Lecture notes for the master course

# Algebraic Topology

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# Lecture 1: Basic notions and motivation

Algebraic Topology studies topological spaces and continuous maps through algebraic invariants. One example for such an algebraic invariant is the fundamental group $\pi_1(X, x_0)$ of a topological space $X$ with basepoint $x_0$. In this course, we will introduce another type of algebraic invariants of topological spaces, the *homology groups*. We will also develop tools to compute them.

The actual definition of homology groups is more complicated than the definition of the fundamental group and will be given in the next lecture. In this lecture, we will study some elementary notions that are relevant for homology, give motivation, and introduce some of the building blocks needed for the definition of homology.

## 1.1 Homeomorphisms and homotopy equivalences

We recall some notions about continuous maps and topological spaces.

**Definition 1.2.** A continuous map $f\colon X \to Y$ is a *homeomorphism* (or *topological equivalence*) if there is a continuous map $g\colon Y \to X$ such that $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_Y$. If there exists a homeomorphism $f\colon X \to Y$, we say that $X$ and $Y$ are homeomorphic and sometimes write $X \cong Y$.

The notion of a homeomorphism is too rigid for many purposes. To relax it, we need the notion of homotopic maps:

**Definition 1.3.** Two continuous maps $f_0, f_1\colon X \to Y$ are *homotopic* if there is a continuous map $H\colon X \times [0,1] \to Y$ such that $H|_{X\times\{i\}} = f_i$ for $i = 0,1$. In this situation, $H$ is called a *homotopy*, and we sometimes write $f_0 \simeq f_1$.

A homotopy can be thought of as a continuous family of maps $H|_{X\times\{t\}} = f_t$, $0 \leq t \leq 1$, interpolating between $f_0$ and $f_1$. However, we stress that the condition that the homotopy $H\colon X \times [0,1] \to Y$ is continuous with respect to the product topology in the source is stronger than requiring that each $H|_{X\times\{t\}}$ is continuous. It is easy to see that "homotopy" defines an equivalence relation on the set of continuous maps from $X$ to $Y$.

**Definition 1.4.** A continuous map $f\colon X \to Y$ is a *homotopy equivalence* if there is a continuous map $g\colon Y \to X$ such that $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$. When there exists a homotopy equivalence $f\colon X \to Y$, we say that $X$ and $Y$ are homotopy equivalent and sometimes write $X \simeq Y$.

Every homeomorphism is a homotopy equivalence, but not the other way round (as we will see in the examples below). We also stress that while the inverse of a homeomorphism is unique, a given map $f\colon X \to Y$ may admit many different $g\colon Y \to X$ such that $gf$ and $fg$ are homotopic to the identity.

To discuss examples, we introduce the notation

$$D^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq 1\}$$

for the $n$-dimensional disk and

$$S^n = \partial D^{n+1} = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

for the $n$-dimensional sphere (which may be viewed as the boundary of the $n+1$-disk).

**Example 1.5.** (i) Let $i = \mathrm{incl}_0 \colon \{0\} \to D^n$ be the inclusion of the origin. Then $i$ is a homotopy equivalence with inverse $r = \mathrm{const}_0 \colon D^n \to \{0\}$, the constant map with value 0. Indeed, we have $ri = \mathrm{id}_{\{0\}}$, and $ir$ is homotopic to $\mathrm{id}_{D^n}$ via the homotopy

$$H \colon D^n \times [0,1] \to D^n, \quad (x,t) \mapsto (1-t)x.$$

(ii) The inclusion $i = \mathrm{incl}_{S^n} \colon S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ is also a homotopy equivalence. To see this, consider

$$r \colon \mathbb{R}^{n+1} \setminus \{0\} \to S^n, \quad x \mapsto \frac{x}{\|x\|}.$$

This is a continuous map with $ri = \mathrm{id}_{S^n}$. A homotopy from $ir$ to id is given by

$$H \colon (\mathbb{R}^{n+1} \setminus \{0\}) \times [0,1] \to \mathbb{R}^{n+1} \setminus \{0\}, \quad (x,t) \mapsto (1-t)\frac{x}{\|x\|} + tx.$$

**Remark 1.6.** These two examples for homotopy equivalences are somewhat special in that one composite is the identity and the homotopy is constant on the image of the inclusion. In the terminology for this situation we will introduce later, the example shows that $\{0\}$ is a *deformation retract* of $D^n$ and that $S^n$ is a deformation retract of $\mathbb{R}^{n+1} \setminus \{0\}$.

We will now outline that homology helps to answer rather basic questions about homeomorphisms and homotopy equivalences. For example, one may ask:

(1) Can $S^m$ and $S^n$ be homotopy equivalent when $m \neq n$?

(2) Can $\mathbb{R}^m$ and $\mathbb{R}^n$ be homeomorphic when $m \neq n$?

The answer to these to questions turns out to be "no" in both cases. However, it is not clear how to come to this answer by just using the definition of a homeomorphism or a homotopy equivalence. The problem is that there are simply too many continuous maps between these spaces to check for every continuous map whether it is a homeomorphism or a homotopy equivalence. (The presence of continuous maps such as the surjective *Peano curve* $[0,1] \to [0,1] \times [0,1]$ highlights that continuous maps can get rather complicated.)

In the next lecture we will introduce *homology groups* $H_n(X;\mathbb{Z})$ for every topological space $X$ and integer $n \geq 0$. They have the following properties that we will establish in later lectures.

(i) A continuous map $f \colon X \to Y$ induces a group homomorphism $f_* \colon H_n(X;\mathbb{Z}) \to H_n(Y;\mathbb{Z})$.

(ii) For composable continuous maps $f \colon X \to Y$ and $g \colon Y \to Z$, we have $(gf)_* = g_* f_*$ as maps $H_n(X;\mathbb{Z}) \to H_n(Z;\mathbb{Z})$.

(iii) The identity $\mathrm{id} \colon X \to X$ induces the identity $H_n(X;\mathbb{Z}) \to H_n(X;\mathbb{Z})$.

(iv) If $f_0, f_1 \colon X \to Y$ are homotopic, then $(f_0)_* = (f_1)_*$ as maps $H_n(X;\mathbb{Z}) \to H_n(Y;\mathbb{Z})$.

(v) For $m, n \geq 1$, we have $H_n(S^m;\mathbb{Z}) \cong \mathbb{Z}$ if $m = n$ and $H_n(S^m;\mathbb{Z}) \cong 0$ if $m \neq n$.

Parts (i) to (iii) can be rephrased by saying that $H_n$ is a *functor* from topological spaces to abelian groups. Except (v), these properties of homology are analogous to those of the fundamental group, which satisfies (i) - (iv) for based maps and basepoint preserving homotopies.

The following important statement is a consequence of properties (i)-(iv).

**Corollary 1.7.** *A homotopy equivalence* $f\colon X \to Y$ *induces an isomorphism* $f_*\colon H_n(X;\mathbb{Z}) \to H_n(Y;\mathbb{Z})$.

*Proof.* Let $g\colon Y \to X$ be a homotopy inverse. Then $g_* f_* = (gf)_* = (\mathrm{id}_X)_* = \mathrm{id}$. Similarly, $f_* g_* = \mathrm{id}$. ☐

Together with property (v), we can now easily resolve the above questions (1) and (2). For simplicity, we ignore cases involving $S^0$, $\mathbb{R}^0$ or $\mathbb{R}^1$. (These can easily be dealt with using the set of path components.)

**Corollary 1.8.** *If* $f\colon S^m \to S^n$ *is a homotopy equivalence between positive dimensional spheres, then* $m = n$.

*Proof.* Since $f$ induces an isomorphism $H_n(S^m;\mathbb{Z}) \to H_n(S^n;\mathbb{Z})$, this follows from (iv). ☐

**Corollary 1.9.** *If* $m, n \geq 2$ *and* $f\colon \mathbb{R}^m \to \mathbb{R}^n$ *is a homeomorphism, then* $m = n$.

*Proof.* By composing $f$ with a translation, we may assume that $f(0) = 0$. In this case, $f$ induces a homeomorphism $\tilde{f}\colon \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$. Now let $l \geq 2$ and $k \geq 1$ be integers. Corollary 1.7, property (iv), and the homotopy equivalence $\mathrm{incl}\colon S^{l-1} \to \mathbb{R}^l \setminus \{0\}$ imply that $H_k(\mathbb{R}^l \setminus \{0\};\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ if $k = l-1$ and isomorphic to the trivial group otherwise. Thus $\tilde{f}$ can only induce an isomorphism $H_{m-1}(\mathbb{R}^m \setminus \{0\};\mathbb{Z}) \to H_{m-1}(\mathbb{R}^n \setminus \{0\};\mathbb{Z})$ if $m = n$. ☐

## Linearization

To prepare for the definition of homology groups, we introduce a linearization construction which produces abelian groups from sets.

**Definition 1.10.** Let $A$ be an abelian group and let $M$ be a set. Then

$$A[M] = \{f\colon M \to A \mid f^{-1}(A \setminus \{0\}) \text{ is finite }\}$$

is called the *$A$-linearization of $M$*.

The set $A[M]$ inherits an abelian group structure from the abelian group $(A, +, 0)$: For $f, g \in A[M]$, we define $f + g$ to be the function $M \to A$ given by $(f+g)(x) = f(x) + g(x)$. The map $\mathrm{const}_0\colon M \to A$ which has constant value 0 is the neutral element.

Informally, one may think of an element $f \in A[M]$ as formal sum

$$a_1 x_1 + \cdots + a_m x_m$$

where the $x_i$ run through the $x \in M$ with $f(x) \neq 0$ and $a_i$ is $f(x_i)$. Then the addition in $A[M]$ is given adding formal sums modulo the relation $ax + a'x = (a + a')x$ and modulo leaving out terms of the form $0\,x$.

Examples for $A$ to keep in mind are $\mathbb{Z}, \mathbb{Z}/2$, or $\mathbb{Q}$. If $k$ is a field and $k[M]$ is formed using the additive structure of $k$, then one can check that $k[M]$ has the structure of a $k$-vector space whose dimension is $|M|$ if $M$ is finite. (Here $|M|$ denotes the cardinality of $M$.)

We also note that the construction $A[M]$ is functorial in $M$. In other words, if $\gamma\colon M \to N$ is a map of sets, there is an induced group homomorphism

$$\gamma_*\colon A[M] \to A[N] \quad \text{with} \quad \gamma_*(f)(y) = \sum_{x \in \gamma^{-1}(y)} f(x).$$

That is, $\gamma_*$ takes $a_1 x_1 + \cdots + a_m x_m$ to $a_1 \gamma(x_1) + \cdots + a_m \gamma(x_m)$.

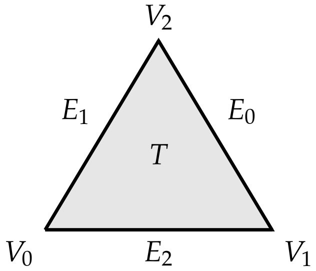

Figure 1.1: Filled triangle

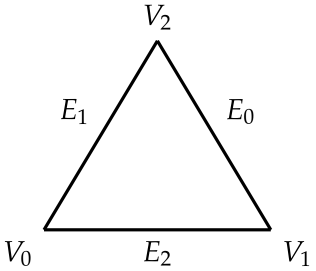

Figure 1.2: Boundary of triangle

### A motivating example for homology

Consider the triangle in Figure 1.1 with vertices $V_0, V_1, V_2$, with edges $E_0, E_1, E_2$, and with interior $T$. Our convention is that the edge $E_i$ lies opposite to the vertex $V_i$, and we will view the set of vertices as an ordered set with respect to the order indicated by the numbering.

