Rational homotopy theory and Poincaré duality

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1. Singular cochains and homotopy types
   - Whitehead's problem
   - Differential graded algebras
   - E_∞-algebras and homotopy types
   - Cochains theories

2. Sullivan models
   - Realization functors
   - Minimal models
   - Homotopy groups
   - Formality

Motivations

D. Sullivan in Postcript (2004) of "Geometric topology" MIT notes (1970)

Problem 2. Construct an algebraic model of a simply-connected closed topological manifold as an integral chain complex with a hierarchy of chain homotopies expressing its structure as an infinitely homotopy associative, graded commutative, Poincaré duality algebra. (All dimensions).

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Motivated by computations in string topology and factorization homology.

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"Classify the homotopy types of polyhedra X, Y,..., by algebraic data. Compute the set of homotopy classes of maps, $[X, Y]$, in terms of the classifying data for X and Y. Moreover, compute the group of homotopy equivalences, $Aut(X)$."

Categorical formulation

Find an "algebraic" category **A** and a functor

$$\mathcal{M} : Ho(Top_*) \to \mathbf{A}$$

(1) Faithful, i.e. $[X, Y] \to [\mathcal{M}(X), \mathcal{M}(Y)]$ is injective,
(2) Full, i.e. $[X, Y] \to [\mathcal{M}(X), \mathcal{M}(Y)]$ is surjective,
(3) Essentially surjective, i.e. each object A of **A** is isomorphic to an object of the form $\mathcal{M}(X)$.

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Let $f : (X, x) \to (Y, f(x))$ be a continuous map between two connected CW-complexes then f is a homotopy

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Corollary

Let $f : (X, x) \to (Y, f(x))$ be a continuous map between two connected and 1-connected CW-complexes then f is a homotopy equivalence iff it induces an isomorphism in singular homology $H_*(-; \mathbb{Z})$.

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Example

Let us consider the continuous map

$$S^1 \times S^1 \times S^1 \xrightarrow{c} S^3 \xrightarrow{\eta} S^2$$

where c is a degree 1 map and η is the Hopf maps is trivial on homotopy groups, on homology group BUT it is

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D. Puppe sequences

Let $X \to Y \to Y/X$ be a cofibration and $F \to E \to B$ be a fibration we have two long exact

$$\ldots [\Sigma Y, K] \to [\Sigma X, K] \to [Y/X, K] \to [Y, K] \to [X, K],$$

$$\ldots [K, \Omega E] \to [K, \Omega B] \to [K, F] \to [K, E] \to [K, B].$$

Example 1 : $[S^3 \times S^3, S^3]$

We use the cofiber sequence : $S^3 \vee S^3 \to S^3 \times S^3 \to S^6$, and we consider the long exact sequence

$$[\Sigma(S^3 \times S^3), S^3] \to [\Sigma(S^3 \vee S^3), S^3] \to [S^6, S^3] \to [S^3 \times S^3, S^3] \to [S^3 \vee S^3, S^3]$$

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$\Sigma(X \vee Y) \simeq \Sigma X \vee \Sigma Y$ and $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$.

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We get a non-trivial extension :

$$\pi_6(S^3) \cong \mathbb{Z}/12\mathbb{Z} \hookrightarrow [S^3 \times S^3, S^3] \twoheadrightarrow \mathbb{Z} \oplus \mathbb{Z} \cong [S^3 \vee S^3, S^3].$$

Example 2 : $[S^1 \times S^1 \times S^1, S^2]$

A Postnikov tower for a connected space X is a tower of fibrations :

$$\dots X_n \overset{p_n}{\twoheadrightarrow} X_{n-1} \overset{p_{n-1}}{\twoheadrightarrow} X_{n-2} \dots X_1 \overset{p_1}{\twoheadrightarrow} X_0$$

- $\pi_k(X_n) = \pi_k(X)$ if $k \leq n$,
- $\pi_k(X_n) = 0$ if $k > n$,
- the fiber of p_n is a $K(\pi_n(X), n)$.

When the space is 1-connected, we have $X \simeq \varprojlim X_n$.

