ON THE ALGEBRAIC K-THEORY OF MODEL CATEGORIES

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ABSTRACT. Waldhausen defined the algebraic K-theory of categories with cofibrations and weak equivalences. In many examples, these categories arise as full subcategories of model categories in the sense of Quillen. We use parts of the additional structure coming from model categories to compare the algebraic K-theory of different categories. The main tool for this is a generalization of Waldhausen's Approximation Theorem.

1. INTRODUCTION

Higher algebraic K-theory was introduced by Quillen [8] to extend the existing theory of the lower K-groups of rings. Waldhausen [16] gave a more general definition of algebraic K-theory that accepts a more flexible input and can be applied to a broader class of situations.

The language of model categories, also introduced by Quillen [7], provides an abstract setup for homotopy theory. Waldhausen did not use model categories for his definition of algebraic K-theory since he was interested in applications like "simple maps" [16, 3.1] which do not fit into the context of model categories [16, 1.2]. Nevertheless, most examples for the categories with cofibrations and weak equivalences which serve as an input for his K-theory machine arise as full subcategories of model categories.

We are concerned with the question whether a functor of categories with cofibrations and weak equivalences induces an equivalence in the algebraic K-theory. Waldhausen's Approximation Theorem [16, Theorem 1.6.7] gives sufficient conditions for this to hold. In a recent preprint [11, A.2 Theorem] Schlichting shows that Waldhausen's original hypothesis on the existence of a cylinder functor can be replaced by a weaker condition of (non functorial) factorization of maps.

Our approach is to replace additionally the approximation property App2 of the Approximation Theorem [16, Theorem 1.6.7] by a weaker condition. That is, we have to find a certain factorization of a map only when its codomain is fibrant in a model category which contains the category with cofibrations and weak equivalences in question. Since model category morphisms mapping into fibrant objects are easier to handle than arbitrary ones, this condition is much easier to verify. Theorem 2.8 below is the resulting variant of the Approximation Theorem.

The first application is Theorem 3.1. There we describe a class of model categories for which the inclusion of the subcategory of finite cofibrant objects into the subcategory of homotopy finite cofibrant objects induces an equivalence in the algebraic K-theory. The conditions of the theorem are particularly satisfied by a pointed simplicial model category which is finitely generated in an appropriate sense. Examples are provided by the category of simplicial sets or by categories of spectra based on simplicial sets like symmetric spectra [6].

Another interesting application is Theorem 3.3. It states that an exact functor $F : \mathcal{C} \to \mathcal{D}$ of categories with cofibrations and weak equivalences inducing an

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equivalence of homotopy categories also induces an equivalence in the algebraic K-theory if $\mathcal{C}$ and $\mathcal{D}$ arise from model categories in a certain way. The latter condition on $\mathcal{C}$ and $\mathcal{D}$ is the following: They should be full subcategories of model categories consisting only of cofibrant objects, with the cofibrations and weak equivalences of $\mathcal{C}$ and $\mathcal{D}$ induced by those of the model categories. And they are asked to contain all cofibrant objects of the model categories that are weakly equivalent to an object of the subcategory. As it is known that two categories with cofibrations and weak equivalences having equivalent homotopy categories may have distinct K-theory [12], Theorem 3.3 is probably the strongest statement one can expect.

Several related results can be found in the literature, all of them also exploiting more structure than just the cofibrations and the weak equivalences. Thomason and Trobaugh [13, Theorem 1.9.8] show that a functor of "complicial biWaldhausen categories" induces an equivalence in the algebraic K-theory if it induces an equivalence of homotopy categories. In [3, Corollary 2.7], a similar statement is shown for "Waldhausen homotopy categories" which for example arise from model categories with all objects fibrant.

Dugger and Shipley [2, Corollary 3.8] prove the statement closest to our second application. They only make the slightly stronger additional assumption that the exact functor is induced by a Quillen equivalence of the model categories containing the categories with cofibrations and weak equivalences. Anyhow, their methods are not suitable to obtain our first application.