The $\mathbb{Q}$-linearizations of the sets of vertices, edges, and interiors provides a sequence of $\mathbb{Q}$-vector spaces and vector space homomorphisms

$$0 \to \mathbb{Q}[\{T\}] \xrightarrow{\partial_2} \mathbb{Q}[\{E_0, E_1, E_2\}] \xrightarrow{\partial_1} \mathbb{Q}[\{V_0, V_1, V_2\}] \to 0.$$

In this diagram, $\partial_1$ and $\partial_2$ are the linear maps that send every face of the triangle to the alternating sum of its sub-faces. More explicitly, they are determined by

$$\partial_2(T) = E_0 - E_1 + E_2, \quad \partial_1(E_0) = V_2 - V_1, \quad \partial_1(E_1) = V_2 - V_0, \text{ and } \quad \partial_1(E_2) = V_1 - V_0.$$

The sum notation here refers to the "informal notation" for the linearization discussed above, and we also omit the coefficient 1 in $\mathbb{Q}$. For example, as an element of $\mathbb{Q}[\{E_0, E_1, E_2\}]$ the $E_0$ denotes the function $\{E_0, E_1, E_2\} \to \mathbb{Q}$ sending $E_0$ to 1 and both $E_1$ and $E_2$ to 0.

We emphasize that since $\partial_1$ and $\partial_2$ involve sums and negatives, they can only be defined on the linearizations of these sets and not on the sets themselves. This is the reason why we have to work with linearizations. (Of course, we could use any abelian group instead of $\mathbb{Q}$ in this example.)

Since

$$(\partial_1 \circ \partial_2)(T) = (V_2 - V_1) - (V_2 - V_0) + (V_1 - V_0) = 0$$

and $\mathbb{Q}[\{T\}]$ is one-dimensional with basis $T$, it follows that the composite $\partial_1 \circ \partial_2$ is the trivial map. In particular, $\operatorname{im} \partial_2 \subseteq \ker \partial_1$, and we can form the quotient vector space

$$\ker \partial_1 / \operatorname{im} \partial_2.$$

However, a simple dimension count shows that this quotient vector space is trivial: Since $\partial_2$ is not the trivial map and its source has dimension 1, we have $\dim \operatorname{im} \partial_2 = 1$ and hence $\dim \ker \partial_1$ is at least 1. Since the vectors $\partial_1(E_0)$ and $\partial_1(E_1)$ are linearly independent in the vector space $\mathbb{Q}[\{V_0, V_1, V_2\}]$, the image of $\partial_1$ has dimension at least 2. So the dimension formula

$$\dim \operatorname{im} \partial_1 + \dim \ker \partial_1 = \dim \mathbb{Q}[\{E_0, E_1, E_2\}] = 3$$

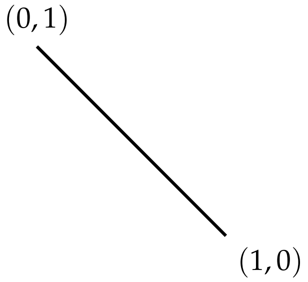

Figure 1.3: The 1-simplex $\Delta^1$

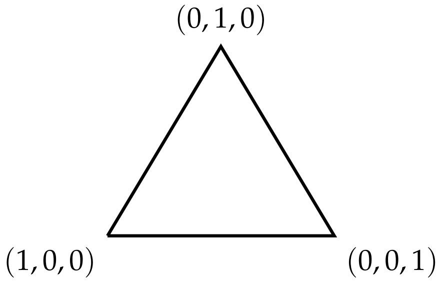

Figure 1.4: The 2-simplex $\Delta^2$

implies that $\dim\ker\partial_1 = 1$. This shows that $(\ker\partial_1)/(\operatorname{im}\partial_2)$ is the trivial vector space.

Although it may up to this point look as if this is only a complicated construction of a trivial vector space, something interesting happens if we remove the interior of our triangle and consider its boundary (as displayed in Figure 1.2). Applying the same construction as before, this leads to a sequence of vector spaces

$$0 \to 0 \xrightarrow{\partial_2'} \mathbb{Q}[\{E_0, E_1, E_2\}] \xrightarrow{\partial_1} \mathbb{Q}[\{V_0, V_1, V_2\}] \to 0$$

where $\partial_1$ is defined as above and $\partial_2'$ is the trivial map. And in this case, the quotient vector space

$$\ker\partial_1 / \operatorname{im}\partial_2'$$

is one dimensional.

The upshot of this discussion is that these quotient vector spaces can distinguish between a triangle and its boundary. This indicates that this type of construction is a candidate for an algebraic invariant of spaces.

## Simplices

Our aim is to generalize the previous example to an arbitrary topological space $X$ and to higher dimensional "triangles". The problem is that $X$ will usually not come with a "triangulation" that allows to consider vertices, edges, and (higher dimensional) triangles in $X$. In order to resolve this, we will construct a more general structure on $X$ by mapping generalized triangles to $X$. These generalized triangles are called "simplices".

**Definition 1.11.** For a natural number $n \geq 0$, we let

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum_{i=0}^n t_i = 1\}$$

be the *standard n-simplex.*

We view $\Delta^n$ as a topological space via the subspace topology inherited from $\mathbb{R}^{n+1}$. Inspecting the definition, we see that $\Delta^0$ is a one-point space, that $\Delta^1$ is homeomorphic to an interval, and that $\Delta^2$ is a ("filled") triangle. The last two cases are displayed in Figures (1.3) and (1.4). It is easy to see that the standard basis vectors $e_0, \dots, e_n$ of $\mathbb{R}^{n+1}$ lie in $\Delta^n$, and one can check that $\Delta^n$ is the convex hull of $\{e_0, \dots, e_n\}$.

$$\begin{tikzcd}
\Delta^0 \arrow[r, "\delta^1"] \arrow[d, "\delta^0"'] & \Delta^1 \arrow[d, "\delta^0"] \\
\Delta^1 \arrow[r, "\delta^2"] & \Delta^2
\end{tikzcd}$$

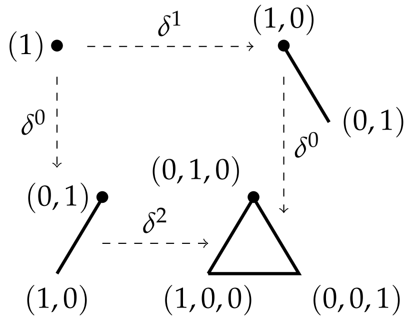

Figure 1.5: The relation $\delta^2\delta^0 = \delta^0\delta^1$. The • marks the images of $\Delta^0$ in $\Delta^1$ and in $\Delta^2$.

Any map of sets $\alpha\colon \{0,\dots,m\} \to \{0,\dots,n\}$ induces a continuous map

$$\alpha_*\colon \Delta^m \to \Delta^n, \quad (t_0,\dots,t_m) \mapsto \left( \sum_{i\in\alpha^{-1}(0)} t_i, \dots, \sum_{i\in\alpha^{-1}(n)} t_i \right).$$

This map $\alpha_*$ sends the basis vector $e_i$ to the basis vector $e_{\alpha(i)}$.

**Definition 1.12.** For natural numbers $n \geq 0$ and $0 \leq i \leq n$, we let

$$\delta_i\colon \{0,\dots,n-1\} \to \{0,\dots,n\}$$

be the unique order preserving injection that does not have $i$ in its image. The induced map of simplices $(\delta_i)_*\colon \Delta^{n-1} \to \Delta^n$ will also be denoted by $\delta_i$.

In order to avoid too many indices, we have not indicated the dimension of the source or the target in the notation for $\delta_i$. In particular, different choices for $n$ provide different maps that are denoted by $\delta_i$.

Explicitly, $\delta_i\colon \Delta^{n-1} \to \Delta^n$ is given by

$$\delta_i(t_0,\dots,t_{n-1}) = (t_0,\dots,t_{i-1},0,t_i,\dots,t_{n-1}).$$

Geometrically, it is the inclusion of the $n-1$-dimensional face of $\Delta^n$ that is opposite to the vertex $e_i$.

**Lemma 1.13.** *For $n \geq 2$ and $0 \leq j < i \leq n$, we have $\delta_i \circ \delta_j = \delta_j \circ \delta_{i-1}$ as maps $\Delta^{n-2} \to \Delta^n$.*

The case $n = 2$, $i = 2$ and $j = 0$ of the lemma is displayed in Figure (1.5).

*Proof.* Both composites are given by the map that is induced by the order preserving injection $\{0,\dots,n-2\} \to \{0,\dots,n\}$ that hits neither $j$ nor $i$. More explicitly, this is the map

$$(\underset{0}{t_0},\dots,\underset{n-2}{t_{n-2}}) \mapsto (\underset{0}{t_0},\dots,\underset{j-1}{t_{j-1}},\underset{j}{0},\underset{j+1}{t_j},\dots,\underset{i-1}{t_{i-2}},\underset{i}{0},\underset{i+1}{t_{i-1}},\dots,\underset{n}{t_{n-2}})$$

that inserts a 0 in the entries indexed by $i$ and $j$ and keeps the order of the remaining entries. (Here the numbers below the entries of these vectors indicate their position.) The index $i-1$ in the composite $\delta_j \circ \delta_{i-1}$ appears because the 0 inserted at the second position is shifted one step to the right when we afterwards insert a 0 at an earlier position. ☐

# Lecture 2: The definition of singular homology

We recall that for a natural number $n \geq 0$, we write

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum_{i=0}^{n} t_i = 1\}$$

for the *standard n-simplex* and view it as a subspace of $\mathbb{R}^{n+1}$. For each $n \geq 0$ and $0 \leq i \leq n$, there is a continuous map

$$\delta_i \colon \Delta^{n-1} \to \Delta^n, \quad (t_0, \dots, t_{n-1}) \mapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$$

that satisfy the relation $\delta_i \circ \delta_j = \delta_j \circ \delta_{i-1}$ for $n \geq 2$ and $0 \leq j < i \leq n$.

We can now define a notion of "generalized triangles" in a space $X$.

**Definition 2.1.** Let $X$ be a topological space and let $n \geq 0$ be a natural number. A *singular n-simplex* in $X$ is a continuous map $\sigma \colon \Delta^n \to X$, and

$$\mathcal{S}(X)_n = \{\sigma \colon \Delta^n \to X \mid \sigma \text{ is continuous }\}$$

denotes the set of singular $n$-simplices.

Since maps from a one-point space to $X$ correspond to points in $X$, the set of singular 0-simplices in $X$ can be identified with the underlying set of $X$. Since $\Delta^1$ is homeomorphic to an interval, the singular 1-simplices in $X$ are paths in $X$ (whose start and end points are not required to coincide). The 1-simplices can be thought of as generalized "edges" in $X$. In contrast to what one usually thinks of as "edges", it is allowed that a 1-simplex is the constant map to a point.

We also point out that the set of singular $n$-simplices is usually very big: While one can view the interval $[0, 1]$ as a space consisting of 2 vertices and one edge, the set of singular 1-simplices is uncountable since there uncountably many continuous functions $\Delta^1 \to [0, 1]$. The advantage of considering all singular 1-simplices is that we do not have to make any choices.

## The singular chain complex

We will now generalize the linearization construction from the motivating example to the singular simplices.

**Definition 2.2.** Let $X$ be a topological space and let $A$ be an abelian group. Then

$$C_n(X; A) = A[\mathcal{S}(X)_n]$$

is called the group of singular $n$-chains (with coefficients in $A$). Here $A[\mathcal{S}(X)_n]$ denotes the $A$-linearization of $\mathcal{S}(X)_n$ defined in the last lecture.

We note that any continuous map $f \colon X \to Y$ induces a map $\mathcal{S}(X)_n \to \mathcal{S}(Y)_n$ sending $\sigma \colon \Delta^n \to X$ to the composite $f \circ \sigma$. Hence there is an induced group homomorphism

$$f_* \colon C_n(X; A) \to C_n(Y; A).$$

One can check from the definitions that $C_n(-; A)$ defines a functor from the category of topological spaces to the category of abelian groups. This means that $\mathrm{id}_X$ induces the identity on $C_n(X; A)$, and that $(g \circ f)_* = g_* \circ f_*$ holds for composable morphisms $f$ and $g$.

