# PROGRAM FOR THE SEMINAR ON "THE STABLE PARAMETRIZED $h$-COBORDISM THEOREM" (AFTER WALDHAUSEN-JAHREN-ROGNES)

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## Introduction

The stable parametrized $h$-cobordism theorem says that for a compact CAT-manifold $M$, where CAT = DIFF, PL, TOP there is a homotopy equivalence

$$\mathcal{H}^{\mathrm{CAT}}(M) \simeq \Omega \mathrm{Wh}^{\mathrm{CAT}}(M).$$

Here DIFF stands for *differentiable*, PL for *piecewise linear*, TOP for *topological*, $\mathcal{H}^{\mathrm{CAT}}(M)$ is the *stable CAT h-cobordism space* of $M$, which is closely related to automorphism groups of CAT-manifolds, and $\Omega \mathrm{Wh}^{\mathrm{CAT}}(M)$ is the loop space of the CAT-*Whitehead space of* $M$, which is closely related to the algebraic $K$-theory of $M$. The theorem provides a bridge between geometric topology and algebraic $K$-theory and makes it possible to use calculational tools for algebraic $K$-theory to obtain information about geometric questions.

The primary references for the seminar are the paper by Waldhausen, Jahren, and Rognes [WJR08] and Waldhausen's foundational paper on algebraic $K$-theory of spaces [Wal85]. Sections 0 and 1 of [WJR08] and Rognes' lecture notes [Rog06] are a good source for motivation, applications and the basic structure of the proof of the theorem.

The seminar will occupy 6 afternoon sessions, each of which has 2 talks of at most 75 minutes length. It divides into 4 parts:

(1) Statement of main results and applications (Talks 1.1. and 1.2)
(2) Relation to algebraic $K$-theory (Talks 2.1 to 3.2)
(3) The *non-manifold part* of the proof (Talks 4.1 to 5.1)
(4) The *manifold part* of the proof (Talks 5.2 to 6.2)

The parts (2)-(4) are more or less independent of each other in that they use different mathematical methods. In particular, preparing (or following) a talk from one of these parts doesn't rely on having a complete knowledge of the material from the earlier parts. When going through the proof of the theorem, we will focus on the PL-case of the theorem because this implies the DIFF- and TOP-cases.

## Session 1: Overview and applications (April 07, 2011)

**Talk 1.1: Statement of the main results (Steffen Sagave).** This talk should start with recalling the classical $h$- and $s$-cobordism theorem. After that, the speaker should define the stable $h$-cobordism space $\mathcal{H}^{\mathrm{CAT}}(M)$ of a compact manifold $M$, introduce $s\mathcal{C}^h(X)$ as a model for the loop space of the PL-Whitehead space of a space $X$, and state the stable parametrized $h$-cobordism theorem. Defining the

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PL-Whitehead space as a sum construction $sN_\bullet\mathcal{C}^h_f(X)$ as in [Wal85, §3.1], one can motivate the stable parametrized version by the classical $h$-cobordism theorem. Building on the second display on [WJR08, p.15], a rough outline of the proof and the methods used in it should be given.

The ordinary Whitehead group is related to $K$-theory of rings, and the Whitehead space is related to the algebraic $K$-theory $A(X)$ of a space $X$. This analogy should be explained, postponing the precise definition of $A(X)$ to later talks. The talk should end by explaining the relation of the stable parametrized $h$-cobordism theorem to algebraic $K$-theory of spaces and formulating [WJR08, Theorem 0.2 and Theorem 0.3].

**Talk 1.2: Automorphism groups (Moritz Rodenhausen).** This talk should show how to apply the stable parametrized $h$-cobordism theorem to the calculation of homotopy groups of the automorphism spaces of manifolds.

First the speaker should explain the homotopy equivalence between the space of concordances on a manifold $M$ and the loop space of the space of $h$-cobordisms on $M$ that stabilizes to a homotopy equivalence $\mathcal{C}^{\mathrm{CAT}}(M) \simeq \Omega\mathcal{H}^{\mathrm{CAT}}(M)$, see [Wal82, p. 3-4] or [Rog06, p. 6-7].

