THE TATE CONSTRUCTION AND SPECTRAL SEQUENCES

ALICE HEDENLUND

Abstract. These are my notes for a talk on topic that I gave at the European Autumn School in Topology in 2018.

Introduction

The goal of this talk is to introduce the concepts of homotopy orbits, homotopy fixed points, and the Tate construction on a G-spectrum. We will also construct spectral sequences that compute the homotopy groups of these constructions. In this way we have a natural division of the talk directly indicated by the title:

(1) The Tate construction.
(2) Spectral sequences.

Many of the standard references for these topics work in a chosen 1-categorical model for spectra, like orthogonal spectra or S-modules, plus a model structure on these categories. However, as the previous talks have already introduced ∞-categories, and in particular the ∞-category of spectra, we have here opted to give an ∞-categorical approach to the subject. We will freely use the notation and terminology from Lurie's two bibles [Lur09, Lur17].

1. The Tate construction

In this section we define the Tate construction on a G-spectrum following [NS17, Section I.1]. We will do this only for finite groups, but indicate that this can be done more generally for all topological groups, and in particular for compact Lie groups.

1.1. Homotopy orbits and homotopy fixed points.

1.1. Homotopy orbits and homotopy fixed points. Before we describe what we mean by homotopy orbits, homotopy fixed points, and the Tate construction, we have to introduce the category of G-spectra. We will denote the ∞-category of spectra by Sp throughout.

Definition 1.1. Let G be a topological group and consider a fixed classifying space BG. The stable ∞-category of G-spectra is defined as the functor category

$$\mathrm{Sp}^G = \mathrm{Fun}(BG, \mathrm{Sp}).$$

Remark. In equivariant stable homotopy theory we often consider another type of G-spectra, so called genuine G-spectra. We warn the reader that above definition is different from these genuine G-spectra, and could be referred as naïve G-spectra. However, the constructions that we are interested in for this talk only depend on the naïve equivariant homotopy type of the spectra we consider, so this is an issue that we will completely disregard.

Date: September 26, 2018.

We can consider the limit and colimits of a functor $BG \to \mathrm{Sp}$ and so get the homotopy fixed points and homotopy orbits.

Definition 1.2. The homotopy orbits and homotopy fixed points of a G-spectrum X are defined as

$$X_{hG} = \operatorname*{colim}_{BG} X \quad \text{and} \quad X^{hG} = \lim_{BG} X.$$

An important example is when G is a finite group and M is a G-module. The Eilenberg-Mac Lane spectrum HM is then a G-spectrum and we can consider the homotopy orbits and homotopy fixed points of this G-spectrum. In fact, we have

$$\pi_*(HM_{hG}) \cong H_*(G; M) \quad \text{and} \quad \pi_*(HM^{hG}) \cong H^{-*}(G; M)$$

so that the homotopy orbits recover group homology and homotopy fixed points recover group cohomology.

1.2. **The norm map and the Tate construction.** To give a motivation for the Tate construction, we recall Tate cohomology as defined in [CE56, Chapter XII]. Classically, Tate cohomology is a way to patch together group homology and group cohomology into a single cohomology theory. If G is a finite group and M is a G-module, we may consider the G-orbits M_G and the G-fixed points M^G. We leave it to the reader to check that the norm map

$$\mathrm{Nm}_G : M_G \to M^G, \quad m \mapsto \sum_{g \in G} gm$$

is a well defined map. The Tate cohomology groups are defined as

$$\hat{H}^i(G; M) = \begin{cases} H^i(G; M) & i \geq 1, \\ \mathrm{coker}(\mathrm{Nm}_G) & i = 0, \\ \ker(\mathrm{Nm}_G) & i = -1, \\ H_{-i-1}(G; M) & i \leq -2. \end{cases}$$

One can check that Tate cohomology enjoys all of the convenient features that ordinary group cohomology does; for example, a short exact sequence of G-modules give rise to a long exact sequence in Tate cohomology, and the cup product in group cohomology extends to a cup product on Tate cohomology.

