**Spectra of units for periodic ring spectra**

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One can associate a spectrum of units $\mathrm{gl}_1 E$ to a commutative structured ring spectrum $E$. This is analogous to forming the abelian group of units $\mathrm{GL}_1 R$ in the underlying multiplicative monoid of an ordinary commutative ring $R$. The spectrum $\mathrm{gl}_1 E$ is useful because it controls the orientation theory of $E$.

**The spectrum of graded units.** The various equivalent definitions for $\mathrm{gl}_1 E$ in the literature have the disadvantage that they do not see the difference between a periodic ring spectrum and its connective cover. Our aim is to define a spectrum of *graded* units which detects periodicity. It is a functor

$$\mathrm{gl}_1^{\mathcal{J}} \colon \mathcal{C}\mathrm{Sp}^{\Sigma} \to \Gamma^{\mathrm{op}}\text{-}\mathcal{S}/b\mathcal{J}, \qquad E \mapsto \mathrm{gl}_1^{\mathcal{J}} E$$

from the category of commutative symmetric ring spectra $\mathcal{C}\mathrm{Sp}^{\Sigma}$ to the category of $\Gamma$-spaces augmented over a certain $\Gamma$-space $b\mathcal{J}$.

Commutative symmetric ring spectra are one possible incarnation of a category of commutative structured ring spectra. They are strictly commutative monoids with respect to the smash product of symmetric spectra. Segal's category of $\Gamma$-spaces $\Gamma^{\mathrm{op}}\text{-}\mathcal{S}$ is a convenient way to encode connective spectra. The $\Gamma$-space $b\mathcal{J}$ arises from a symmetric monoidal category $\mathcal{J}$ and represents the sphere spectrum. So $\mathrm{gl}_1^{\mathcal{J}} E$ may be viewed as a connective spectrum over the sphere spectrum.

Below we outline some aspects of the definition of $\mathrm{gl}_1^{\mathcal{J}} E$ and explain why we think of it as *graded* units. Before that, we discuss how it relates to the ordinary units $\mathrm{gl}_1 E$.

**Graded units and ordinary units.** Let $\mathrm{bgl}_1^* E$ be the spectrum associated with the homotopy cofiber of the augmentation $\mathrm{gl}_1^{\mathcal{J}} E \to b\mathcal{J}$.

**Theorem 1.** [2] *Let $E$ be a positive fibrant commutative symmetric ring spectrum. The ordinary spectrum of units $\mathrm{gl}_1 E$ is the connective cover of $\Omega \mathrm{bgl}_1^* E$, and $\mathrm{bgl}_1^* E$ is connective.*

*The bottom homotopy group $\pi_0(\mathrm{bgl}_1^* E)$ is isomorphic to $\mathbb{Z}/n_E\mathbb{Z}$ where $n_E \in \mathbb{N}_0$ is the* periodicity *of $E$. By definition, $n_E = 0$ (and $\mathbb{Z}/n_E\mathbb{Z} \cong \mathbb{Z}$) if all units of the multiplicative graded monoid $\pi_*(E)$ have degree $0$, and $n_E$ is the smallest positive degree of a unit in $\pi_*(E)$ otherwise.*

In other words, $\mathrm{bgl}_1^* E$ is a not necessarily connected delooping of the ordinary spectrum of units whose bottom homotopy group detects periodicity. The periodic and connective complex $K$-theory spectra $KU$ and $ku$ illustrate this: The map $ku \to KU$ that exhibits $ku$ as the connective cover of $KU$ induces the surjection

$$\mathbb{Z} \cong \pi_0(\mathrm{bgl}_1^*(ku)) \to \pi_0(\mathrm{bgl}_1^*(KU)) \cong \mathbb{Z}/2.$$

In contrast, the induced map of ordinary units $\mathrm{gl}_1(ku) \to \mathrm{gl}_1(KU)$ is a stable equivalence.