Next the speaker should explain the *Hatcher spectral sequence* whose input is $\pi_*\mathcal{C}^{\mathrm{CAT}}(M)$ with a certain canonical involution and which converges (in a range) to the homotopy groups of the automorphism space $\widetilde{\mathrm{CAT}}(M)/\mathrm{CAT}(M)$. Here $\widetilde{\mathrm{CAT}}(M)$ is the space of block CAT-automorphisms of $M$ and $\mathrm{CAT}(M)$ is the space of "honest" CAT-automorphisms of $M$. See [Rog06, p.3-6] or [Hat78, §2].

For $M = D^n$ and CAT = DIFF, the rational homotopy groups of $\widetilde{\mathrm{DIFF}}_\partial(D^n)$ are known from surgery theory while those of $\mathrm{Wh}^{\mathrm{DIFF}}(D^n)$ are known from $K$-theory computations. The speaker should use these results to calculate the rational homotopy groups of $\mathrm{DIFF}_\partial(D^n)$. See [FH78], [Rog06, §3.2], or [WW01, §6.1]. Similarly, for $M$ a negatively curved manifold and CAT = TOP the speaker should plug in the known results about the homotopy groups of the spaces $\widetilde{\mathrm{TOP}}(M)$ and $\mathrm{Wh}^{\mathrm{TOP}}(M)$ to calculate the homotopy groups of $\mathrm{TOP}(M)$, see [Far02, §5].

## Session 2: Algebraic $K$-theory of spaces (April 21,2011)

**Talk 2.1: The definition of $A(X)$ (Karol Szumilo).** The objective of this talk is to introduce the *algebraic $K$-theory of spaces* $A(X)$ in the sense of Waldhausen [Wal85, §1]. After defining what a *category with cofibrations and weak equivalences* (or *Waldhausen-category*) $\mathcal{C}$ is, the algebraic $K$-theory of $\mathcal{C}$ can be constructed as the loop space of the realization of a certain bisimplicial set $wS_\bullet\mathcal{C}$ associated with $\mathcal{C}$. Applying this to an appropriate category $\mathcal{R}_f(X)$ associated with a space $X$ defines $A(X)$. One can motivate this using Euler-Characteristics [Wal84, p. 17].

The speaker should also present (but most likely not prove) the *Additivity theorem* [Wal85, Proposition 1.3.2]. The *Eilenberg-Swindle* should be discussed as one immediate consequence of the additivity theorem. It emphasizes the need for finiteness condition in order to get an interesting definition of $K$-theory. Another important application is that $A(X)$ is in fact the infinite loop space associated with a (positive fibrant symmetric) *spectrum*.

Besides Waldhausen's foundational paper [Wal85, §1], Weibel's book project [Wei] is a good source for this material. The brief outline of the definition of $A(X)$ given in [Wal84, §1] might also be helpful.

**Talk 2.2: $K$-theory theorems (Katja Hutschenreuter).** This talk should focus on some of the main theorems about Waldhausen's $K$-theory construction, namely the *Fibration theorem* [Wal85, Theorem 1.6.4] and the *Approximation theorem* [Wal85, Theorem 1.6.7]. As an application of the Approximation theorem, equivalent definitions of $A(X)$ using finite and homotopy finite retractive spaces or the Kan loop group should be discussed.

Besides that, the speaker should (at least for $X = *$) construct the linearization map $A(X) \to K(\mathbb{Z}[G(X)])$ and mention that it is a rational equivalence [Wal78, Proposition 2.2]. However, the proof of the latter fact might be skipped because this result seems to be difficult to establish in the model for $A(X)$ used here. Depending on the interests of the speaker, it is also possible to briefly discuss how the algebraic $K$-theory of structured ring spectra is defined and to use this language to express the linearization map.

It does not seem to be realistic to give full proofs of the Fibration and Approximation theorems within one talk. So the speaker should focus on introducing the terminology needed to state them and only give the main idea behind the proofs. When studying the proofs of these theorems, it might also be helpful to look at [Sch06, Appendix A]. The linearization map is discussed in [Wal78, §2], [Wal85, §2.3], and also in [Hüt06].

