Rigidification of homotopy coherent commutative multiplications

Steffen Sagave

Universität Bonn

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http://www.math.uni-bonn.de/people/sagave/

Symmetric monoidal categories

Definition
A symmetric monoidal category $(\mathcal{A}, \otimes)$ is a category $\mathcal{A}$ with

- a product $\otimes\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$,
- a unit object $1 \in \mathcal{A}$, and
- associativity, unitality, and symmetry isomorphisms satisfying appropriate coherence conditions.

Example

- (*Sets*, $\times$)
- (R-modules, $\oplus$)

Definition
A *permutative* category is a symmetric monoidal category with strict associativity and unitality.

E_∞ structures on classifying spaces

- Let $\widetilde{\Sigma}_k$ be the translation category of the symmetric group Σ_k.
- The space $B\widetilde{\Sigma}_k = E\Sigma_k$ is the kth space of the E_∞ operad $\mathcal{E}$ known as the Barratt–Eccles operad.

Lemma (May)
Let $(\mathcal{A}, \otimes)$ be a permutative category. Its classifying space is $B\mathcal{A}$ is an $\mathcal{E}$-algebra, i.e., an E_∞ space.

Proof.
The functor

$$\widetilde{\Sigma}_k \times \mathcal{A}^{\times k} \to \mathcal{A}, \quad (\sigma; a_1, \ldots, a_k) \mapsto a_{\sigma^{-1}(1)} \otimes \ldots \otimes a_{\sigma^{-1}(k)}.$$

induces the action

$$B\widetilde{\Sigma}_k \times (B\mathcal{A})^{\times k} \to B\mathcal{A}.$$

□

$\mathcal{I}$-spaces

Definition
Let $\mathcal{I}$ be the category with

- objects the finite sets $\mathbf{m} = \{1, \ldots, m\}$, $m \geq 0$, and
- morphisms the injections.

It is permutative with respect to the concatenation

$$\sqcup\colon \mathcal{I} \times \mathcal{I} \to \mathcal{I}, \qquad (\mathbf{m}, \mathbf{n}) \mapsto \mathbf{m} \sqcup \mathbf{n} = \mathbf{m} + \mathbf{n}.$$

- The unit is $\mathbf{0} = \emptyset$.
- The symmetry isomorphism is the block permutation $\tau_{\mathbf{m},\mathbf{n}}\colon \mathbf{m} \sqcup \mathbf{n} \to \mathbf{n} \sqcup \mathbf{m}$.

Definition

- An *$\mathcal{I}$-space* is a functor $\mathcal{I} \to \mathcal{S}$ from $\mathcal{I}$ to the category of spaces $\mathcal{S}$.
- We write $\mathcal{S}^{\mathcal{I}}$ for the category of $\mathcal{I}$-spaces.

THE $\boxtimes$-PRODUCT OF $\mathcal{I}$-SPACES

Let X and Y be $\mathcal{I}$-spaces.

DEFINITION
Let $X \boxtimes Y \colon \mathcal{I} \to \mathcal{S}$ be the left Kan extension of

$$\mathcal{I} \times \mathcal{I} \xrightarrow{X \times Y} \mathcal{S} \times \mathcal{S} \xrightarrow{\times} \mathcal{S}$$

along $\mathcal{I} \times \mathcal{I} \xrightarrow{\sqcup} \mathcal{I}$.

More explicitly,

$$(X \boxtimes Y)(\mathbf{m}) = \operatorname{colim}_{\mathbf{k} \sqcup \mathbf{l} \to \mathbf{m}} X(\mathbf{k}) \times Y(\mathbf{l})$$

where the colimit is taken over the comma category $- \sqcup - \downarrow \mathbf{m}$.

COMMUTATIVE $\mathcal{I}$-SPACE MONOIDS I

- $\boxtimes \colon \mathcal{S}^{\mathcal{I}} \times \mathcal{S}^{\mathcal{I}} \to \mathcal{S}^{\mathcal{I}}$ is a symmetric monoidal product. (It is an instance of a Day-convolution product.)
- The unit is $\mathcal{I}(\mathbf{0}, -) \cong \mathrm{const}_{\mathcal{I}}(*)$.