**Definition 2.3.** Let $X$ be a topological space and let $A$ be an abelian group. We write

$$d_i\colon C_n(X;A) \to C_{n-1}(X;A)$$

for the group homomorphism which is induced by the map of sets

$$\mathcal{S}(X)_n \to \mathcal{S}(X)_{n-1}, \quad (\sigma\colon \Delta^n \to X) \mapsto \sigma \circ \delta_i.$$

As before, we have not indicated the dimension of the source or the target in the notation $d_i$.

We recall that if $A$ and $B$ are abelian groups, then the set of group homomorphisms from $A$ to $B$ inherits an abelian group structure. The sum of $h\colon A \to B$ and $k\colon A \to B$ is the map $h + k\colon A \to B$ given by $(h + k)(a) = h(a) + k(a)$, and the negative of $h\colon A \to B$ is the group homomorphism $-h\colon A \to B$ given by $(-h)(x) = -h(x)$. Since the linearization $C_n(X;A) = A[\mathcal{S}(X)_n]$ comes with an abelian group structure, this allows us to form the alternating sum over the maps $d_i$.

**Definition 2.4.** The *singular boundary operator* is the group homomorphism

$$\partial = \partial_n = \sum_{i=0}^{n} (-1)^i d_i\colon C_n(X;A) \to C_{n-1}(X;A).$$

**Lemma 2.5.** *We have $\partial_{n-1} \circ \partial_n = 0$ as maps $C_n(X;A) \to C_{n-2}(X;A)$.*

*Proof.* We first note that for $0 \leq j < i \leq n$, the identity $\delta_i \circ \delta_j = \delta_j \circ \delta_{i-1}$ established above implies that

$$(d_j \circ d_i)(\sigma) = \sigma \circ\ \delta_i \circ \delta_j = \sigma \circ \delta_j \circ \delta_{i-1} = (d_{i-1} \circ d_j)(\sigma)$$

for every singular $n$-simplex $\sigma\colon \Delta^n \to X$. Therefore, $d_j \circ d_i$ and $d_{i-1} \circ d_j$ coincide as maps $C_n(X,A) \to C_{n-2}(X;A)$, and the following calculation implies the claim:

$$\begin{aligned}
\partial_{n-1} \circ \partial_n &= \sum_{j=0}^{n-1} \sum_{i=0}^{n} (-1)^{i+j} d_j \circ d_i \\
&= \sum_{0 \leq i \leq j \leq n-1} (-1)^{i+j} d_j \circ d_i + \sum_{0 \leq j < i \leq n} (-1)^{i+j} d_j \circ d_i \\
&= \sum_{0 \leq i \leq j \leq n-1} (-1)^{i+j} d_j \circ d_i + \sum_{0 \leq j < i \leq n} (-1)^{i+j} d_{i-1} \circ d_j \\
&= \sum_{0 \leq i \leq j \leq n-1} (-1)^{i+j} d_j \circ d_i - \sum_{0 \leq j \leq i' \leq n-1} (-1)^{i'+j} d_{i'} \circ d_j \\
&= 0
\end{aligned}$$

Here we have set $i' = i - 1$ to get from line 3 to line 4. □

The abelian groups $C_n(X;A)$ together with the maps $\partial$ form the *singular chain complex* of $X$ with coefficients in $A$. By the previous lemma, this is indeed a chain complex in the following more general sense:

**Definition 2.6.** A *chain complex* $C$ is a sequence of abelian groups $C_n$ for $n \geq 0$ together with group homomorphisms $\partial = \partial_n\colon C_n \to C_{n-1}$ for $n \geq 1$ such that $\partial_{n-1} \circ \partial_n = 0$. Here $\partial$ is called the *differential*, elements of $C_n$ are called *n-chains*, elements of $\ker \partial_n$ are called *n-cycles*, and elements of $\operatorname{im} \partial_{n+1}$ are called *n-boundaries*.

## Singular homology groups

The relation $\partial_{n-1} \circ \partial_n = 0$ required in the definition of a chain complex does in particular imply that $\operatorname{im} \partial_{n+1} \subseteq \ker \partial_n$. (Or, in the above terminology, that all $n$-boundaries are $n$-cycles.) This enables us to state the next definition.

**Definition 2.7.** Let $C$ be a chain complex. Then the quotient group

$$H_n(C) = \ker \partial_n / \operatorname{im} \partial_{n+1}$$

is called the $n$-th homology group of $C$. This quotient is only defined for $n \geq 1$, and we set $H_0(C) = C_0 / \operatorname{im} \partial_1$ to also have a 0-th homology group.

We can now define our main object of study.

**Definition 2.8.** Let $X$ be a topological space, let $A$ be an abelian group, and let $n \geq 0$ be an integer. The *n-th singular homology group* of $X$ (with coefficients in $A$) is

$$H_n(X; A) = H_n(C(X; A)).$$

It is rarely possible to determine the homology groups of a space by directly working with the definition. In the subsequent lectures, we will develop various methods that will help us to make homology computations in examples.

For the moment, we only calculate the homology groups for the one-point space $*$. In this case, there is only one singular $n$-simplex $\sigma_n \colon \Delta^n \to *$ for every $n$.

**Lemma 2.9.** *Let $A$ be an abelian group. Then the composite of the canonical maps*

$$A \xrightarrow{\cong} A[\{\sigma_0\}] = C_0(*; A) \to H_0(*; A)$$

*is an isomorphism $A \xrightarrow{\cong} H_0(*; A)$. The group $H_n(*; A)$ is isomorphic to the trivial group if $n \geq 1$.*

*Proof.* We have isomorphisms $A \xrightarrow{\cong} A[\{\sigma_n\}] = C_n(*; A)$ for all $n$, and under these isomorphisms each $d_i$ is the identity map. This implies that

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \text{id} & \text{if } n \text{ is even} \end{cases}$$

It follows that $C(*; A)$ is isomorphic to the chain complex

$$\dots A \xrightarrow{0} A \xrightarrow{\text{id}} A \xrightarrow{0} A$$

which has a copy of $A$ in each degree and whose differentials alternate between the trivial map and the identity. This description implies that the composite of $A \xrightarrow{\cong} C_0(*; A) \to H_0(*; A)$ is an isomorphism. The homology groups $H_n(*; A)$ vanish for even $n \geq 2$ since $\ker \partial_n = 0$ in this case and a quotient of the trivial group is trivial. The homology groups $H_n(*; A)$ vanish for odd $n$ since in this case $\operatorname{im} \partial_{n+1} = \ker \partial_n$ and the quotient group is trivial. □

## Functoriality of homology groups

In order to explain how continuous maps induce group homomorphism in homology, we will use the following notion.

**Definition 2.10.** Let $C$ and $D$ be chain complexes. A sequence of group homomorphisms $g_n \colon C_n \to D_n$ is a *chain map* if the squares

$$\begin{tikzcd} C_n \arrow[r, "g_n"] \arrow[d, "\partial_n"'] & D_n \arrow[d, "\partial_n"] \\ C_{n-1} \arrow[r, "g_{n-1}"] & D_{n-1} \end{tikzcd}$$

commute for all $n \geq 1$. (Writing $\partial^C$ for the differential of $C$ and $\partial^D$ for the differential of $D$, this means that we require $\partial_n^D \circ g_n = g_{n-1} \circ \partial_n^C$ .)

The compatibility of the group homomorphisms and the differentials stated in this definition ensures that there are induced maps of homology groups:

**Lemma 2.11.** *Any chain map $g \colon C \to D$ induces group homomorphisms $H_n(C) \to H_n(D)$ for all $n \geq 0$.*

*Proof.* Let $x \in \ker \partial_n \subseteq C_n$ represent a class $[x] \in H_n(C) = \ker \partial_n / \operatorname{im} \partial_{n+1}$ in the quotient group. Then

$$\partial_n(g_n(x)) = g_{n-1}(\partial_n(x)) = g_{n-1}(0) = 0$$

implies that $g_n(x)$ is in the kernel of the differential of $D$. Next we show that the homology class of $g_n(x)$ in $H_n(D)$ does not depend on the choice of the representative of $[x]$. A different representative of $[x]$ is of the form $x + \partial_{n+1}(y)$ for some $y \in C_{n+1}$. Since

$$g_n(x + \partial_{n+1}(y)) = g_n(x) + \partial_{n+1}(g_{n+1}(y)),$$

the cycle $g_n(x + \partial_{n+1}(y))$ also represents $[g_n(x)]$. The fact that this defines a group homomorphism $H_n(C) \to H_n(D)$ is an immediate consequence of the definitions. □

**Lemma 2.12.** *Let $f \colon X \to Y$ be a continuous map of topological spaces and let $A$ be an abelian group. Then the induced group homomorphisms $f_* \colon C_n(X; A) \to C_n(Y; A)$ form a chain map.*

*Proof.* Recall that $\partial_n = \sum_{i=0}^n (-1)^i d_i$ where $d_i$ is the group homomorphism induced by the map of sets sending $\sigma$ to $\sigma \circ \delta_i$. This description implies that it is enough to verify that the square of sets

$$\begin{tikzcd} \mathcal{S}(X)_n \arrow[r] \arrow[d] & \mathcal{S}(Y)_n \arrow[d] \\ \mathcal{S}(X)_{n-1} \arrow[r] & \mathcal{S}(Y)_{n-1} \end{tikzcd}$$

commutes in which the horizontal maps are given by post-composition with $f$, while the vertical maps are given by pre-composition with $\delta_i$. The square commutes since we have $(f \circ \sigma) \circ \delta^i = f \circ (\sigma \circ \delta^i)$. □

**Corollary 2.13.** *Let $f \colon X \to Y$ be a continuous map of topological spaces and let $A$ be an abelian group. Then $f$ induces group homomorphisms $f_* \colon H_n(X; A) \to H_n(Y; A)$ for all $n \geq 1$.*

*Proof.* This follows by combining the last two lemmas. $\square$

The induced maps on homology are functorial, that is, we have $(\mathrm{id}_X)_* = \mathrm{id}_{H_n(X)}$ and $(g \circ f)_* = g_* \circ f_*$ for composable continuous maps $f\colon X \to Y$ and $g\colon Y \to Z$. In other words, $H_n(-;A)$ defines a functor from the category of topological spaces and continuous maps to the category of abelian groups and group homomorphisms.

### Singular homology of disjoint unions

Recall that a topological space $X$ is *path connected* if for every pair of points $x_0, x_1 \in X$, there exists a continuous map $f\colon [0,1] \to X$ with $f(0) = x_0$ and $f(1) = x_1$. A non-empty subspace $X' \subseteq X$ is a *path component* if $X'$ is path connected and if any path connected subspace $Y \subseteq X$ with $X' \subseteq Y$ satisfies $X' = Y$. The underlying set of every topological space $X$ can be written as the disjoint union of its path components.

The next lemma shows that the computation of homology groups can be carried out by considering one path component of a space at a time.

**Lemma 2.14.** *Let $X$ be a topological space, let $(X_j)_{j\in J}$ be the path components of $X$, and let $A$ be an abelian group. Then the inclusions $X_j \to X$ induce a natural isomorphism*

$$\bigoplus_{j\in J} H_n(X_j;A) \to H_n(X;A)$$

*for all $n$.*

*Proof.* Since the standard $n$-simplex is a path connected space, the image of every continuous map $\sigma\colon \Delta^n \to X$ lies in one of the path components of $X$. This implies that the inclusions $X_j \to X$ induce a bijection

$$\coprod_{j\in J} \mathcal{S}(X_j)_n \to \mathcal{S}(X)_n$$

from the disjoint union of the sets of singular $n$-simplices of the $X_j$ to the set of singular $n$-simplices of $X$. Since the $A$-linearization of a disjoint union is isomorphic to the direct sum of the $A$-linearizations, the inclusions $X_j \to X$ induce an isomorphism of abelian groups

$$\bigoplus_{j\in J} C_n(X_j;A) \to C_n(X;A).$$

Under these isomorphisms for varying $n$, the direct sum of the boundary operators of the chain complexes $C(X_j;A)$ is isomorphic to the boundary operator of $C(X;A)$. Hence the inclusions $X_j \to X$ induce an isomorphism of chain complexes

$$\bigoplus_{j\in J} C(X_j;A) \to C(X;A)\,.$$

By Exercise 2.2, this implies the isomorphism of homology group in the statement of the lemma. $\square$

The lemma also applies in the slightly more general situation where $(X_j)_{j\in J}$ is a decomposition of $X$ into a disjoint union of subsets $X_j$ such that each $X_j$ is a union of path components of $X$.

