LOGARITHMIC STRUCTURES ON K-THEORY SPECTRA

Steffen Sagave

Universität Bonn

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http://www.math.uni-bonn.de/people/sagave/

PREAMBLE

DEFINITION

A *pre-log structure* on a commutative ring A is a commutative monoid M together with a monoid map $\alpha\colon M \to (A, \cdot)$.
The triple (A, M, α) is a called a *pre-log ring*.

- Any $a \in A$ generates a pre-log structure $\langle a \rangle \to (A, \cdot)$.

DEFINITION

A pre-log ring (A, M, α) is a *log ring* if the map $\widetilde{\alpha}$ in the pullback

$$\begin{tikzcd} \alpha^{-1}(A^\times) \arrow[r, "\widetilde{\alpha}"] \arrow[d] & A^\times \arrow[d] \\ M \arrow[r, "\alpha"] & (A, \cdot) \end{tikzcd}$$

is an isomorphism. The logification of a pre log-structure (M, α) is $M^a = M \coprod_{\alpha^{-1}(A^\times)} A^\times \to (A, \cdot)$. It is a log structure.

- $A \setminus \{0\} \hookrightarrow (A, \cdot)$ is a log structure if A is an integral domain
- $A^\times \hookrightarrow (A, \cdot)$ defines the trivial log structure on A

Overview

Central Aim

- Introduce pre-log and log structures on structured ring spectra.
- Outline the logarithmic versions of TAQ and THH.

Guiding Example

Consider the connective complex K-theory spectrum ku and the Bott class $u \in \pi_*(ku) \cong \mathbb{Z}[u]$. What is a good notion of a pre-log structure generated by u?

Contents of the talk

1. Motivation: Why is this interesting?
2. Log ring spectra: Rognes' definition and a modification
3. Logarithmic topological Andre-Quillen homology
4. Logarithmic topological Hochschild homology
   (joint work in progress with J. Rognes and C. Schlichtkrull)

ALGEBRAIC K-THEORY OF LOCAL FIELDS

Let A be a complete discrete valuation ring with field of fractions K and residue field k. Assume that $\operatorname{char} K = 0$ and that k is perfect with $\operatorname{char} k = p > 2$.

EXAMPLE

$A = \mathbb{Z}_p, K = \mathbb{Q}_p$, and $k = \mathbb{F}_p$

In their computation of $K_*(K, \mathbb{Z}/p^v)$, Hesselholt-Madsen use a map of cofibration sequences

$$\begin{tikzcd} K(k) \arrow[r] \arrow[d] & K(A) \arrow[r] \arrow[d] & K(K) \arrow[r] \arrow[d] & \Sigma K(k) \arrow[d] \\ \mathrm{THH}(k) \arrow[r] & \mathrm{THH}(A) \arrow[r] & \mathrm{THH}(A|K) \arrow[r] & \Sigma\, \mathrm{THH}(k). \end{tikzcd}$$

The $\mathrm{THH}(A|K)$ in the bottom sequence is not equivalent to $\mathrm{THH}(K)$. (It is defined as the THH of a certain linear Waldhausen category.)

Log differentials and tame ramification

For a pre-log ring (A, M, α), there is an A-module $\Omega^1_{(A,M)}$ of log Kähler differentials (to be defined later). For $m \in M$, it has additional generators $d\log m$ with $d\alpha(m) = \alpha(m) d\log m$.

Hesselholt-Madsen show $\pi_1(\mathrm{THH}(A|K)) \cong \Omega^1_{(A, A\setminus\{0\})}$.
This suggests that $\mathrm{THH}(A|K)$ is a candidate for the THH of the log ring $(A, A \setminus \{0\}) = (A, \langle \text{uniformizer} \rangle^a)$.

One motivation behind the algebraic geometry of log rings is to extend the range of smooth and étale maps. In particular, tamely ramified extensions become log étale.

