Graded units of ring spectra and R-module Thom spectra

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(joint work with Samik Basu and Christian Schlichtkrull)

Classically one can form the Thom spectrum associated with a continuous map $f\colon X \to BO$ to the classifying space of the orthogonal group $O = \bigcup_{n\in\mathbb{N}} O(n)$ or with a continuous map $f\colon X \to BF$ to the classifying space of stable spherical fibrations. In the language of structured ring spectra, BF may be identified with $B\mathrm{GL}_1\mathbb{S}$, the classifying space of the units of the sphere spectrum, and a spectrum may be viewed as an $\mathbb{S}$-module spectrum. Replacing the sphere spectrum $\mathbb{S}$ by a general A_∞ ring spectrum R, Ando, Blumberg, Gepner, Hopkins, and Rezk [1] generalized the classical construction of Thom spectra by building R-module Thom spectra associated with continuous maps $f\colon X \to B\mathrm{GL}_1 R$. They also provide a variant of their Thom spectrum functor that respects actions of E_∞ operads.

The aim of the present project is to implement an R-module Thom spectrum functor in the context of symmetric spectra and to define more general R-module Thom spectra associated with continuous maps to a suitable classifying space of the *graded* units of a symmetric ring spectrum R. The graded units $\mathrm{GL}_1^{\mathcal{J}} R$ of a commutative symmetric ring spectrum R were introduced in [3]. By definition, $\mathrm{GL}_1^{\mathcal{J}} R$ is a a space-valued symmetric monoidal functor on a certain symmetric monoidal indexing category $\mathcal{J}$. The diagram $\mathrm{GL}_1^{\mathcal{J}} R$ is built from the loop spaces $\Omega^{n_2} R_{n_1}$ on the levels of the symmetric spectrum R. In contrast to the ordinary units $\mathrm{GL}_1^{\mathcal{I}} R$ of R which are indexed by the category of finite sets and injections $\mathcal{I}$, the graded units $\mathrm{GL}_1^{\mathcal{J}} R$ also detect units of non-zero degree in the graded ring of homotopy groups $\pi_*(R)$ of R.

Symmetric R-module Thom spectra. (joint with C. Schlichtkrull) If R is a commutative symmetric ring spectrum, we define classifying spaces $B\mathrm{GL}_1^{\mathcal{I}} R$ and $B\mathrm{GL}_1^{\mathcal{J}} R$ of the units and the graded units. These are E_∞ spaces over the Barratt–Eccles operad. Building on this we define R-module Thom spectrum functors

$$T\colon \mathcal{S}/B\mathrm{GL}_1^{\mathcal{I}} R \to R\text{-Mod} \qquad \text{and} \qquad T\colon \mathcal{S}/B\mathrm{GL}_1^{\mathcal{J}} R \to R\text{-Mod}$$

on the categories of spaces augmented over these classifying spaces. These functors have many desirable properties: They send weak equivalences over the classifying spaces to stable equivalences of R-module spectra, they preserve homotopy colimits, and they preserve actions of operads augmented over the Barratt–Eccles operad. In particular, the Thom spectrum associated with a map of topological monoids inherits an associative R-algebra structure.

The Thom spectrum functor for graded units extends the one for ordinary units: There is a natural morphism $\iota\colon B\mathrm{GL}_1^{\mathcal{I}} R \to B\mathrm{GL}_1^{\mathcal{J}} R$ such that the restriction of the Thom spectrum functor for maps to $B\mathrm{GL}_1^{\mathcal{J}} R$ along ι coincides with the one for maps to $B\mathrm{GL}_1^{\mathcal{I}} R$. The map ι turns out to be a 0-connected cover map. The extra information about the non-zero degree units of $\pi_*(R)$ in $\mathrm{GL}_1^{\mathcal{J}} R$ is reflected in $B\mathrm{GL}_1^{\mathcal{J}} R$ by the fact that its monoid of path components is $\mathbb{Z}/d\mathbb{Z}$, where d is the periodicity of R, i.e., d is the smallest positive degree of a unit in $\pi_*(R)$ if there exists a unit of positive degree, and zero otherwise. If for example R is the sphere spectrum, then $B\mathrm{GL}_1^{\mathcal{J}}\mathbb{S}$ has the homotopy type of $\mathbb{Z} \times BF$ and may be viewed as a classifying space of virtual spherical fibrations. Hence our work provides a Thom spectrum functor defined on maps that classify virtual spherical fibrations. Its advantage over a naive extension of the classical Thom spectrum functor by shifting to the zero component and suspending or desuspending the resulting spectrum accordingly is that it preserves E_∞ structures.

Topological Hochschild homology of Thom spectra. (joint with S. Basu and C. Schlichtkrull) Our Thom spectrum functors are set up in a way that allows for an immediate generalization of the main result of Blumberg, Cohen, and Schlichtkrull [2] to R-based topological Hochschild homology: If $f\colon A \to B\mathrm{GL}_1^{\mathcal{J}}R$ is a map of topological monoids with A grouplike and well based, then the R-based topological Hochschild homology $\mathrm{THH}^R(T(f))$ of $T(f)$ is stably equivalent to the Thom spectrum associated with a certain morphism $L^\eta(Bf)$. If f is in addition assumed to be a 3-fold loop map, then $\mathrm{THH}^R(T(f))$ is stably equivalent to $T(f) \wedge (BA)_+$. This provides a new tool for computations of R-based topological Hochschild homology groups.

References

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