If we consider a set $M$ as a topological space with the discrete topology, then the lemma and our computation of $H_n(*; A)$ imply that $H_0(M; A) \cong \bigoplus_M A \cong A[M]$ and that $H_n(M; A) \cong 0$ if $n \geq 1$.

### Homology in degree 0

Let $X$ be a topological space. We write $\pi_0(X)$ for its set of path components and note that there is a canonical map of sets $X \to \pi_0(X)$ sending a point $x$ to its path component $[x]$. If $x \in X$ is a point, we write $\widetilde{x}$ for the singular 0-simplex with value $x$. Then $X \to \mathcal{S}(X)_0, x \mapsto \widetilde{x}$ defines a bijection of sets and thus induces an isomorphism $A[X] \to A[\mathcal{S}(X)_0] = C_0(X; A)$. Composing this isomorphism with the quotient map $C_0(X; A) \to H_0(X; A)$ provides a group homomorphism

$$\phi_X \colon A[X] \to H_0(X; A) \ .$$

**Lemma 2.15.** *This map factors as a composite of the homomorphism $A[X] \to A[\pi_0(X)]$ induced by the map $X \to \pi_0(X)$ and a homomorphism $\psi_X \colon A[\pi_0(X)] \to H_0(X; A)$.*

*Proof.* If $x_0$ and $x_1$ lie in the same path component of $X$, then there exists a singular one-simplex $\sigma \colon \Delta^1 \to X$ with $\sigma(0, 1) = x_0$ and $\sigma(1, 0) = x_1$. Hence for any $a \in A$, we have $\partial(a \cdot \sigma) = ad_0(\sigma) - ad_1(\sigma) = a \cdot \widetilde{x}_0 - a \cdot \widetilde{x}_1$. This implies that $\phi_X(ax_0) = \phi_X(ax_1)$. Thus $\phi_X$ factors over $A[X] \to A[\pi_0(X)]$ and induces a homomorphism $\psi_X \colon A[\pi_0(X)] \to H_0(X; A)$ by setting $\psi_X(a[x]) = \phi_X(ax)$ and extending $\psi_X$ linearly to arbitrary elements of $A[\pi_0(X)]$. □

Let $*$ denote the one-point space. The unique map $X \to *$ and the canonical isomorphism $H_0(*; A) \cong A$ induce a homomorphism $\varepsilon \colon H_0(X; A) \to H_0(*; A) \xrightarrow{\cong} A$ that is sometimes called an *augmentation*.

**Theorem 2.16.** *Let $X$ be a non-empty topological space. Then the homomorphism $\psi_X \colon A[\pi_0(X)] \to H_0(X; A)$ from the last lemma is an isomorphism. If $X$ is path-connected, then the augmentation $\varepsilon \colon H_0(X; A) \to A$ is also an isomorphism.*

*Proof.* First assume that $X$ is path connected and choose $x \in X$. We claim that the map $A \to H_0(X; A), a \mapsto \psi_X(a[x])$ is inverse to $\varepsilon$. Indeed, it follows directly from the definitions that $\varepsilon(\psi_X(a[x])) = a$, and $\psi_X(\varepsilon[c]) = [c]$ holds because Lemma 2.15 and the assumption that $X$ is path connected allow us to assume that the homology class of an arbitrary 0-cycle $c = a_1\widetilde{x_1} + \cdots + a_n\widetilde{x_n}$ is represented by $(a_1 + \cdots + a_n)\widetilde{x}$. Hence both $\psi_X$ and $\varepsilon$ are isomorphisms.

The statement about $\psi_X$ for general $X$ follows from Lemma 2.14 and the case of path connected spaces that we just established. □

# Lecture 3: The fundamental group and homology in degree 1

## Review of the fundamental group

We briefly review some basics about the *fundamental group* of a topological space $X$ with basepoint $x_0$. Writing $I = [0,1]$ for the unit interval, a path in $X$ is a continuous map $f\colon I \to X$, and a *loop* in $X$ is a path $f$ with $f(0) = f(1)$. The fundamental group is defined to be the set of equivalences classes

$$\pi_1(X, x_0) = \{f\colon I \to X \mid f \text{ is continuous and } f(0) = x_0 = f(1)\}/\sim$$

of loops in $X$ that start and end at the basepoint $x_0$, modulo the equivalence relation "basepoint preserving homotopy". This means that $f$ and $g$ are equivalent if and only if there exists a continuous map $H\colon I \times I \to X$ with $H(t,0) = f(t)$ and $H(t,1) = g(t)$ for all $t \in I$ as well as $H(0,t) = x_0 = H(1,t)$ for all $t \in I$.

For paths $f, g\colon I \to X$ with $f(1) = g(0)$ we set

$$(f * g)(t) = \begin{cases} f(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

In particular, this defines a composition of loops that start and end in $x_0$. This composition law is not associative. However, if $f$ and $g$ represent elements $[f]$ and $[g]$ in $\pi_1(X, x_0)$, then setting

$$[f] \cdot [g] = [f * g]$$

provides a well-defined group structure on the set $\pi_1(X, x_0)$. The unit is the class represented by the constant path with value $x_0$, and the inverse of $[f]$ is represented by the loop $f^{-1}$ given by $f^{-1}\colon I \to X, t \mapsto f(1-t)$.

## Fundamental group and first homology group with $\mathbb{Z}$-coefficients

For the rest of this lecture we restrict our attention to homology with coefficients in the abelian group $\mathbb{Z}$ and write $H_n(X)$ for $H_n(X;\mathbb{Z})$. There is a canonical map $\mathcal{S}(X)_n \to C_n(X;\mathbb{Z})$ sending a singular $n$-simplex $\sigma$ to the 1-chain $1 \cdot \sigma$. Here $1 \cdot \sigma$ is a notation for the element in $\mathbb{Z}[\mathcal{S}(X)_n]$ represented by the function that is 1 on $\sigma$ and 0 else. In the sequel, we will just write $\sigma$ for $1 \cdot \sigma$.

Moreover, we let $\iota\colon \Delta^1 \to I$ be the homeomorphism given by $\iota(1-t, t) = t$. Using $\iota$, every path $f\colon I \to X$ gives rise to a singular 1-simplex $f \circ \iota\colon \Delta^1 \to X$. We write again $\widetilde{x}_0$ for the constant 0-simplex with value $x_0$. If $f\colon I \to X$ is a loop that starts and ends at $x_0$, then the associated 1-chain $f \circ \iota$ is a 1-cycle since

$$\partial(f \circ \iota) = \widetilde{x}_0 - \widetilde{x}_0 = 0\,.$$

**Lemma 3.1.** *Setting $\phi([f]) = [f \circ \iota]$ provides a well-defined function $\phi\colon \pi_1(X, x_0) \to H_1(X)$.*

It is important to note that $[-]$ has two meanings here: While $[f]$ means "homotopy class of the loop $f$", the notation $[f \circ \iota]$ means "homology class of the 1-cycle $f \circ \iota$".

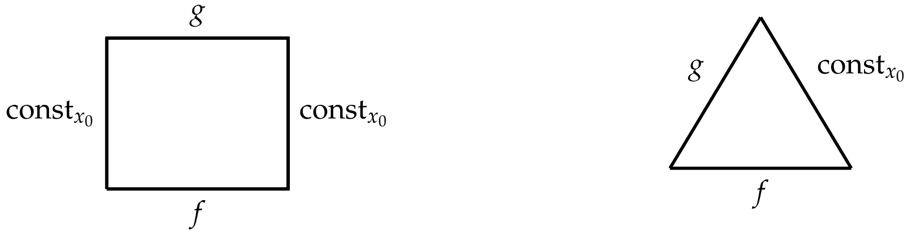

Figure 3.1: The homotopy $H$ and the singular simplex $\sigma$ in the proof of Lemma 3.1

*Proof.* Let $H\colon I \times I \to X$ be a basepoint preserving homotopy from $f\colon I \to X$ to $g\colon I \to X$. Since $H$ has the constant value $x_0$ on the subspace $\{0\} \times I$ of $I \times I$, it factors over the quotient space $(I \times I)/(\{0\} \times I)$ obtained by collapsing this subspace to the point. We write

$$H'\colon (I \times I)/(\{0\} \times I) \to X$$

for the induced map on the quotient. There exists a homeomorphism $(I \times I)/(\{0\} \times I) \to \Delta^2$ that maps

- $I \times \{0\}$ homeomorphically to the edge $\delta_1(\Delta^1) \subset \Delta^2$,
- $I \times \{1\}$ homeomorphically to the edge $\delta_2(\Delta^1) \subset \Delta^2$, and
- $\{1\} \times I$ homeomorphically to the edge $\delta_0(\Delta^1) \subset \Delta^2$;

compare also the discussion in the next remark. As outlined in Figure 3.1, composing $H'$ with this homeomorphism provides a singular 2-simplex $\sigma\colon \Delta^2 \to X$ with the property

$$\partial(\sigma) = (\mathrm{const}_{x_0} \circ \iota) - (f \circ \iota) + (g \circ \iota)\,.$$

Since $(\mathrm{const}_{x_0} \circ \iota)$ is the boundary of the constant 2-simplex with value $x_0$, it follows that the difference of the cycles $f \circ \iota$ and $g \circ \iota$ is in the image of $\partial\colon C_2(X;\mathbb{Z}) \to C_1(X;\mathbb{Z})$. Hence they represent the same homology class. □

**Remark 3.2.** The homeomorphism $(I \times I)/(\{0\} \times I) \to \Delta^2$ used in the last proof may be constructed as follows: Let $T \subset \mathbb{R}^2$ be the triangle spanned by the 3 vertices $(0,0), (0,1)$ and $(1,1)$. The continuous map $I \times I \to T, \quad (s,t) \mapsto (s, st)$ sends the subspace $\{0\} \times I$ to the point $(0,0)$ and thus induces a continuous map $(I \times I)/(\{0\} \times I) \to T$. The latter map can be checked to be a bijection, and it is a homeomorphism since its source is compact and its target is a Hausdorff space. Composing this homeomorphism with an affine linear map $T \to \Delta^2$ that sends $(0,0)$ to $(1,0,0)$, $(1,1)$ to $(0,1,0)$, and $(0,1)$ to $(0,0,1)$ provides the desired homeomorphism $(I \times I)/(\{0\} \times I) \to \Delta^2$.

In view of the argument given in the proof of the previous lemma, one can also wonder if collapsing the edge $\{1\} \times I$ in the square or using a different homeomorphism between the triangles $T$ and $\Delta^2$ would show that $f + g$ is a boundary. It turns out that this is not possible, and that $f + g$ will in general not be a boundary. The reason is that if we for example rotate the vertices of $\Delta^2$ clockwise, then the orientation of the edges labeled 0 and $f$ changes and our argument again shows that $-g \circ \iota - \mathrm{const}_{x_0} \circ \iota + f \circ \iota$ is a boundary.

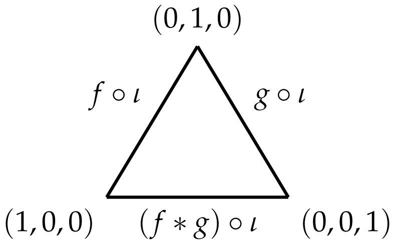

Figure 3.2: The singular simplex $\sigma$ in the proof of Lemma 3.3

**Lemma 3.3.** *Let $f, g\colon I \to X$ be paths in $X$ with $f(1) = g(0)$. Then the 1-chain $(g \circ \iota) + (f \circ \iota) - ((f * g) \circ \iota)$ is a boundary.*

*Proof.* In this situation, there is a singular simplex $\sigma\colon \Delta^2 \to X$ whose restriction to the edge

- $\delta_0(\Delta^1) \subset \Delta^2$ is $g \circ \iota$,
- $\delta_1(\Delta^1) \subset \Delta^2$ is $(f * g) \circ \iota$, and
- $\delta_2(\Delta^1) \subset \Delta^2$ is $f \circ \iota$.

