LOGARITHMIC STRUCTURES IN HOMOTOPY THEORY

Steffen Sagave

Universität Bonn

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http://www.math.uni-bonn.de/people/sagave/

Aim of the talk

1. Illustrate why logarithmic structures are interesting from the point of view of homotopy theory.
2. Explain what the homotopy theoretic analog of a log structure is and introduce *logarithmic topological Hochschild homology* as a sample application of this concept.

Most of the definitions and results about log structures in homotopy theory to be presented are due to John Rognes.

We may get to joint work in progress of the speaker with Rognes towards the end.

K-THEORY OF LOCAL FIELDS

Let A be a complete discrete valuation ring with field of fractions K of characteristic 0 and perfect residue field k of characteristic $p > 2$.

Hesselholt and Madsen compute the Quillen K-theory $K_*(K, \mathbb{Z}/p^v)$ with $\mathbb{Z}/p^v$-coefficients if k is finite.

We will review the very first steps of their calculation.

Trace maps

Let R be a ring. There is a *Bökstedt trace map*

$$K_i(R) \to \mathrm{THH}_i(R).$$

into *topological Hochschild homology*.

It detects more information than the Dennis trace map $K \to \mathrm{HH}$ to ordinary Hochschild homology. For example,

$$\mathbb{Z}/48 \cong K_3(\mathbb{Z}) \twoheadrightarrow \mathrm{THH}_3(\mathbb{Z}) \cong \mathbb{Z}/2$$

(In fact, *THH* is a building block for an much stronger tool to compute K-theory, the *topological cyclic homology* TC.)

First step of Hesselholt-Madsen computation

Quillen's long exact localization sequence for algebraic K-theory can be compared to a long exact sequence in THH via the Bökstedt trace map:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & K_i(k) & \longrightarrow & K_i(A) & \longrightarrow & K_i(K) & \longrightarrow & K_{i-1}(k) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & \mathrm{THH}_i(k) & \longrightarrow & \mathrm{THH}_i(A) & \longrightarrow & \mathrm{THH}_i(A|K) & \longrightarrow & \mathrm{THH}_{i-1}(k) & \longrightarrow \cdots
\end{array}$$

The term $\mathrm{THH}_i(A|K)$ sitting in the long exact sequence is not isomorphic to $\mathrm{THH}_i(K)$!

The definition of $\mathrm{THH}_i(A|K)$ is very special to this situation. It is manufactured to fit into the long exact sequence, and uses THH of linear Waldhausen categories.

THH AND LOG KÄHLER DIFFERENTIALS

There is an isomorphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{THH}_1(A) & \longrightarrow & \mathrm{THH}_1(A|K) & \longrightarrow & \mathrm{THH}_0(k) & \longrightarrow & 0 \\ & & \cong\uparrow & & \cong\uparrow & & \cong\uparrow & & \\ 0 & \longrightarrow & \Omega^1_A & \longrightarrow & \Omega^1_{(A,M)} & \longrightarrow & k & \longrightarrow & 0 \end{array}$$

where $M = (A \setminus \{0\}, \cdot) \hookrightarrow (A, \cdot)$ is the standard log structure and

$$\Omega^1_{(A,M)} = (\Omega^1_A \oplus (A \otimes_{\mathbb{Z}} M^{\mathrm{gp}}))/\langle d\alpha(m) - \alpha(m) \otimes m | m \in M \rangle$$

is the A-module of log Kähler differentials.

(Hesselholt and Madsen express the whole $\pi_*(\mathrm{THH}(A, K), \mathbb{Z}/p)$ using the de Rham complex with log poles.)

QUESTIONS

GUIDING QUESTION

Is there another way to define $\mathrm{THH}(A, M)$ for a general commutative ring with a log structure so that

- the $\mathrm{THH}(A|K)$ arises as a special case and
- the new construction accepts even more general input data?

The answer is ‘Yes’, and this will be the *logarithmic topological Hochschild homology*.

What is 'ordinary' THH?

Short answer

Topological Hochschild homology THH is Hochschild homology with the *sphere spectrum* $\mathbb{S}$ as the ground ring.

Here $\mathbb{S}$ is the sphere spectrum of algebraic topology.

