DG-ALGEBRAS AND DERIVED A_∞-ALGEBRAS

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Initial Question

Let A be a differential graded algebra over a commutative ring k, possibly unbounded and with homological grading.

Its homology $H_*(A)$ is a graded k-algebra.

Question

Is there some additional structure on $H_*(A)$ which allows us to recover the quasi-isomorphism type of A from $H_*(A)$?

If k is a field (or, more generally, $H_*(A)$ is k-projective), a minimal A_∞-structure on $H_*(A)$ provides such a structure.

MINIMAL MODELS

THEOREM (KADEISHVILI)

Let A be a dga with $H_*(A)$ k-projective. There exist a minimal A_∞-structure on $H_*(A)$ and a quasi-isomorphism $f \colon H_*(A) \to A$.

This means:

- There are k-linear maps $m_j \colon H_*(A)^{\otimes j} \to H_*(A)[2-j]$ with $j \geq 2$ satisfying appropriate relations.
  m_2 is the algebra multiplication.
- There are k-linear maps $f_j \colon H_*(A)^{\otimes j} \to A[1-j]$ with $j \geq 1$ satisfying appropriate relations.
  f_1 is a cycle selection map.

One can recover the quasi-isomorphism type of A from the minimal A_∞-algebra $H_*(A)$.

A NON-EXAMPLE

Let $k = \mathbb{Z}/p^2$.

Let A be the dga $k \xrightarrow{\cdot p} k$ concentrated in degrees 0 and 1.

$\rightsquigarrow$ $0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \xrightarrow{\cdot p} \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$ is exact, so $H_*(A) = \mathbb{Z}/p$ if $* = 0, 1$ and 0 otherwise.

$\rightsquigarrow$ There is no k-linear cycle selection $f_1 : H_0(A) \to A_0$, hence the statement of Kadeishvili's theorem does not hold for A.

An additional grading

How to overcome this?

Allow to resolve the dga in the direction of an additional grading and look for higher multiplications on a resolution of $H_*(A)$!

Consider $(\mathbb{N}, \mathbb{Z})$-bigraded k-modules
$\mathbb{N}$-grading = ‘horizontal’ direction
$\mathbb{Z}$-grading = ‘vertical’ direction

For E and F bigraded k-modules,

- $E[st]_{ij} = E_{i-s,t-j}$
- $(E \otimes F)_{uv} = \bigoplus_{\substack{i+p=u \\ j+q=v}} E_{ij} \otimes E_{pq}$

Definition of dA_∞-algebras

Definition

A *derived A_∞-algebra* (or *dA_∞-algebra*) is a $(\mathbb{N}, \mathbb{Z})$-bigraded k-module E with a unit element $1_E \in E_{0,0}$ and structure maps

$$m_{ij} : E^{\otimes j} \to E[i, 2-(i+j)] \text{ with } i \geq 0, j \geq 1$$

satisfying

$$\sum_{\substack{i+p=u \\ r+q+t=v \\ r+1+t=j}} (-1)^{rq+t+pj} m_{ij}(\mathbf{1}^{\otimes r} \otimes m_{pq} \otimes \mathbf{1}^{\otimes t}) = 0$$

for all $u \geq 0$ and $v \geq 1$ (and a unit condition).

- A_∞-algebras are dA_∞-algebras concentrated in horizontal degree 0.
- dgas may be viewed as dA_∞-algebras in the same way.

FORMULAS FOR dA_∞-ALGEBRAS

A dA_∞-algebra has structure maps starting with:

$m_{01}: E \to E[0,1]$ $\quad m_{11}: E \to E[1,0]$ $\quad m_{21}: E \to E[2,-1]$

$m_{02}: E^{\otimes 2} \to E$ $\quad m_{12}: E^{\otimes 2} \to E[1,-1]$

$m_{03}: E^{\otimes 3} \to E[0,-1]$

The first six relations are:

$$\begin{aligned}
m_{01}m_{01} &= 0 \\
m_{01}m_{02} &= m_{02}(\mathbf{1} \otimes m_{01}) + m_{02}(m_{01} \otimes \mathbf{1}) \\
m_{01}m_{11} &= m_{11}m_{01} \\
m_{02}(m_{02} \otimes \mathbf{1}) &= m_{01}m_{03} + m_{02}(\mathbf{1} \otimes m_{02}) + m_{03}(m_{01} \otimes \mathbf{1}^{\otimes 2}) \\
&\quad + m_{03}(\mathbf{1} \otimes m_{01} \otimes \mathbf{1}) + m_{03}(\mathbf{1}^{\otimes 2} \otimes m_{01}) \\
m_{11}m_{02} &= m_{01}m_{12} + m_{12}(\mathbf{1} \otimes m_{01}) + m_{12}(m_{01} \otimes \mathbf{1}) \\
&\quad + m_{02}(\mathbf{1} \otimes m_{11}) + m_{02}(m_{11} \otimes \mathbf{1}) \\
m_{11}m_{11} &= m_{01}m_{21} + m_{21}m_{01}
\end{aligned}$$