## Session 3: Whitehead spaces and $K$-theory (May 05, 2011)

**Talk 3.1: Simple maps and the Whitehead space (Martin Langer).** The main objective of this talk is to present [Wal85, Theorem 3.1.7]. This theorem gives an alternative description of the PL-Whitehead space introduced in Talk 1.1. in terms of algebraic $K$-theory of spaces. That is, there is a Waldhausen category associated with a simplicial set $X$ so that its algebraic $K$-theory space is homotopy equivalent to PL-Whitehead space of $X$. After setting up the notation, this reads as

$$sN_\bullet \mathcal{C}^h_f(X) \simeq sS_\bullet R^h_f(X^{\Delta^\bullet})$$

The speaker should also discuss [Wal85, Theorem 3.2.1] and the terminology needed to state it. This theorem is important for the next talk and explains to some extent why it may be interesting to consider the category $sS_\bullet R_f(X^{\Delta^\bullet})$. There will not be enough time to prove both theorems in this talk, and the emphasis should lie on the ideas behind theorem [Wal85, Theorem 3.1.7].

The talk is mostly based on [Wal85, §3], but as explained in [WJR08, Remark 1.4.5] it also needs some results from [WJR08]. Some of the facts needed about simple maps will be discussed in Talk 4.1.

**Talk 3.2: The fibration sequences relating Whitehead spaces and $K$-theory (Wolfgang Steimle).** This talk should start by explaining [Wal85, Theorem 3.3.1], which provides a homotopy fiber sequence relating the algebraic $K$-theory of a space $X$ and its PL-Whitehead space. In particular, this explains how the $K$-theory formulation of the PL-stable parametrized $h$-cobordism theorem in [WJR08, Theorem 0.2] can be derived from the theorem itself. The speaker should agree with the speaker of the previous talk how exactly to divide the material from [Wal85, §3.2].

In the second part of the talk the speaker should explain how the DIFF-Whitehead space is defined and outline the arguments that reduce the DIFF-case of the stable parametrized $h$-cobordism to the PL case. This is surveyed at the end of [WJR08, §1.3] and builds on various deep results which we are unable to study in full detail in this seminar.

## Session 4: Simple maps and spaces of simplicial sets (May 19, 2011)

**Talk 4.1: Simple maps and non-singular simplicial sets (Henrik Rüping).** This talk is the first one out of three dealing with the *non-manifold part* of the stable parametrized $h$-cobordism theorem. The main result of the non-manifold part says that the PL Whitehead space of $X$, defined in terms of simplicial sets and simple maps, can be alternatively defined in terms Serre fibrations of compact polyhedra and simple maps. Since the geometric realization of a simplicial set is, in general, not a polyhedron, the passage from simplicial sets to polyhedra requires, as an intermediate step, to consider *non-singular* simplicial sets.

The material needed for this talk is contained in [WJR08, Sections 2.1 to 2.5]. The objective is to prove [WJR08, Proposition 2.5.1] which says that, up to simple maps, any finite simplicial set can be replaced by a finite non-singular one. To do that, the speaker should start by discussing the basic properties of simple maps [WJR08, Proposition 2.1.3]. However, the main focus of the talk are the two constructions *normal subdivision* and *reduced mapping cylinder*, a "non-singular analogue" of the ordinary mapping cylinder. The speaker should survey on the results that are relevant for the proof of [WJR08, Proposition 2.5.1]. The statement of [WJR08, Proposition 2.4.12] might serve as an illustration of the advantage of the reduced mapping cylinder over the ordinary mapping cylinder. The speaker should also mention the main results of [WJR08, Section 2.3].

**Talk 4.2: Spaces of simplicial sets (Irakli Patchkoria).** In technical language, the non-manifold part of the stable parametrized $h$-cobordism theorem compares the geometric realizations of certain simplicial categories. Two basic categories of interest are the category $\mathcal{C}$ of finite simplicial sets (and simplicial maps) and the full subcategory $\mathcal{D}$ of non-singular simplicial sets, and their subcategories $h\mathcal{C}, s\mathcal{C}, h\mathcal{D}, s\mathcal{D}$ of homotopy equivalences resp. simple maps. The first objective of this talk is to use the results from the last talk and Quillen's theorem A to prove that the inclusions

$$h\mathcal{D} \to h\mathcal{C} \quad \text{and} \quad s\mathcal{D} \to s\mathcal{C}$$

are homotopy equivalences [WJR08, Section 3.1]. (See the two lower lines of Figure 1.)