DEFINITION
A *commutative $\mathcal{I}$-space monoid* is a commutative monoid object in $(\mathcal{S}^{\mathcal{I}}, \boxtimes)$.

COMMUTATIVE $\mathcal{I}$-SPACE MONOIDS II

Spelling out definitions, a commutative $\mathcal{I}$-space monoid M is an $\mathcal{I}$-space $M \colon \mathcal{I} \to \mathcal{S}$ together with

- multiplications $M(\mathbf{k}) \times M(\mathbf{l}) \to M(\mathbf{k} \sqcup \mathbf{l})$ and
- a unit $* \in M(\mathbf{0})$

that are associative, unital, and commutative.

Commutativity means that

$$\begin{array}{ccc} M(\mathbf{k}) \times M(\mathbf{l}) & \longrightarrow & M(\mathbf{k} \sqcup \mathbf{l}) \\ \downarrow{\scriptstyle \text{twist}} & & \downarrow{\scriptstyle (\tau_{\mathbf{m},\mathbf{n}})_*} \\ M(\mathbf{l}) \times M(\mathbf{k}) & \longrightarrow & M(\mathbf{l} \sqcup \mathbf{k}) \end{array}$$

commutes.

E_∞ STRUCTURE ON $M_{h\mathcal{I}}$

We denote the Bousfield–Kan homotopy colimit of $X \colon \mathcal{I} \to \mathcal{S}$ by

$$X_{h\mathcal{I}} = \operatorname{hocolim}_{\mathcal{I}} X = \left| [s] \mapsto \coprod_{\mathbf{k_0} \leftarrow \ldots \leftarrow \mathbf{k_s}} X(\mathbf{k_s}) \right| .$$

LEMMA
Let M be a commutative $\mathcal{I}$-space monoid. Then $M_{h\mathcal{I}}$ is an E_∞ space over the Barratt–Eccles operad.

PROOF.
Let $\mathcal{I}(M)$ be the ($\mathcal{S}$-) category with

- objects $(\mathbf{m}, x)$ where $\mathbf{m} \in \mathcal{I}$ and $x \in M(\mathbf{m})$ and
- morphisms $\alpha \colon (\mathbf{m}, x) \to (\mathbf{n}, y)$ with $\alpha \colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{I}$ and $\alpha_*(x) = y$.

Then $B\mathcal{I}(M) \cong M_{h\mathcal{I}}$. Since M is commutative, $\mathcal{I}(M)$ is permutative. Hence $M_{h\mathcal{I}}$ is an $\mathcal{E}$-algebra. $\square$

EXAMPLES OF COMMUTATIVE $\mathcal{I}$-SPACE MONOIDS I

EXAMPLE
The collection of spaces

$$\mathbf{n} \mapsto BO(n)$$

is a commutative $\mathcal{I}$-space monoid under block sum.

Since $BO_{h\mathcal{I}} \xleftarrow{\simeq} BO_{h\mathcal{N}} \simeq BO$, it models the E_∞ space BO.

WARNING
For general $X \in \mathcal{S}^{\mathcal{I}}$, the hocolims over $\mathcal{I}$ and the natural numbers $\mathcal{N} \subset \mathcal{I}$ differ.

EXAMPLES OF COMMUTATIVE $\mathcal{I}$-SPACE MONOIDS II

EXAMPLE (SCHLICHTKRULL)
Let R be a ring. Then

$$\mathbf{n} \mapsto B\mathrm{GL}_n(R)$$

is a commutative $\mathcal{I}$-space monoid under block sum.

It models Quillen's plus construction, i.e.,

$$(B\mathrm{GL}_\bullet(R))_{h\mathcal{I}} \simeq BGL(R)^+.$$

EXAMPLES OF COMMUTATIVE $\mathcal{I}$-SPACE MONOIDS III

EXAMPLE

- If E is a commutative symmetric ring spectrum, then

$$\mathbf{n} \mapsto \Omega^{\mathcal{I}}(E)(\mathbf{n}) = \Omega^n E_n$$

  is a commutative $\mathcal{I}$-space monoid.
- It models the multiplicative E_∞ space of E.
- There is a sub commutative $\mathcal{I}$-space monoid of units

$$\mathrm{GL}_1^{\mathcal{I}}(R) \to \Omega^{\mathcal{I}}(E)$$

EXAMPLES OF COMMUTATIVE $\mathcal{I}$-SPACE MONOIDS IV

EXAMPLE (SCHLICHTKRULL)
Let Z be a based space.