SOME TERMINOLOGY FOR dA_∞-ALGEBRAS

- A more concise definition of dA_∞-algebras and their maps in terms of tensor coalgebras will be given later.
- A map of dA_∞-algebras is a family of k-module maps $f_{ij} : E^{\otimes j} \to E[i, 1-(i+j)]$ satisfying appropriate relations.
- A dA_∞-algebra is *minimal* if $m_{01}^E = 0$.
- A map $f : E \to F$ is an E_2*-equivalence* if it induces isomorphisms in the iterated homology with respect to m_{01} and m_{11}. Possible since we require

$$m_{01}m_{01} = 0 \quad \text{and} \quad m_{01}m_{21} + m_{21}m_{01} = m_{11}m_{11}.$$

Main theorems

Theorem 1
Let A be a dga over a commutative ring k. There exists a k-projective minimal dA_∞-algebra E together with an E_2-equivalence $E \to A$ of dA_∞-algebras.

- This *minimal dA_∞-algebra model E* of A is well defined up to E_2-equivalences between k-projective minimal dA_∞-algebras.
- (E, m_{11}^E, m_{02}^E) is a k-projective resolution of the graded k-algebra $H_*(A)$.

Theorem 2
The quasi-isomorphism type of A can be recovered from E.

Proof of Theorem 1 (sketch)

Definition

A bidga B is a dA_∞-algebra with $m_{ij}^B = 0$ if $i + j \geq 3$. Maps of bidgas have $f_{ij} = 0$ for $i + j \geq 2$.

- Equivalently: Monoids in the category of bicomplexes.
- dgas are bidgas concentrated in horizontal degree 0

1st step of proof

Given A, there is a bidga B and an E_2-equivalence $B \to A$ of bidgas such that $H_*(B, m_{01}^B)$ is k-projective.

2nd step of proof

Set $E = H_*(B, m_{01}^B)$. There exist a minimal dA_∞-structure on E and an E_2-equivalence $E \to B$.

Application and example: Ext-algebras

Let M be a k-module and P be a k-projective resolution of M.

The endomorphism dga $A = \mathrm{Hom}_k(P, P)$ of P has homology

$$H_*(A) = \mathrm{Ext}_k^{-*}(M, M).$$

A minimal dA_∞-algebra model of A is a resolution of the Yoneda algebra $\mathrm{Ext}_k^*(M, M)$ together with structure maps m_{ij}. This data encodes the quasi-isomorphism type of the endomorphism dga.

Example
Let $k = \mathbb{Z}$ and $M = \mathbb{Z}/p$. Then $H_*(A) = \Lambda^*_{\mathbb{Z}/p}(w)$ with $|w| = -1$.

EXAMPLE: $\mathrm{Ext}^*_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/p)$

Let $E = \Lambda^*(a, b)$ with $|a| = (0, -1)$ and $|b| = (1, 0)$.

$$\begin{array}{cccccc} & & & 0 & & 1 \\ 0 & \mathbb{Z}/p & \leftarrow\!-\,-\,- & \mathbb{Z}\{\iota\} & \xleftarrow[\cdot p]{\quad} & \mathbb{Z}\{b\} \\ -1 & \mathbb{Z}/p\{w\} & \leftarrow\!-\,- & \mathbb{Z}\{a\} & \xleftarrow[\cdot p]{\quad} & \mathbb{Z}\{ab\} \end{array}$$

- The given data specifies the m_{11} and m_{02} of a minimal dA_∞-algebra.
- m_{12} satisfies $m_{12}(a \otimes b) = \iota$, $m_{12}(a \otimes ab) = a$, $m_{12}(ab \otimes b) = -b$, $m_{12}(ab \otimes ab) = -ab$ and is 0 elsewhere.
- All other m_{ij} vanish.

This is a complete description of a minimal dA_∞-model for A.

(Derived) Hochschild cohomology classes

Let $H_*(A)$ be k-projective and $m_i, i \geq 2$, a minimal A_∞-structure.

$$\gamma_A := [m_3] \in \mathrm{HH}^{3,-1}(H_*(A))$$

is a well defined Hochschild-cohomology class.