Example 1.3. Let $G = C_p = \langle g \mid g^p = e \rangle$ and $M = \mathbb{Z}$ as a trivial C_p-module. Group (co)homology of G with coefficients in M is computed via the projective resolution

$$\cdots \longrightarrow \mathbb{Z}C_p \xrightarrow{e-g} \mathbb{Z}C_p \xrightarrow{N} \mathbb{Z}C_p \xrightarrow{e-g} \mathbb{Z}C_p \longrightarrow 0\ ,$$

so all that is left to figure out is the cokernel and the kernel of the norm map. The norm map evaluated on a class m is

$$\mathrm{Nm}_{C_p}(m) = \sum_{i=0}^{p-1} g^i m = \sum_{i=0}^{p-1} m = pm$$

since we are working with the trivial action. We hence have

$$\mathrm{coker}(\mathrm{Nm}_{C_p}) = \mathbb{Z}/p\mathbb{Z} \quad \text{and} \quad \ker(\mathrm{Nm}_{C_p}) = 0$$

and additively we conclude that

$$\hat{H}^i(C_p;\mathbb{Z}) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{even } i \\ 0 & \text{odd } i \end{cases}.$$

We remark that the multiplicative structure (that we do not have time to discuss) can be computed to be

$$\hat{H}^*(C_p;\mathbb{Z}) = (\mathbb{Z}/p\mathbb{Z})[t^{\pm}], \quad |t| = 2.$$

Our goal in this section is to give an ∞-categorical analogue of this norm map and use this to define the Tate construction in spectra. To simplify, we will only do this for finite G. We start with giving an alternative description of the homotopy fixed points and homotopy orbits. Consider the map $f : BG \to *$. The map gives rise to a pullback functor $f^* : \mathrm{Sp} \to \mathrm{Sp}^G$ that gives an ordinary spectrum the trivial G-action. This functor has left and right adjoints, $f_!$ and f_*, and these are just different names for the homotopy orbits and homotopy fixed points:

$$f_! = (-)_{hG} \quad \text{and} \quad f_* = (-)^{hG}.$$

We will use that these are left and right adjoints to the same map in order to get a transformation $f_! \to f_*$. Here is a step-by-step construction:

(1) Consider the pullback diagram

$$\begin{array}{ccc} BG \times BG & \xrightarrow{p} & BG \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ BG & \xrightarrow{f} & * \end{array}$$

where p and q denote projection onto the first and second coordinate, respectively. Since this is a pullback diagram the canonical transformation

$$f^* f_* \to p_* q^*$$

is also equivalence by [Lur17, Lemma 6.1.6.3].

(2) Consider the diagonal map $\delta : BG \to BG \times BG$ and observe that the induced functors $\delta_* : \mathrm{Sp}^{BG} \to \mathrm{Sp}^{BG\times BG}$ and $\delta_! : \mathrm{Sp}^{BG} \to \mathrm{Sp}^{BG\times BG}$ are equivalent. Explicitly, they are both given by

$$\delta_*, \delta_! : X \mapsto \bigoplus_{g\in G} X$$

where the target is a $G \times G$-spectrum by letting one factor act on the indexing set and the other one via the action on X. Here, we have implicitly used that G is a finite group, and that finite coproducts and products in Sp are the same.

(3) The unit $1 \to \delta_*\delta^*$ and the counit $\delta_!\delta^* \to 1$ can be put together with the equivalence of Step (2) to give the transformation

$$p^* \to \delta_*\delta^* p^* \simeq \delta_* \simeq \delta_! \simeq \delta_!\delta^* q^* \to q^*.$$

(4) By adjunction we have a map $1 \to p_* q^*$ so by the equivalence of Step (1) there is a map $1 \to f^* f_*$. Now we use adjunction again to obtain the wanted norm map

$$\mathrm{Nm}_G : f_! \to f_*.$$

Remark. The composition

$$X \longrightarrow X_{hG} \xrightarrow{\mathrm{Nm}_G} X^{hG} \longrightarrow X$$

can informally be given as $\sum_{g\in G} \rho_g$ where $\rho_g : X \to X$ denotes the action of the element g in G.

We now have everything in place to define the Tate construction.

Definition 1.4. The Tate construction on a G-spectrum X is defined as the cofiber

$$X^{tG} = \mathrm{cofib}(\mathrm{Nm}_G : X_{hG} \to X^{hG}).$$

If M is a G-module, then HM^{tG} recovers Tate cohomology in the sense that

$$\pi_*(HM^{tG}) \cong \hat{H}^{-*}(G; M).$$

In particular, since the Tate construction is defined as a cofiber, we have a cofiber sequence

$$X_{hG} \xrightarrow{\mathrm{Nm}_G} X^{hG} \xrightarrow{\mathrm{can}} X^{tG}$$

where the last map is the canonical map. Informally, we can say that the Tate construction measures how close the norm map is to being an equivalence.

