European Autumn School in Topology 2018

Preparatory talks

The following 5 talks of 60 minutes length are supposed to provide foundations for the series of talks held by Tyler Lawson and Thomas Nikolaus. They will make up the program of the first day of the autumn school.

The talks are

1. Introduction to ∞-categories
2. The ∞-category of spectra
3. Bar constructions in algebra
4. The Tate construction and spectral sequences
5. Operads

If you volunteer for a talk, the first and especially the third and fifth talk should be the easier ones. Below you find a detailed outline of the contents of these talks.

Talk 1: Introduction to ∞-categories Recall the definition of an ∞-category in the sense of a quasi-category [Lur09, Chapter 1]. Give the example of the ∞-category of spaces $\mathcal{S}$ and of the ∞-category of (small) ∞-categories Cat_∞. These can be defined via the simplicial nerve construction (see Section 1.1.5 and Definitions 1.2.16.1 and 3.0.0.1 in [Lur09]). You might contemplate just given the low-dimensional simplices of $\mathcal{S}$ and Cat_∞ in detail.[1]

Next discuss (co)limits in an ∞-category following [Lur09, Chapter 1]. Dicuss in particular pushouts and pullbacks. State that in Top if one leg of the diagram is a cofibration respectively a fibration, then the pushout respectively pullback in $\mathcal{S}$ can be calculated as a (1-categorical) pushouts respetively pullback in Top.[2] Use this to show that the unreduced suspension $\Sigma X = CX \cup_X CX$ defines a pushout of $* \leftarrow X \rightarrow *$ in $\mathcal{S}$ and similarly that the loop space ΩX for a pointed space defines a pullback of $* \rightarrow X \leftarrow *$ in $\mathcal{S}$ (using path spaces).

If time remains, discuss adjunctions between ∞-categories. The definition of adjunction in [Lur09] is a bit hard to understand at first. You can use Definition 2.1.1 from [RV18] instead.[3] Mention [RV18, Theorem 2.4.2].

[1] For a topological space X, its singular comples $\mathrm{Sing}(X)$ is alwyays a Kan set. Viewing the 1-category Top as a simplicial category (with discrete mapping spaces), we obtain a functor from Top to the simplicial category of Kan complexes; taking (simplicial) nerves produces a functor $N\,\mathrm{Top} \rightarrow \mathcal{S}$. One can show that this functor is the universal one that sends weak homotopy equivalences to equivalences.

[2] This can be proved as follows if you know some model category theory. By [Lur09, Proposition 4.2.4.4] pushouts in $\mathcal{S}$ can be computed as homotopy pushout in the model category of simplicial sets. As $\mathrm{Sing}\colon \mathrm{Top} \rightarrow \mathrm{sSet}$ is the right adjoint of a Quillen equivalence and every object in Top is fibrant, Sing preserves homotopy pushout squares; here we use the Quillen model structure on Top, where the weak homotopy equivalences are the weak equivalences, and the Kan–Quillen model structure on sSet. For example by [Lur09, Proposition A2.4.4] a pushout diagram in Top is a homotopy pushout diagram if one of the legs is a cofibration in the Quillen–Serre model structure. But this is also true if we just require the leg to be cofibration (in the sense of the homotopy extension property) by [BV73, Appendix, Proposition 4.8b] and replacing the original diagram up to weak homotopy equivalence by one where one leg is a cofibration in the Quillen model structure. The argument for pullbacks and fibrations is similar, but does not need this last step.

[3] For our purposes, please ignore what they say about ∞-cosmoi before this definition; just interpret the words ∞-category and functor as in [Lur09] and a natural transformation is a 1-simplex in $\mathrm{Fun}(A, B)$. Their notation $h\,\mathrm{Fun}(A, B)$ is the homotopy category of $\mathrm{Fun}(A, B)$.

Talk 2: The ∞-category of spectra The goal of this talk is to introduce the ∞-category of spectra, discuss some of its basic properties and introduce the smash product. A line to do this is the following.

Begin by recalling that Eilenberg–MacLane spaces represent singular cohomology on CW-complexes or reduced singular cohomology on pointed CW-complexes. The suspension isomorphism corresponds to $\Omega K(G,n) \simeq K(G,n-1)$. A sequence of pointed spaces X_n with chosen weak homotopy equivalences $\Omega X_n \simeq X_{n-1}$ is called an Ω-*spectrum*. These represent cohomology theories and by Brown representability every cohomology theory is represented by one. Here you can also mention the example of (complex) K-theory.