To make this slogan precise, we need to explain:

- In what sense is $\mathbb{S}$ a commutative ring?
- What are algebras over $\mathbb{S}$, and in which way is a commutative ring an algebra over $\mathbb{S}$?

Dictionary algebra $\rightsquigarrow$ topology

sets	$\rightsquigarrow$	spaces
abelian groups $\mathcal{A}b$	$\rightsquigarrow$	symmetric spectra Sp^Σ
linearization $\mathbb{Z}[-]$	$\rightsquigarrow$	stabilization / suspension spectrum $\mathbb{S}[-]$
tensor product $\otimes_{\mathbb{Z}}$	$\rightsquigarrow$	smash product $\wedge$ in Sp^Σ
commutative rings (=$\mathbb{Z}$-algebras)	$\rightsquigarrow$	commutative symmetric ring spectra (=$\mathbb{S}$-algebras)

Symmetric spectra

The *smash product* of based spaces K and L is

$$K \wedge L = K \times L/(K \times \{*\} \cup \{*\} \times L)$$

Basic example: Smash products of spheres $S^m \wedge S^n \cong S^{m+n}$

Definition

A *symmetric spectrum* E is

- a sequence of based spaces E_m with basepoint preserving Σ_m-actions for $m \geq 0$ and
- structure maps $E_m \wedge S^1 \to E_{m+1}$

such that the composite $E_m \wedge S^n \to E_{m+1} \wedge S^{n-1} \to \cdots \to E_{m+n}$ is $\Sigma_m \times \Sigma_n$-equivariant. This gives a category Sp^Σ

Examples

- The sphere spectrum $\mathbb{S}$ with $\mathbb{S}_n = S^n$ and structure map $S^n \wedge S^1 \xrightarrow{\cong} S^{n+1}$.
- Let K be a space. The suspension spectrum $\mathbb{S}[K]$ with $\mathbb{S}[K]_m = K_+ \wedge S^m$ is a symmetric spectrum.

Homotopy theory of symmetric spectra

The category Sp^Σ admits a *model category structure* in the sense of Quillen. One can form its homotopy category $\mathrm{Ho}(\mathrm{Sp}^\Sigma)$, and $\mathrm{Ho}(\mathrm{Sp}^\Sigma)$ is a triangulated category (the *stable homotopy category*).

If E is a symmetric spectrum, then

$$K \mapsto E^*(K) = \mathrm{Ho}(\mathrm{Sp}^\Sigma)(\mathbb{S}[K], E)^*$$

is a generalized cohomology theory, and every generalized cohomology theory is represented by a symmetric spectrum.

Examples

- Complex K-theory KU
- Real K-theory KO
- Complex cobordism MU

Eilenberg Mac Lane spectra

Example

Let A be an abelian group.
$H^*(-, A)$ (ordinary cohomology with coefficients in A)
is represented by the *Eilenberg-Mac Lane spectrum HA*.

It admits an explicit description as a symmetric spectrum:

$$HA_m = A[S^m] = A \otimes \mathbb{Z}[S^m]$$

Smash product

Symmetric spectra E and F have a smash product $E \wedge F$, characterized by a similar universal property than $\otimes$:

Maps $E \wedge F \to D$ in Sp^Σ correspond to compatible families of maps $E_m \wedge F_n \to D_{m+n}, m, n \geq 0$.

Theorem

$(Sp^\Sigma, \wedge)$ *is a closed symmetric monoidal category with the sphere spectrum* $\mathbb{S}$ *as the monoidal unit.*

Definition

A commutative $\mathbb{S}$-algebra is a commutative monoid in $(\mathrm{Sp}^\Sigma, \wedge)$.

Giving a commutative $\mathbb{S}$-algebra A is the same as giving $A \in \mathrm{Sp}^\Sigma$ with maps $A \wedge A \to A$ and $\mathbb{S} \to A$ satisfying associativity, commutativity, and unitality.

(Equivalently: Maps $A_m \wedge A_n \to A_{m+n}$ and $S^m \to A_m$ satisfying ...)