Let A be a dga with minimal dA_∞-model E. The complex

$$C^{qt}(E) = \bigoplus_{r+s=q} \mathrm{Hom}_k(E^{\otimes r}, E[s,t])$$

has a differential $C^{qt} \to C^{q+1,t}$ induced from m_{11} and a Hochschild differential. Its cohomology is the derived Hochschild cohomology $\mathrm{dHH}^{qt}(H_*(A))$.

Proposition

$\gamma_A := [m_{03} + m_{12} + m_{21}] \in \mathrm{dHH}^{3,-1}(H_*(A))$ is well defined.

PROPERTIES OF γ_A

- As with the Hochschild-class, $\gamma_A \in \mathrm{dHH}^{3,-1}(H_*(A))$ knows all triple Massey products, but not in a k-linear fashion. (Compare with Benson, Krause and Schwede (2004)).
- Assume $H_*(A) = 0$ for $* < 0$. Then

$$\mathrm{dHH}^{3,-1}(H_*(A)) \to \mathrm{dHH}^3(H_0(A), H_1(A))$$

maps γ_A to the first k-invariant of A.

Twisted chain complexes

Definition

A *twisted chain complex* E is an $(\mathbb{N},\mathbb{Z})$-graded k-module with differentials $d_i^E : E \to E[i, 1-i]$ for $i \geq 0$ satisfying

$$\sum_{i+p=u} (-1)^i d_i d_p = 0 \text{ for } u \geq 0.$$

Maps are families of k-module maps $f_i : E \to F[i, -i]$ satisfying $\sum_{i+p=u} (-1)^i f_i d_p^E = \sum_{i+p=u} d_i^F f_p$.
Composition of maps: $(gf)_u = \sum_{i+p=u} g_i f_p$

Slogan

A_∞-algebras	$\leftrightarrow$	chain complexes
dA_∞-algebras	$\leftrightarrow$	twisted chain complexes

- If E is a dA_∞-algebra, then $(E, m_{i1}, i \geq 0)$ is a twisted chain complex.

dA_∞-STRUCTURES AND THE TENSOR COALGEBRA

Let E be a dA_∞-algebra, $SE = E[0,1]$, and $\overline{T}SE = \bigoplus_{j\geq 1} SE^{\otimes j}$

$$\begin{aligned} m_{ij} &\leftrightarrow \widetilde{m}^1_{ij} \colon SE^{\otimes j} \to SE[i, 1-i] \\ &\leftrightarrow \widetilde{m}^q_{ij} = \sum_{\substack{r+1+t=q \\ r+s+t=j}} \mathbf{1}^{\otimes r} \otimes \widetilde{m}^1_{is} \otimes \mathbf{1}^{\otimes t} \colon SE^{\otimes j} \to SE^{\otimes q}[i, 1-i] \\ &\leftrightarrow \widetilde{m}_i \colon \overline{T}SE \to \overline{T}SE[i, 1-i] \text{ with components } \widetilde{m}^q_{ij}. \end{aligned}$$

LEMMA
dA_∞-relations $\Leftrightarrow$ $(\overline{T}SE, \widetilde{m}_i)$ is a twisted chain complex

LEMMA
dA_∞-algebra maps $E \to F$ correspond to maps $(\overline{T}SE, \widetilde{m}_i) \to (\overline{T}SF, \widetilde{m}_i)$ of twisted chain complexes

Proof of Theorem 2 (sketch)

- The category of modules over a dA_∞-algebra E is enriched in twisted chain complexes.
- The endomorphism object $\underline{\mathrm{Hom}}_E(E, E)$ is a monoid in tCh_k.
- $\mathrm{Tot}\,\underline{\mathrm{Hom}}_E(E, E)$ is a dga.
- If E has E_2-homology concentrated in horizontal degree 0, there is an E_2-equivalence $E \to \mathrm{Tot}\,\underline{\mathrm{Hom}}_E(E, E)$.
- If E and F are E_2-quasi-isomorphic, then $\mathrm{Tot}\,\underline{\mathrm{Hom}}_E(E, E)$ and $\mathrm{Tot}\,\underline{\mathrm{Hom}}_F(F, F)$ are quasi-isomorphic as dgas.

Proof of Theorem 2.
Apply the last statement to $E \to A$. □

In other words:
The dA_∞-algebras with E_2-homology concentrated in horizontal degree 0 model quasi-isomorphism types of dgas.