For $\mathcal{S}_*$ the ∞-category of pointed spaces and Ω the loop functor, we define the ∞-category Sp of spectra as the limit of the diagram

$$\cdots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*$$

in the ∞-category Cat_∞ of ∞-categories. This is not the definition used in [Lur17], but it is equivalent to it by [Lur17, Proposition 1.4.2.24].

An object in Sp is the same as an arrow $* \to \mathrm{Sp}$ from the one-object ∞-category. Discuss how the universal property of the limit identifies this exactly with an Ω-spectrum.

A lot of spectra do not arise naturally as Ω-spectrum. A *sequential spectrum* is a sequence of pointed space X_i with maps $\Sigma X_i \to X_{i+1}$. By adjunction we obtain maps $X_i \to \Omega X_{i+1}$. Given a sequential spectrum (X_i) we define

$$RX_i = \mathrm{colim}_n \left(\cdots \to \Omega^n X_{n+i} \to \Omega^{n+1} X_{n+1+i} \to \cdots \right).$$

Here, the colimit is taken in the ∞-category $\mathcal{S}_*$. In general, Ω commutes with directed colimits in $\mathcal{S}_*$.[4] This easily implies that (RX_i) forms an Ω-spectrum and thus an object in Sp. Define the suspension spectrum $\Sigma^\infty X$ of a pointed space X by $(R\Sigma^i X)$. The sphere spectrum $\mathbb{S}$ is the suspension spectrum $\Sigma^\infty S^0$.

The ∞-category Sp enjoys many pleasant properties.[5] The most important ones are that it has all small limits and colimits[6] and that it is *stable* in the sense of [Lur17, Definition 1.1.1.9], i.e. that cofiber and fiber sequence agree [Lur17, Corollary 1.4.2.17]. This implies that the loop and suspension functor are mutually inverse to each other on Sp (see the discussion on p.23 and p.24 of [Lur17]). There you also find explained why the homototopy category Ho(Sp) is additive. You can mention that it is actually triangulated, but you should not give the full definition of a triangulated category.

The last vital item is the smash product that defines a pairing $\wedge\colon \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$. The most important properties for us are

- $\mathbb{S} \wedge X \simeq X$
- $\Sigma^\infty X \wedge \Sigma^\infty Y \simeq \Sigma^\infty (X \wedge Y)$

[4] This can be proven by the compactness of S^1 in the category of simplicial sets or in the category of topological spaces (with respect to directed systems of closed inclusions of T_1-spaces) and the fact that colimits in $\mathcal{S}_*$ can be computed as homotopy colimits in these categories; for the general ∞-categorical statement see [Lur09, Proposition 5.3.3.3].

[5] After the steep hills of unstable homotopy theory, it feels like a valley of joy.

[6] Limits of Ω-spectra are formed space-wise. Colimits are a bit more subtle: One can form them in sequential spectra and then apply our functor R. See [Lur17, Proposition 1.4.3.7] for a different argument using that $\mathcal{S}$ is compactly generated (which is essentially contained in [Lur09, Example 5.5.1.8]).

- For every $X \in \mathrm{Sp}$ the functor $X \wedge -\colon \mathrm{Sp} \to \mathrm{Sp}$ commutes with colimits.
- The smash product endows Ho(Sp) with the structure of a symmetric monoidal category.

The last point can actually be refined to the statement that $\wedge$ is part of the structure of a symmetric monoidal ∞-category on Sp [Lur17, Corollary 4.8.2.19], but defining this would lead to far.

Talk 3: Bar constructions in algebra The aim of this talk to introduce several homology and cohomology theories occurring in algebra in the unified framework of bar constructions: Group (co)homology, André-Quillen (co)homology and Hochschild homology.

Introduce group (co)homology first as a derived functor and compute the example of the group $\mathbb{Z}/n$. Then introduce the bar construction and say that it is a second way to compute group (co)homology.

This is a special case of cotriple homology.[7] This is done in Sections 8.6 and 8.7 of [Wei94]. In particular, introduce the chain complex associated to a simplicial object. You can skip most of the examples here, but do Example 8.7.2 and relate it to group (co)homology.

