LOGARITHMIC TOPOLOGICAL HOCHSCHILD HOMOLOGY

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(joint with John Rognes and Christian Schlichtkrull)

The cyclic bar construction

Let $(\mathcal{A}, \otimes, \mathbf{1})$ be a symmetric monoidal category and let A be a monoid in $\mathcal{A}$.

Definition

The *cyclic bar construction* of A is the simplicial object

$$B^{\mathrm{cy}}_\bullet(A) \colon \Delta^{\mathrm{op}} \to \mathcal{A}, \qquad [k] \mapsto \underbrace{A \otimes \ldots \otimes A}_{k+1 \text{ copies}}.$$

The face and degeneracy maps are as follows:

$$d_i(a_0 \otimes \ldots \otimes a_k) = \begin{cases} a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_k & \text{if i<k} \\ a_k a_0 \otimes \ldots \otimes a_{k-1} & \text{if i=k} \end{cases}$$

$$s_i(a_0 \otimes \ldots \otimes a_k) = a_0 \otimes \ldots \otimes a_i \otimes \mathbf{1} \otimes a_{i+1} \otimes \ldots \otimes a_k$$

Via cyclic permutation of $\otimes$-factors, $B^{\mathrm{cy}}_\bullet(A)$ extends to a cyclic object $\Lambda^{\mathrm{op}} \to \mathcal{A}$.

Topological Hochschild homology

The smash product of symmetric spectra is symmetric monoidal. Its unit is the sphere spectrum $\mathbb{S}$. Monoids in $(\mathrm{Sp}^{\Sigma}, \wedge, \mathbb{S})$ are known as (symmetric) ring spectra.

Definition

The *topological Hochschild homology* of a (sufficiently cofibrant) symmetric ring spectrum A is

$$\mathrm{THH}(A) = |B_{\bullet}^{\mathrm{cy}}(A)|,$$

the realization of the cyclic bar construction of A in $(\mathrm{Sp}^{\Sigma}, \wedge, \mathbb{S})$.

Example

Any discrete ring R gives rise to a symmetric ring spectrum HR, the Eilenberg–Mac Lane spectrum of R. The topological Hochschild homology of R is defined by $\mathrm{THH}(R) = \mathrm{THH}(HR)$.

TRACE MAPS

Let A be a ring spectrum. Topological Hochschild homology is useful because there are trace maps

$$\begin{array}{ccc} & & \mathrm{TC}(A) \\ & \overset{\mathrm{trc}}{\nearrow} & \downarrow \\ K(A) & \xrightarrow{\mathrm{tr}} & \mathrm{THH}(A). \end{array}$$

- $K(A)$ is the algebraic K-theory of A. For many A, it is both hard and interesting to compute $K(A)$.
  ($K(\mathbf{S})$ is Waldhausen's $A(*)$ and $K(HR)$ is Quillen's $K(R)$.)
- $\mathrm{TC}(A)$ is the *topological cyclic homology* of A, a refinement of $\mathrm{THH}(A)$ constructed from fixed point information of an S^1-action on $\mathrm{THH}(A)$.
- In some examples of interest, $\mathrm{trc}\colon K(A) \to \mathrm{TC}(A)$ is close to being an equivalence.

Trace maps for periodic ring spectra?

When trying to understand how algebraic K-theory of ring spectra interacts with localization and étale descent, it is natural to also consider $K(A)$ for periodic A (or, more general, for non-connective A).

Easiest examples

$A = KU$, $A = KO$, $A = L$ (the p-local Adams summand)

Problem

The trace map $K(A) \to \mathrm{THH}(A)$ is less useful for periodic A.

One indication: If A is commutative, $\mathrm{THH}(A)$ is an A-module spectrum.