Another closely related result is proved by Toën and Vezzosi [14]. They work in a dual setup of categories with fibrations and weak equivalences that admit well behaved embeddings into model categories and show that an (abstract) equivalence of the hammock localizations of these categories implies that their K-theory spectra are equivalent.

Throughout the paper, we freely use Waldhausen's K-theory machine [16] and Quillen's language of model categories [7]. Following Hovey's treatment of model categories [5], we assume all model categories to have functorial factorizations as well as small limits and colimits.

2. An Approximation Theorem

Definition 2.1. Let $\mathcal{C}$ be a category with cofibrations and weak equivalences in the sense of Waldhausen [16, 1.2]. We say that $\mathcal{C}$ is equipped with *special objects* if the following data are given:

(i) A full subcategory $\overline{\mathcal{C}}$ of $\mathcal{C}$ whose objects are called the *special objects* of $\mathcal{C}$.
(ii) An endofunctor Q of $\mathcal{C}$ with image in $\overline{\mathcal{C}}$, referred to as the *special replacement.*
(iii) A natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \to Q$ such that for every object X in $\mathcal{C}$ the map $\eta_X : X \to Q(X)$ is both a cofibration and a weak equivalence.

As a trivial example, we can consider a category with cofibrations and weak equivalences $\mathcal{C}$ as a category with cofibrations, weak equivalences, and special objects by setting $\overline{\mathcal{C}} = \mathcal{C}, Q = \mathrm{id}_{\mathcal{C}}$, and $\eta : \mathrm{id}_{\mathcal{C}} \to Q$ as the identity. In this case we get nothing new. As we will see next, model categories give rise to non-trivial examples.

Let $\mathcal{M}$ be a model category which is pointed, i.e., equipped with a distinguished zero object. Suppose $\mathcal{C}$ is a full subcategory of $\mathcal{M}$ such that all objects of $\mathcal{C}$ are cofibrant in $\mathcal{M}$ and $\mathcal{C}$ contains the zero object, pushouts along cofibrations in $\mathcal{C}$, and all cofibrant objects of $\mathcal{M}$ weakly equivalent to an object of $\mathcal{C}$. As the Gluing Lemma for cofibrant objects in a model category holds [5, Lemma 5.2.6], the model structure of $\mathcal{M}$ restricts to the structure of a category with cofibrations and weak equivalences on $\mathcal{C}$.

Definition 2.2. We call a full subcategory $\mathcal{C}$ of a pointed model category $\mathcal{M}$ an *exhaustive* category with cofibrations and weak equivalences if it satisfies the conditions above.

Since we assume our model categories to have functorial factorizations, the fibrant objects and the fibrant replacement of a model category give rise to special objects in an exhaustive category with cofibrations and weak equivalences.

An important part of Waldhausen's definition of K-theory is the $S_\bullet$-construction [16, 1.3] which constructs a simplicial category with cofibrations and weak equivalences $S_\bullet\mathcal{C}$ out of a category with cofibrations and weak equivalences $\mathcal{C}$. The next proposition shows that structure of special objects is compatible with this construction.

Proposition 2.3. *Let $\mathcal{C}$ be a category with cofibrations, weak equivalences, and special objects. Then the special objects of $\mathcal{C}$ give rise to special objects in $S_n\mathcal{C}$.*

Proof. It suffices to define the additional structure on the category $F_n\mathcal{C}$ of sequences of cofibrations of length n since $F_n\mathcal{C}$ is equivalent to $S_{n+1}\mathcal{C}$ [16, 1.1]. As the category of special objects $\overline{F_n\mathcal{C}}$ we choose the full subcategory of $F_n\mathcal{C}$ whose objects are the sequences

$$A_0 \rightarrowtail A_1 \rightarrowtail \cdots \rightarrowtail A_n$$

with A_i in $\overline{\mathcal{C}}$ for every $0 \leq i \leq n$. We set $Q(A)_0 := Q(A_0)$ and define $Q(A)_i$ inductively by

$$\begin{array}{ccccc}
A_{i-1} & \rightarrowtail & A_i & & \\
{\sim}\downarrow & & {\sim}\downarrow & \searrow^{\sim} & \\
Q(A)_{i-1} & \rightarrowtail & Q(A)_{i-1} \cup_{A_{i-1}} A_i & \overset{\sim}{\rightarrowtail} & Q(Q(A)_{i-1} \cup_{A_{i-1}} A_i) =: Q(A)_i
\end{array}$$

for $1 \leq i \leq n$. The natural transformation associated to the special replacement is given by the collection of maps $A_i \to Q(A)_i$ indicated in the diagram. □

