

DG-ALGEBRAS AND DERIVED A_∞ -ALGEBRAS

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1/16

A_∞ -ALGEBRAS

DEFINITION (STASHEFF)

An A_∞ -algebra is a graded k -module A together with a unit element $1_A \in A_0$ and k -linear maps $m_j: A^{\otimes j} \rightarrow A[2-j]$ for $j \geq 1$, satisfying

$$\sum_{r+s+t=n} (-1)^{rs+t} m_{r+1+t}(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) = 0$$

for $n \geq 1$ (and a unit condition).

- A map $f: A \rightarrow B$ of A_∞ -algebras is a family of k -linear maps $f_j: A^{\otimes j} \rightarrow B[1-j]$ satisfying appropriate relations.
- An A_∞ -algebra is *minimal* if $m_1 = 0$.
- dgas are A_∞ -algebras with m_1 the differential, m_2 the multiplication, and all other $m_j = 0$.

3/16

INITIAL QUESTION

Let A be a differential graded algebra over a commutative ring k , possibly unbounded and with homological grading.

Its homology $H_*(A)$ is a graded k -algebra.

QUESTION

Is there some additional structure on $H_*(A)$ which allows us to recover the quasi-isomorphism type of A from $H_*(A)$?

If k is a field (or, more generally, $H_*(A)$ is k -projective), a minimal A_∞ -algebra structure on $H_*(A)$ provides an answer.

2/16

MINIMAL MODELS

THEOREM (KADEISHVILI)

Let A be a dga with $H_*(A)$ k -projective. There exist a minimal A_∞ -algebra structure on $H_*(A)$ and a quasi-isomorphism $f: H_*(A) \rightarrow A$, where the m_2 of $H_*(A)$ is the algebra multiplication.

One can recover the quasi-isomorphism type of A from the minimal A_∞ -algebra $H_*(A)$.

In other words: The higher multiplications $m_j, j \geq 3$, on the graded algebra $(H_*(A), m_2)$ provide the data we are asking for.

4/16

RESOLUTIONS

The statement of Kadeishvili's theorem does in general not hold if $H_*(A)$ is not k -projective.

WHAT TO DO INSTEAD?

Look for higher multiplications on a resolution of $H_*(A)$!

We consider (\mathbb{N}, \mathbb{Z}) -bigraded k -modules with the \mathbb{N} -grading the 'horizontal' direction and the \mathbb{Z} -grading the 'vertical' direction.

For E and F bigraded k -modules,

- $E[st]_{ij} = E_{i-s, t-j}$
- $(E \otimes F)_{uv} = \bigoplus_{\substack{i+p=u \\ j+q=v}} E_{ij} \otimes_k E_{pq}$

5 / 16

STRUCTURE MAPS & FORMULAS FOR dA_∞ -ALGEBRAS

A dA_∞ -algebra has structure maps starting with:

$$m_{01}: E \rightarrow E[0, 1] \quad m_{11}: E \rightarrow E[1, 0] \quad m_{21}: E \rightarrow E[2, -1]$$

$$m_{02}: E^{\otimes 2} \rightarrow E \quad m_{12}: E^{\otimes 2} \rightarrow E[1, -1]$$

$$m_{03}: E^{\otimes 3} \rightarrow E[0, -1]$$

The first six relations are:

$$m_{01}m_{01} = 0$$

$$m_{01}m_{02} = m_{02}(\mathbf{1} \otimes m_{01}) + m_{02}(m_{01} \otimes \mathbf{1})$$

$$m_{01}m_{11} = m_{11}m_{01}$$

$$m_{02}(m_{02} \otimes \mathbf{1}) = m_{01}m_{03} + m_{02}(\mathbf{1} \otimes m_{02}) + m_{03}(m_{01} \otimes \mathbf{1}^{\otimes 2}) \\ + m_{03}(\mathbf{1} \otimes m_{01} \otimes \mathbf{1}) + m_{03}(\mathbf{1}^{\otimes 2} \otimes m_{01})$$

$$m_{11}m_{02} = m_{01}m_{12} + m_{12}(\mathbf{1} \otimes m_{01}) + m_{12}(m_{01} \otimes \mathbf{1}) \\ + m_{02}(\mathbf{1} \otimes m_{11}) + m_{02}(m_{11} \otimes \mathbf{1})$$

$$m_{11}m_{11} = m_{01}m_{21} + m_{21}m_{01}$$

7 / 16

DEFINITION OF dA_∞ -ALGEBRAS

DEFINITION

A *derived A_∞ -algebra* (or *dA_∞ -algebra*) is a (\mathbb{N}, \mathbb{Z}) -bigraded k -module E with a unit element $1_E \in E_{0,0}$ and structure maps

$$m_{ij}: E^{\otimes j} \rightarrow E[i, 2 - (i + j)] \text{ with } i \geq 0, j \geq 1$$

satisfying

$$\sum_{\substack{i+p=u \\ r+q+t=v \\ r+1+t=j}} (-1)^{rq+t+pj} m_{ij}(\mathbf{1}^{\otimes r} \otimes m_{pq} \otimes \mathbf{1}^{\otimes t}) = 0$$

for all $u \geq 0$ and $v \geq 1$ (and a unit condition).