Example 2 : using the Postnikov tower of S^2

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Let us consider the map $S^2 \to X_3$ its fiber is 3-connected, by obstruction theory we get that

$$[S^1 \times S^1 \times S^1, S^2] \cong [S^1 \times S^1 \times S^1, X_3].$$

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Let us consider the fibration $K(\mathbb{Z},3) \to X_3 \to K(\mathbb{Z},2)$ we an exact sequence

$$[S^1 \times S^1 \times S^1, K(\mathbb{Z},3)] \to [S^1 \times S^1 \times S^1, X_3] \to [S^1 \times S^1 \times S^1, K(\mathbb{Z},2)]$$

In fact we get the short exact sequence

$$H^3(S^1 \times S^1 \times S^1, \mathbb{Z}) \hookrightarrow [S^1 \times S^1 \times S^1, X_3] \twoheadrightarrow H^2(S^1 \times S^1 \times S^1, \mathbb{Z}).$$

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Not enough structure ! Consider the singular cochains as a functor :

$$C^*(-;\mathbb{Z}) : Top^{op} \to dga_{\mathbb{Z}},$$

we have taken into account the cup product.

Adams-Hilton models

We can detect the Hopf map !

$$[S^3, S^2] \to [C^*(S^2;\mathbb{Z}), C^*(S^3;\mathbb{Z})]_{dga}$$

we need a cofibrant model for $C^*(S^3;\mathbb{Z})$ one can take :

$$T(u_2, u_3 \mapsto u_2 \otimes u_2, \dots).$$

In fact
Adams-Hilton+Husemoller-Moore-Stasheff+Félix-Halperin-Thomas : if X is 1-connected of finite type ($H_k(X;\mathbb{Z})$ is finitely generated), we have

$$[C^*(X;\mathbb{Z}), C^*(S^n;\mathbb{Z})]_{dga} \cong H_{n-1}(\Omega X;\mathbb{Z}).$$

up to a sign we get Hurewicz morphism for ΩX :

$$[S^n, X] \cong [S^{n-1}, \Omega X] \to [C^*(X;\mathbb{Z}), C^*(S^n;\mathbb{Z})]_{dga}.$$

dga is not enough !

We have Steenrod squares acting on $H^*(X;\mathbb{Z}/p)$ they are not encoded only by the cup product.
Need to take into account the cup-i products.

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Solution : Operad theory !

E-algebras

We have a family of cochain complexes of natural multilinear operations $\{E(k)\}_{k\geq 0}$ acting on the singular cochains

$$E(k)\otimes C^*(-)^{\otimes k}\to C^*(-).$$

$C^*(X)$ is a E-algebra.

The operad E is resolution of the operad *Com* i.e. we have a quasi-iso $E\to Com$.

E-algebras are good !

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- we have an inclusion $[X, Y] \to [C^*(Y;\mathbb{Z}), C^*(X;\mathbb{Z})]_{E-dga}$.

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E-algebras and cdgas

Over a field of $\mathbb{F}$ of characteristic zero we have a Quillen adjoint pair :

$$Com \otimes_E - : E - dgas \leftrightarrows cdgas : U$$

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Thus we can replace by singular cochains by $X \mapsto Com \otimes_E F_X$ where F_X is a natural cofibrant resolution of $C^*(X;\mathbb{F})$.

Thom-Whitney Polynomial forms

Let Δ^k be the standard k-simplex we define the algebra polynomial forms on Δ^k as the algebra :

$$A^*_{PL}(\Delta^k) = S(t_0, \ldots, t_k; dt_0, \ldots, dt_k)/(\sum t_i = 1, \sum dt_i = 0).$$

This functor gives a contravariant functor :

$$A^*_{PL} : sets^{op} \to cdgas.$$

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We have $H^*(A^*_{pl}(Sing(X))) \cong H^*(X; \mathbb{Q})$ (iso of graded algebras).

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Two ingredients :
- $A^*_{pl}(\Delta^n) \twoheadrightarrow A^*_{pl}(\partial\Delta^n)$
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Consequences :
- A^*_{pl} sends cofibrations to fibrations,
- It preserves weak-equivalences.