- Permutation of coordinates, insertion of $*$, and concatenation provide a commutative $\mathcal{I}$-space monoid

$$\mathbf{n} \mapsto Z^\bullet(\mathbf{n}) = Z^{\times n}.$$

- If Z is connected, then $(Z^\bullet)_{h\mathcal{I}} \simeq Q(Z) = \Omega^\infty \Sigma^\infty Z$.

EXAMPLES OF COMMUTATIVE $\mathcal{I}$-SPACE MONOIDS V

EXAMPLE (SOLBERG)

Let $(\mathcal{A}, \otimes)$ be a symmetric monoidal category.

- Let $\Phi(\mathcal{A})(\mathbf{n})$ be the category with objects the n-tuples $(a_1, \dots, a_n)$ of objects in $\mathcal{A}$ and morphism sets

  $$\mathrm{Hom}((a_1, \dots, a_n), (b_1, \dots, b_n)) = \mathcal{A}(a_1 \otimes \dots \otimes a_n, b_1 \otimes \dots \otimes b_n).$$

- Permutation of entries and insertion of the unit object of $\mathcal{A}$ makes $\Phi(\mathcal{A})$ functorial in $\mathcal{I}$.
- The symmetric monoidal structure of $\mathcal{A}$ makes $B\Phi(\mathcal{A})$ a commutative $\mathcal{I}$-space monoid.
- $B\Phi(\mathcal{A})$ models the E_∞ space $B\mathcal{A}$.

E_∞ SPACES AND COMMUTATIVE $\mathcal{I}$-SPACE MONOIDS

KEY FACT

Up to weak equivalence, every E_∞ space is of the form $M_{h\mathcal{I}}$ for a commutative $\mathcal{I}$-space monoid M.

MODEL STRUCTURES ON $\mathcal{I}$-SPACES

DEFINITION

A map of $\mathcal{I}$-spaces $X \to Y$ is an $\mathcal{I}$*-equivalence* if

$$X_{h\mathcal{I}} \to Y_{h\mathcal{I}}$$

is a weak equivalence of spaces.

- The category of $\mathcal{I}$-spaces admits a *positive* $\mathcal{I}$*-model structure* with weak equivalences the $\mathcal{I}$-equivalences.
- For any operad $\mathcal{D}$ in spaces, the category $\mathcal{S}^{\mathcal{I}}[\mathcal{D}]$ of $\mathcal{D}$-algebras in $\mathcal{S}^{\mathcal{I}}$ admits a model structure where a map is a fibration or weak equivalence if the underlying map in $\mathcal{S}^{\mathcal{I}}_{\mathrm{pos}}$ is.

QUILLEN EQUIVALENCE TO E_∞ SPACES

THEOREM (S–SCHLICHTKRULL)

Let $\mathcal{E}$ be the Barratt–Eccles operad (or any E_∞ operad), and let $\Phi\colon \mathcal{E} \to \mathcal{C}$ be the canonical morphism to the commutativity operad.
Then there is a chain of Quillen equivalences

$$\mathcal{S}^{\mathcal{I}}_{\mathrm{pos}}[\mathcal{C}] \underset{\Phi^*}{\overset{\Phi_*}{\leftrightarrows}} \mathcal{S}^{\mathcal{I}}_{\mathrm{pos}}[\mathcal{E}] \underset{\mathrm{const}_{\mathcal{I}}}{\overset{\mathrm{colim}_{\mathcal{I}}}{\rightleftarrows}} \mathcal{S}[\mathcal{E}]\ .$$

- As in symmetric spectra, "positive" means that there is no homotopical information in degree 0.

Rigidification of E_∞ spaces

Corollary

Let X be an E_∞ space. Then there exists a commutative $\mathcal{I}$-space monoid M and a natural chain of $\mathcal{I}$-equivalences between E_∞ $\mathcal{I}$-spaces

$$M \xleftarrow{\sim} Y \xrightarrow{\sim} \mathrm{const}_{\mathcal{I}} X.$$

Proof.