The singular simplex $\sigma$ is obtained by requiring that $\sigma$ is constant on the intersections of $\Delta^2$ with lines that are orthogonal to the line through $(1,0,0)$ and $(0,0,1)$. It is displayed in Figure 3.2. By construction, we have

$$\partial(\sigma) = (g \circ \iota) - ((f * g) \circ \iota) + (f \circ \iota)\,.$$

□

**Corollary 3.4.** *The map $\phi\colon \pi_1(X, x_0) \to H_1(X)$ is a group homomorphism.*

*Proof.* The last lemma implies the equation

$$\phi([f]) + \phi([g]) = [g \circ \iota] + [f \circ \iota] = [(f * g) \circ \iota] = \phi([f] \cdot [g])$$

in $H_1(X)$. □

The lemma also implies that up to adding a boundary, the inverse of a singular 1-simplex in the abelian group structure of $C_1(X; \mathbb{Z})$ is obtained by “reversing its orientation”:

**Corollary 3.5.** *Let $\omega \in \mathcal{S}(X)_1$ be a singular 1-simplex, and let $\bar{\omega} \in \mathcal{S}(X)_1$ be the singular 1-simplex given by $\bar{\omega}(t, 1-t) = \omega(1-t, t)$. Then $\omega + \bar{\omega}$ is a boundary, that is, there 2-chain $c \in C_2(X; \mathbb{Z})$ with $\partial(c) = \omega + \bar{\omega}$.*

*Proof.* Applying Lemma 3.3 to $f = \omega \circ \iota^{-1}$ and $g = \bar{\omega} \circ \iota^{-1}$ shows that

$$\omega + \bar{\omega} - [((\omega \circ \iota^{-1}) * (\bar{\omega} \circ \iota^{-1})) \circ \iota]$$

is a boundary. By construction, the path $(\omega \circ \iota^{-1}) * (\bar{\omega} \circ \iota^{-1})$ is a contractible closed loop that starts and ends at $x_0 = \omega(1,0) = (\omega \circ \iota^{-1})(0)$. Hence we know from the previous corollary that

$$[((\omega \circ \iota^{-1}) * (\bar{\omega} \circ \iota^{-1})) \circ \iota] = \phi([(\omega \circ \iota^{-1}) * (\bar{\omega} \circ \iota^{-1})]) = \phi([\mathrm{const}_{\omega(1,0)}]) = 0$$

in $H_1(X)$. Thus $[((\omega \circ \iota^{-1}) * (\bar{\omega} \circ \iota^{-1})) \circ \iota]$ is a boundary. □

One may wonder how close the map $\phi\colon \pi_1(X, x_0) \to H_1(X)$ is to being an isomorphism. There are two obvious obstacles to this. The first restriction is that since that the fundamental group with respect to the basepoint $x_0$ only detects information about the path component of $x_0$, we can only expect $\phi$ to detect information if $X$ is path connected. The second restriction comes from the fact that $H_1(X)$ is an *abelian* group, while $\pi_1(X, x_0)$ can be non-abelian. (In fact, one can show that *every* group arises as the fundamental group of a topological space.) In some sense, the next theorem states that these are the only two reasons that keep $\phi$ from being an isomorphism.

To phrase this result, we recall an important notion from group theory. If $G$ is a group, then $[G, G]$ is defined to be the smallest normal subgroup of $G$ that contains all elements of the form $ghg^{-1}h^{-1}$ for $g, h \in G$. This subgroup is called the *commutator subgroup* of $G$. The quotient group $G^{\mathrm{ab}} = G/[G, G]$ is abelian, and it has the property that any group homomorphism $\alpha\colon G \to A$ to an abelian group factors as

$$\begin{tikzcd}
G \arrow[r, "\alpha"] \arrow[d] & A \\
G^{\mathrm{ab}} \arrow[ur, dashed, "\alpha^{\mathrm{ab}}"'] &
\end{tikzcd}$$

with a unique group homomorphism $\alpha^{\mathrm{ab}}$.

Applying this to the group homomorphism $\phi\colon \pi_1(X, x_0) \to H_1(X)$ from the last lemma provides a homomorphism

$$\phi^{\mathrm{ab}}\colon \pi_1(X, x_0)^{\mathrm{ab}} \to H_1(X) .$$

**Theorem 3.6.** *Let $X$ be a path connected space. Then $\phi^{\mathrm{ab}}$ is an isomorphism.*

Before proving the theorem, we note that it implies several useful computations of first homology groups. For example, $\pi_1(S^1, x_0) \cong \mathbb{Z}$ implies that $H_1(S^1) \cong \mathbb{Z}$. Moreover, the fact that $\pi_1(S^n, x_0) \cong 0$ for $n > 1$ implies that $H_1(S^n) \cong 0$ for $n > 1$. The real projective space $\mathbb{R}P^n$ is the space all 1-dimensional sub vector spaces in $\mathbb{R}^{n+1}$. One can show that $\mathbb{R}P^n$ has fundamental group $\mathbb{Z}/2$ if $n > 1$. Hence we have an isomorphism $H_1(\mathbb{R}P^n) \cong \mathbb{Z}/2$ in this case.

*Proof of Theorem 3.6.* As a first step, we choose a path $\ell_x$ from $x$ to $x_0$ for every point $x \in X$. If $x = x_0$, we require that $\ell_x$ is the constant path.

Let $\omega\colon \Delta^1 \to X$ be a singular 1-simplex. By the definition of $\iota\colon \Delta^1 \to I$, the path $\omega \circ \iota^{-1}$ starts at $(\omega \circ \iota^{-1})(0) = \omega(1, 0)$ and ends at $(\omega \circ \iota^{-1})(1) = \omega(0, 1)$. Hence we obtain a closed loop that starts and ends at $x_0$ if we first run through the inverse of the path $\ell_{\omega(1,0)}$, then through the path $\omega \circ \iota^{-1}$, and lastly through the path $\ell_{\omega(0,1)}$ to get back to $x_0$. Since our convention for the composition of paths is to go through paths from the left to the right, we denote the homotopy class of this path by

$$[\ell^{-1}_{\omega(1,0)} * (\omega \circ \iota^{-1}) * \ell_{\omega(0,1)}] .$$

(Because the operation $*$ is not associative, it is slightly imprecise to apply it to more than two paths without indicating which paths to compose first. However, as long as we are only interested in the resulting homotopy class, this makes no difference.)

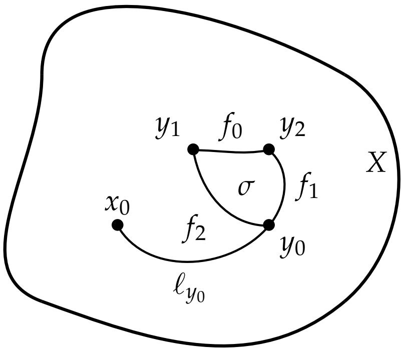

Figure 3.3: The path built from $\sigma$ in the proof of Theorem 3.6

By composing with the quotient map $\pi_1(X,x_0) \to \pi_1(X,x_0)^{\mathrm{ab}}$, we may view the this loop as an element of $\pi_1(X,x_0)^{\mathrm{ab}}$. Altogether, this defines a function

$$\widetilde{\psi}\colon \mathcal{S}(X)_1 \to \pi_1(X,x_0)^{\mathrm{ab}}, \qquad \omega \mapsto [\ell^{-1}_{\omega(1,0)} * (\omega \circ \iota^{-1}) * \ell_{\omega(0,1)}] \,.$$

Since every map of sets from $\mathcal{S}(X)_1$ to an abelian group extends over the $\mathbb{Z}$-linearization of $\mathcal{S}(X)_1$ in a unique way, we obtain an induced map

$$\bar{\psi}\colon \mathbb{Z}[\mathcal{S}(X)_1] \to \pi_1(X,x_0)^{\mathrm{ab}}.$$

We now show that $\bar{\psi}$ takes the image of $\partial_2\colon C_2(X;\mathbb{Z}) \to C_1(X;\mathbb{Z}) = \mathbb{Z}[\mathcal{S}(X)_1]$ to the class of the constant loop in $\pi_1(X,x_0)^{\mathrm{ab}}$. For this we choose a 2-simplex $\sigma\colon \Delta^2 \to X$. For $0 \leq k \leq 2$, we write $y_k = \sigma(e_k)$ for the image of the vertex $e_k$ of $\Delta^2$ under $\sigma$ and $f_k = \sigma \circ \delta_k \circ \iota^{-1}$ for the path in $X$ that is given by restricting $\sigma$ to its $k$-th edge. This means that $f_0$ is a path from $y_1$ to $y_2$, that $f_1$ is a path from $y_0$ to $y_2$, and that $f_2$ is a path from $y_0$ to $y_1$ (see Figure 3.3).

Since the simplex $\sigma$ provides a null homotopy for the path $f_2 * f_0 * f_1^{-1}$, we have

$$\begin{aligned}
\bar{\psi}(\partial\sigma) &= \widetilde{\psi}(\sigma \circ \delta_0) - \widetilde{\psi}(\sigma \circ \delta_1) + \widetilde{\psi}(\sigma \circ \delta_2) = \widetilde{\psi}(\sigma \circ \delta_2) + \widetilde{\psi}(\sigma \circ \delta_0) - \widetilde{\psi}(\sigma \circ \delta_1) \\
&= [\left(\ell^{-1}_{y_0} * f_2 * \ell_{y_1}\right) * \left(\ell^{-1}_{y_1} * f_0 * \ell_{y_2}\right) * \left(\ell^{-1}_{y_0} * f_1 * \ell_{y_2}\right)^{-1}] \\
&= [\ell^{-1}_{y_0} * (f_2 * f_0 * f_1^{-1}) * \ell_{y_0}] = [\ell^{-1}_{y_0} * \ell_{y_0}] = [\mathrm{const}_{x_0}] \,.
\end{aligned}$$

It follows that $\bar{\psi}$ induces a map $\psi\colon H_1(X) \to \pi_1(X,x_0)^{\mathrm{ab}}$. We claim that this map is inverse to $\phi^{\mathrm{ab}}$. The composite

$$\pi_1(X,x_0)^{\mathrm{ab}} \xrightarrow{\phi^{\mathrm{ab}}} H_1(X) \xrightarrow{\psi} \pi_1(X,x_0)^{\mathrm{ab}}$$

is the identity since our assumption $\ell_{x_0} = x_0$ implies that the 1-chain represented by a closed loop $f$ that starts and ends at $x_0$ is mapped to the class of $f$.