Let (A, K, k) and (B, L, l) be DVRs as above, and assume that $A \to B$ is a finite tamely ramified extension. Then

$$HB \wedge_{HA} \mathrm{THH}(A|K) \simeq_p \mathrm{THH}(B|L).$$

Thinking of these as log THH-terms, this is compatible with $(A, A \setminus \{0\}) \to (B, B \setminus \{0\})$ being formally log étale.

ALGEBRAIC K-THEORY OF TOPOLOGICAL K-THEORY

Let ku_p be the p-complete connective complex K-theory spectrum. Blumberg-Mandell established homotopy cofiber sequences

$$\begin{array}{ccccccc} K(\mathbb{Z}_p) & \longrightarrow & K(ku_p) & \longrightarrow & K(KU_p) & \longrightarrow & \Sigma K(\mathbb{Z}_p) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{THH}(\mathbb{Z}_p) & \rightarrow & \mathrm{THH}(ku_p) & \rightarrow & \mathrm{THH}(ku_p|KU_p) & \rightarrow & \Sigma\,\mathrm{THH}(\mathbb{Z}_p), \end{array}$$

and similarly for the p-complete Adams summand ℓ_p.

Again, $\mathrm{THH}(ku_p|KU_p)$ is not equivalent to $\mathrm{THH}(KU_p)$.
(It is defined as the THH of a certain Waldhausen category.)

Understanding $\mathrm{THH}(ku_p|KU_p)$ is desirable: It captures information about the algebraic K-theory of the non-connective ring spectrum KU_p.

Tame ramification for structured ring spectra

The above analogy between DVRs and K-theory spectra suggests that $\mathrm{THH}(ku_p|KU_p)$ and $\mathrm{THH}(\ell_p|L_p)$ may arise as the log THH of appropriate *log ring spectra* $(ku_p, ?)$ and $(\ell_p, ?)$.

On homotopy groups, the inclusion $\ell_p \to ku_p$ induces the map

$$\mathbb{Z}_p[v_1] \to \mathbb{Z}_p[u], \qquad v_1 \mapsto u^{p-1}.$$

Thinking of v_1 and u as uniformizers, $\ell_p \to ku_p$ is tamely ramified on homotopy groups.

Computations of $\mathrm{THH}(\ell_p)$ and $\mathrm{THH}(ku_p)$ by McClure-Staffeldt and Ausoni show that $\ell_p \to ku_p$ behaves like a tamely ramified extension on THH.

Hence a suitable extension of $\ell_p \to ku_p$ to a map of log ring spectra $(\ell_p, ?) \to (ku_p, ?)$ should be a *formally log étale map*.

Pre-log ring spectra

In the context of structured ring spectra, E_∞ ring spectra play the role of commutative rings, and E_∞ spaces play the role of commutative monoids.

Definition (Rognes)

A *pre-log ring spectrum* (A, M, α) is an E_∞ ring spectrum A together with an E_∞ space M and an E_∞ map $\alpha\colon M \to \Omega^\infty_\otimes(A)$. Here $\Omega^\infty_\otimes(A)$ is the underlying multiplicative E_∞ space of A.

By definition, $\mathrm{GL}_1(A) \subseteq \Omega^\infty_\otimes A$ is the sub E_∞ space of invertible path components (corresponding to $\pi_0(A)^\times \subseteq \pi_0(A)$).

Definition (Rognes)

A pre-log ring spectrum (A, M, α) is a log ring spectrum if $\widetilde{\alpha}\colon \alpha^{-1}(\mathrm{GL}_1(A)) \to \mathrm{GL}_1(A)$ is a weak equivalence.

- $\mathrm{GL}_1(A) \hookrightarrow \Omega^\infty_\otimes A$ defines the trivial log structure on A

LOG STRUCTURES ON *ku*

What are interesting log structures on *ku*?