$\mathbb{S}$-ALGEBRAS

Every commutative $\mathbb{S}$-algebra A has
a graded commutative ring of stable homotopy groups

$$\pi_* A = \mathrm{Ho}(\mathrm{Sp}^\Sigma)(\mathbb{S}, A).$$

EXAMPLES FOR COMMUTATIVE $\mathbb{S}$-ALGEBRAS

- Sphere spectrum $\mathbb{S}$ with $\pi_*\mathbb{S} = \pi_*^{\mathrm{st}} S^0$.
- Eilenberg-Mac Lane spectrum HR
  for a commutative ring R with $\pi_*(HR) \cong R$.
- Complex K-theory KU with $\pi_* KU = \mathbb{Z}[u^{\pm 1}], |u| = 2$
- Real K-theory KO with $\pi_* KO$ being 8-periodic
- Complex cobordism MU

The cyclic bar construction

Let A be a monoid in a symmetric monoidal category $(\mathcal{A}, \boxtimes, \mathbf{1}_{\mathcal{A}})$.

Definition

The *cyclic bar construction* $B^{\mathrm{cy}}_{\bullet}(A)$ is the simplicial object

$$[k] \mapsto A \boxtimes \cdots \boxtimes A$$

with face operators

$$d_i(a_0 \boxtimes \cdots \boxtimes a_k) = \begin{cases} a_0 \boxtimes \cdots \boxtimes a_i a_{i+1} \boxtimes \cdots \boxtimes a_k & 0 \leq i \leq k-1 \\ a_k a_0 \boxtimes \cdots \boxtimes a_{k-1} & i = k \end{cases}$$

and degeneracy operators

$$s_i(a_0 \boxtimes \cdots \boxtimes a_k) = a_0 \boxtimes \cdots \boxtimes a_i \boxtimes \mathbf{1}_{\mathcal{A}} \boxtimes a_{i+1} \boxtimes \cdots \boxtimes a_k \quad 0 \leq i \leq k$$

(Strictly speaking, A has no ‘elements’ a_i. One has to use $A \boxtimes A \to A$ and $1_{\mathcal{A}} \to A$ as indicated by the formulas.)

Topological Hochschild homology

Let R be a commutative ring. Then HR is a commutative monoid in $(\mathrm{Sp}^{\Sigma}, \wedge, \mathbb{S})$.

Definition

$$\mathrm{THH}(R) = B^{\mathrm{cy}}(HR) \quad \text{and} \quad \mathrm{THH}_i(R) = \pi_i \,\mathrm{THH}(R)$$

- $B^{\mathrm{cy}}(HR)$ is the realization of the simplicial object $B^{\mathrm{cy}}_{\bullet}(HR)$
- Also applies to give $\mathrm{THH}(A)$ for A a commutative $\mathbb{S}$-algebra.
- A ‘projective resolution’ (cofibrant replacement) is implicit in the definition.
- Alternative definition: Let $A^e = A \wedge A^{\mathrm{op}}$ be the enveloping algebra of A. Then A is an A^e-module, and

  $$\mathrm{THH}(A) = A \wedge^{\mathbb{L}}_{A^e} A$$

  where $\wedge^{\mathbb{L}}$ is a *derived* smash product.

Examples

$\mathrm{THH}_*(\mathbb{F}_p) = \mathbb{F}_p[x]$ with $|x| = 2$ $\qquad$ $\mathrm{THH}(MU) = MU \wedge \mathbb{S}[SU]$

How to define log $\mathbb{S}$-algebras?

For many purposes, commutative $\mathbb{S}$-algebras are the ‘right’ homotopy theoretic generalizations of commutative rings.

Question

What are logarithmic structures on commutative $\mathbb{S}$-algebras?

This needs a notion of *commutative monoids* compatible with commutative $\mathbb{S}$-algebras.

First guess: Commutative monoids in spaces, i.e., commutative topological (or simplicial) monoids.

This does not work. (Good analogy: Commutative dgas over fields of positive characteristic have shortcomings.)

$\mathcal{I}$-SPACES

Let $\mathcal{I}$ be the category with objects the finite sets $\mathbf{n} = \{1, \ldots, n\}$ for $n \geq 0$ and morphisms the injective maps.

DEFINITION

Let $\mathcal{S}$ be the category of spaces. An $\mathcal{I}$*-space* is a functor $\mathcal{I} \to \mathcal{S}$. We write $\mathcal{S}^{\mathcal{I}}$ for the category of $\mathcal{I}$-spaces.