Remark. We can also define a suitable norm map Nm_G when G is a compact Lie group and so get the Tate construction with respect to an action of any compact Lie group. In this case, the norm map is a natural transformation

$$\mathrm{Nm}_G : (D_G \otimes X)_{hG} \to X^{hG}$$

where D_G is the so called dualizing spectrum of G [Kle01]. Concretely, we have that the dualizing spectrum of a Lie group is $D_G = \mathbb{S}^{\mathfrak{g}}$, the representation sphere of the adjoint representation $\mathfrak{g}$ on G. An important case is when $G = \mathbb{T}$, the circle group. Observe that $\mathbb{T}$ is abelian so the adjoint representation is trivial with dimension the same as $\mathbb{T}$, namely 1. Hence, we get a norm map

$$\mathrm{Nm}_{\mathbb{T}} : \Sigma X_{h\mathbb{T}} \to X^{h\mathbb{T}}.$$

2. Spectral sequences

Spectral sequences are incredibly useful tools in modern mathematics. In this section we give a short introduction to the topic using the language of ∞-categories. In particular, we construct the homotopy orbit, homotopy fixed point, and Tate spectral sequence. This is a condensed version of [HKN]. We end the section by using the homotopy fixed point spectral sequence to compute the homotopy groups of the homotopy fixed point of complex topological K-theory with the conjugation action.

2.1. Towers in spectra.

The spectral sequences we consider will be produced by towers in spectra. To make sure no confusion arises, we fix some notation. The 1-category $\mathbb{Z}$ is the category whose objects are integers, and where $\mathbb{Z}(m, n)$ is a single element if $m \leq n$ and otherwise empty.

Definition 2.1. Let $\mathscr{C} \in \{\mathrm{Sp}, \mathrm{Sp}^G\}$. The ∞-category of towers in $\mathscr{C}$ is the functor category

$$\mathrm{Tow}(\mathscr{C}) = \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathscr{C}).$$

We agree to denote a tower on the form $X^\bullet$, indicating that the image of the functor at the integer n is to be denoted by X^n. We visualise a tower as a diagram

$$\cdots \to X^{n+1} \to X^n \to X^{n-1} \to \cdots .$$

The colimit and limit of the tower $X^\bullet$ are denoted

$$X^{-\infty} = \operatorname{colim} X^\bullet \quad \text{and} \quad X^{\infty} = \lim X^\bullet,$$

and to shorten notation we will also write

$$X^m/X^n = \operatorname{cofib}(X^n \to X^m)$$

for $m \le n$.

Example 2.2. Any (G-)spectrum E gives rise to a tower $\operatorname{White}(E)^\bullet$ by killing off homotopy groups under a certain degree:

$$\operatorname{White}(E)_n = \tau_{\ge n} E.$$

Observe that

$$\operatorname{White}(E)^{-\infty} \simeq E \quad \text{and} \quad \operatorname{White}(E)^{\infty} \simeq 0.$$

This construction makes sense in any stable ∞-category with a t-structure.

Example 2.3. Given a G-spectrum E we can consider the Whitehead filtration $\operatorname{White}(E)$. If we postcompose this functor with homotopy fixed points we get a tower $\operatorname{White}(E)^{hG}$ which in every degree is

$$(\operatorname{White}(E)^{hG})^n = (\tau_{\ge n} E)^{hG}.$$

Using [NS17, Lemma I.2.6] we see that taking homotopy fixed points interacts well with Whitehead covers, and we have

$$(\operatorname{White}(E)^{hG})^{-\infty} \simeq E^{hG} \quad \text{and} \quad (\operatorname{White}(E)^{hG})^{\infty} \simeq 0.$$

The same discussion goes through for both homotopy orbits and the Tate construction.

2.2. **Spectral sequences from towers.** Instead of constructing spectral sequences via exact sequences as in [Boa99], we will here construct them following [Lur17, Section 1.1.2.2], but in the dualized setting. Observe that given a tower $X^\bullet$, for any triple $i \ge j \ge k$ the cofiber sequence

$$X^j/X^i \to X^k/X^i \to X^k/X^j$$

gives rise to a long exact sequence in homotopy

$$\cdots \to \pi_n(X^j/X^i) \to \pi_n(X^k/X^i) \to \pi_n(X^k/X^j) \to \pi_{n-1}(X^j/X^i) \to \cdots .$$