- Choose a cofibrant replacement in $\mathcal{S}^{\mathcal{I}}_{\mathrm{pos}}[\mathcal{E}]$:

$$Y = (\mathrm{const}_{\mathcal{I}} X)^{\mathrm{cof}} \to \mathrm{const}_{\mathcal{I}} X$$

- Set $M = \Phi_*(Y)^{\mathrm{fib}}$
- Since (Φ_*, Φ^*) is a Quillen equivalence, the canonical map

$$Y \to \Phi^*(\Phi_*(Y)^{\mathrm{fib}}) = \Phi^*(M) = M$$

is an $\mathcal{I}$-equivalence.

□

Advantages of commutative $\mathcal{I}$-space monoids

Let M be a commutative $\mathcal{I}$-space monoid.

- The comma category $\mathcal{S}^{\mathcal{I}}/M$ inherits a symmetric monoidal structure with

$$(U \to M) \boxtimes (V \to M) = (U \boxtimes V \to M \boxtimes M \to M)$$

- The category of $\mathcal{S}^{\mathcal{I}}_M$ of M-modules in $(\mathcal{S}^{\mathcal{I}}, \boxtimes)$ inherits a symmetric monoidal product $\boxtimes_M$ with

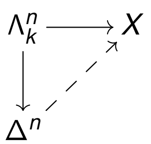
$$X \boxtimes_M Y = \mathrm{coequalizer}(X \boxtimes M \boxtimes Y \rightrightarrows X \boxtimes Y)$$

These two constructions don't work if M is only E_∞.

Quasi-categories

Definition

A simplicial set X is a *weak Kan complex* or a *quasi-category* if for every $0 < k < n$ and every horn inclusion $\Lambda^n_k \to X$, the extension in the following diagram exists:



- Weak Kan complexes are one model for ∞-categories.

The Joyal model structure

There is a Joyal model structure on sSet where

- the cofibrations are the monomorphisms,
- the weak equivalences are the Joyal equivalences, and
- the fibrant objects are the weak Kan-complexes.

Every Joyal equivalence is a weak homotopy equivalence.
($\mathrm{sSet}_{\mathrm{Kan}}$ is a left Bousfield localization of $\mathrm{sSet}_{\mathrm{Joyal}}$)

Proposition (Kodjabachev–S)

The Joyal model structure also induces positive model structures on $\mathrm{sSet}^{\mathcal{I}}$ and $\mathrm{sSet}^{\mathcal{I}}[\mathcal{D}]$.

(Caution: Avoid Bousfield–Kan formula for $\mathrm{hocolim}$ in $\mathrm{sSet}_{\mathrm{Joyal}}$.)

Rigidification of E_∞ quasi-categories

- The Barratt–Eccles operad $\mathcal{E}$ is also an E_∞ operad in $\mathrm{sSet}_{\mathrm{Joyal}}$.

THEOREM (KODJABACHEV–S)
There is a chain of Quillen equivalences

$$\mathrm{sSet}^{\mathcal{I}}_{\text{pos-Joyal}}[\mathcal{C}] \underset{\Phi^*}{\overset{\Phi_*}{\leftrightarrows}} \mathrm{sSet}^{\mathcal{I}}_{\text{pos-Joyal}}[\mathcal{E}] \underset{\mathrm{const}_{\mathcal{I}}}{\overset{\mathrm{colim}_{\mathcal{I}}}{\rightleftarrows}} \mathrm{sSet}_{\mathrm{Joyal}}[\mathcal{E}] .$$

COROLLARY
Let X be an $\mathcal{E}$-algebra in $\mathrm{sSet}_{\mathrm{Joyal}}$. Then there exists an M in $\mathrm{sSet}^{\mathcal{I}}[\mathcal{C}]$ and a natural chain of Joyal $\mathcal{I}$-equivalences in $\mathrm{sSet}^{\mathcal{I}}_{\text{pos-Joyal}}[\mathcal{E}]$:

$$M \xleftarrow{\sim} Y \xrightarrow{\sim} \mathrm{const}_{\mathcal{I}} X.$$

Symmetric monoidal ∞-categories

$\mathcal{E}$-algebras in $\mathrm{sSet}_{\mathrm{Joyal}}$ are one possible model for the *symmetric monoidal ∞-categories* introduced by Lurie.

