative cell-complexes in **cdga** built from such inclusions $\mathbb{Q} \hookrightarrow \Lambda(t, dt)$, we first consider pushouts of such. Given a pushout of cdgas

$$\begin{tikzcd} \mathbb{Q} \arrow[r] \arrow[d, tail, "\sim"'] & A \arrow[d, tail] \\ \Lambda(t, dt) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & B \end{tikzcd},$$

then $B = A \otimes \Lambda(t, dt)$ and hence $A \rightarrowtail B$ is a quasi-isomorphism by Künneth (recalling that all the $\Lambda(t, dt)$ are contractible). In particular, $A \rightarrowtail B$ is a trivial cofibration in $\mathbf{Ch}^{\geqslant 0}$.

As for transfinite compositions, we note that the forgetful functor $\mathbf{cdga} \to \mathbf{Ch}^{\geqslant 0}$ preserves all sequential colimits (even all filtered ones). With this and the above observations, when passing to $\mathbf{Ch}^{\geqslant 0}$, every relative cell complex in **cdga** with cells $\mathbb{Q} \hookrightarrow \Lambda(t, dt)$ is mapped to a transfinite composition of trivial cofibrations in $\mathbf{Ch}^{\geqslant 0}$. Since these are stable under transfinite compositions, condition (b) from the above theorem follows.

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