The second objective is to introduce a new simplicial direction, by considering $k$-parameter families of objects and homotopy equivalences resp. simple maps between these. The resulting simplicial categories are easier to compare with the space of $h$-cobordisms. There are two types of simplicial categories,

$$h\mathcal{D}_\bullet \quad \text{and} \quad h\tilde{\mathcal{D}}_\bullet,$$

where an object in simplicial degree $n$ is a *PL bundle*, resp. *Serre fibration*, of a finite non-singular simplicial set over $\Delta^n$. The remainder of the talk should be devoted to the proof of [WJR08, Proposition 3.5.1] which says that the inclusions

$$s\mathcal{C} \to s\tilde{\mathcal{C}}_\bullet, \quad s\mathcal{D} \to s\tilde{\mathcal{D}}_\bullet, \quad s\mathcal{D} \to s\mathcal{D}_\bullet$$

of the zero-simplices (and similarly with $s$ replaced by $h$) are homotopy equivalences. This needs material from [WJR08, Sections 2.6 and 2.7]. The central result is [WJR08, Proposition 2.7.6] which relates Serre fibrations over $\Delta^q$ with "trivializations up to simple maps".

## Session 5: From simplicial sets to PL-manifolds (June 9, 2011)

**Talk 5.1: Polyhedral realization and homotopy fiber sequences (Tibor Macko).** This is the last talk of the non-manifold part. The geometric realization of a non-singular simplicial set has a canonical PL structure. Thus, letting $\mathcal{E}$ be the category of compact polyhedra and PL maps, there is a *polyhedral realization* functor $r\colon \mathcal{D} \to \mathcal{E}$. The first objective of this talk is to show that this functor induces homotopy equivalences

$$s\mathcal{D}_\bullet \to s\mathcal{E}_\bullet, \quad s\tilde{\mathcal{D}}_\bullet \to s\tilde{\mathcal{E}}_\bullet$$

between the corresponding simplicial categories, and likewise with $s$ replaced by $h$ [WJR08, Proposition 3.4.4].

The second part of the talk deals with the relative categories such as $s\mathcal{C}^h(X)$ (which is the geometric model for $\Omega\mathrm{Wh}^{\mathrm{PL}}(X)$). The speaker should discuss homotopy invariance properties of these [WJR08, Proposition 3.2.2] and survey on the identification of these relative categories with the homotopy fibers of the corresponding forgetful functors that view simple maps as homotopy equivalences [WJR08, Proposition 3.3.1]. In other words, the three lower lines in Figure 1 are homotopy fibration sequences.

The outcome of the talk should be that the inclusion of non-singular into general simplicial sets, the inclusion of 0-simplices, and the polyhedral realization induce homotopy equivalences

$$\Omega\mathrm{Wh}^{\mathrm{PL}}(X) \xleftarrow{\simeq} s\mathcal{C}^h(X) \xleftarrow{\simeq} s\mathcal{D}^h(X) \xrightarrow{\simeq} s\tilde{\mathcal{D}}^h_\bullet(X) \xrightarrow{\simeq} s\tilde{\mathcal{E}}^h_\bullet(|X|).$$

This provides the two lower homotopy equivalences in the left hand column of Figure 1.

Figure 1. Scheme of the proof of the main theorem ($M = |X|$)

$$\begin{array}{ccccccc}
\mathcal{H}^{PL}(M) & \to & \mathrm{colim}_n \mathcal{M}^n_\bullet & \to & \mathrm{colim}_n h\mathcal{M}^n_\bullet & & \text{manifolds} \\
\simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & & \\
s\widetilde{\mathcal{E}}^h_\bullet(M) & \longrightarrow & s\widetilde{\mathcal{E}}_\bullet & \longrightarrow & h\widetilde{\mathcal{E}}_\bullet & & \text{polyhedra} \\
\simeq\uparrow & & \simeq\uparrow & & \simeq\uparrow & & \\
s\mathcal{D}^h(X) & \longrightarrow & s\mathcal{D} & \longrightarrow & h\mathcal{D} & & \text{non-singular simplicial sets} \\
\simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & & \\
s\mathcal{C}^h(X) & \longrightarrow & s\mathcal{C} & \longrightarrow & h\mathcal{C} & & \text{simplicial sets}
\end{array}$$