We say that a category with cofibrations, weak equivalences, and special objects satisfies the saturation axiom if the two-out-of-three property for weak equivalences holds [16, 1.2]. A functor F from a category with cofibrations and weak equivalences into a category with cofibrations, weak equivalences, and special objects is exact if it preserves the zero object, cofibrations, weak equivalences, and pushouts along cofibrations. In particular, exactness does not involve the special objects. This ensures that $d_0 : S_n\mathcal{C} \to S_{n-1}\mathcal{C}$ [16, 1.3] is an exact functor. The functor F is said to have the property App1 [16, 1.6] if a map is a weak equivalence if and only if its image under F is.

Definition 2.4. An exact functor $F : \mathcal{C} \to \mathcal{D}$ from a category with cofibrations and weak equivalences into a category with cofibrations, weak equivalences, and special objects has the *special approximation property (SAP)* if the following holds: Given an object A in $\mathcal{C}$, a special object B in $\overline{\mathcal{D}} \subset \mathcal{D}$, and a map $f : F(A) \to B$ in $\mathcal{D}$, there is an object C in $\mathcal{C}$, a cofibration $g : A \to C$ in $\mathcal{C}$, and a weak equivalence $F(C) \to B$ in $\mathcal{D}$ such that the diagram

$$\begin{array}{ccc}
F(A) & \xrightarrow{\quad f \quad} & B \\
 & {\scriptstyle F(g)}\searrow \quad \nearrow{\scriptstyle \sim} & \\
 & F(C) &
\end{array}$$

commutes.

The special approximation property replaces Waldhausen's approximation property App2 [16, 1.6] which requires the same kind of factorizations for maps with arbitrary codomain.

Lemma 2.5. *The special approximation property is preserved under the $S_\bullet$-construction. That is, given an exact functor $F : \mathcal{C} \to \mathcal{D}$ with the SAP, the induced functor $S_n F : S_n\mathcal{C} \to S_n\mathcal{D}$ has the SAP for every $n \geq 1$.*

Proof. The proof is the same as for the corresponding statement about the property App2 in [16, Lemma 1.6.6]. □

The algebraic K-theory of a category with cofibrations, weak equivalences, and special objects is defined as the algebraic K-theory of the underlying category with cofibrations and weak equivalences. Although the special objects play no part in the definition of the K-theory, they are very useful to compare the K-theory of different categories. To use the additional structure of special objects for an adapted version of the Approximation Theorem, we need the following generalization of Quillen's Theorem A [8, §1] which the author learned from Stefan Schwede. The proof is deferred to section 4.

Lemma 2.6. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor, $\overline{\mathcal{B}} \subset \mathcal{B}$ a full subcategory, Q an endofunctor of $\mathcal{B}$ with image in $\overline{\mathcal{B}}$, and η a natural transformation from $\mathrm{id}_{\mathcal{B}}$ to Q. Assume that for every object B in $\overline{\mathcal{B}}$ the left fiber F/B is contractible. Then the map F induces a homotopy equivalence of the nerves of the categories $\mathcal{A}$ and $\mathcal{B}$.*

Definition 2.7. [11, A.5 Definition] A category with cofibrations and weak equivalences $\mathcal{C}$ has *factorizations* if every map in $\mathcal{C}$ can be factored as a cofibration followed by a weak equivalence.

A category with cofibrations and weak equivalences $\mathcal{C}$ has factorizations if and only if $\mathrm{id}_{\mathcal{C}}$ has the property App2. Since App2 is preserved under the $S_\bullet$-construction [16, Lemma 1.6.6], $S_n\mathcal{C}$ has factorizations if $\mathcal{C}$ has.