It remains to analyze the composite

$$H_1(X) \xrightarrow{\psi} \pi_1(X,x_0)^{\mathrm{ab}} \xrightarrow{\phi^{\mathrm{ab}}} H_1(X).$$

By Lemma 3.7 below and the fact that $\psi$ and $\phi^{\mathrm{ab}}$ are group homomorphisms, it is enough to check this for a homology class represented by a 1-simplex $\omega \in \mathcal{S}(X)_1$ with $\omega(1,0) = \omega(0,1)$. In this case we note that

$$[(\ell_{\omega(1,0)}^{-1} * \ell_{\omega(1,0)}) \circ \iota] = \phi([(\ell_{\omega(1,0)}^{-1} * \ell_{\omega(1,0)})]) = 0$$

since $\ell_{\omega(1,0)}^{-1} * \ell_{\omega(1,0)}$ is contractible and $\phi$ is a group homomorphism. Using this, Lemma 3.3 implies

$$\begin{aligned}\phi^{\mathrm{ab}}(\psi([\omega]) &= \phi^{\mathrm{ab}}(\widetilde{\psi}(\omega)) = \phi^{\mathrm{ab}}([\ell_{\omega(1,0)}^{-1} * (\omega \circ \iota^{-1}) * \ell_{\omega(0,1)}]) \\ &= [(\ell_{\omega(1,0)}^{-1} * (\omega \circ \iota^{-1}) * \ell_{\omega(0,1)}) \circ \iota] = [\ell_{\omega(1,0)}^{-1} \circ \iota] + [\omega] + [\ell_{\omega(0,1)} \circ \iota] \\ &= [\omega] + [(\ell_{\omega(1,0)}^{-1} * \ell_{\omega(1,0)}) \circ \iota] = [\omega].\end{aligned}$$

Altogether, we have shown that $\phi^{\mathrm{ab}}$ is an isomorphism. □

The following lemma was used in the previous proof:

**Lemma 3.7.** *Let $X$ be a topological space. Every homology class in $[c] \in H_1(X)$ is represented by a 1-chain of the form $\omega_1 + \cdots + \omega_k$ where the $\omega_i \in \mathcal{S}(X)_1$ are 1-simplices with $\omega_i(1,0) = \omega_i(0,1)$.*

*Proof.* Using Corollary 3.5, we can get rid of negative entries in the representation of $c$ as a linear combination of singular 1-simplices and assume that $c$ has the form $\omega_1 + \cdots + \omega_k$ with $\omega_i \in \mathcal{S}(X)_1$. Now we argue by induction over $k$: If $\omega_i(0,1) \neq \omega_i(1,0)$ for some $i$, then $\partial(c) = 0$ implies that there is a $j \neq 0$ with $\omega_j(1,0) = \omega_i(0,1)$, and applying Lemma 3.3 enables us to represent $\omega_i + \omega_j$ by a single 1-simplex. □

# Lecture 4: Relative homology groups

In this lecture we construct the relative homology groups of a pair of spaces $X' \subset X$ and the long exact sequence in homology.

## Exact sequences

We begin by reviewing basic results about exact sequences.

**Definition 4.1.** A sequence of abelian groups and group homomorphisms

$$\dots \xrightarrow{f_{i+2}} A_{i+1} \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} A_{i-1} \xrightarrow{f_{i-1}} \dots$$

is *exact at $A_i$* if $\operatorname{im} f_{i+1} = \ker f_i$. It is *exact* if it is exact at every $A_i$.

**Example 4.2.** A chain complex $C$ is exact at $C_n$ if and only if $H_n(C) \cong 0$.

**Example 4.3.** Let $f\colon A \to B$ be a homomorphism of abelian groups.

(i) The sequence $A \xrightarrow{f} B \to 0$ is exact if and only if $f$ is surjective.

(ii) The sequence $0 \to A \xrightarrow{f} B$ is exact if and only if $f$ is injective.

**Example 4.4.** One of our most important application of exact sequences will be the following observation: Suppose we are given an exact sequence

$$A_{i+1} \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} A_{i-1} \xrightarrow{f_{i-1}} A_{i-2}$$

in which $A_{i+1}$ and $A_{i-2}$ are trivial groups. Then the previous example implies that $f_i$ is both injective and surjective, and therefore an isomorphism. Hence the triviality of appropriate groups in an exact sequence can be used to detect whether a map is an isomorphism.

**Example 4.5.** One can also use the exactness of a sequence to show that certain abelian groups are trivial: If

$$A_{i+1} \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} A_{i-1} \xrightarrow{f_{i-1}} A_{i-2} \xrightarrow{f_{i-2}} A_{i-3}$$

is an exact sequence with $f_{i+1}$ a surjective map and $f_{i-2}$ an injective map, then $A_{i-1}$ is isomorphic to the trivial group. To see this, choose $x \in A_{i-1}$. Then $f_{i-2}(f_{i-1}(x)) = 0$ by exactness at $A_{i-1}$. Since $f_{i-2}$ is injective, $f_{i-1}(x) = 0$. Exactness at $A_{i-1}$ implies that there exists an element $y \in A_i$ with $f_i(y) = x$. Since $f_{i+1}$ is surjective, there is an element $z \in A_{i+1}$ with $f_{i+1}(z) = y$. Exactness at $A_i$ implies our claim $x = f_i(f_{i+1}(z)) = 0$.

The following statement, known as the *Five Lemma*, is a frequently used tool when arguing with exact sequences.

**Lemma 4.6.** *Let*

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"'] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"'] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"'] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r, "\beta_1"'] & B_2 \arrow[r, "\beta_2"'] & B_3 \arrow[r, "\beta_3"'] & B_4 \arrow[r, "\beta_4"'] & B_5
\end{tikzcd}$$

*be a commutative diagram of abelian groups and group homomorphisms in which both rows are exact. Then the following statements hold:*

*(i) If $f_2$ and $f_4$ are injective and $f_1$ is surjective, then $f_3$ is injective.*

*(ii) If $f_2$ and $f_4$ are surjective and $f_5$ is injective, then $f_3$ is surjective.*

*(iii) If $f_2$ and $f_4$ are isomorphisms, $f_1$ is surjective, and $f_5$ is injective, then $f_3$ is an isomorphism.*

*Proof.* This is Exercise 3.1. □

## The long exact sequence in homology

Exact sequences of the shape $0 \to \bullet \to \bullet \to \bullet \to 0$ play an important role and have a separate name:

**Definition 4.7.** An exact sequence of abelian groups and group homomorphisms of the form

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] & A \arrow[r] & \overline{A} \arrow[r] & 0 \end{tikzcd}$$

is called a *short exact sequence*. A sequence of chain complexes and chain maps

$$\begin{tikzcd} 0 \arrow[r] & C' \arrow[r] & C \arrow[r] & \overline{C} \arrow[r] & 0 \end{tikzcd}$$

is called a *short exact sequence* if it is levelwise short exact, that is, if

$$\begin{tikzcd} 0 \arrow[r] & C'_n \arrow[r] & C_n \arrow[r] & \overline{C}_n \arrow[r] & 0 \end{tikzcd}$$

is a short exact sequence of abelian groups for all $n$.

**Example 4.8.** Short exact sequences of chain complexes arise in the following way: Given a chain map $i\colon C' \to C$ such that each $i_n\colon C'_n \to C_n$ is injective, then there is a quotient complex $\overline{C} = C/C'$ given by $\overline{C}_n = C_n/\operatorname{im} i_n$. It comes with a canonical chain map $C \to \overline{C}$ (compare Exercise 3.3) that together with $i$ gives a short exact sequence.

We consider a short exact sequence of chain complexes

$$\begin{tikzcd} 0 \arrow[r] & C' \arrow[r, "i"] & C \arrow[r, "p"] & \overline{C} \arrow[r] & 0\,. \end{tikzcd}$$

For the next construction, it is advisable to consult Figure (4.1) that displays the relevant individual levels of this short exact sequence.

Let $n \geq 1$, let $[\overline{x}] \in H_n(\overline{C})$ be a homology class, let $\overline{x} \in \ker \overline{\partial}_n \subseteq \overline{C}_n$ be a representing element, and let $x \in C_n$ be a preimage of $\overline{x}$ under the surjective map $p_n\colon C_n \to \overline{C}_n$. Then $p_{n-1}(\partial_n(x)) = \overline{\partial}_n(p_n(x)) = \overline{\partial}_n(\overline{x}) = 0$ since $p$ is a chain map and $\overline{x}$ lies in the kernel of $\overline{\partial}_n$. By exactness at $C_{n-1}$, this implies that $\partial_n(x)$ is in the image of $i_{n-1}\colon C'_{n-1} \to C_{n-1}$, and we choose $x' \in C'_{n-1}$ with $i_{n-1}(x') = \partial_n(x)$. Now we consider the map

$$\delta\colon H_n(\overline{C}) \to H_{n-1}(C'), \quad [\overline{x}] \mapsto [x']\,.$$

**Lemma 4.9.** *This map is a well-defined group homomorphism that is natural in the short exact sequence.*

$$\begin{tikzcd}
 & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \\
0 \arrow[r] & C'_{n+1} \arrow[r, "i_{n+1}"] \arrow[d, "\partial'_{n+1}"'] & C_{n+1} \arrow[r, "p_{n+1}"] \arrow[d, "\partial_{n+1}"'] & \overline{C}_{n+1} \arrow[r] \arrow[d, "\overline{\partial}_{n+1}"] & 0 \\
0 \arrow[r] & C'_{n} \arrow[r, "i_{n}"] \arrow[d, "\partial'_{n}"'] & C_{n} \arrow[r, "p_{n}"] \arrow[d, "\partial_{n}"'] & \overline{C}_{n} \arrow[r] \arrow[d, "\overline{\partial}_{n}"] & 0 \\
0 \arrow[r] & C'_{n-1} \arrow[r, "i_{n-1}"] \arrow[d, "\partial'_{n-1}"'] & C_{n-1} \arrow[r, "p_{n-1}"] \arrow[d, "\partial_{n-1}"'] & \overline{C}_{n-1} \arrow[r] \arrow[d, "\overline{\partial}_{n-1}"] & 0 \\
0 \arrow[r] & C'_{n-2} \arrow[r, "i_{n-2}"] \arrow[d] & C_{n-2} \arrow[r, "p_{n-2}"] \arrow[d] & \overline{C}_{n-2} \arrow[r] \arrow[d] & 0 \\
 & \cdots & \cdots & \cdots &
\end{tikzcd}$$

Figure 4.1: A short exact sequence of chain complexes

*Proof.* For $[x']$ to be a homology class, we have to first check that $\partial'_{n-1}(x') = 0$. Since $i_{n-2}$ is injective, this follows from

$$i_{n-2}(\partial'_{n-1}(x')) = \partial_{n-1}(i_{n-1}(x')) = \partial_{n-1}(\partial_n(x)) = 0\,.$$

Next we assume that $y \in C_n$ is another element with $p_n(y) = \overline{x}$. Then $p_n(y-x) = p_n(y) - p_n(x) = 0$ and thus there exists $z \in C'_n$ with $i_n(z) = y - x$. Then

$$i_{n-1}(x' + \partial'_n(z)) = \partial_n(x) + \partial_n(y-x) = \partial_n(y)\,.$$

Thus $x' + \partial'_n(z)$ is the unique preimage of $\partial(y)$ under the injection $i_{n-1}$. By the definition of homology as a quotient, $x' + \partial'_n(z)$ and $x'$ represent the same homology class in $H_{n-1}(C')$. So we have shown that our map $\delta$ does not depend on the choice of the preimage of $\overline{x}$ and of $p_n$.

Now we let $\overline{y} \in \ker \overline{\partial}_n \subseteq \overline{C}_n$ be another element representing $[\overline{x}] \in H_n(\overline{C})$. Then there is a $\overline{z} \in \overline{C}_{n+1}$ with $\overline{\partial}_{n+1}(\overline{z}) = \overline{y} - \overline{x}$. Using the surjectivity of $p_{n+1}$, we choose $z \in C_{n+1}$ with $p_{n+1}(z) = \overline{z}$. Then

$$p_n(x + \partial_{n+1}(z)) = \overline{x} + \overline{\partial}_{n+1}(p_{n+1}(z)) = \overline{y}\,.$$

Thus $x + \partial_{n+1}(z))$ is a lift of $y$ along $p_n$. Since $\partial_n(x + \partial_{n+1}(z)) = \partial_n(x)$, they give rise to the same element in $C_{n-1}$ and thus also to the same cycle in $C'_{n-1}$. Hence the definition of $\delta$ does not depend on the choice of the representing cycle of $[\overline{x}]$.

The map is a group homomorphism because given homology classes $[\overline{x}_1]$ and $[\overline{x}_2]$, the choices of elements $\overline{x}_i$, $x_i$, $x'_i$ for $i = 1, 2$ as above provide choices $\overline{x}_1 + \overline{x}_2$, $x_1 + x_2$ and $x'_1 + x'_2$ for the sum $[\overline{x}_1] + [\overline{x}_2]$.

