European Autumn School in Topology 2017

Preparatory talks

The following 5 talks of 60 minutes length are supposed to provide foundations for the series of talks held by Greg Arone and David Chataur. They will make up the program of the first day of the autumn school.

Talk 1: Model categories State the definition of model categories and Quillen functors and briefly explain how one can construct the homotopy category of a model category. Discuss the following examples of model categories: Topological spaces with the "Quillen" model structure, simplicial sets, chain complexes. (Due to time constraints, it will not be possible to verify the model category axioms in these examples.) Good references for this are the survey articles [DS95] and [GS07] which provide concise introductions to the subject that cover this material. Hovey's book [Hov99] is a more extensive account on this subject, and Quillen's original book [Qui67] is also still relevant.

Talk 2: Homotopy colimits and limits First discuss the basic idea behind homotopy colimits in the example of homotopy pushouts, following [DS95, §10]. Then discuss the projective model structure for diagrams of spaces of more general shape [Dug16, §13.1] and the resulting "derived functor" approach to homotopy colimits. Explicitly mention sequential homotopy colimits and homotopy orbits. Then turn to the Bousfield–Kan formula for homotopy colimits [Dug16, Definition 4.5] and outline that it is equivalent to the "derived functor approach". For this last point, you can use either [Dug16, Corollary 9.8] or the elegant formulation of the Bousfield–Kan formula in terms of a two sided bar construction [Dug16, Example 11.7]. End the talk by a brief discussion of homotopy limits via injective model structures [Dug16, §13.3].

Talk 3: Localization Discuss the abstract concept of a *left Bousfield localization* of a model category, for example following [Hir03, Chapter 3]. Then turn to the localization of simplicial sets with respect to homology theories [Bou75, §10–11]. Also discuss rationalization and explain how it arises as an explicit construction in the case of CW-complexes (see [Moe17, §2.1])

Talk 4: Spectra and the stable homotopy category Introduce the category of sequential spectra discussed in [BF78]. Construct the level model structure then the stable model structure via the approach to left Bousfield localizations given in [Bou01, §9]. The homotopy category associated with this stable model structure is known as the *stable homotopy category*, and you should try to explain why we care about it, for example by mentioning Brown representability. Also discuss examples for spectra like the sphere spectrum, suspension spectra, Eilenberg–Mac Lane spectra and complex or unoriented bordism. For these examples, you can consult [Sch, Chapter 1.2] and ignore the additional symmetric group actions discussed there. If time permits, you could also sketch how the localization results from Talk 3 carry over to spectra [Bou79] and mention how this gives rise to the p-local stable homotopy category and its chromatic localizations.

Talk 5: The Dold–Kan correspondence and commutative dgas Explain the Dold–Kan correspondence relating the categories of simplicial R-modules and non-negative chain complexes of R-modules [GJ99, Chapter III.2]. Outline that under this equivalence, the model structure on simplicial R-modules that is transferred from the one on chain complexes has weak equivalences and fibrations detected by the underlying maps of simplicial sets. Discuss the model structure on differential graded algebras and on commutative differential graded algebras over $\mathbb{Q}$ (see e.g. [Moe17, Appendix, Example 5.2]).

References

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