**Talk 5.2: Spaces of ($PL$-) manifolds (Philipp Kühl).** This is the first of three talks dealing with the manifold part of the theorem [WJR08, Theorem 1.1.7]. It states that for a compact PL-manifold $M$ there is a homotopy equivalence $\mathcal{H}^{PL}(M) \simeq s\widetilde{\mathcal{E}}^h_\bullet(M)$. Here $\mathcal{H}^{PL}(M)$ is again the stable parametrized $h$-cobordism space, and $s\widetilde{\mathcal{E}}^h_\bullet(M)$ is the space considered in the previous talk.

The aim of this first talk is to prove the [WJR08, Theorem 1.1.7] assuming [WJR08, Theorem 4.1.14]. First of all, it turns out that it is enough to prove the theorem for stably framed manifolds. After that the strategy of the proof can be read off the first two lines of Figure 1: The top line contains the space of stable $h$-cobordisms $\mathcal{H}^{PL}(M)$ and displays it within a homotopy fiber sequence with the stabilized version of the spaces $\mathcal{M}^n_\bullet$ and $h\mathcal{M}^n_\bullet$. Here $\mathcal{M}^n_\bullet$ is the space of stably framed $n$-dimensional manifolds and $h\mathcal{M}^n_\bullet$ is the realization of the simplicial category of stably framed $n$-dimensional manifolds and homotopy equivalences between them. The second line in Figure 1 is the homotopy fiber sequence discussed in the previous talk, and the strategy of the proof is to show middle and right vertical map relating the sequences are homotopy equivalences.

After explaining the reduction to the stably framed case in [WJR08, Remark 1.1.8] the speaker should introduce the spaces and the homotopy fiber sequence in the top line of Figure 1 and prove what is outlined in the last paragraph, including [WJR08, Proposition 4.1.15]. This material is contained in [WJR08, §4.1]. The diagram [WJR08, (4.1.13)] explains the structure of the argument in more detail.

## Session 6: Thickenings (July 07, 2011 )

**Talk 6.1: Spaces of Thickenings (Martin Olbermann).** At this stage it remains to prove [WJR08, Theorem 4.1.14]. It says that the comparison map $w$ from the stabilized version of the space $\mathcal{M}^n_\bullet$ of PL stably framed $n$-manifold bundles to the stabilized version of the space $s\mathcal{E}_\bullet$ of PL-bundles of polyhedra is a homotopy equivalence. The main aim of this talk is to relate the homotopy fiber of $w$ over a polyhedron $K$ to the space $\mathrm{colim}_n T^n(K)$ where $T^n(K)$ is the space of $n$-manifold thickenings of $K$. The identification goes via another space $\mathrm{colim}_n S^n(K)$. The definitions of these spaces should be given and the main ideas of the identification as well. This includes the "Proof of Theorem 4.1.14 assuming Proposition 4.2.2" in [WJR08, §4.2], which is quite formal, and the three following lemmas which have more geometric content.

Now the proof is reduced to showing that $\mathrm{colim}_n T^n(K)$ is contractible for any $K$. In this talk the first building block for the proof of this statement should be discussed, namely [WJR08, Proposition 4.2.8] which says that for $n \geq 6$ the space $T^n(*)$ is at least $(n-2)$-connected.

**Talk 6.2: Straightening of the thickenings (Johannes Ebert).** This talk contains the last step of the proof of [WJR08, Theorem 1.1.7], which is [WJR08, Theorem 4.3.1]. The latter theorem states that the space of thickenings $T^n(K)$ of a $k$-dimensional polyhedron $K$ is at least $(n-2k-6)$-connected.

The proof requires the definition of the space of thickenings relative to a fixed thickening on a subpolyhedron, denoted $T^n(L \to K, N)$. The theorem about the connectivity of $T^n(K)$ has a relative analogue, which is [WJR08, Theorem 4.3.5], and this is what needs to be proved. The proof proceeds by induction on the dimension of $K$. Both the induction start and induction step are non-trivial to prove and require different set of ideas. For time reasons, the speaker might be unable to supply all details.

## References


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