Approximation Theorem 2.8. *Let $\mathcal{C}$ be a category with cofibrations and weak equivalences that satisfies the saturation axiom and has factorizations. Let $\mathcal{D}$ be a category with cofibrations, weak equivalences, and special objects satisfying the saturation axiom. Consider an exact functor $F : \mathcal{C} \to \mathcal{D}$ which has the property App1 and the special approximation property. Then F induces an equivalence in the algebraic K-theory of the categories.*

Proof. The strategy of the proof is the same as for [16, Theorem 1.6.7] or Schlichting's version [11, A.2 Theorem]. Like there, by the realization lemma for bisimplicial sets and Lemma 2.5, we are done with showing that the induced map $w\mathcal{C} \to w\mathcal{D}$ is a homotopy equivalence.

The additional structure of special objects in $\mathcal{D}$ provides everything we need to apply Lemma 2.6 to the induced functor $w\mathcal{C} \to w\mathcal{D}$: We define the required subcategory $\overline{w\mathcal{D}}$ as the full subcategory of $w\mathcal{D}$ whose objects are the special objects in $\mathcal{D}$. An application of the saturation axiom shows that Q restricts to an endofunctor of $w\mathcal{D}$, and by construction its image lies in $\overline{w\mathcal{D}}$. The natural transformation from the identity functor to this restricted functor needed for Lemma 2.6 is just the natural transformation $\eta : \mathrm{id}_{\mathcal{D}} \to Q$ of $\mathcal{D}$. Lemma 2.6 says that it remains to show that for every special object Y of $w\mathcal{D}$ the left fiber of the induced functor $w\mathcal{C} \to w\mathcal{D}$ over Y is contractible.

But from this point on we can follow Schlichting's proof line by line and notice that the only application of the property App2 is made with Y as target object. Hence the SAP is sufficient. □

Schlichting uses the factorizations in $\mathcal{C}$ to replace the cylinder functor on $\mathcal{C}$ needed for Waldhausen's original Approximation Theorem [16, Theorem 1.6.7] by a weaker condition. If there happens to be a cylinder functor on $\mathcal{C}$, on could either use the cylinder functor to get factorizations or modify the proof of Waldhausen's original Approximation Theorem in the same way to replace the property App2 by the special approximation property.

3. Applications

Homotopy finiteness. The algebraic K-theory of the whole subcategory of cofibrant objects of a model category is trivial by a version of the Eilenberg swindle. That is, the Additivity Theorem [16, Theorem 1.4.2] implies that the identity functor on the algebraic K-theory of the category is null-homotopic if there exist infinite sums. Only under some sort of finiteness condition will a non-trivial K-theory be obtained.

We call an object A in a category $\mathcal{A}$ with small colimits *finite* if the functor $\mathcal{C}(A,-)$ commutes with directed colimits. Some other authors also use the term "finitely presented" instead of finite for this condition. For example, the finite objects in the category of sets are the finite sets, and the finite objects in the category of simplicial sets are the finite simplicial sets, i.e., the simplicial sets with finitely many non-degenerated simplices. We notice that a finite colimit of finite objects is finite again.

If $\mathcal{M}$ is a pointed model category, we denote the full subcategory of finite cofibrant objects by $\mathcal{M}^f_{\mathrm{cof}}$. Using the Gluing Lemma for cofibrant objects in a model category [5, Lemma 5.2.6], it is easy to check that the cofibrations and weak equivalences of $\mathcal{M}$ induce the structure of a category with cofibrations and weak equivalences on $\mathcal{M}^f_{\mathrm{cof}}$.

We call an object of $\mathcal{M}$ *homotopy finite* if it is weakly equivalent, i.e., isomorphic in the homotopy category, to a finite cofibrant object of $\mathcal{M}$. Equivalently, a cofibrant object X is homotopy finite if there are a finite cofibrant object X', a cofibrant and fibrant object $\tilde{X}$, and a chain of weak equivalences $X \xrightarrow{\sim} \tilde{X} \xleftarrow{\sim} X'$. The full subcategory of homotopy finite cofibrant objects of $\mathcal{M}$ is denoted by $\mathcal{M}^{\mathrm{hf}}_{\mathrm{cof}}$.

