EAST 2023 – Preparatory Talks

(1) **Differential graded Lie algebras and differential graded coalgebras (60 minutes)**

A good source for this is Section 10 of Berglund's lecture notes [Ber]. Explain the basic definitions of dg Lie algebras, dg coalgebras, and the universal enveloping algebra of a dg Lie algebra. A key point here is what is nowadays often called 'Koszul duality' between Lie algebras and coalgebras, which is the content of Section 10.4 (specifically Corollary 10.16 and Theorem 10.17). Explain the dg Lie algebra that Quillen associates to a space, as in Section 10.3. Section 10.5 contains some nice examples.

It might also be nice to consult the original source, which is Appendix B to Quillen's paper on rational homotopy theory [Qui69]. Another possible source is Part IV of the book of Félix–Halperin–Thomas [FHT12].

(2) **Rationalization and rational completion (30 minutes)** Define the rationalization of a space following [Ber, p.10]. Define the $\mathbb{Q}$-completion of a space following [BK72, I.2-4] by specializing to $R = \mathbb{Q}$. Mention that $\mathbb{Q}$-completion is a rationalization for simply-connected spaces. (See e.g. [BK72, V.4] for the more general case of nilpotent spaces. See also [MP12] for the notion of nilpotent spaces.)

(3) **Quadratic forms (30 minutes)**

Following [Sch85, Section 1.1], quickly recall the notion of a bilinear and a quadratic form and explain their equivalence over a field of characteristic not 2. Introduce the Grothendieck and the Witt ring of quadratic forms [Sch85, Section 2.1] and explain their relevance for the classification of quadratic forms. Mention the basic invariants dimension and determinant/discriminant from [Sch85, Section 2.2]. Over the real numbers, bilinear forms are classified by the Sylvester inertia theorem; explain how this computes the Grothendieck and Witt group (cf. [Sch85, Section 2.4]). Give as further examples the computation of the Witt group in the cases of an algebraically closed field, finite fields, rational numbers and integers [Sch85, Sections 2.3 (in particular 3.1 and 3.11), 5.3, 5.4]; you won't have time to explain the proofs, but mention at least that the congruence conditions in 3.11 correspond to whether -1 is a square or not.

As an alternative source for some of the material you can also consult [Lam05].

(4) **Linear algebraic groups (60 minutes)**

Fix a field k, which in doubt you can always assume to be of characteristic zero. Define an algebraic group as in [Mil17, Chapter 1.a] and give some examples. Mention that every algebraic group is an algebraic subgroup of GL_n for some n [Mil17, Corollary 4.10].

Introduce the definition of arithmetic subgroups following [Ser79]. Mention examples (1)-(3), and hence that every finitely generated abelian group is arithmetic, and (8). This should motivate us to care about arithmetic and algebraic groups.

Follow [Mil17] in introducing unipotent algebraic groups; mention the two first equivalent characterizations from [Mil17, Theorem 14.5]. Also introduce what is essentially the "opposite" of a unipotent group, namely a reductive algebraic group (see [Mil17, 6.46]) and give examples (see e.g. [MT11, Example 6.17]). An important theorem is that every finite-dimensional algebraic representation of a reductive group is semisimple ([Mil17, Corollary 22.43]).

If time remains, describe the equivalence of unipotent groups to nilpotent Lie algebras from [Mil17, Chapter 14].

(5) **Localization and the unstable motivic homotopy category (60 minutes)**

The main objective of this talk is to define the unstable motivic homotopy category and the $\mathbf{P}^1$-stable motivic homotopy category (both over some fixed base S). This requires a discussion of the Nisnevich topology, (pre)sheaves of spaces, $\mathbf{A}^1$-invariance, and the basics of left Bousfield localization. There are many different approaches to this material, both through model categories (as for example in [Mor04, Lev16]) or through ∞-categories (as for example in [Dég]). Do note that some of these only work over a field, others over a more general base.

Background on Bousfield localization can for example be found in Hovey's book [Hov07] (for model categories) or in Lurie's work [Lur09, Section 5.5.4] for presentable ∞-categories.

(6) **Chow rings (60 minutes)**

The aim is to provide an overview of the Chow group of a variety and classical intersection theory. This includes the definition of Chow groups of a variety, the Chow ring of a smooth variety (ie intersection product), proper pushforward, pullback for maps of smooth varieties and Chern classes of vector bundles.

This is covered in [O'G, Sections 1.1, 1.2] and [Ges, Lecture 1]. For more background, see [Har77] for the classic case of divisors and [Ful98] for a full-blown treatment.

References

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