A dga may be viewed as a dA_∞ -algebra concentrated in horizontal degree 0, with m_{01} the differential, m_{02} the multiplication, and all other $m_{ij} = 0$.

6 / 16

SOME TERMINOLOGY FOR dA_∞ -ALGEBRAS

- A map of dA_∞ -algebras is a family of k -module maps $f_{ij}: E^{\otimes j} \rightarrow E[i, 1 - (i + j)]$ satisfying appropriate relations.
- A dA_∞ -algebra is *minimal* if $m_{01} = 0$.
- A map $f: E \rightarrow F$ is an E_2 -*equivalence* if it induces isomorphisms in the iterated homology with respect to m_{01} and m_{11} . This is possible since we require

$$m_{01}m_{01} = 0 \quad \text{and} \quad m_{01}m_{21} + m_{21}m_{01} = m_{11}m_{11}.$$

8 / 16

DERIVED HOCHSCHILD COHOMOLOGY CLASS

Let A be a dga with minimal dA_∞ -model E . The complex

$$C^{qt}(E) = \bigoplus_{r+s=q} \text{Hom}_k(E^{\otimes r}, E[s, t])$$

has a differential $C^{qt} \rightarrow C^{q+1, t}$ induced from m_{11} and a Hochschild differential. Its cohomology is the *derived Hochschild cohomology* $\text{dHH}^{qt}(H_*(A))$.

PROPOSITION

$\gamma_A := [m_{03} + m_{12} + m_{21}] \in \text{dHH}^{3, -1}(H_*(A))$ is a well defined cohomology class depending only on the quasi-isomorphism type of A .

If $H_*(A) = 0$ for $* < 0$, then

$$\text{dHH}^{3, -1}(H_*(A)) \rightarrow \text{dHH}^3(H_0(A), H_1(A))$$

maps γ_A to the first k -invariant of A .

13 / 16

dA_∞ -STRUCTURES AND THE TENSOR COALGEBRA

Let E be a dA_∞ -algebra, $SE = E[0, 1]$, and $\bar{T}SE = \bigoplus_{j \geq 1} SE^{\otimes j}$

$$\begin{aligned} m_{ij} &\leftrightarrow \tilde{m}_{ij}^1: SE^{\otimes i} \rightarrow SE[i, 1 - i] \\ &\leftrightarrow \tilde{m}_{ij}^q = \sum_{\substack{r+1+t=q \\ r+s+t=j}} \mathbf{1}^{\otimes r} \otimes \tilde{m}_{is}^1 \otimes \mathbf{1}^{\otimes t}: SE^{\otimes j} \rightarrow SE^{\otimes q}[i, 1 - i] \\ &\leftrightarrow \tilde{m}_j: \bar{T}SE \rightarrow \bar{T}SE[i, 1 - i] \text{ with components } \tilde{m}_{ij}^q. \end{aligned}$$

LEMMA

dA_∞ -relations $\Leftrightarrow (\bar{T}SE, \tilde{m}_i)$ is a twisted chain complex

LEMMA

dA_∞ -algebra maps $E \rightarrow F$ correspond to maps $(\bar{T}SE, \tilde{m}_i^E) \rightarrow (\bar{T}SF, \tilde{m}_i^F)$ of twisted chain complexes

15 / 16

TWISTED CHAIN COMPLEXES

DEFINITION

A *twisted chain complex* E is an (\mathbb{N}, \mathbb{Z}) -graded k -module with differentials $d_i^E: E \rightarrow E[i, 1 - i]$ for $i \geq 0$ satisfying

$$\sum_{i+p=u} (-1)^i d_i d_p = 0 \text{ for } u \geq 0.$$

Maps are families of k -module maps $f_i: E \rightarrow F[i, -i]$ satisfying

$$\sum_{i+p=u} (-1)^i f_i d_p^E = \sum_{i+p=u} d_i^F f_p.$$

Composition of maps: $(gf)_u = \sum_{i+p=u} g_i f_p$

SLOGAN

A_∞ -algebras \leftrightarrow chain complexes

dA_∞ -algebras \leftrightarrow twisted chain complexes

If E is a dA_∞ -algebra, then $(E, m_{i1}, i \geq 0)$ is a twisted chain complex.

14 / 16

PROOF OF THEOREM 2 (SKETCH)

- The category of modules over a dA_∞ -algebra E is enriched in twisted chain complexes.
- The endomorphism object $\underline{\text{Hom}}_E(E, E)$ is a monoid in tCh_k .
- $\text{Tot } \underline{\text{Hom}}_E(E, E)$ is a dga.
- If E has E_2 -homology concentrated in horizontal degree 0, there is an E_2 -equivalence $E \rightarrow \text{Tot } \underline{\text{Hom}}_E(E, E)$.
- If E and F are E_2 -quasi-isomorphic, then $\text{Tot } \underline{\text{Hom}}_E(E, E)$ and $\text{Tot } \underline{\text{Hom}}_F(F, F)$ are quasi-isomorphic as dgas.

PROOF OF THEOREM 2.

Apply the last statement to $E \rightarrow A$. □

IN OTHER WORDS:

The dA_∞ -algebras with E_2 -homology concentrated in horizontal degree 0 model quasi-isomorphism types of dgas.

16 / 16