We define the pages of a spectral sequence associated to $X^\bullet$ by setting the pages to be

$$E^r_{p,q} = \operatorname{im}(\pi_{p+q}(X^q/X^{q+r-1}) \to \pi_{p+q}(X^{q-r+2}/X^{q+1})).$$

for every $r \ge 2$. The differential $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ in the spectral sequence is characterized as being the map making the diagram

$$\begin{array}{ccccc} \pi_{p+q}(X^q/X^{q+r-1}) & \longrightarrow & E^r_{p,q} & \longrightarrow & \pi_{p+q}(X^{q-r+2}/X^{q+1}) \\ \downarrow & & \downarrow{\scriptstyle d^r} & & \downarrow \\ \pi_{p+q-1}(X^{q+r-1}/X^{q+2r-2}) & \longrightarrow & E^r_{p-r,q+r-1} & \longrightarrow & \pi_{p+q-1}(X^{q+1}/X^{q+r}) \end{array}$$

commute. The so constructed spectral sequence converges conditionally to $\pi_*(X^{-\infty}/X^{\infty})$. In particular we observe that the second page of the spectral sequence associated to $X^\bullet$ is given as

$$E^2_{p,q} = \pi_{p+q}(X^q/X^{q+1})$$

and the d_2-differential is induced by connecting homomorphism $d : X^q/X^{q+1} \to \Sigma X^{q+1}/X^{q+2}$ of the pushout square

$$\begin{array}{ccc} X^{q+1}/X^{q+2} & \longrightarrow & X^q/X^{q+2} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & X^q/X^{q+1}. \end{array}$$

Example 2.4. Let us check what the second page of the spectral sequence associated to the Whitehead tower $\mathrm{White}(E)$ of some (G-)spectrum E looks like. The cofiber sequences in the Whitehead tower are given as

$$\tau_{\geq q+1}(E) \to \tau_{\geq q}(E) \to \Sigma^q H\pi_q(E),$$

so the second page of the Whitehead spectral sequence is

$$E^2_{p,q} = \pi_{p+q}(\Sigma^q H\pi_q(E)) = \begin{cases} \pi_q(E) & \text{if } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

This is perhaps not incredibly exciting, but it provides a good foundation for constructing other spectral sequences as we will see in the next example.

Example 2.5. The homotopy fixed point spectral sequence is constructed by using the tower $\mathrm{White}(E)^{hG}$. Since taking homotopy fixed points is an exact functor we have cofiber sequences

$$(\tau_{\geq q+1}(E))^{hG} \to (\tau_{\geq q}(E))^{hG} \to \Sigma^q(H\pi_q(E))^{hG}$$

and the second page of the spectral sequence becomes

$$E^2_{p,q} = \pi_{p+q}(\Sigma^q(H\pi_q(E))^{hG}) = \pi_p(H\pi_q(E)^{hG}) \cong H^{-p}(G; \pi_q(E)).$$

The same discussion goes through for homotopy orbits and the Tate construction and we conclude that we have spectral sequences

$$\begin{aligned} E^2_{p,q} &\cong H_p(G; \pi_q(E)) \Rightarrow \pi_{p+q}(E_{hG}) \\ E^2_{p,q} &\cong H^{-p}(G; \pi_q(E)) \Rightarrow \pi_{p+q}(E^{hG}) \\ E^2_{p,q} &\cong \hat{H}^{-p}(G; \pi_q(E)) \Rightarrow \pi_{p+q}(E^{tG}) \end{aligned}$$

that converges conditionally the homotopy groups of all of these constructions.

2.3. **Computation.** To end this talk we consider a classic application of the homotopy fixed point spectral sequence: the computation of $\pi_*(\mathrm{KU}^{hC_2})$ where KU denotes complex topological K-theory spectrum and $C_2 = \langle g \mid g^2 = e\rangle$ acts on KU by complex conjugation. The homotopy fixed point spectral sequence for complex conjugation on KU takes the form