Cochain theory

Definition

A cochain theory is a contravariant functor

$$F : Top^{op} \to \mathbb{M}$$

where ($\mathbb{M}$ can be $R-dgmod$, $R-dga$, $E-dga$, $cdga$) such that :
(1) F preserves weak equivalences,
(2) F sends cofiber sequences $X \to Y \to X/Y$ to a homotopy pull-back i.e.

$$F(X/Y) \simeq hofiber(F(Y) \to F(X))$$

(3) $F(\coprod_\alpha)X_\alpha \stackrel{\sim}{\to} \prod_\alpha F(X_\alpha)$.
(4) $H^*(F(pt)) = R$ iff $* = 0$ and 0 if $* \neq 0$.

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We have a zig-zag of quasi-isos of E-dgas :

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Just set $T^*(\Delta^n) = C^*(\Delta^n;\mathbb{F}) \otimes A^*_{pl}(\Delta^n)$ and extend T^* to simplicial sets and then

Realization functors

Let us start with a complete category C and a functor

$$F : \Delta^{op} \to C$$

using limits we extend it to a functor

$$F : Ssets^{op} \to C.$$

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This functor has an adjoint :

$$| - | : C^{op} \to Ssets$$

given by $|b|_k = C(F(\Delta^k), b)$. We get the adjunction formula :

$$Ssets(X, |b|) \cong C(b, F(X)).$$

Application : computation of $[X, Y]$

Using singular cochains and simplicial realization we get a pair of adjoint functors

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Now use the fact that $|T(u_n)| \sim K(\mathbb{Z}, n)$ and that $|-|$ sends cofibration to fibration to prove

$$|T(u_2, u_3 \mapsto u_2 \otimes u_2)| \sim X_3$$

where X_3 is the third stage of the Postnikov tower of S^2.

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We get that

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Let us suppose that $C^*(X;\mathbb{Z})$ is formal then we get that

$$[X, X_3] = \{(a, b) \in H^2(X;\mathbb{Z}) \times H^3(X;\mathbb{Z}) : a \cup a = 0\}.$$

cdgas vs simplicial sets

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Take $X = K(\mathbb{Z}, n)$ then we know that

$$H^*(K(\mathbb{Z}, n), \mathbb{Q}) \cong S(u_n)$$

proof : use Serre spectral sequences.

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In general : tensor the Postinok tower with $\mathbb{Q}$!

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Ingredients :
1) Realizations of free objects are rational Eilenberg-MacLane spaces,
2) Realization is a Quillen functor, in particular it sends cofibrations on fibrations and homotopy cofibers on fibers.
3) On the other side if we start with a continuous map $f : X \to Y$, if Y is 1-connected the homotopy cofiber of the

$$A^*_{pl}(Y) \to A^*_{pl}(X)$$

is weakly-equivalent to $A^*_{pl}(Hofiber(f))$.

Minimal model

Any cdga A has a cofibrant replacement :

$$\mathcal{M}_A \twoheadrightarrow A$$

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When A is 1-connected one can construct a cofibrant resolution such that the differential of $\mathcal{M}_A$ satisfies a

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A minimal model is unique up to isomorphisms of cdgas.

Examples

- Spheres,
- Complex projective spaces,
- $S^4 \vee S^4$:

$$S(a_4, b_4; a_7 \mapsto a_4^2, b_7 \mapsto b_4^2, c_7 \mapsto a_4.b_4;$$

$$d_{10} \mapsto a_7.b_4 - a_4.c_7, e_{10} \mapsto a_4.b_7 - c_7.b_4;$$

$$f_{13} \mapsto a_4.d_{10} - a_7.b_7, g_{13} \mapsto b_4.e_{10} - c_7.b_7$$

$$h_{13} \mapsto a_7.b_7 - a_4.e_{10} + b_4.d_{10}, \dots)$$

Indecomposable elements

Let A be an augmented cdga a decomposable elements is an element in $A^+.A^+$.
We have a functor $Ind : A \mapsto A^+/A^+.A^+$ we can derive this functor

$$\mathbb{L}Ind : Ho(cdga) \to Ho(dg - \mathbb{Q} - evs)$$

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We define

$$H_*^Q(A) = H_*(\mathbb{L}Ind(A)).$$

When A has a minimal model $S(V)$ we get that $H_*^Q(S(V)) \cong V$.