One would like to have that the restriction of the model structure on $\mathcal{M}$ turns $\mathcal{M}^{\mathrm{hf}}_{\mathrm{cof}}$ into a category with cofibrations and weak equivalences whose K-theory is equivalent to the one of $\mathcal{M}^f_{\mathrm{cof}}$. Unfortunately it is not clear that the pushout of homotopy finite objects is homotopy finite again. The next theorem gives sufficient conditions on $\mathcal{M}$ for this to hold.

Theorem 3.1. *Let $\mathcal{M}$ be a pointed model category. Suppose the following holds:*

(i) Every map $U \to V$ of finite cofibrant objects has a factorization $U \rightarrowtail W \xrightarrow{\sim} V$ as a cofibration followed by a weak equivalence with a finite object W.
(ii) Every map $X \to Y$ from a finite cofibrant object X to a homotopy finite cofibrant and fibrant object Y factors as $X \to Z \xrightarrow{\sim} Y$ with a finite cofibrant object Z.

Then $\mathcal{M}^{\mathrm{hf}}_{\mathrm{cof}}$ has the structure of a category with cofibrations and weak equivalences, and the inclusion $\mathcal{M}^f_{\mathrm{cof}} \to \mathcal{M}^{\mathrm{hf}}_{\mathrm{cof}}$ induces an equivalence in the algebraic K-theory.

Proof. We first verify the non-trivial part of the first statement. Given a diagram $Y \leftarrowtail X \to Z$ of homotopy finite cofibrant objects, we can use the hypothesis and the model category axioms to build up a commutative diagram

$$\begin{array}{ccccc} Y & \leftarrowtail & X & \longrightarrow & Z \\ \sim\downarrow & & \sim\downarrow & & \downarrow\sim \\ \tilde{Y} & \leftarrowtail & \tilde{X} & \longrightarrow & \tilde{Z} \\ \sim\uparrow & & \sim\uparrow & & \uparrow\sim \\ Y' & \leftarrowtail & X' & \longrightarrow & Z' \end{array}$$

of cofibrant objects with all objects in the lower row are finite. Applying the Gluing Lemma twice to the diagram shows that $Y \cup_X Z$ is weakly equivalent to $Y' \cup_{X'} Z'$, and thus homotopy finite.

Now $\mathcal{M}^{\mathrm{hf}}_{\mathrm{cof}}$ is an exhaustive category with cofibrations and weak equivalences. Hence the fibrant objects give rise to special objects in $\mathcal{M}^{\mathrm{hf}}_{\mathrm{cof}}$, and an application of the Approximation Theorem 2.8 shows the second part. □

As we will see next, the assumptions of the theorem above are satisfied by a broad class of model categories.

Proposition 3.2. *Let $\mathcal{M}$ be a pointed simplicial model category which has a set of generating acyclic cofibrations I with finite domains and codomains and the property that the simplicial action $- \otimes \Delta^1$ of Δ^1 preserves finite objects. Then $\mathcal{M}^{\mathrm{hf}}_{\mathrm{cof}}$ has the structure of a category with cofibrations and weak equivalences, and the inclusion $\mathcal{M}^{f}_{\mathrm{cof}} \to \mathcal{M}^{\mathrm{hf}}_{\mathrm{cof}}$ induces an equivalence in the algebraic K-theory.*

Proof. We construct the factorizations needed to apply the previous theorem. The ordinary mapping cylinder $X \otimes \Delta^1 \cup_X Y$ of a map $f : X \to Y$ provides the desired factorizations for maps of finite cofibrant objects.