Ordered concatenation $\mathbf{m} \sqcup \mathbf{n}$ makes $\mathcal{I}$ a symmetric monoidal category. This induces a symmetric monoidal structure on $\mathcal{S}^{\mathcal{I}}$:

$$(X \boxtimes Y)(\mathbf{n}) = \operatorname{colim}_{\mathbf{k} \sqcup \mathbf{l} \to \mathbf{n}} X(\mathbf{k}) \times Y(\mathbf{l})$$

DEFINITION

A *commutative* $\mathcal{I}$*-space monoid* is commutative monoid in $(\mathcal{S}^{\mathcal{I}}, \boxtimes)$. Write $\mathcal{CS}^{\mathcal{I}}$ for the resulting category.

Commutative $\mathcal{I}$-space monoids are one ‘correct’ model for spaces with a commutative multiplication.

For $M \in \mathcal{CS}^{\mathcal{I}}$, the path components $\pi_0(M)$ form a commutative monoid.

Dictionary algebra $\rightsquigarrow$ topology continued

commutative monoid C	$\rightsquigarrow$	commutative $\mathcal{I}$-space monoid M
commutative ring R	$\rightsquigarrow$	commutative $\mathbb{S}$-algebras A
$R \mapsto (R, \cdot)$	$\rightsquigarrow$	$A \mapsto \Omega^{\mathcal{I}}(A)$ *underlying multiplicative infinite loop space* of A
$C \mapsto \mathbb{Z}[C]$	$\rightsquigarrow$	$M \mapsto \mathbb{S}^{\mathcal{I}}[M]$ (‘*spherical group ring*’)
$C^\times \subset C$	$\rightsquigarrow$	$M^\times \subset M$ (‘*units* of M’)
$R^\times \subset (R, \cdot)$	$\rightsquigarrow$	$\mathrm{GL}_1^{\mathcal{I}} A \subset \Omega^{\mathcal{I}} A$ (‘*units* of A’)

Here

$$\mathbb{S}^{\mathcal{I}}[-] \colon \mathcal{CS}^{\mathcal{I}} \rightleftarrows \mathcal{C}\mathrm{Sp}^{\Sigma} \colon \Omega^{\mathcal{I}}$$

is an adjunction with $\Omega^{\mathcal{I}}(A)(\mathbf{n}) = \Omega^n A_n$, and

$M^\times \subset M$ is the sub-object of invertible path components, i.e., $\pi_0(M^\times) = (\pi_0 M)^\times \subset \pi_0 M$.

Logarithmic $\mathbb{S}$-algebras

Definition (Rognes)

A *pre-log $\mathbb{S}$-algebra* (A, M) is a commutative $\mathbb{S}$-algebra A together with a commutative $\mathcal{I}$-space monoid M and a map of commutative $\mathcal{I}$-space monoids $\alpha\colon M \to \Omega^{\mathcal{I}} A$.

Definition (Rognes)

A pre-log $\mathbb{S}$-algebra (A, M) is a *log $\mathbb{S}$-algebra* if the map $\widetilde{\alpha}$ in the (homotopy) pullback square

$$\begin{array}{ccc} \alpha^{-1}(\mathrm{GL}_1^{\mathcal{I}} A) & \xrightarrow{\widetilde{\alpha}} & \mathrm{GL}_1^{\mathcal{I}} A \\ \downarrow & & \downarrow \\ M & \xrightarrow{\alpha} & \Omega^{\mathcal{I}} A \end{array}$$

is an equivalence of $\mathcal{I}$-spaces.

Lemma

Every pre-log $\mathbb{S}$-algebra (A, M) has a well defined logification.

Examples for log $\mathbb{S}$-algebras

- If (R, C) is a log ring in the algebraic sense, the Eilenberg-Mac Lane spectrum HR and the constant discrete commutative $\mathcal{I}$-space monoid C form a log symmetric ring spectrum (HR, C).
- Direct and inverse image log structures.
- Let A be a commutative symmetric ring spectrum. A map $a\colon S^k \to (\Omega^{\mathcal{I}} A)(\mathbf{n})$ represents $[a] \in \pi_k A$ and gives rise to the free pre-log structure

$$\mathbb{C}F^{\mathcal{I}}_{\mathbf{n}} S^k \to \Omega^{\mathcal{I}} A$$

  generated by a.