Rational homotopy groups

When X is 1-connected of $\mathbb{Q}$-finite type we have that

$$[S^n, X] \otimes \mathbb{Q} \cong [S^n, |A^*_{pl}(X)|]$$

by adjunction, we get

$$[S^n, X] \otimes \mathbb{Q} \cong [A^*_{pl}(X), A^*_{pl}(S^n)] \cong [A^*_{pl}(X), H^*(S^n; \mathbb{Q})]$$

replace $A^*_{pl}(X)$ by its minimal model $S(V)$

$$[S^n, X] \otimes \mathbb{Q} \cong [S(V), H^*(S^n; \mathbb{Q})] \cong Hom(V_n; \mathbb{Q}) \cong Hom(H^Q_n(A^*_{pl}(X)), \mathbb{Q}).$$

Koszul duality

$sH_*^Q(A)$ is a coLie coalgebra,

$$H_{k-1}^Q(A) \to \oplus_{l+m=k} H_l^Q(A) \otimes H_m^Q(A)$$

it comes from Koszul duality of operads. Better think of $Hom(sH_*^Q(A), \mathbb{Q})$ as a graded Lie algebra. The bracket are determined by the quadratic part of the differential.

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it comes from Koszul duality of operads. Better think of $Hom(sH_*^Q(A), \mathbb{Q})$ as a graded Lie algebra. The bracket are determined by the quadratic part of the differential.
For example in the minimal model of $S^4 \vee S^4$ we have

$$a_7 \mapsto a_4^2, b_7 \mapsto b_4^2, c_7 \mapsto a_4.b_4$$

we view the differential as a cobracket and we dualize everything :

$$a_7 \leftrightsquigarrow [\alpha_3, \alpha_3], b_7 \leftrightsquigarrow [\beta_3, \beta_3], c_7 \leftrightsquigarrow [\alpha_3, \beta_3].$$

More examples

- Odd spheres,
- Even spheres : $\mathbb{L}(v_{n-1})/(\mathbb{L}(v_n)^+)^{\geq 3}$

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- Even spheres : $\mathbb{L}(v_{n-1})/(\mathbb{L}(v_n)^+)^{\geq 3}$
- Complex projective spaces $\mathbb{C}P^n$ with $n > 1$: $\mathbb{L}(v_1, v_{2n-2})$.

Whitehead product

On $\pi_k(\Omega X) \otimes \mathbb{Q} = \pi_{k+1}(X) \otimes \mathbb{Q}$ we have a graded Lie algebra structure given

$$S^{k+l-1} \to S^k \vee S^l \to S^k \times S^l.$$

We set $[f, g]$ via the composition :

$$S^{k+l-1} \to S^k \vee S^l \xrightarrow{f \vee g} X.$$

Baues-Lemaire conjecture

M. Majewski

The two graded Lie algebras $Hom(sH_*^Q(A_{pl}^*(X)), \mathbb{Q})$ and $\pi_*(\Omega(X)) \otimes \mathbb{Q}$ are

Dichotomy

Elliptic vs hyperbolic spaces

Let X be a 1-connected compact CW-complex, then we have the following alternative :

- $dim(\oplus_k \pi_k(X) \otimes \mathbb{Q}) < \infty$ (elliptic space),
- $dim_k(\pi_k(X) \otimes \mathbb{Q})$ grows exponentially (hyperbolic space), i.e. $\exists N$ such that for $n > N$

$$\sum_{k=1}^{n} dim_k(\pi_k(X) \otimes \mathbb{Q}) \geq C^n$$

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The connected sum $\mathbb{C}P^2 \boxplus \mathbb{C}P^2$ is hyperbolic. Because when X is an elliptic space of $dim(X) = n$ then $dim(H^*(X; \mathbb{Q})) \leq 2^n$.

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A Kähler manifold is a complex manifold X with a Hermitian metric h whose associated 2-form ω is closed. Where the 2-form is given by

$$\omega(u, v) = Reh(iu, v)$$

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They prove formality over $\mathbb{R}$. We can always go down to $\mathbb{Q}$.

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Let X be a $(p-1)$-connected space, $p \geq 2$, of dimension $\leq 3p-2$. Then X is formal.

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Examples in dimension 7 :

1) pull-back of

$$S^2 \times S^2 \to S^4 \leftarrow S^7$$

2) $(S^2 \times S^5) \boxplus (S^2 \times S^5)$.