Since $b\mathcal{J}$ represents the sphere spectrum, the definition of $\mathrm{bgl}_1^*E$ provides a map $\mathbb{S} \to \mathrm{bgl}_1^*E$. The induced map

$$\mathbb{Z}/2 \cong \pi_1(\mathbb{S}) \to \pi_1(\mathrm{bgl}_1^*E) \cong \pi_0(\mathrm{gl}_1 E) \cong (\pi_0(E))^\times$$

is the sign action of the additive group structure on $\pi_0(E)$. This implies that the first $k$-invariant of $\mathrm{bgl}_1^*E$ is non-trivial as soon as $\{\pm 1\}$ acts non-trivially on $\pi_0(E)$.

**Graded $E_\infty$ spaces.** The definition $\mathrm{gl}_1^{\mathcal{J}}E$ builds on the diagram space of graded units $\mathrm{GL}_1^{\mathcal{J}}E$ that we introduced in joint work with Schlichtkrull [3], and we will now summarize the relevant material from [3].

In the same way as commutative symmetric ring spectra model $E_\infty$ spectra, one can give strictly commutative models for $E_\infty$ spaces: The category $\mathcal{S}^{\mathcal{I}}$ of space valued functors on the category of finite sets and injections $\mathcal{I}$ has a symmetric monoidal product $\boxtimes$ such that all homotopy types of $E_\infty$ spaces are represented by commutative monoids in $(\mathcal{S}^{\mathcal{I}}, \boxtimes)$ [3, Theorem 1.2]. We call these commutative monoids in $(\mathcal{S}^{\mathcal{I}}, \boxtimes)$ *commutative $\mathcal{I}$-space monoids*. For example, the underlying multiplicative $E_\infty$ space of a commutative symmetric ring spectrum $E$ arises as commutative $\mathcal{I}$-space monoid $\Omega^{\mathcal{I}}E$ in a natural way. The value of $\Omega^{\mathcal{I}}E$ at the finite set $\mathbf{m} = \{1, \dots, m\}$ is the space $\Omega^m E_m$.

In this terminology, the construction of the ordinary units $\mathrm{gl}_1 E$ goes as follows: The commutative $\mathcal{I}$-space monoid $\Omega^{\mathcal{I}}E$ has a sub commutative $\mathcal{I}$-space monoid $\mathrm{GL}_1^{\mathcal{I}}E$ of invertible path components, and Schlichtkrull [4] showed how to build a $\Gamma$-space $\mathrm{gl}_1 E$ from $\mathrm{GL}_1^{\mathcal{I}}E$. Defining $\mathrm{gl}_1 E$ using $\mathrm{GL}_1^{\mathcal{I}}E$ explains why $\mathrm{gl}_1 E$ does not detect periodicity: The inclusion $\mathrm{GL}_1^{\mathcal{I}}E \to \Omega^{\mathcal{I}}E$ corresponds to the inclusion $\pi_0(E)^\times \to \pi_0(E)$, and both $\mathrm{GL}_1^{\mathcal{I}}E$ and $\Omega^{\mathcal{I}}E$ do not detect multiplicative units of $\pi_*(E)$ in non-zero degrees because they are build from the spaces $\Omega^m E_m$ which do not carry information about the negative dimensional homotopy groups of $E$.

The key idea is now to pass to a more elaborate indexing category in order to include information about units in all degrees of $\pi_*(E)$. Let $\mathcal{J}$ be the Quillen localization construction $\Sigma^{-1}\Sigma$ on the category of finite sets and bijections $\Sigma$. This is a symmetric monoidal category whose objects are pairs of finite sets $(\mathbf{m_1}, \mathbf{m_2})$. Its classifying space has the homotopy type of $QS^0$. As for $\mathcal{I}$, we obtain a symmetric monoidal category of space valued functors $(\mathcal{S}^{\mathcal{J}}, \boxtimes)$. We call the commutative monoids in $(\mathcal{S}^{\mathcal{J}}, \boxtimes)$ *commutative $\mathcal{J}$-space monoids* and write $\mathcal{C}\mathcal{S}^{\mathcal{J}}$ for the resulting category.