EXAMPLE

Let A be an integral domain with quotient field K. Let $K^\times \to (K, \cdot)$ be the trivial log structure. Forming the pullback of

$$(A, \cdot) \to (K, \cdot) \leftarrow K^\times,$$

we obtain the log ring $(A, A \setminus \{0\})$ considered earlier.

EXAMPLE

Replacing A by ku and K by KU, the pullback of

$$\Omega^\infty_\otimes(ku) \to \Omega^\infty_\otimes(KU) \leftarrow \mathrm{GL}_1(KU),$$

only provides the trivial log structure $\mathrm{GL}_1(ku) \to \Omega^\infty_\otimes(ku)$ on ku.

PROBLEM

Both $\Omega^\infty_\otimes(ku) \to \Omega^\infty_\otimes(KU)$ and $\mathrm{GL}_1(ku) \to \mathrm{GL}_1(KU)$ are equivalences, but $\pi_*(KU) = \mathbb{Z}[u^{\pm 1}]$ has more units than $\pi_*(ku)$.

Graded and ungraded monoids

It would be better to use a homotopical version of the adjunction

$$\begin{array}{c}(\mathbb{Z}\text{-graded com. monoids}) \\ \uparrow\downarrow \\ (\mathbb{Z}\text{-graded com. rings})\end{array}$$

The adjunction

$$\Sigma^\infty(-)_+ \colon (E_\infty \text{ spaces}) \rightleftarrows (E_\infty \text{ ring spectra}) : \Omega^\infty_\otimes$$

models only its "ungraded" counterpart.

To do so, we will next define the "underlying graded multiplicative E_∞ space" of an E_∞ ring spectrum and its subobject of "graded units". These capture information about the graded commutative monoids $(\pi_*(A), \cdot)$ and $(\pi_*(A), \cdot)^\times$.

Commutative $\mathcal{J}$-space monoids

Definition

Let $\mathcal{J}$ be the category with objects $(\mathbf{m_1}, \mathbf{m_2})$ where the $\mathbf{m_i} = \{1, \dots, m_i\}$ are finite sets. A morphism $(\alpha_1, \alpha_2, \rho)\colon (\mathbf{m_1}, \mathbf{m_2}) \to (\mathbf{n_1}, \mathbf{n_2})$ is a pair of injective maps $\alpha_i\colon \mathbf{m_i} \to \mathbf{n_i}$ together with a bijection $\rho\colon \mathbf{n_1} \setminus \alpha_1 \to \mathbf{n_2} \setminus \alpha_2$.

$\mathcal{J}$ is symmetric monoidal under entry-wise concatenation.

Remark

$\mathcal{J}$ is equivalent to Quillen's localization construction $\Sigma^{-1}\Sigma$ on the category of finite sets and bijections Σ. Hence $B\mathcal{J} \simeq QS^0$.

Definition

A *commutative $\mathcal{J}$-space monoid M* is a functor $M\colon \mathcal{J} \to \mathcal{S}$ to the category of unbased spaces together with a unit $1 \in M(\mathbf{0}, \mathbf{0})$ and coherent multiplication maps

$$M(\mathbf{n_1}, \mathbf{n_2}) \times M(\mathbf{n_1'}, \mathbf{n_2'}) \to M(\mathbf{n_1} \sqcup \mathbf{n_1'}, \mathbf{n_2} \sqcup \mathbf{n_2'}).$$

$\mathcal{J}$-SPACES AND SYMMETRIC SPECTRA

Let $\mathcal{CS}^{\mathcal{J}}$ be the category of commutative $\mathcal{J}$-space monoids. The category $\mathcal{J}$ is chosen so that there is an adjunction

$$\mathbb{S}^{\mathcal{J}}[-]\colon \mathcal{CS}^{\mathcal{J}} \rightleftarrows \mathcal{C}\mathrm{Sp}^{\Sigma}\colon \Omega^{\mathcal{J}} \qquad (*)$$