To obtain the other factorization, we fix two finite cofibrant objects X and Y of $\mathcal{M}$ and denote by QY the fibrant replacement of Y arising from an application of Quillen's small object argument [5, Theorem 2.1.14] using the set I. Then every map $X \to QY$ factors via $Y' \xrightarrow{\sim} QY$ where the object Y' is obtained from Y by attaching finitely many maps from I. Hence Y' is a finite cofibrant object again. If $f : X \to Z$ is a map from X to an arbitrary homotopy finite cofibrant and fibrant object Z, we can find a finite cofibrant object Y and a weak equivalence $u : Z \xrightarrow{\sim} QY$ such that the map uf factors via Y' as above. Since both QY and Z are cofibrant and fibrant, u has a homotopy inverse $v : QY \to Z$, and the homotopy relating id_Z and vu gives rise to a weak equivalence from the mapping cylinder of $X \to Y'$ to Z. This provides the desired factorization $X \to X \otimes \Delta^1 \cup_X Y' \xrightarrow{\sim} Z$. □

Equivalences of homotopy categories. Let $\mathcal{M}$ and $\mathcal{N}$ be pointed model categories and let $\mathcal{C} \subset \mathcal{M}$ and $\mathcal{D} \subset \mathcal{N}$ be subcategories which are exhaustive categories with cofibrations and weak equivalences. Then the factorizations of the model categories give rise to factorizations in $\mathcal{C}$ and special objects in $\mathcal{D}$.

We define the homotopy categories $\mathrm{Ho}(\mathcal{C})$ and $\mathrm{Ho}(\mathcal{D})$ of $\mathcal{C}$ and $\mathcal{D}$ to be the full subcategories of the homotopy categories of the model categories $\mathcal{M}$ and $\mathcal{N}$ that contain precisely the objects of $\mathcal{C}$ and $\mathcal{D}$. These homotopy categories are the localizations of $\mathcal{C}$ and $\mathcal{D}$ with respect to the weak equivalences.

Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor that induces an equivalence $\mathrm{Ho}(F) : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$. As a first consequence, the functor F has the property App1: A map in $\mathcal{C}$ or $\mathcal{D}$ is a weak equivalence if and only if its image in the homotopy category is an isomorphism, and a map in a category is a isomorphism if and only if its image under an equivalence of categories is.

An argument used in the proof of [14, Lemma 3.5] shows that F also has the special approximation property: Let $f : F(X) \to Y$ be a test map with X cofibrant in $\mathcal{C} \subset \mathcal{M}$ and Y cofibrant and fibrant in $\mathcal{D} \subset \mathcal{N}$. Using the assumptions on X

and Y and the fact that F induces an equivalence of homotopy categories, we can find an object Z in $\mathcal{C}$, a cofibration $u : X \rightarrowtail Z$ in $\mathcal{C}$, and a weak equivalence $v : F(Z) \xrightarrow{\sim} Y$ such that $vF(u)$ and f coincide in $\mathrm{Ho}(\mathcal{D})$. Now we can use [4, Corollary 7.3.12(1)] to replace the weak equivalence v by another weak equivalence w satisfying $f = wF(u)$ actually in the category $\mathcal{N}$.

Hence we can apply the Approximation Theorem 2.8 to obtain the following.

Theorem 3.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be full subcategories of pointed model categories $\mathcal{M}$ and $\mathcal{N}$ that are exhaustive categories with cofibrations and weak equivalences. Then an exact functor $F : \mathcal{C} \to \mathcal{D}$ inducing an equivalence of the homotopy categories also induces an equivalence in the K-theory of the categories.*

Example 3.4. Let $\mathcal{S}_*$ be the category of pointed simplicial sets and let Sp be the category of spectra in the sense of Bousfield and Friedlander [1]. Proposition 3.2 shows that each of the inclusions $(\mathcal{S}_*)^f_{\mathrm{cof}} \to (\mathcal{S}_*)^{\mathrm{hf}}_{\mathrm{cof}}$ and $(Sp)^f_{\mathrm{cof}} \to (Sp)^{\mathrm{hf}}_{\mathrm{cof}}$ induces an equivalence in the algebraic K-theory. In the case of simplicial sets, this recovers [16, Proposition 2.1.1] and gives two equivalent ways to define $A(*)$, Waldhausen's algebraic K-theory of a point.