The most important other example for us in Andre-Quillen homology, treated in Sections 8.8 of [Wei94]. Definitely do the definition and the example of polynomial rings. Exercise 8.8.3 is also important. State the second exact sequence in 8.8.6 and treat how it extends the fundamental exact sequence as a motivation for viewing André–Quillen homology as (non-abelian) derived functors of Ω^1.

In general, André–Quillen homology groups are very difficult to calculate. You can mention Theorem 8.8.9 (without explicitly defining which summand it is) as one motivation to go to Hochschild homology instead. Then define Hochschild homology as in Section 9.1 and give at least one example.

Talk 4: The Tate construction and spectral sequences The goal of this talk is to introduce first spectral sequence for homotopy fixed points and homotopy orbits. Then you should introduce the Tate construction. There should be examples.

For the homotopy fixed point spectral sequence see e.g. [Dug03]. The homotopy orbit spectral sequence is an analogous spectral sequence

$$H_q(G; \pi_p X) \Rightarrow \pi_{p+q}(X_{hG})$$

if X is a spectrum with an action by a group G. This is also induced by the cellcular filtration of EG.

The Tate spectrum X^{tG} is the cofiber of the norm map relating homotopy orbits and homotopy fixed points. We recommend the treatment of [Rog08]. Please replace all occurrences of 'E-local S-module' by 'spectrum'. We furthermore recommend to concentrate on the finite groups of equivariance, but please mention the compact Lie group case as well.

If M is an abelian group with a G-action (for G finite), one can identify $\pi_* HM^{tG}$ with the Tate cohomology $\widehat{H}^{-i}(G, M)$, which is a mixture of group homology and group cohomology (see e.g. [Bro94, Chapter VI] for an introduction to Tate cohomology). Give the example of $G = C_p$ and $M = \mathbb{Z}$ with the trivial action. If you know how

[7] Mention that cotriples are more commonly called monads.

to do it, you can also introduce the spectral sequence from Tate cohomology to the homotopy groups of the Tate construction.

One of the classic examples for the homotopy fixed point spectral sequence is the C_2-action on complex K-theory KU by complex conjugation. Recall that $\pi_* KU = \mathbb{Z}[u^{\pm 1}]$ with u in degree 2; this corresponds to the Bott periodicity fact that $\widetilde{KU}^0(S^i)$ is $\mathbb{Z}$ if i is even 0 if i is odd. The generator for $i = 2$ is the tautological bundle on $\mathbb{CP}^1 = S^2$ and the isomorphism $\widetilde{KU}^0(S^2) \to \mathbb{Z} = H^2(S^2;\mathbb{Z})$ is given by the first Chern class. This implies that complex conjugation acts by -1 on u. An easy calculation shows that the E_2-term of the homotopy fixed point spectral sequence is isomorphic to $\mathbb{Z}[u^{\pm 2}, x]/2x$ with $x \in H^1(C_2; \pi_2 KU)$. Use now that x is a permanent cycle that represents the image of the element η along the unit map $\mathbb{S} \to KU$.[8] As $\eta^4 = 0$, there must be a differential and deduce that the only possibility is $d_3(u^2) = \eta^3$. This implies by the multiplicativity of the spectral sequence (see [Dug03]) a whole bunch of other d_3-differentials and on the remaining E_4-page there can be no further differentials for degree reasons. The resulting homotopy groups of KU^{hC_2} agree with the known values of $\pi_* KO$. In contrast, KU^{tC_2} vanishes as can be seen from the Tate spectral sequence. Thus, the norm map $KU_{hC_2} \to KU^{hC_2}$ is an equivalence in this case.

Talk 5: Operads This talk is an introduction to the theory of operads.

Introduce in an example-based way what a (topological) operad and an algebra over it are. The most important examples for us are the E_n-operads (up to $n = \infty$) – other examples one might consider are the commutative or the Lie operad. You can follow [Bel17]. More background can be found in [MSS02], Sections 1.2, 1.4, 2.1 and 2.2, or [MS04]. The recognition principle for n-fold loop spaces should be mentioned (and maybe used as a motivation), but you do not have to treat it in detail.

References

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[8] There are several ways to see this, but none that fits easily into this talk.

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