relating $\mathcal{CS}^{\mathcal{J}}$ to the category of commutative symmetric ring spectra $\mathcal{C}\mathrm{Sp}^{\Sigma}$. For $A \in \mathcal{C}\mathrm{Sp}^{\Sigma}$ we have

$$\Omega^{\mathcal{J}}(A)(\mathbf{m_1}, \mathbf{m_2}) = \Omega^{m_2}(A_{m_1}).$$

A map $(\alpha_1, \alpha_2, \rho)\colon (\mathbf{m_1}, \mathbf{m_2}) \to (\mathbf{n_1}, \mathbf{n_2})$ sends $f\colon S^{m_2} \to A_{m_1}$ to

$$S^{\mathbf{n_2}} \xrightarrow[(\alpha_2)_*]{\cong} S^{\mathbf{m_2}} \wedge S^{\mathbf{n_2} \setminus \alpha_2} \xrightarrow{f \wedge \rho_*^{-1}} A_{\mathbf{m_1}} \wedge S^{\mathbf{n_1} \setminus \alpha_1} \xrightarrow{\sigma} A_{\mathbf{m_1} \sqcup (\mathbf{n_1} \setminus \alpha_1)} \xrightarrow[(\alpha_1)_*]{\cong} A_{\mathbf{n_1}}$$

The $\Omega^{\mathcal{J}}(A)$ captures information about $\pi_i(A)$ for all $i \in \mathbb{Z}$. The adjunction $(*)$ will serve as the homotopical version of

$$(\mathbb{Z}\text{-graded com. monoids}) \rightleftarrows (\mathbb{Z}\text{-graded com. rings})\,.$$

Graded E_∞ spaces

Theorem (S-Schlichtkrull)

The category $\mathcal{CS}^{\mathcal{J}}$ admits a cofibrantly generated proper simplicial model structure where

- *$M \to N$ is a weak equivalence if* $\mathrm{hocolim}_{\mathcal{J}} M \to \mathrm{hocolim}_{\mathcal{J}} N$ *is a weak equivalence in $\mathcal{S}$*
- *M is fibrant if all $(\alpha_1, \alpha_2, \rho)\colon (\mathbf{m_1}, \mathbf{m_2}) \to (\mathbf{n_1}, \mathbf{n_2})$ with $m_1 \geq 1$ induce a weak equivalence of (fibrant) spaces $M(\mathbf{m_1}, \mathbf{m_2}) \to M(\mathbf{n_1}, \mathbf{n_2})$.*

Corollary

$(\mathbb{S}^{\mathcal{J}}[-], \Omega^{\mathcal{J}})$ *is a Quillen adjunction with respect to this model structure and the positive stable model structure on $\mathcal{C}Sp^{\Sigma}$.*

Theorem (S-Schlichtkrull)

There is a chain of Quillen equivalences relating $\mathcal{CS}^{\mathcal{J}}$ and the category of E_∞-spaces over $B\mathcal{J} \simeq QS^0$.

We view $\Omega^{\mathcal{J}}(A)$ as the underlying graded E_∞ space of A.

Graded pre-log ring spectra

Definition

A *graded pre-log ring spectrum* (A, M, α) is a commutative symmetric ring spectrum A together with a commutative $\mathcal{J}$-space monoid M and a map $\alpha\colon M \to \Omega^{\mathcal{J}}(A)$ in $\mathcal{CS}^{\mathcal{J}}$.

Definition

A graded pre-log ring spectrum (A, M, α) is a log ring spectrum if $\widetilde{\alpha}\colon \alpha^{-1}(\mathrm{GL}_1^{\mathcal{J}}(A)) \to \mathrm{GL}_1^{\mathcal{J}}(A)$ is a weak equivalence in $\mathcal{CS}^{\mathcal{J}}$.

The previous definition uses

Definition

Let A be a positive fibrant in $\mathcal{C}\mathrm{Sp}^{\Sigma}$. The *graded units* of A is the sub commutative $\mathcal{J}$-space monoid $\mathrm{GL}_1^{\mathcal{J}}(A) \subseteq \Omega^{\mathcal{J}}(A)$ corresponding to the submonoid $(\pi_*(A))^{\times}$ of $(\pi_*(A), \cdot)$.