$$H^{-p}(C_2; \pi_q(\mathrm{KU})) \Rightarrow \pi_{p+q}(\mathrm{KU}^{hC_2}).$$

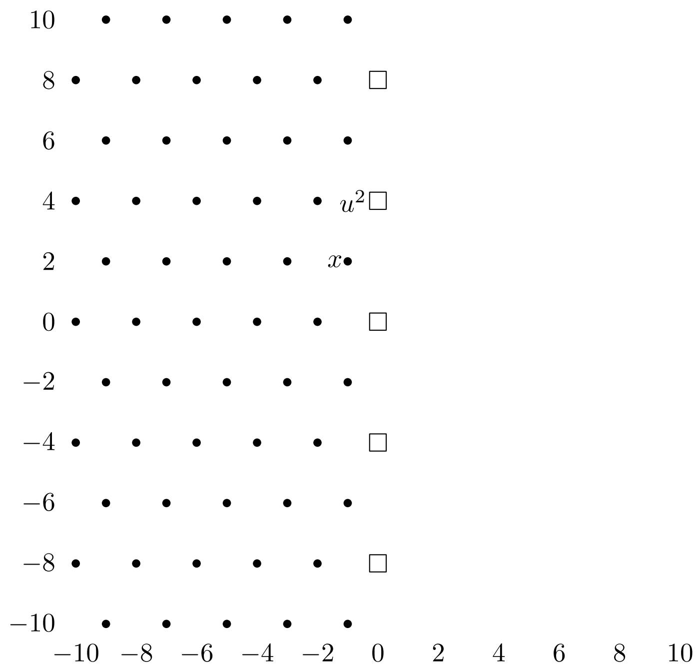
Recall that the homotopy groups of KU are 2-periodic

$$\pi_*(\mathrm{KU}) = \mathbb{Z}[u^{\pm 1}], \quad |u| = 2$$

where u is the so called Bott element. The complex conjugation action on KU is exhibited on homotopy groups by $gu = -u$. The second page of the spectral sequence can now be computed using the projective resolution we saw in Example 1.3:

$$E^2_{**} = \mathbb{Z}[x, u^{\pm 2}]/(2x)$$

where $|u^2| = (0, 4)$ and $|x| = (-1, 2)$. To makes this a bit more clear we give a picture of the E^2-term:



Here, bullet represents $\mathbb{Z}/2\mathbb{Z}$ and box represents $\mathbb{Z}$. We have marked the groups generated by u^2 and x, respectively. For degree reasons $d^2 = 0$, so we focus our attention on the possible non-zero d^3-differential. Fact:

$$d^3(u^2) = x^3.$$ [1]

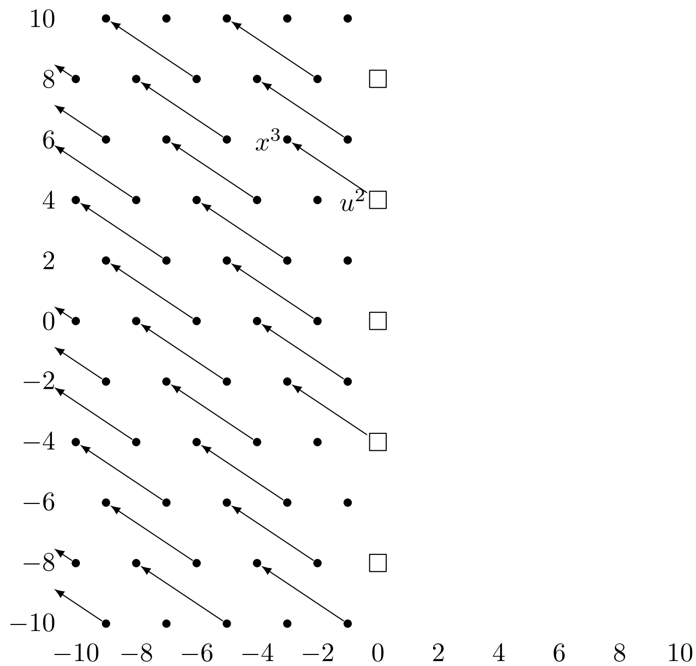
Since the homotopy fixed point spectral sequence is multiplicative (which we have not proved in this talk, but does hold from the Whitehead tower being multiplicative and the homotopy fixed point construction being lax symmetric monoidal) this

[1]There is no easy way of justifying this differential without introducing more sophisticated machinery. In the context of this talk we will have to take a leap of faith and just believe that the d^3-differential is given in this way. For the people who want a little more justification: the differential can be figured out by using another spectral sequences, namely the Adams-Novikov spectral sequence:

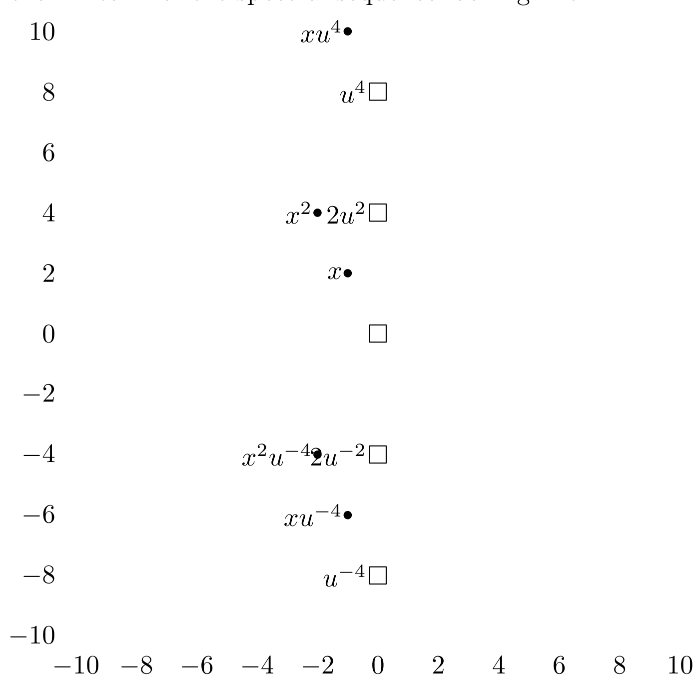
$$\mathrm{Ext}^{p,q}_{\mathrm{MU}_*(\mathrm{MU})}(\mathrm{MU}_*, \mathrm{MU}_*) \Longrightarrow \pi_{q-p}(\mathbb{S}).$$

There is a map of spectral sequences from the Adams-Novikov spectral sequence for $\mathbb{S}$ to the homtopy fixed point spectral sequence for KU. This map is an isomorphism in bidegree $(-1, 2)$, where the Hopf element η is detected in the Adams-Novikov spectral sequence. Since $\eta^4 = 0$ in the Adams-Novikov spectral sequence, we conclude that $x^4 = 0$ in the homotopy fixed point spectral sequence. The only differential that can kill off x^3 is precisely the differntial $d^3(xu^2) = x^4$, which is equivalent to the d^3-differential we have specified, by multiplicativity.

differential propagates through the entire E^3-page. We give a picture of the E^3-term with the d^3-differential:



This leaves the E^4-term of the spectral sequence looking like:



Which more compactly can be written

$$E^4_{**} = \mathbb{Z}[x, \alpha, u^{\pm 4}]/(2x, x^3, x\alpha, \alpha^2 - 4u^4)$$

with $|u^4| = (0,8)$, $|x| = (-1,2)$ and $|\alpha| = (0,4)$. The element α represents $2u^2$. Since there is no room for further differentials, we conclude that this is the E^∞-term. Nor is there room for any extension problems, so we conclude that

$$\pi_*(\mathrm{KU}^{hC_2}) = \mathbb{Z}[x,\alpha,\beta]/(2x, x^3, x\alpha, \alpha^2 - 4\beta)$$

with $|x| = 1$, $|\alpha| = 4$, and $|\beta| = 8$.

We can import the differentials from the homotopy fixed point spectral sequence to the Tate spectral sequence. By multiplicativity, the d^3-differential described above propagates through the entire spectral sequence, killing off everything. We conclude that

$$\pi_*(\mathrm{KU}^{tC_2}) = 0.^2$$

References

[AMS98] M. Ando, J. Morava, and H. Sadofsky, *Completions of* $\mathbf{Z}/(p)$*-Tate cohomology of periodic spectra*, Geom. Topol. **2** (1998), 145–174.

[Boa99] J. M. Boardman, *Conditionally convergent spectral sequences*, Contemp. Math. **239** (1999), 49-84.

[CE56] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton University Press, Princeton, N. J., 1956.

[HKN] A. Hedenlund, A. Krause, and T. Nikolaus, *What is convergence of spectral sequences?*. In preparation.

[Lur09] J. Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, vol. 170, 2009.

[Lur17] ———, *Higher Algebra*, 2017. http://www.math.harvard.edu/~lurie/papers/HA.pdf.

[Kle01] J. R. Klein, *The dualizing spectrum of a topological group*, Math. Ann. **319** (2001), 421–456.

[NS17] T. Nikolaus and P. Scholze, *On topological cyclic homology* (2017). https://arxiv.org/abs/1707.01799.

[2]Compare this to the homotopy groups of the Tate construction on KU with *trivial* C_2-action. Complex topological K-theory is an example of a complex oriented cohomology theory, so we can use the well-known formula [AMS98, Lemma 2.1] to conclude that

$$\pi_*(\mathrm{KU}^{tC_2}) \cong \mathrm{KU}_*((x))/[2](x) \cong \mathrm{KU}_*((x))/(x^2 - 2x)$$

if C_2 acts trivially on KU.