DEFINITION
A *symmetric monoidal model category* is a symmetric monoidal category $(\mathcal{M}, \otimes)$ with a compatible model structure, i.e., a model structure that satisfies the pushout product axiom.

EXAMPLES
Spaces, symmetric spectra, chain complexes, ...

LEMMA (LURIE)
Via the homotopy coherent nerve, symmetric monoidal model categories give rise to symmetric monoidal ∞-categories.

Combinatorial model categories

DEFINITION
An ∞-category is *presentable* if it is accessible and admits all small colimits.

(“accessible” means “generated by a small subcategory”)

THEOREM (LURIE, DUGGER)
An ∞-category is presentable if and only if it is equivalent to the homotopy coherent nerve of a combinatorial model category.

Presentably symmetric monoidal ∞-categories

QUESTION
Is every symmetric monoidal ∞-category equivalent to the homotopy coherent nerve of a symmetric monoidal model category?

DEFINITION
A symmetric monoidal ∞-category $\mathcal{C}$ is *presentably symmetric monoidal* if $\mathcal{C}$ is presentable and $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves colimits in each variable.

The underlying ∞-category of a combinatorial symmetric monoidal model category is always presentably symmetric monoidal.

EXISTENCE RESULT

THEOREM (NIKOLAUS–S)

Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category. There is a simplicial, combinatorial and left proper symmetric monoidal model category $\mathcal{M}$ whose underlying symmetric monoidal ∞-category is equivalent to $\mathcal{C}$.

- Lurie has an analogous result in the associative case.
- Commutative algebras in the ∞-category $\mathcal{C}$ are equivalent to (strictly) commutative monoids in $\mathcal{M}$.

STRATEGY OF PROOF

- First reduce to the case where $\mathcal{C} = P(\mathcal{D})$ is the category of presheaves on a small symmetric monoidal ∞-category $\mathcal{D}$.
- Choose an $\mathcal{E}$-algebra X in $\mathrm{sSet}_{\mathrm{Joyal}}$ representing $\mathcal{D}$.
- By the ∞-categorical Grothendieck construction, the *contravariant* model structure on sSet/X is a model for the ∞-category $P(\mathcal{D})$.
- The above rigidification $M \xleftarrow{\sim} Y \xrightarrow{\sim} \mathrm{const}_{\mathcal{I}}X$ induces a chain of Quillen equivalences

$$\mathrm{sSet}^{\mathcal{I}}/M \leftrightarrows \mathrm{sSet}^{\mathcal{I}}/Y \rightleftarrows \mathrm{sSet}^{\mathcal{I}}/\,\mathrm{const}_{\mathcal{I}}\,X \rightleftarrows \mathrm{sSet}/X.$$

  with respect to certain contravariant $\mathcal{I}$-model structures.
- $\mathrm{sSet}^{\mathcal{I}}/M$ is symmetric monoidal since M is commutative.
- By naturality, $\mathrm{sSet}^{\mathcal{I}}/M$ captures the correct symmetric monoidal structure on $P(\mathcal{D}) = \mathcal{C}$.

APPLICATIONS

- Lurie shows that topological operads with the Boardman–Vogt tensor product form a symmetric monoidal ∞-category.
- By our theorem, there is a symmetric monoidal model category representing this symmetric monoidal ∞-category.

Thank you for your attention!

REFERENCES

- Diagram spaces and symmetric spectra
  (with C. Schlichtkrull)
  Adv. Math. 231 (2012), no. 3–4, 2116–2193
  DOI:10.1016/j.aim.2012.07.013
- Strictly commutative models for E_∞ quasi-categories
  (with D. Kodjabachev)
  Homology Homotopy Appl. 17 (2015), no. 1, 121–128
  DOI:10.4310/HHA.2015.v17.n1.a5
- Presentably symmetric monoidal ∞-categories are represented by symmetric monoidal model categories
  (with T. Nikolaus)
  arXiv:1506.01475