The naturality claim is that if

$$\begin{tikzcd}
0 \arrow[r] & C' \arrow[r, "i"] \arrow[d, "f"'] & C \arrow[r, "p"] \arrow[d, "g"'] & \overline{C} \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & D' \arrow[r, "j"] & D \arrow[r, "q"] & \overline{D} \arrow[r] & 0
\end{tikzcd}$$

is a diagram of chain complexes and chain maps with exact rows, then the resulting square

$$\begin{tikzcd}
H_n(\overline{C}) \arrow[r, "\delta"] \arrow[d, "h_*"'] & H_{n-1}(C') \arrow[d, "f_*"] \\
H_n(\overline{D}) \arrow[r, "\delta"] & H_{n-1}(D')
\end{tikzcd}$$

commutes. This follows from the observation that the choices of the elements $\overline{x}, x$ and $x'$ for the map $\delta$ in the upper row provide choices $h_n(\overline{x})$, $g_n(x)$ and $f_{n-1}(x')$ of the corresponding elements for the map $\delta$ associated with the lower row.

Lastly, we point out that for our arguments to be also valid in the case $n = 1$, we let $C'_{-1}$, $C_{-1}$ and $\overline{C}_{-1}$ be the trivial groups. ☐

**Definition 4.10.** The map $\delta\colon H_n(\overline{C}) \to H_{n-1}(C')$ is called the *connecting homomorphism* of the short exact sequence of chain complexes.

**Theorem 4.11.** *Let*

$$0 \longrightarrow C' \xrightarrow{i} C \xrightarrow{p} \overline{C} \longrightarrow 0 .$$

*be a short exact sequence of chain complexes. Then the following sequence is exact:*

$$\begin{aligned}\cdots \to H_n(C) \xrightarrow{p_*} H_n(\overline{C}) \xrightarrow{\delta} H_{n-1}(C') \xrightarrow{i_*} H_{n-1}(C) \to \dots \\ \cdots \to H_0(C') \xrightarrow{i_*} H_0(C) \xrightarrow{p_*} H_0(\overline{C}) \to 0 . \quad (4.1)\end{aligned}$$

*Proof.* For this proof it is again advisable to consult Figure (4.1).

We start with exactness at $H_{n-1}(C')$. If $[x'] \in H_{n-1}(C')$ is in the image of the connecting homomorphism $\delta\colon H_n(\overline{C}) \to H_{n-1}(C')$, then the definition of $\delta$ implies that $i_{n-1}(x') = \partial_n(x)$ for a some element $x \in C_n$. Thus

$$i_*([x']) = [i_{n-1}(x')] = [\partial(x)] = 0 .$$

If $x' \in C'_{n-1}$ is a cycle such that $i_*([x']) = 0$, then $i_{n-1}(x')$ is a boundary in $C_n$. Thus there is an $x \in C_n$ with $\partial_n(x) = i_{n-1}(x')$. We set $\overline{x} = p_n(x)$. Then we have

$$\overline{\partial}_n(\overline{x}) = \overline{\partial}_n(p_n(x)) = p_{n-1}(\partial_n(x)) = p_{n-1}(i_{n-1}(x')) = 0 .$$

Hence $\overline{x}$ represents a homology class $[\overline{x}] \in H_n(\overline{C})$, and by definition of $\delta$ we have $\delta([\overline{x}]) = [x']$. So we have shown that the sequence is exact at $H_{n-1}(C')$.

Next we treat the exactness at $H_n(\overline{C})$. First we show that the composite $\delta \circ p_*$ is trivial, which is equivalent to $\operatorname{im} p_* \subseteq \ker \delta$. For this, we let $x \in \ker \partial_n \subseteq C_n$ be a cycle. Then $p_n(x)$ represents the homology class $p_*([x])$, and $x$ itself is an element in the preimage of $p_n(x)$ under $p_n$. Since $\partial_n(x) = 0$, it follows from the definition of $\delta$ that $\delta(p_*([x])) = 0$. Next we let $\overline{x} \in \ker \overline{\partial}_n \subseteq \overline{C}_n$ be a cycle such that $[\overline{x}]$ is in the kernel of $\delta$. Let $x \in C_n$ be an element with $p_n(x) = \overline{x}$ and let $x' \in C'_{n-1}$ be an element with $i_{n-1}(x') = \partial_n(x)$. Since $\delta([\overline{x}]) = [x']$ by definition, our assumption $\delta([\overline{x}]) = 0$ implies that there is an element $y' \in C'_n$ with $\partial'_n(y') = x'$. Since $i$ is a chain map,

$$\partial_n(i_n(y')) = i_{n-1}(\partial'_n(y')) = i_{n-1}(x') = \partial_n(x) .$$

Then $\partial_n(x - i_n(y')) = 0$ and $p_n(x - i_n(y')) = p_n(x) = \overline{x}$. Hence $x - i_n(y')$ represents a homology class $[x - i_n(y')] \in H_n(C)$ with $p_*([x - i_n(y')] = [\overline{x}]$.

The last case is exactness at $H_n(C)$. The composite $p_* \circ i_*$ is trivial since for a cycle $x' \in C'_n$, we have $(p_* \circ i_*)[x'] = [p_n(i_n(x'))] = [0] = 0$. This shows that $\operatorname{im} i_* \subseteq \ker p_*$. Now let $x \in \ker \partial_n \subseteq C_n$ be a cycle with $p_*([x]) = 0$. Then there exists a $\overline{y} \in \overline{C}_{n+1}$ with $\overline{\partial}_{n+1}(\overline{y}) = p_n(x)$. Let $y \in C_{n+1}$ be an element with $p_{n+1}(y) = \overline{y}$. Then the cycle $x - \partial_{n+1}(y)$ also represents $[x]$ and has the property

$$p_n(x - \partial_{n+1}(y)) = p_n(x) - p_n(\partial_{n+1}(y)) = p_n(x) - \partial_{n+1}(p_{n+1}(y)) = 0\,.$$

Hence $x - \partial_{n+1}(y)$ is in the image of $i_n$. By injectivity of $i_{n-1}$, it follows that $x - \partial_{n+1}(y)$ is the image of a cycle in $C'_n$. Thus $[x - \partial_{n+1}(y)]$ is in the image of $i_*$.

□

## Relative homology groups of spaces

**Definition 4.12.** A pair of topological spaces $(X, X')$ is a topological space $X$ together with a subspace $X' \subseteq X$. A morphism of pairs $f\colon (X, X') \to (Y, Y')$ is a continuous map $f\colon X \to Y$ that restricts to $X' \to Y'$, that is, $f$ satisfies $f(X') \subseteq Y'$.

Let $A$ be an abelian group. If $(X, X')$ is a pair of topological spaces, then there are canonical injections $\mathcal{S}(X')_n \to \mathcal{S}(X)_n$ which induce a chain map $C(X'; A) \to C(X; A)$ that is an injective homomorphism of abelian groups in each level.

**Definition 4.13.** Let $(X, X')$ be a pair of topological spaces and let $A$ be an abelian group. Then the quotient complex

$$C(X, X'; A) = C(X; A)/C(X'; A)$$

is called the *relative chain complex* of $(X, X')$. Its homology groups

$$H_n(X, X'; A) = H_n(C(X, X'; A))$$

are called the *relative homology groups* of $(X, X')$.

**Corollary 4.14.** *Let $(X, X')$ be a pair of topological spaces and let $A$ be an abelian group. Then there is an exact sequence*

$$\begin{aligned} \dots \to H_n(X; A) \to H_n(X, X'; A) \xrightarrow{\delta} H_{n-1}(X'; A) \to H_{n-1}(X; A) \to \dots \\ \dots \to H_0(X'; A) \to H_0(X; A) \to H_0(X, X'; A) \to 0\,. \end{aligned}$$

As an immediate corollary, we observe that relative homology groups measure if the inclusion $X' \to X$ induces isomorphisms in homology groups:

**Corollary 4.15.** *Let $(X, X')$ be a pair of topological spaces and let $A$ be an abelian group. Then the following statements are equivalent.*

*(i) The inclusion induces an isomorphism $H_n(X'; A) \to H_n(X; A)$ for all $n \geq 0$.*

*(ii) The relative homology group $H_n(X, X'; A)$ is trivial for all $n \geq 0$.*

*Proof.* This follows by applying Examples 4.4 and 4.5 to suitable parts of the long exact sequence of the pair $(X, X')$. $\square$

We now give another application of the long exact sequence. For this, we consider a morphism $f\colon (X, X') \to (Y, Y')$ of pairs of spaces. Then $f$ restricts to a map $f|_{X'}\colon X' \to Y'$, and we obtain commutative squares

$$\begin{tikzcd} \ldots \arrow[r] & H_n(X';A) \arrow[r] \arrow[d] & H_n(X;A) \arrow[r] \arrow[d] & H_n(X,X';A) \arrow[r] \arrow[d] & H_{n-1}(X';A) \arrow[r] \arrow[d] & \ldots \\ \ldots \arrow[r] & H_n(Y';A) \arrow[r] & H_n(Y;A) \arrow[r] & H_n(Y,Y';A) \arrow[r] & H_{n-1}(Y';A) \arrow[r] & \ldots \end{tikzcd} \tag{4.2}$$

relating the two exact sequences. (Here we use that the connecting homomorphism is natural with respect to maps of pairs and that the chain maps $f_*\colon C(X;A) \to C(Y;A)$ and $f_*\colon C(X';A) \to C(Y';A)$ induce a chain map between the quotient complexes.)

**Lemma 4.16.** *If two out of the three vertical maps*

$$H_n(X';A) \to H_n(Y';A), \qquad H_n(X;A) \to H_n(Y;A), \qquad H_n(X,X';A) \to H_n(Y,Y';A)$$

*in diagram* (4.2) *are isomorphisms for all $n \geq 0$, then so is the third.*

*Proof.* This follows directly from the long exact sequences and the Five Lemma. $\square$

## Excision and homotopy invariance

The *excision theorem* is one of the most important features of singular homology. We will prove it in one of the later lectures. We state it without proof here to be able to compute homology groups in an important example.

**Theorem 4.17.** *Let $(X, X')$ be a pair of spaces and let $Y \subset X'$ be a subspace with the property that* $\mathrm{closure}(Y) \subseteq \mathrm{interior}(X')$. *Then the inclusion $X \setminus Y \to X$ induces isomorphisms of relative homology groups*

$$H_n(X \setminus Y, X' \setminus Y; A) \to H_n(X, X'; A)$$

*for all $n \geq 0$ and all abelian coefficient groups $A$.*

Recall that two continuous maps $f, f'\colon X \to Y$ are *homotopic* if there exists a continuous map $H\colon X \times [0,1] \to Y$ such that $H|_{X\times\{0\}} = f$ and $H|_{X\times\{1\}} = f'$. We sometimes use the notation $f \simeq f'$ to indicate that $f$ and $f'$ are homotopic. A continuous map $f\colon X \to Y$ is a *homotopy equivalence* if there exists a continuous map $g\colon Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. We say that a topological space is *contractible* if the unique map $X \to *$ to the one point space is a homotopy equivalence.

Another central feature of singular homology is the following *homotopy invariance* result that we will also prove in a later lecture.

**Theorem 4.18.** *If $f, f'\colon X \to Y$ are two homotopic continuous maps, then $f_*$ and $f'_*$ coincide as maps $H_n(X;A) \to H_n(Y;A)$ for all $n \geq 0$ and all abelian coefficient groups $A$.*

**Corollary 4.19.** *Homotopy equivalences induce isomorphisms on homology groups.*

*Proof.* If $g\colon Y \to X$ is homotopy inverse to $f\colon X \to Y$, then $f_* \circ g_* = (f \circ g)_* = \mathrm{id}_{H_n(Y;A)}$ and $g_* \circ f_* = (g \circ f)_* = \mathrm{id}_{H_n(X;A)}$. ☐

This corollary and the two out of three property from Lemma 4.16 have the following consequence:

**Lemma 4.20.** *Let $f\colon (X, X') \to (Y, Y')$ be a morphism of pairs of spaces. If $f\colon X \to Y$ and $f|_{X'}\colon X' \to Y'$ are homotopy equivalences, then $f$ induces isomorphisms*

$$H_n(X, X'; A) \to H_n(Y, Y'; A)$$

*for all $n$ and $A$.* ☐

## The homology groups of the spheres

Let

$$S^m = \{x \in \mathbb{R}^{m+1} \mid \|x\| = 1\}$$

be the $m$-dimensional sphere and let

$$D^m = \{x \in \mathbb{R}^{m+1} \mid \|x\| = 1 \text{ and } x_{m+1} \leq 0\}$$

be the subspace of $S^m$ that may be viewed as the "southern hemisphere". We use the notation $D^m$ for this space since $D^m$ is homeomorphic to the usual $m$-disk $\{x \in \mathbb{R}^m \mid \|x\| \leq 1\}$ via the homeomorphism that forgets the last coordinate of $x \in D^m$. Moreover, we let

$$S^{m-1} = \{x \in D^m \mid x_{m+1} = 0\}$$

be the boundary sphere of $D^m$ that we also view as the "equator" in $S^m$. Finally, let $x = (0, \ldots, 0, 1)$ and $y = (0, \ldots, 0, -1)$ be the points that we think of as the "north pole" and the "south pole" in $S^m$.