Localization sequences

Blumberg and Mandell established compatible homotopy cofiber sequences

$$\begin{array}{ccccccc} K(\mathbb{Z}) & \longrightarrow & K(ku) & \longrightarrow & K(KU) & \longrightarrow & \Sigma K(\mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{THH}(\mathbb{Z}) & \rightarrow & \mathrm{THH}(ku) & \rightarrow & \mathrm{THH}(ku|KU) & \rightarrow & \Sigma\,\mathrm{THH}(\mathbb{Z}). \end{array}$$

The relative THH-term $\mathrm{THH}(ku|KU)$ is defined using localization techniques and THH of Waldhausen categories. $\mathrm{THH}(ku|KU)$ is **not** equivalent to $\mathrm{THH}(KU)$.

A motivation for logarithmic THH

We like to give an alternative construction of relative THH-terms such as THH($ku|KU$) which is

- more accessible to computations and
- takes *logarithmic ring spectra* as input data.

Discrete log rings

Definition

A discrete *pre-log ring* (A, M) is a commutative ring A and a commutative monoid M together with a monoid homomorphism

$$\alpha \colon M \to (A, \cdot)$$

to the multiplicative monoid of A.

The inclusion of the units $A^\times \to A$ induces a pullback square

$$\begin{array}{ccc} \alpha^{-1}(A^\times) & \longrightarrow & A^\times \\ \downarrow & & \downarrow \\ M & \xrightarrow{\alpha} & A. \end{array}$$

Definition

A pre-log ring (A, M) is a *log ring* if $\alpha^{-1}(A^\times) \to A^\times$ is an isomorphism.

Example for discrete log rings

Let A be an integral domain with quotient field K.

- $(A, A^\times)$ and $(K, K^\times)$ are (trivial) log rings.
- $(A, A \setminus \{0\})$ is a log ring that sits in a factorization

$$(A, A^\times) \to (A, A \setminus \{0\}) \to (K, K^\times).$$

It is useful to think of $A \setminus \{0\}$ as $(A \to K)^*(K^\times)$.

Topological generalizations of log rings

- The classical notions of *multiplicative E_∞ spaces* and *units of ring spectra* lead to a version of logarithmic ring spectra.
- However, this framework makes it difficult to produce interesting topological examples lying beyond Eilenberg–Mac Lane spectra.
- To generalize log rings to log ring spectra in a more interesting way, we need *graded* notions of *multiplicative monoids* and *units* for ring spectra that detect units in non-zero degree.

Commutative $\mathcal{J}$-space monoids

Let $\mathcal{J} = \Sigma^{-1}\Sigma$ be Quillen's localization construction on the category Σ of finite sets and bijections. The category $\mathcal{J}$ is symmetric monoidal under concatenation $\sqcup$, and $B\mathcal{J} \simeq QS^0$.

Definition

A $\mathcal{J}$-space is a functor $X\colon \mathcal{J} \to \mathcal{S}$ to the category of spaces $\mathcal{S}$.

The functor category $\mathcal{S}^{\mathcal{J}}$ inherits a symmetric monoidal convolution product $\boxtimes$ from the product of $\mathcal{J}$. By definition, $X \boxtimes Y$ is the left Kan extension of

$$\mathcal{J} \times \mathcal{J} \xrightarrow{X \times Y} \mathcal{S} \times \mathcal{S} \xrightarrow{\times} \mathcal{S}$$

along $\sqcup\colon \mathcal{J} \times \mathcal{J} \to \mathcal{J}$.

Definition

A *commutative $\mathcal{J}$-space monoid* is a commutative monoid in $(\mathcal{S}^{\mathcal{J}}, \boxtimes)$.

Graded E_∞ spaces

The category of commutative $\mathcal{J}$-space monoids $\mathcal{C}\mathcal{S}^{\mathcal{J}}$ admits a model structure where $f\colon M \to N$ is a weak equivalence iff

$$\operatorname{hocolim}_{\mathcal{J}} f\colon \operatorname{hocolim}_{\mathcal{J}} M \to \operatorname{hocolim}_{\mathcal{J}} N$$

is a weak homotopy equivalence in $\mathcal{S}$.