The suspension spectrum functor from pointed simplicial sets to spectra induces an exact functor $(\mathcal{S}_*)^f_{\mathrm{cof}} \to (Sp)^f_{\mathrm{cof}}$. As outlined in [15, §1] and elaborated in [9], this functor induces an equivalence in the algebraic K-theory since suspension induces a self-equivalence on $(\mathcal{S}_*)^f_{\mathrm{cof}}$ [16, Proposition 1.6.2]. Hence one can also use $(Sp)^f_{\mathrm{cof}}$ or $(Sp)^{\mathrm{hf}}_{\mathrm{cof}}$ to define $A(*)$.

The category of symmetric spectra Sp^Σ, introduced in [6], is a model category that is Quillen equivalent to the category Sp. It has the advantage that there is a smash product of symmetric spectra that induces the smash product in the stable homotopy category. Proposition 3.2 shows that $(Sp^\Sigma)^f_{\mathrm{cof}}$ and $(Sp^\Sigma)^{\mathrm{hf}}_{\mathrm{cof}}$ have equivalent algebraic K-theory, and an application of Theorem 3.3 to the left Quillen functor $V : Sp \to Sp^\Sigma$ [6, Theorem 4.2.5] shows that $(Sp^\Sigma)^f_{\mathrm{cof}}$ and $(Sp^\Sigma)^{\mathrm{hf}}_{\mathrm{cof}}$ can also be used to define $A(*)$.

This description of $A(*)$ is the symmetric spectrum analogue to a result about S-modules [3, Theorem VI.8.2], and it makes it possible to apply localization techniques in the stable homotopy category to the calculation of $A(*)$, see [15]. A detailed proof of the statements above can be found in [10].

4. Proof of Lemma 2.6

We recall that the nerve construction is a functor from the category of small categories to simplicial sets such that a natural transformation of functors gives rise to a simplicial homotopy of the maps of simplicial sets induced by the functors. We call two functors homotopic if the geometric realization of their nerves are. This is for example the case if there is a natural transformation relating them. In particular, it makes sense to ask a functor to be a homotopy equivalence or a category to be contractible.

Proof. First we define two auxiliary categories needed for the proof. Let $\mathcal{C}$ be the category whose objects are the tuples $(A, F(A) \to \overline{B} \leftarrow B)$ with A in $\mathcal{A}, B$ in $\mathcal{B}$, and $\overline{B}$ in $\overline{\mathcal{B}}$. A morphism from $(A, F(A) \to \overline{B} \leftarrow B)$ to $(A', F(A') \to \overline{B}' \leftarrow B')$ is a collection of morphisms $A \to A', \overline{B} \to \overline{B}', B \to B'$ making the obvious diagram commutative. Further we define $\mathcal{D}$ as the full subcategory of $\mathrm{Ar}\,\mathcal{B}$ consisting of those morphisms whose codomain is in $\overline{\mathcal{B}}$.

We obtain a diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{G} & \mathcal{D} \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle K} \\ \mathcal{A} & \xrightarrow{F} & \mathcal{B} \end{array} \qquad \text{with} \qquad \begin{array}{c} G(A, F(A) \to \overline{B} \leftarrow B) = B \to \overline{B}, \\ H(A, F(A) \to \overline{B} \leftarrow B) = A, \quad \text{and} \\ K(B \to \overline{B}) = B \end{array}$$

denoting the forgetful functors. The two compositions FH and KG do not coincide, but they are both homotopic to the functor

$$T : \mathcal{C} \to \mathcal{B}, \quad (A, F(A) \to \overline{B} \leftarrow B) \mapsto \overline{B}.$$

The homotopies are induced by the natural transformations

$$\begin{array}{lll} FH \to T, & (A, F(A) \to \overline{B} \leftarrow B) \mapsto (F(A) \to \overline{B}) & \quad \text{and} \\ KG \to T, & (A, F(A) \to \overline{B} \leftarrow B) \mapsto (B \to \overline{B}). & \end{array}$$

Hence FH is a homotopy equivalence if and only if KG is, and so it suffices to show that H, G, and K are homotopy equivalences.