Exact and replete maps

Drawback

Integral, *fine*, or *saturated* commutative monoids do not seem to have a good analog in the topological context.

We *do* have a good notion of group completion $M \to M^{\text{gp}}$ for commutative $\mathcal{I}$-space monoids.

Definition

A map $\varepsilon\colon M \to N$ in $\mathcal{CS}^{\mathcal{I}}$ is

- *exact* if the square

$$\begin{array}{ccc} M & \longrightarrow & M^{\text{gp}} \\ \downarrow & & \downarrow \\ N & \longrightarrow & N^{\text{gp}} \end{array}$$

  is a homotopy pullback square,
- *virtual surjective* if $\pi_0 M^{\text{gp}} \to \pi_0 N^{\text{gp}}$ is surjective, and
- *replete* if it is exact and virtual surjective.

REPLETION

DEFINITION

Let $\varepsilon\colon M \to N$ in $\mathcal{CS}^{\mathcal{I}}$ be virtual surjective. Its *repletion* $M^{\mathrm{rep}} \to N$ is given by the the homotopy pullback square on the right hand side:

$$\begin{array}{ccccc} M & \longrightarrow & M^{\mathrm{rep}} & \longrightarrow & M^{\mathrm{gp}} \\ \downarrow & & \downarrow & & \downarrow \\ N & \xrightarrow{=} & N & \longrightarrow & N^{\mathrm{gp}} \end{array}$$

LEMMA

$M^{\mathrm{rep}} \to N$ is replete.

EXAMPLE

For $M \in \mathcal{CS}^{\mathcal{I}}$, we can form the cyclic bar construction $B^{\mathrm{cy}}(M)$ in $(\mathcal{CS}^{\mathcal{I}}, \boxtimes)$. It admits a virtual surjective augmentation map $B^{\mathrm{cy}}(M) \to M$ in $\mathcal{CS}^{\mathcal{I}}$.

DEFINITION

The *replete bar construction* $B^{\mathrm{rep}}(M) \to M$ is the repletion of $B^{\mathrm{cy}}(M) \to M$

Logarithmic THH

Let (A, M) be a pre-log $\mathbb{S}$-algebra.

Then $\alpha\colon M \to \Omega^{\mathcal{I}} A$ is adjoint to $\mathbb{S}^{\mathcal{I}}[M] \to A$, and we get a map

$$\mathbb{S}^{\mathcal{I}}[B^{\mathrm{cy}} M] \cong B^{\mathrm{cy}} \mathbb{S}^{\mathcal{I}}[M] \to B^{\mathrm{cy}} A = \mathrm{THH}\, A$$

of commutative $\mathbb{S}$-algebras.

Definition

The *logarithmic topological Hochschild homology* $\mathrm{THH}(A, M)$ of a pre-log $\mathbb{S}$-algebra (A, M) is given by the following (homotopy) pushout in commutative $\mathbb{S}$-algebras:

$$\begin{array}{ccc} \mathbb{S}^{\mathcal{I}}[B^{\mathrm{cy}} M] & \longrightarrow & \mathbb{S}^{\mathcal{I}}[B^{\mathrm{rep}} M] \\ \downarrow & & \downarrow \\ \mathrm{THH}(A) & \longrightarrow & \mathrm{THH}(A, M) \end{array}$$

One can view $(\mathrm{THH}(A, M), B^{\mathrm{rep}} M)$ as the repletion of $(\mathrm{THH}(A), B^{\mathrm{cy}} M) \to (A, M)$.

Log THH of log rings

Back to the example at the beginning:

Let A a be discrete valuation ring with uniformizer π, residue field k and fraction field K.

Then A has a pre-log structure $\langle \pi \rangle$, and we get a pre-log $\mathbb{S}$-algebra $(HA, \langle \pi \rangle)$.