Theorem (S.–Schlichtkrull)

There is a chain of Quillen equivalences

$$\mathcal{C}\mathcal{S}^{\mathcal{J}} \simeq E_\infty\text{-spaces}/QS^0$$

sending a commutative $\mathcal{J}$-space monoid M to

$$\operatorname{hocolim}_{\mathcal{J}} M \to \operatorname{hocolim}_{\mathcal{J}} \operatorname{const}_{\mathcal{J}}(*) = B\mathcal{J} \simeq QS^0.$$

We view the augmentation $\operatorname{hocolim}_{\mathcal{J}} M \to QS^0$ as a grading of the E_∞ space $\operatorname{hocolim}_{\mathcal{J}} M$.

Graded E_∞ spaces and Thom spectra

There is a Quillen-adjunction

$$\mathbb{S}^{\mathcal{J}} \colon \mathcal{C}\mathcal{S}^{\mathcal{J}} \rightleftarrows \mathcal{C}\mathrm{Sp}^{\Sigma} \colon \Omega^{\mathcal{J}}$$

relating $\mathcal{C}\mathcal{S}^{\mathcal{J}}$ to commutative symmetric ring spectra.

- $\Omega^{\mathcal{J}}(A)$ models the graded multiplicative E_∞ space of A.
- There is a notion of units $\mathrm{GL}_1^{\mathcal{J}}(A) \subset \Omega^{\mathcal{J}}(A)$ that captures $\pi_*(A)^\times \subset \pi_*(A)$.
- $\mathbb{S}^{\mathcal{J}}[M]$ models the graded spherical monoid ring of M.

Theorem (S.–Schlichtkrull)

If M is sufficiently cofibrant, then $\mathbb{S}^{\mathcal{J}}[M]$ is equivalent to the Thom spectrum of the virtual vector bundle classified by

$$\mathrm{hocolim}_{\mathcal{J}} M \to \mathrm{hocolim}_{\mathcal{J}} \mathrm{const}_{\mathcal{J}}(*) \simeq QS^0 \to \mathbb{Z} \times BO.$$

LOGARITHMIC RING SPECTRA

DEFINITION

A *pre-log ring spectrum* (A, M) is a commutative symmetric ring spectrum A together with a commutative $\mathcal{J}$-space monoid M and a map $\alpha \colon M \to \Omega^{\mathcal{J}}(A)$ in $\mathcal{CS}^{\mathcal{J}}$.

DEFINITION

A pre-log ring spectrum (A, M) is a *log ring spectrum* if $\alpha^{-1}(\mathrm{GL}_1^{\mathcal{J}}(A)) \to \mathrm{GL}_1^{\mathcal{J}}(A)$ is a weak equivalence in $\mathcal{CS}^{\mathcal{J}}$.

Every commutative symmetric ring spectrum A gives rise to the trivial log ring spectrum $(A, \mathrm{GL}_1^{\mathcal{J}}(A))$.

Examples for logarithmic ring spectra

Let E be a d-periodic commutative symmetric ring spectrum, let $x \in \pi_d(E)$ be a unit of minimal positive degree, and let $j\colon e \to E$ be the connective cover of E.

Consider the pullback $j_*(\mathrm{GL}_1^{\mathcal{J}}(E))$ of

$$\mathrm{GL}_1^{\mathcal{J}}(E) \to \Omega^{\mathcal{J}}(E) \leftarrow \Omega^{\mathcal{J}}(e).$$

We write $(e, \langle x \rangle)$ for the log ring spectrum $(e, j_*(\mathrm{GL}_1^{\mathcal{J}}(E))$.

This log ring spectrum comes with a factorization

$$(e, \mathrm{GL}_1^{\mathcal{J}}(e)) \to (e, \langle x \rangle) \to (E, \mathrm{GL}_1^{\mathcal{J}}(E)).$$

Example

The Bott class $u \in \pi_2(KU)$ gives rise to a factorization

$$(ku, \mathrm{GL}_1^{\mathcal{J}}(ku)) \to (ku, \langle u \rangle) \to (KU, \mathrm{GL}_1^{\mathcal{J}}(KU)).$$

The replete bar construction

Let M be a commutative $\mathcal{J}$-space monoid.