In [3] we develop a homotopy theory for commutative $\mathcal{J}$-space monoids that clarifies their relationship to ordinary $E_\infty$-spaces:

**Theorem 2.** [3, Theorem 1.7] *The category of commutative $\mathcal{J}$-space monoids admits a model structure such that it is Quillen equivalent to the category of $E_\infty$ spaces over $B\mathcal{J}$.*

Thinking of $E_\infty$ spaces as a homotopical generalization of commutative monoids and of commutative symmetric ring spectra as a generalization of commutative rings, the following analogy explains this statement: An ordinary $\mathbb{Z}$-graded commutative monoid may be defined as a commutative monoid with a map to the additive monoid of the initial commutative ring $\mathbb{Z}$. By the theorem, a commutative $\mathcal{J}$-space monoid is up to homotopy an $E_\infty$ space with a map to the underlying additive $E_\infty$ space $B\mathcal{J} \simeq QS^0$ of the initial commutative ring spectrum $\mathbb{S}$. Therefore we think of commutative $\mathcal{J}$-space monoids as graded $E_\infty$ spaces.

Exploiting a close relationship between $\mathcal{J}$ and the combinatorics of symmetric spectra, a commutative symmetric ring spectrum $E$ has an associated commutative $\mathcal{J}$-space monoid $\Omega^{\mathcal{J}}E$ that is defined by $(\Omega^{\mathcal{J}}E)(\mathbf{m_1}, \mathbf{m_2}) = \Omega^{m_2}E_{m_1}$ on the objects of $\mathcal{J}$. This description indicates that $\Omega^{\mathcal{J}}(E)$ also captures information about negative dimensional homotopy groups of $E$. In view of the above discussion, $\Omega^{\mathcal{J}}E$ is a model for the graded multiplicative $E_\infty$-space of $E$.

**Grouplike graded $E_\infty$ spaces.** A classical theorem in stable homotopy theory states that grouplike $E_\infty$ spaces are equivalent to connective spectra. This has a counterpart for graded $E_\infty$ spaces:

**Theorem 3.** [2] *There is a chain of Quillen equivalences that induces an equivalence between the homotopy category of grouplike commutative $\mathcal{J}$-space monoids and the homotopy category of connective spectra over the sphere spectrum.*

If $E$ is a commutative symmetric ring spectrum, one can form a grouplike sub commutative $\mathcal{J}$-space monoid $\mathrm{GL}_1^{\mathcal{J}}E$ of *graded units* in $\Omega^{\mathcal{J}}E$ that corresponds to the inclusion of the graded commutative group of units $\pi_*(E)^\times$ into the underlying graded multiplicative monoid of $\pi_*(E)$. The augmented $\Gamma$-space of units $\mathrm{gl}_1^{\mathcal{J}}E$ discussed above is constructed from $\mathrm{GL}_1^{\mathcal{J}}E$, and the theorem makes clear why a spectrum associated with $\mathrm{GL}_1^{\mathcal{J}}E$ should be a connective spectrum over the sphere spectrum.

The last theorem is also a key ingredient for

**Theorem 4.** [2] *The functor* $\mathrm{Ho}(\mathcal{C}\mathrm{Sp}^\Sigma) \to \mathrm{Ho}(\Gamma^{\mathrm{op}}\text{-}\mathcal{S}/b\mathcal{J})$ *induced by* $\mathrm{gl}_1^{\mathcal{J}}$ *is a right adjoint.*

A similar statement about the ordinary units is proven by Ando, Blumberg, Gepner, Hopkins, and Rezk [1, Theorem 3.2].

## References


[1] M. Ando, A. J. Blumberg, D. J. Gepner, M. J. Hopkins, and C. Rezk. Units of ring spectra and Thom spectra, 2008. `arXiv:0810.4535`.

[2] S. Sagave. Spectra of units for periodic ring spectra. `arXiv:1111.6731`.

[3] S. Sagave and C. Schlichtkrull. Diagram spaces and symmetric spectra. `arXiv:1103.2764`.

[4] C. Schlichtkrull. Units of ring spectra and their traces in algebraic $K$-theory. *Geom. Topol.*, 8:645–673 (electronic), 2004.