Proposition

There is a long exact sequences

$$\ldots \to \mathrm{THH}_i(k) \to \mathrm{THH}_i(A) \to \mathrm{THH}_i(HA, \langle \pi \rangle) \to \mathrm{THH}_{i-1}(k) \to \ldots$$

Particularly, $\mathrm{THH}_i(HA, \langle \pi \rangle)$ is isomorphic to the $\mathrm{THH}(A|K)$ of Hesselholt-Masdsen and makes $\mathrm{THH}(A|K)$ the special case of a general definition.

Algebraic K-theory of $\mathbb{S}$-algebras

One can define the algebraic K-theory $K(A)$ of a commutative $\mathbb{S}$-algebra A.

If R is a ring, $K(HR)$ is equivalent to Quillen's $K(R)$.

By work of Waldhausen and others, algebraic K-theory of commutative $\mathbb{S}$-algebras has deep connections to diffeomorphism groups of high dimensional manifold.

In the algebraic K-theory of commutative $\mathbb{S}$-algebras, certain stable homotopy classes like the Bott class $u \in \pi_2 KU$ play a similar role as primes.

Example (Blumberg-Mandell)

There are long exact sequences

$$\ldots \to K_i(\mathbb{Z}) \to K_i(ku) \to K_i(KU) \to K_{i-1}(\mathbb{Z}) \to \ldots$$
$$\ldots \to K_i(\mathbb{Z}_p) \to K_i(\ell_p) \to K_i(L_p) \to K_{i-1}(\mathbb{Z}_p) \to \ldots$$

(L_p is the Adams summand of KU_p, ℓ_p is its connective cover.)

K-THEORY OF THE ADAMS SUMMAND

Known computational tools for K-theory of $\mathbb{S}$-algebras (e.g. THH) tend not to work well for periodic $\mathbb{S}$-algebras like KU or L_p.

Computations by Ausoni, Ausoni-Rognes, and Hesselholt suggest that there should exist a map of long exact sequences

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & K_i(\mathbb{Z}_p) & \longrightarrow & K_i(\ell_p) & \longrightarrow & K_i(L_p) & \longrightarrow & K_{i-1}(\mathbb{Z}_p) & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & \mathrm{THH}_i(\mathbb{Z}_p) & \longrightarrow & \mathrm{THH}_i(\ell_p) & \longrightarrow & \mathrm{THH}_i(\ell_p|L_p) & \longrightarrow & \mathrm{THH}_{i-1}(\mathbb{Z}_p) & \longrightarrow \cdots \end{array}$$

with the conjectural term $\mathrm{THH}_i(\ell_p|L_p)$ generalizing $\mathrm{THH}_i(A|K)$.

QUESTION

Is there a log structure M on ℓ_p such that $\mathrm{THH}(\ell_p|L_p) = \mathrm{THH}(\ell_p, M)$ defines the desired term?

K-THEORY OF LOG $\mathbb{S}$-ALGEBRAS

Rognes has a conjectural picture of how to understand K-theory of $\mathbb{S}$-algebras using Galois descent and localization.

Difficulty: There is no good notion of a faction field within commutative $\mathbb{S}$-algebras.
However, Ausoni and Rognes computed in the example of ℓ_p what the K-theory of such a fraction field should be.

Rognes conjectures that these fraction fields may be realized as log $\mathbb{S}$-algebras, so that the K-theory of the fraction field is the log K-theory of a log $\mathbb{S}$-algebra.

DEFINITION (ROGNES-SAGAVE)

Let (A, M) be a log $\mathbb{S}$-algebra. Its *log K-theory* $K(A, M)$ is the Waldhausen K-theory of the subcategory of compact objects in the stabilization of the category of replete augmented (A, M)-log algebras.

OPEN QUESTION

Which log structures give the desired K-theory?

GRADED LOG $\mathbb{S}$-ALGEBRAS

To realize the log $\mathbb{S}$-algebras described above, a further generalization might be necessary:

BASIC OBSERVATION

The stable homotopy groups $\pi_* A$ of a commutative $\mathbb{S}$-algebra form a *graded commutative ring* rather than a commutative ring.

One may argue that a log structure on a graded commutative ring should have a *graded commutative monoid* part of its data.

Accordingly, we may need to consider log $\mathbb{S}$-algebras having *graded commutative spaces* as part of their data.

A notion of *graded commutative spaces* serving for this purpose has been developed by the speaker in joint work with C. Schlichtkrull.