Definition

Let $B^{\mathrm{cy}}(M) = |B^{\mathrm{cy}}_{\bullet}(M)|$ be the realization of the cyclic bar construction of M in $(\mathcal{S}^{\mathcal{J}}, \boxtimes)$.

Definition

The replete bar construction of M is the (homotopy) pullback

$$\begin{array}{ccc} B^{\mathrm{rep}}(M) & \longrightarrow & B^{\mathrm{cy}}(M^{\mathrm{gp}}) \\ \downarrow & & \downarrow \\ M & \longrightarrow & M^{\mathrm{gp}} \end{array}$$

in commutative $\mathcal{J}$-space monoids.

- $M \to M^{\mathrm{gp}}$ is the group completion of M.
- There is a canonical *repletion map* $\rho\colon B^{\mathrm{cy}}(M) \to B^{\mathrm{rep}}(M)$.

Replete bar construction of $\mathbb{N}$

One can also consider B^{cy} and B^{rep} for discrete monoids.

$$B^{\mathrm{cy}}(\mathbb{N}) = \{*\} \amalg \coprod_{k\geq 1} S^1$$

$$B^{\mathrm{cy}}(\mathbb{Z}) = \coprod_{k\in\mathbb{Z}} S^1$$

$$B^{\mathrm{rep}}(\mathbb{N}) = \coprod_{k\geq 0} S^1$$

In homology, the repletion map $B^{\mathrm{cy}}(\mathbb{N}) \to B^{\mathrm{rep}}(\mathbb{N})$ takes the form

$$\rho_*\colon P(x)\otimes E(dx) \to P(x)\otimes E(\mathrm{dlog}x), \quad \rho_*(x) = x, \rho_*(dx) = x\cdot \mathrm{dlog}x$$

where P denotes a polynomial algebra, E denotes an exterior algebra, and the generators have degrees

$$|x| = (0,1),\ |dx| = (1,1),\ \text{ and } \ |\mathrm{dlog}x| = (1,0).$$

Definition of logarithmic THH

Let (A, M) be a (cofibrant) pre-log ring spectrum. The repletion and the adjoint $\mathbb{S}^{\mathcal{J}}[M] \to A$ of $M \to \Omega^{\mathcal{J}}(A)$ induce a diagram of commutative symmetric ring spectra

$$\mathrm{THH}(A) \leftarrow \mathrm{THH}(\mathbb{S}^{\mathcal{J}}[M]) \xleftarrow{\cong} \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)] \to \mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(M)]$$

Definition

The *logarithmic topological Hochschild homology* is defined to be the pushout

$$\mathrm{THH}(A, M) = \mathrm{THH}(A) \wedge_{\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)]} \mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(M)]$$

in commutative symmetric ring spectra.

Example

For trivial log ring spectra, we have

$$\mathrm{THH}(A) \xrightarrow{\sim} \mathrm{THH}(A, \mathrm{GL}_1^{\mathcal{J}}(A)).$$

Localization sequences for log *THH*

Let E be a d-periodic commutative symmetric ring spectrum with periodicity class $x \in \pi_d(E)$ and connective cover $e \to E$. We write $e[0, d\rangle$ for the dth Postnikov section of e.

Theorem (Rognes–S.–Schlichtkrull)

There is a localization homotopy cofiber sequence

$$\mathrm{THH}(e) \to \mathrm{THH}(e, \langle x \rangle) \to \Sigma\, \mathrm{THH}(e[0, d\rangle).$$

The resulting homotopy cofiber sequence

$$\mathrm{THH}(ku) \to \mathrm{THH}(ku, \langle u \rangle) \to \Sigma\, \mathrm{THH}(\mathbb{Z})$$

is analogous to the cofiber sequence established by Blumberg–Mandell. We expect the relative THH-terms to be equivalent when both are defined.

