European Autumn School in Topology 2022

Preparatory talks

Paul Goerss and Alexander Kupers

The following 5 talks of 60 minutes length are supposed to provide foundations for the series of talks held by Paul Goerss and Alexander Kupers. They will constitute the program of the first day of the autumn school.

Talk 1: Classifying spaces and characteristic classes The goal of this talk is to define the classifying space BG of a topological group G, describe constructions of it, and discuss the interpretation of its homotopy groups and cohomology groups. Define principal G-bundles, and give some examples (e.g. the frame bundle of a manifold). State the homotopy covering theorem for principal G-bundles over CW-complexes and deduce that $B \mapsto \{$isomorphism classes of principal G-bundles over $B\}$ is represented by any principal G-bundle with weakly contractible total space. The weak homotopy type of the base of such a "universal bundle" is the classifying space BG of G. Explain that the space BG can be constructed as the quotient of a weakly contractible space by sufficiently nice free G-action. Give some examples, including infinite Grassmannians, bar constructions, and Eilenberg–Mac Lane spaces $K(G, 1)$ for discrete G. Conclude by explaining that $\pi_k\, BG \cong \pi_{k-1}\, G$, and how cohomology classes on BG are in bijection with characteristic classes of principal G-bundles.

References: Mitchell's notes tell the above story concisely [Mit11]. It is slightly non-standard: usually one works with arbitrary spaces as a base and obtains BG as a homotopy type, but this requires more technicalities (such as considering "numerable" principal G-bundles). This is the approach of [tD08, Chapter 14] or [May99, §23.8], which are worth consulting. The construction of classifying spaces using bar constructions appeared in [Seg68], see also [May99, §16.5] and [Mala]. An alternative construction, which works slightly more generally, is due to Milnor [Mil56].

Talk 2: Spectra and the Stable Category This talk should begin with the basics of the stable homotopy theory of spectra and the connection to (co-)homology theories. Model categories and/or ∞-categories can be mentioned, but the emphasis should be on the equivalences, which determine the homotopy theory. Commutative ring cohomology theories (that is, those with a good cup product) should be introduced, along with the notion of homotopy commutative ring spectrum. Subtleties around the symmetric monoidal smash product should be mentioned, but not dwelt upon. At this point, we should see that basic examples: ordinary cohomology, topological K-theory, Thom spectra, and cobordism theories. Having emphasized homology theories, we can reasonably ask how much of the stable homotopy category is seen by any one homology theory. This leads to the definition of Bousfield localization. It would be good to have some examples when localization is mild, say for complex cobordism or ordinary cohomology, but also mention it can be quite radical, as in the case of complex K-theory.

References: There are many topological models for the stable homotopy category, but the most basic is to be found in Bousfield and Friedlander [BF78]. All others must be compared to this one, and it suffices for many purposes. Important examples of spectra are discussed early in Chapter III of Adams's classic book [Ada74]. The preferred model with a good symmetric monoidal smash product is orthogonal spectra.

It is difficult to find a concise synopsis of this in the literature; one source is Section 2.1 of [Sch18] with G the trivial group. The survey article [Malb] is useful for keeping the definitions and models straight. Thom spectra have also developed into an elaborate theory, as in [ABG$^+$14], but we will need only what is available from the classical sources, or even from Chapter 2 of [nLa]. For localization, there is no harm in reading the well-written classics. See [Bou79].

Talk 3: Chern Classes, Pontryagin classes, and Complex Oriented Cohomology Theories This talk could begin with orientations for vector bundles, the Thom isomorphism, the Euler class, Chern classes, and Pontryagin classes. This leads naturally to the definition of a complex oriented ring cohomology theory, and thence to formal group laws. The standard examples of ordinary cohomology, complex K-theory, and complex cobordism could be revisited, and Quillen's work on the universality of MU could be summarized. At this point we can reverse the flow of information, constructing new homology theories with the Landweber Exact Functor Theorem.

References: Much of this material is in the classical sources, such as Chapter 2 of [Ada74] or Ravenel's "Green Book" [Rav04]. For a post-modern take, see [Lur]. The LEFT theorem has many expositions. The original source of [Lan76] is still very readable and one can turn to [Mil19] for a particularly compact exposition using ideas from algebraic geometry. For orientations and characteristic classes we need look no further than the every-fresh [MS74].

Talk 4: Cohomology of Groups This talk would cover groups and their modules, including profinite groups and continuous modules. Then cohomology can be defined as derived functors of invariants and as Ext-groups, but when G is discrete also as cohomology of the classifying space BG of G. The (co-)bar complex should be defined and, perhaps, mentioned how the bar complex can be obtained from the bar construction model for BG. Then the emphasis would be on examples: cyclic, dihedral, and quaternion groups and, for the profinite case, the simple example of the p-adic integers would suffice. If time permits, the talks could cover the Lyndon-Hochschild-Serre Spectral Sequence. The example of the LHSSS for the quaternion group of order 8 is very instructive.

References: The classic sources suffice for this material: the first chapter of [AM04], the appropriate chapters of [Bro94], the early chapters of the incomparable book [Ser63, Ser02] by Serre, and for more technical details on the profinite case one can turn to [SW00], although that gets technical fast.

Talk 5: The Smale–Hirsch theorem This talk is independent of the previous ones. Its goal is to state and indicate the proof of the Smale–Hirsch theorem, which classifies immersions between manifolds in homotopy-theoretic terms. Define spaces of immersions $M \looparrowright N$ and vector bundle monomorphisms $TM \to TN$, and explain that derivative gives a map from the former to the latter. State the Smale–Hirsch theorem that this map is an equivalence if $\dim(M) < \dim(N)$, or if $\dim(M) = \dim(N)$ and M has no closed components. Outline the proof by handle induction as given in the two references below. As an application, prove that 2-spheres in $\mathbb{R}^3$ may be everted.

References: The expositions in [Wal16, §6.2] and [Wei20] are great guides for this talk. The original references are [Sma59, Hir59]. The original sphere eversion paper is [Sma58]. You can find many explicit constructions and videos of these online, and may want to show one during the lecture.

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