The functor

$$H' : \mathcal{A} \to \mathcal{C}, \quad A \mapsto (A, F(A) \to Q(F(A)) \xleftarrow{=} Q(F(A)))$$

is a section for H, and the composite $H'H : \mathcal{C} \to \mathcal{C}$ is homotopic to the functor

$$\mathcal{C} \to \mathcal{C}, \quad (A, F(A) \to \overline{B} \leftarrow B) \mapsto (A, F(A) \to Q(\overline{B}) \leftarrow Q(\overline{B}))$$

by means of the natural transformation

$$(A, F(A) \to \overline{B} \leftarrow B) \mapsto \left(\begin{array}{cccc} A & F(A) \longrightarrow & Q(F(A)) \longleftarrow & Q(F(A)) \\ \downarrow & \downarrow & \downarrow & \downarrow \\ A & F(A) \longrightarrow & Q(\overline{B}) \longleftarrow & Q(\overline{B}) \end{array} \right).$$

But $\mathrm{id}_{\mathcal{C}}$ is also homotopic to this endofunctor of $\mathcal{C}$ via the natural transformation given by

$$(A, F(A) \to \overline{B} \leftarrow B) \mapsto \left(\begin{array}{cccc} A & F(A) \longrightarrow & \overline{B} \longleftarrow & B \\ \downarrow & \downarrow & \downarrow & \downarrow \\ A & F(A) \longrightarrow & Q(\overline{B}) \longleftarrow & Q(\overline{B}) \end{array} \right).$$

So $H'H$ is homotopic to $\mathrm{id}_{\mathcal{C}}$. Since we have $HH' = \mathrm{id}_{\mathcal{C}}$ the functor H is a homotopy equivalence.

A similar argument using the section $K' : \mathcal{B} \to \mathcal{D}$, $B \mapsto (B \to Q(B))$ of K shows that K is also a homotopy equivalence.

By Quillen's Theorem A [8, §1], the functor G is a homotopy equivalence if the left fiber $G/(B' \to \overline{B}')$ is contractible for every object $B' \to \overline{B}'$ in $\mathcal{D}$. In view of the hypothesis on the left fibers of F over objects of $\overline{\mathcal{B}}$ it suffices to show that the functor

$$U' : F/\overline{B}' \to G/(B' \to \overline{B}'), \quad (A, F(A) \to \overline{B}') \mapsto \left(A, \begin{array}{ccc} F(A) \longrightarrow & \overline{B}' \longleftarrow & B' \\ & \downarrow & \downarrow \\ & \overline{B}' \longleftarrow & B' \end{array} \right)$$

is a homotopy equivalence. To do so, we use the retraction $U : G/(B' \to \overline{B}') \to F/\overline{B}'$ of U' defined by

$$\left(A, \begin{array}{ccc} F(A) \longrightarrow & \overline{B} \longleftarrow & B \\ & \downarrow & \downarrow \\ & \overline{B}' \longleftarrow & B' \end{array} \right) \mapsto \left(A, F(A) \to \overline{B}' \right).$$

The non-identity composite $U'U : G/(B' \to \overline{B}') \to G/(B' \to \overline{B}')$ is homotopic to the identity on $G/(B' \to \overline{B}')$ by means of the natural transformation whose image on an element of this left fiber is given by

$$\begin{array}{ccccccccc}
 & F(A) & \longrightarrow & & \overline{B}' & \longleftarrow & & B' & \\
 \nearrow & & & \nearrow & \downarrow & & \nearrow & \downarrow & \\
F(A) & & \longrightarrow & \overline{B} & \longleftarrow & B & & & \\
 & & & \downarrow & \overline{B}' & \longleftarrow & & B'. & \\
 & & & \downarrow & \nearrow & & \downarrow & \nearrow & \\
 & & & \overline{B}' & \longleftarrow & B' & & &
\end{array}$$

Hence the left fibers of G are contractible, and the application of Theorem A completes the proof. □

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