Log structures on K-theory spectra

We view the connective cover $ku \to KU$ as a map of positive fibrant objects in $\mathcal{C}\mathrm{Sp}^{\Sigma}$ and form the following pullback in $\mathcal{C}\mathcal{S}^{\mathcal{J}}$:

$$\begin{array}{ccc} i_* \mathrm{GL}_1^{\mathcal{J}}(KU) & \longrightarrow & \Omega^{\mathcal{J}}(ku) \\ \downarrow & & \downarrow \\ \mathrm{GL}_1^{\mathcal{J}}(KU) & \longrightarrow & \Omega^{\mathcal{J}}(KU) \end{array}$$

We get a sequence of graded log ring spectra

$$(ku, \mathrm{GL}_1^{\mathcal{J}}(ku)) \to (ku, i_* \mathrm{GL}_1^{\mathcal{J}}(KU)) \to (KU, \mathrm{GL}_1^{\mathcal{J}}(KU)).$$

So passing to graded log ring spectra with trivial log structures, $ku \to KU$ factors through the "intermediate localization" $(ku, i_* \mathrm{GL}_1^{\mathcal{J}}(KU))$.

This is analogous to the factorization

$$(A, A^{\times}) \to (A, A \setminus \{0\}) \to (K, K^{\times})$$

for an integral domain A with quotient field K.

Log structures and localization

For an integral domain A with quotient field K, we can reconstruct K from the log ring $(A, A \setminus \{0\})$ by forming

$$A[(A \setminus \{0\})^{-1}] \cong \mathbb{Z}[(A \setminus \{0\})^{\mathrm{gp}}] \otimes_{\mathbb{Z}[A \setminus \{0\}]} A.$$

This has a homotopical counterpart:

Theorem (S)

The following square is a homotopy pushout in $\mathcal{C}Sp^{\Sigma}$:

$$\begin{array}{ccc} \mathbb{S}^{\mathcal{J}}[i_* \mathrm{GL}_1^{\mathcal{J}}(KU)] & \longrightarrow & ku \\ \downarrow & & \downarrow \\ \mathbb{S}^{\mathcal{J}}[(i_* \mathrm{GL}_1^{\mathcal{J}}(KU))^{gp}] & \longrightarrow & KU \end{array}$$

- Works more generally for connective $A \in \mathcal{C}\mathrm{Sp}^{\Sigma}$ and $x \in \pi_*(A)$ with $A \simeq A[1/x]_{\geq 0}$.
- For log structures defined with usual E_∞-spaces, this cannot hold because $\Sigma^\infty(M_+)$ is always connective.

Logarithmic derivations

Let (A, M) be a log ring and let X be an A-module. Let $A \oplus X$ be the square zero extension of A by X.

Definition

The square zero extension of (A, M) by X is the log ring $(A \oplus X, M \times (X, +))$ with structure map $(m, x) \mapsto \alpha(m)(1 \oplus x)$.

Definition

A *derivation* of (A, M) with values in X is map of log rings $(A, M) \to (A \oplus X, M \times (X, +))$ over (A, M).

The A-module of *log Kähler* differentials is characterized by

$$\mathrm{Hom}_A(\Omega^1_{(A,M)}, X) \cong \mathrm{Der}((A, M), X).$$

More explicitly,

$$\Omega^1_{(A,M)} \cong \Omega^1_A \oplus (A \otimes M^{\mathrm{gp}}) / (d\alpha(m) \sim \alpha(m) \otimes m)$$

(It is a standard convention to set $d\log m = 1 \otimes m$.)

Derivations of graded log ring spectra

In similar fashion as described by Rognes for log ring spectra, these notions have counterparts in graded log ring spectra:

Let (A, M) be a graded pre-log ring spectrum, let X be an A-module, and let $A \vee X$ be the square zero extension.