TAME RAMIFICATION

Let p be an odd prime, let $ku = ku_{(p)}$ be the p-local connective complex K-theory spectrum, and let $\ell \to ku$ be the inclusion of the connective p-local Adams summand.

On π_*, the map $\ell \to ku$ induces $\mathbb{Z}_{(p)}[v] \to \mathbb{Z}_{(p)}[u]$, $v \mapsto u^{p-1}$.

There are compatible homotopy cofiber sequences

$$\begin{array}{ccccc} \mathrm{THH}(\ell) & \longrightarrow & \mathrm{THH}(\ell, \langle v \rangle) & \longrightarrow & \Sigma\,\mathrm{THH}(\mathbb{Z}_{(p)}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{THH}(ku) & \longrightarrow & \mathrm{THH}(ku, \langle u \rangle) & \longrightarrow & \Sigma\,\mathrm{THH}(\mathbb{Z}_{(p)}) \end{array} .$$

THEOREM (ROGNES–S.–SCHLICHTKRULL)

The diagram induces a stable equivalence

$$ku \wedge_\ell \mathrm{THH}(\ell, \langle v \rangle) \to \mathrm{THH}(ku, \langle u \rangle),$$

*i.e., $\ell \to ku$ is formally log-*THH *étale.*

COMPUTATIONS FOR ℓ AND $ku_{(p)}$

For a spectrum X, let $V(1)_*X = \pi_*(V(1) \wedge X)$ denote the $V(1)$-homotopy groups. (Here

$$V(1) = \mathrm{cone}(v_1 \colon \Sigma^{2p-2} S/p \to S/p)$$

is a Smith–Toda complex of type 2).

THEOREM (BÖKSTEDT)

$$V(1)_* \mathrm{THH}(\mathbb{Z}_{(p)}) \cong E(\overset{2p-1}{\epsilon_1}, \overset{2p-1}{\lambda_1}) \otimes P(\overset{2p}{\mu_1})$$

THEOREM (MCCLURE–STAFFELDT)

$$V(1)_* \mathrm{THH}(\ell) \cong E(\overset{2p-1}{\lambda_1}, \overset{2p^2-1}{\lambda_2}) \otimes P(\overset{2p^2}{\mu_2})$$

THEOREM (ROGNES–S.–SCHLICHTKRULL)

$$V(1)_* \mathrm{THH}(\ell, \langle v \rangle) \cong E(\overset{2p-1}{\lambda_1}, \overset{1}{\mathrm{dlog}} v) \otimes P(\overset{2p}{\kappa_1})$$

COROLLARY (ROGNES–S.–SCHLICHTKRULL)

$$V(1)_* \mathrm{THH}(ku, \langle u \rangle) \cong P_{p-1}(\overset{2}{u}) \otimes E(\overset{2p-1}{\lambda_1}, \overset{1}{\mathrm{dlog}} u) \otimes P(\overset{2p}{\kappa_1})$$

Towards logarithmic TC

Currently there appear to be 3 possible constructions of TC:

(1) The original construction by Bökstedt–Hsiang–Madsen, exploiting the cyclotomic structure on the Bökstedt model for THH.

(2) The approach by Angeltveit–Blumberg–Gerhardt–Hill–Lawson–Mandell building on a property of the geometric fixed points of norms of orthogonal spectra and the Blumberg–Mandell description of cyclotomic spectra.

(3) The Nikolaus-Scholze approach using an S^1-equivariant map to the C_p-Tate construction of $\mathrm{THH}(A)$.

Work in progress

For an interesting class of pre-log ring spectra (A, M), our model of $\mathrm{THH}(A, M)$ is cyclotomic in the sense of (2). The approach (3) is likely to also produce cyclotomic structures on $\mathrm{THH}(A, M)$.