# European Autumn School in Topology 2019

## Preparatory talks

The following 5 talks of 60 minutes length are supposed to provide foundations for the series of talks held by Tobias Barthel and Paolo Salvatore. They will make up the program of the first day of the autumn school.

**Talk 1: Operads** This talk should provide an introduction to operads and algebras over operads. The talk should be based on examples, particularly including the Stasheff associahedra and the little disks/cubes operads. There is a lot of literature on this topic, ranging from short surveys such as [Sta04, Bel17] and introductory papers [MS04] to full books [MSS02, Fre17a, Fre17b].

**Talk 2: Spectra** This talk should be an introduction to spectra and the stable homotopy category. Start with a short prelude why you should not expect a monoidal structure on the category of (sequential) spectra (i.e. on the category of sequences of pointed spaces $X_n$ with maps $\Sigma X_n \to X_{n+1}$).[1] Then the talk should cover the definition of orthogonal spectra, their stable equivalences (aka $\pi_*$-isomorphisms), the smash product and the notions of ring and module spectra. As examples for orthogonal (ring) spectra, the real and complex cobordism spectra $MO$ and $MU$, Eilenberg–Mac Lane spectra, and topological $K$-theory spectra should be mentioned. While it is good to point out that inverting the stable equivalences of orthogonal spectra leads to the stable homotopy category, it is not necessary to go into the details of model categories.

A good source for this is [Sch19]: Section 1 treats the basic definitions and Section 2.1 the basic examples (where you should assume $G$ to be the trivial group). Missing here are the examples of $MO$ and $K$-theory that can either be found in [Sch, Section 1.1] (replacing the symmetric group actions by orthogonal group actions) or [Sch18] (ignoring equivariance) – just mention the existence of a construction for an orthogonal spectrum for $K$-theory without going into details. For navigating the different models of spectra and the reasons for their existence, [Mal] might also be useful.

**Talk 3: The stable homotopy category** Discuss the $\infty$-categorical viewpoint on spectra following e.g. [Gro15, Section 5.1]. Assume here the existence of a nice theory of $\infty$-categories that allows to define homotopy (co)limits and mapping spaces, as explained in the first two sections of [Gro15] or [Lur09, Chapter 1] (without going into any details of the underlying model). Important points are that spectra form a stable $\infty$-category, that the homotopy category of such have canonically the structure of a triangulated category, and that the smash product can be characterized by a universal property [Lur17, 4.8.2]. (You do not have to prove any of these statements; please also do not give the full definition of a triangulated category.) Further background sources for this are [Gep19] and [Lur17].

There are two further topics to discuss: First mention the Brown representability theorem (stated e.g. in [Rav92, A.4.1]). Further define Bousfield localizations. The original source for the latter is [Bou79] and a modern treatment applying to the $\infty$-category Sp of spectra can be found in [Lur10, Lecture 20].

[1] One succint explanation can be found at the bottom of the first page of [HSS00]; their solution are symmetric spectra, but the reasoning applies as well to orthogonal spectra as the symmetric groups sit inside the orthogonal groups.

**Talk 4: Configuration spaces** This talk should be an introduction to the (co)homology and homotopy theory of euclidean configuration spaces. This will provide a computation of the homology of the little disks operad as well since the spaces it is built from are homotopy equivalent to euclidean configuration spaces. The main goal is to explain the statement of Cohen's theorem [Sin06, Theorem 6.3] and discuss the relation to trees and how they exhibit (co)homology classes in configuration spaces. Focus particularly on the cases $d = 1$ and 2, where you can draw pictures easily. Further references are [Sal01] and [CLM76, III.6-III.7]. If you have time you can also mention the case of framed little disk operads and their relation to BV-algebras; see e.g. [CV06, Section 2.1.4].

**Talk 5: Complex orientable ring spectra** Explain the notions of a complex orientation and of a formal group law and that the former gives rise to the latter. (Use the definition of complex orientations as in [Ada74] or [Lur10].) Give ordinary homology, complex K-theory and complex bordism as examples of complex-orientable spectra. Mention that $MU$ is the universal complex-oriented spectrum: If $R$ is a homotopy commutative ring spectrum (i.e. a commutative monoid in the *homotopy category* of spectra), then complex orientations on $R$ are in bijection with ring maps $MU \to R$. Then state the deeper result that the homotopy groups of $MU$ agree with the Lazard ring, i.e. are the universal ring for formal group laws. To prove this one can use the Adams spectral sequence; thus give the general form of the Adams spectral sequence and a very short indication how one might apply it here.

At last, state the Landweber exact functor theorem. The most elementary definition of a choice of the elements $v_i$ appearing is as the coefficient in front of $x^{p^i}$ in the power series $[p]_F(x)$. Discuss the example of K-theory.

Sources: [Ada74, II.1-II.8], [Goe08, Section 4], [Mei, Section 4.1], [Lur10]

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