There is a pre-log ring spectrum $(A \vee X, M \boxtimes (1 + X)^{\mathcal{J}})$ where $(1+X)^{\mathcal{J}}$ is a $\mathcal{J}$-space monoid model for the connective cover of the underlying spectrum of X. ($\boxtimes$ is the coproduct in $\mathcal{CS}^{\mathcal{J}}$.)

Definition

Let $(R, P) \to (A, M)$ be a map of graded pre-log ring spectra. The space $\mathrm{Der}_{(R,P)}((A, M), X)$ of graded log derivations with values in X is the space of maps of graded log ring spectra

$$(A, M) \to (A \vee X, M \boxtimes (1 + X)^{\mathcal{J}})$$

under (R, P) and over (A, M).

Log TAQ

Let $(R, P) \to (A, M)$ be a map of graded pre-log ring spectra. Its *logarithmic topological André-Quillen homology* is the A-module $\mathrm{TAQ}^{(R,P)}(A, M)$ characterized by the property

$$\mathrm{Mod}_A(\mathrm{TAQ}^{(R,P)}(A, M), X) \simeq \mathrm{Der}_{(R,P)}((A, M), X).$$

The inclusion of the p-complete Adams summand extends to

$$(\ell_p, i_* \mathrm{GL}_1^{\mathcal{J}}(L_p)) \to (ku_p, i_* \mathrm{GL}_1^{\mathcal{J}}(KU_p))$$

Theorem (S)

This map is formally log étale, i.e., $\mathrm{TAQ}^{(\ell_p, i_* \mathrm{GL}_1^{\mathcal{J}}(L_p))}(ku_p, i_* \mathrm{GL}_1^{\mathcal{J}}(KU_p))$ *is contractible.*

- The proof is non-computational.
- One may view this of an instance of *tame ramification.*

Definition of log THH

(from here on this is joint work in progress with John Rognes and Christian Schlichtkrull)

Let (A, M) be a graded pre-log ring spectrum with A and M cofibrant.

- $\mathrm{THH}(A)$ can be defined as $A \otimes S^1 \cong B^{\mathrm{cy}}_{\wedge}(A)$.
- Can also form $B^{\mathrm{cy}}_{\boxtimes}(M) \cong M \otimes S^1$ for $M \in \mathcal{CS}^{\mathcal{J}}$. The map $\alpha\colon M \to \Omega^{\mathcal{J}}(A)$ induces a map $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}_{\boxtimes}(M)] \to \mathrm{THH}(A)$.
- The homotopy pullback of

$$M \to M^{\mathrm{gp}} \leftarrow (B^{\mathrm{cy}}_{\boxtimes}(M))^{\mathrm{gp}}$$

defines the *replete bar construction* $B^{\mathrm{rep}}_{\boxtimes}(M)$ on M.

Definition

The *logarithmic topological Hochschild homology* of (A, M) is

$$\mathrm{THH}(A, M) = \mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}_{\boxtimes}(M)] \wedge^{\mathbb{L}}_{\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}_{\boxtimes}(M)]} \mathrm{THH}(A)$$

HOMOTOPY COFIBER SEQUENCES FOR LOG THH

THEOREM (ROGNES-S-SCHLICHTKRULL)

Let A be a commutative symmetric ring spectrum, and let $x \in \pi_n(A)$ be a homotopy class of even positive degree n such that $\pi_*(A) \cong \pi_0(A)[x]$. Then there is a homotopy cofiber sequence of $\mathrm{THH}(A)$*-modules*

$$\mathrm{THH}(\pi_0(A)) \to \mathrm{THH}(A) \to \mathrm{THH}(A, i_* \mathrm{GL}_1^{\mathcal{J}}(A[1/x])) \to \Sigma\, \mathrm{THH}(\pi_0(A))$$

- This applies for example to ku, ku_p, ℓ_p.