**Proposition 4.21.** *Let $m > 0$ and $n > 0$ be integers and let $A$ be an abelian group. Then there is a chain of isomorphisms*

$$\begin{aligned} H_n(D^m, S^{m-1}; A) &\xrightarrow{\cong} H_n(S^m \setminus \{x\}, S^m \setminus \{x, y\}; A) \\ &\xrightarrow{\cong} H_n(S^m, S^m \setminus \{y\}; A) \xleftarrow{\cong} H_n(S^m, \{x\}; A) \xleftarrow{\cong} H_n(S^m; A) \end{aligned}$$

*Proof.* The inclusion $D^m \to S^m \setminus \{x\}$ is a homotopy equivalence that restricts to a homotopy equivalence $S^{m-1} \to S^m \setminus \{x, y\}$. Hence Lemma 4.20 applies to give the first isomorphism. The second isomorphism follows by applying the excision theorem with $X = S^m$, $X' = S^m \setminus \{y\}$ and $Y = \{x\}$. Since the identity of $S^m$ restricts to a homotopy equivalence $\{x\} \to S^m \setminus \{y\}$, another application of Lemma 4.20 gives the third isomorphism. The last isomorphism results from the long exact sequence for the pair $(S^m, \{x\})$ and our computation of the homology groups of the one point space. In the case $n = 1$, the surjectivity of $H_1(S^m; A) \to H_1(S^m, \{x\}; A)$ uses that the connecting map $H_1(S^m, \{x\}; A) \to H_0(\{x\}; A)$ is trivial. This follows from the long exact sequence because our identification of zeroth homology groups with the $A$-linearization of the set of path components and the assumption $m > 0$ imply that $H_0(\{x\}; A) \to H_0(S^m; A)$ is an isomorphism. ☐

The long exact sequence for the pair $(D^m, S^{m-1})$ has the following form:

$$\dots \to H_n(D^m; A) \to H_n(D^m, S^{m-1}; A) \to H_{n-1}(S^{m-1}; A) \to H_{n-1}(D^m; A) \to \dots$$

This gives us another way to relate the relative homology groups $(D^m, S^{m-1})$ to the homology groups of the spheres.

**Lemma 4.22.** *For $n \geq 2$ and $m \geq 1$, the connecting homomorphism in the long exact sequence for $(D^m, S^{m-1})$ is an isomorphism $H_n(D^m, S^{m-1}; A) \to H_{n-1}(S^{m-1}; A)$.*

*Proof.* Since $D^m$ is contractible, we have $H_n(D^m; A) \cong 0$ and $H_{n-1}(D^m; A) \cong 0$ as soon as $n \geq 2$. This implies that $0 \to H_n(D^m, S^{m-1}; A) \to H_{n-1}(S^{m-1}) \to 0$ is exact. Hence the map is an isomorphism. □

In order to use this for computing the homology groups of spheres, we need one more ingredient:

**Lemma 4.23.** *The relative homology group $H_1(D^m, S^{m-1}; A)$ is isomorphic to $A$ if $m = 1$. It is isomorphic to the trivial group if $m > 1$.*

*Proof.* The long exact sequence implies that $H_1(D^m, S^{m-1}; A)$ is isomorphic to the kernel of $H_0(S^{m-1}; A) \to H_0(D^m; A)$. If $m > 1$, this map is an isomorphism since both spaces are connected. If $m = 1$, this map is isomorphic to the map $A \oplus A \to A$ sending $(a, a')$ to $a + a'$. The kernel of the latter map is $\{(a, a') \in A \oplus A \mid a = -a'\}$, which is isomorphic to $A$. □

We can now assemble all these results to the following calculation of the homology groups of spheres:

**Theorem 4.24.** *For $m > 0$ and $n > 0$, we have isomorphisms*

$$H_n(S^m; A) \cong H_n(D^m, S^{m-1}; A) \cong \begin{cases} A & m = n \\ 0 & else \end{cases}$$

*Proof.* The first isomorphism in the theorem was established in Proposition 4.21. If $n = 1$, then last lemma shows the claim. Next we note that since $S^0$ is the disjoint union of two points, Lemmas 2.9 and 2.14 imply that $H_k(S^0; A) \cong 0$ for positive $k$. Hence for $n > 1$ and $m = 1$, Lemma 4.22 shows the claim. If both $m > 1$ and $n > 1$, we argue by induction over $n$ and use that Proposition 4.21 and Lemma 4.22 reduce the computation of $H_n(S^m; A)$ to that of $H_{n-1}(S^{m-1}; A)$. □

# Lecture 5: Homotopy invariance of singular homology

The aim of this lecture is to prove the homotopy invariance of singular homology. To explain our strategy of proof, we recall the different steps in the definition of singular homology:

$$\begin{tikzcd}
\mathrm{Top} \arrow[r, "\substack{\text{singular}\\ \text{complex}}", "\mathcal{S}(-)"'] & \text{"discrete data"} \arrow[r, "A\text{-linearization}", "A[-]"'] & \mathrm{Ch} \arrow[r, "\text{homology}", "H_n(-)"'] & \mathrm{Ab}
\end{tikzcd}$$

We will show that the singular complex $\mathcal{S}(-)$ and the $A$-linearization preserve a suitable notion of homotopy. To make this precise, we need to define what homotopies in the context of the "discrete data" and of chain complexes are. We will also show that the homology functor from chain complexes to abelian groups sends homotopic chain maps to the same maps of abelian groups. Altogether, this will imply the homotopy invariance of singular homology.

## Simplicial sets

We prepare for defining the type of object in which the singular complex takes values.

**Definition 5.1.** Let $[n]$ be the ordered set $(0 < 1 < \cdots < n)$, and let $\Delta$ be the category with objects $[n]$ for $n \geq 0$ and morphisms the order preserving maps.

In this context, "order preserving" means "weakly monotone". In other words, a morphism $\alpha\colon [m] \to [n]$ in $\Delta$ is a map of sets $\alpha\colon \{0,\dots,m\} \to \{0,\dots,n\}$ such that $\alpha(i) \leq \alpha(j)$ if $i \leq j$. (We do not require $\alpha(i) < \alpha(j)$ if $i < j$.) This defines a category since the identity is order preserving and since the composition of order preserving maps is order preserving again.

The following two types of order preserving maps are of special interest to us: The map

$$\delta_i\colon [n-1] \to [n]$$

is the order preserving injection with $i \notin \operatorname{im} \delta_i$, and

$$\sigma_i\colon [n+1] \to [n]$$

is the order preserving surjection which maps two distinct elements to $i$. Both maps are uniquely characterized by these properties.

**Definition 5.2.** A simplicial set $K$ is a contravariant functor $K\colon \Delta \to \mathrm{Set}$.

Explicitly, a simplicial set $K$ is a sequence of sets $K_n$ indexed by non-negative integers $n$ and structure maps $\alpha^*\colon K_n \to K_m$ for each order preserving map $\alpha\colon [m] \to [n]$ such that $(\mathrm{id}_{[m]})^* = \mathrm{id}_{K_m}$ and $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$ for composable $\alpha\colon [m] \to [n]$ and $\beta\colon [n] \to [p]$. Despite of what the name "simplicial set" indicates, a simplicial set is not a set with an extra structure, but rather a collection of sets. We refer to the elements of $K_m$ as the *m-simplices* of $K$.

Recall from Lecture 2 that each map $\alpha\colon [m] \to [n]$ induces a continuous map $\alpha_*\colon \Delta^m \to \Delta^n$ sending the vector $e_i$ from the standard basis of $\mathbb{R}^{m+1}$ to $e_{\alpha(i)}$. Now let $X$ be a topological space. Next we show that the collection of the sets of singular simplices

$$\mathcal{S}(X)_n = \{\sigma\colon \Delta^n \to X \mid \sigma \text{ is continuous}\}$$

forms a simplicial set. The structure map induced by $\alpha\colon [m] \to [n]$ is

$$\alpha^*\colon \mathcal{S}(X)_n \to \mathcal{S}(X)_m, \qquad \sigma \mapsto \sigma \circ \alpha_* \ .$$

With this definition, it is clear that $(\mathrm{id}_{[m]})^* = \mathrm{id}_{\mathcal{S}(X)_m}$, and for order preserving $\alpha\colon [m] \to [n]$ and $\beta\colon [n] \to [p]$ and a singular $p$-simplex $\sigma\colon \Delta^p \to X$ we have

$$(\beta \circ \alpha)^*(\sigma) = \sigma \circ (\beta \circ \alpha)_* = \sigma \circ \beta_* \circ \alpha_* = \alpha^*(\sigma \circ \beta_*) = \alpha^*(\beta^*(\sigma)) = (\alpha^* \circ \beta^*)(\sigma) \ .$$

Hence simplicial sets do indeed provide a setup for the "discrete data" in which the singular complex takes values.

**Remark 5.3.** At this point one may wonder why we restrict our attention to the order preserving maps, and not just to the injective order preserving maps or to the class of all (not necessarily order preserving) maps between the sets $\{0, \dots, m\}$. The reason is that our argument below needs the maps induced by the $\delta_i$ and the $\sigma_i$, and one can show that the order preserving maps form the smallest class of maps between the sets $\{0, \dots, m\}$ that contains the $\delta_i$ and the $\sigma_i$ and is closed under composition.

To formulate that the singular complex preserves homotopies, we first need to introduce morphisms of simplicial sets.

**Definition 5.4.** A morphism of simplicial sets $h\colon K \to L$ is a collection of maps of sets $h_m\colon K_m \to L_m$ such that the square

$$\begin{tikzcd} K_n \arrow[r, "\alpha^*"] \arrow[d, "h_n"'] & K_m \arrow[d, "h_m"] \\ L_n \arrow[r, "\alpha^*"] & L_m \end{tikzcd}$$

commutes for all order preserving $\alpha\colon [m] \to [n]$.

In the language of categories and functors, a morphism of simplicial sets is a natural transformation of contravariant functors $\Delta \to \mathrm{Set}$.

An isomorphism of simplicial sets is a map of simplicial sets $K \to L$ for which there is a map of simplicial sets $L \to K$ such that both composites equal the respective identity maps. One can check that a map of simplicial sets $K \to L$ is an isomorphism if and only if $K_m \to L_m$ is bijective for all $m \geq 0$.

**Lemma 5.5.** *A continuous map $f\colon X \to Y$ induces a morphism of simplicial sets $f_*\colon \mathcal{S}(X) \to \mathcal{S}(Y)$ with $(f_*)_n$ given by $(f_*)_n(\sigma\colon \Delta^n \to X) = f \circ \sigma$. The induced morphisms satisfy $(\mathrm{id}_X)_* = \mathrm{id}_{\mathcal{S}(X)}$ and $(g \circ f)_* = g_* \circ f_*$ for composable maps $f$ and $g$. In other words, the singular complex is a covariant functor $\mathcal{S}\colon \mathrm{Top} \to \mathrm{sSet}$ from the category of topological spaces to the category of simplicial sets.*

*Proof.* For $\alpha\colon [m] \to [n]$ and $\sigma\colon \Delta^n \to X$, we have

$$\alpha^*((f_*)_n(\sigma)) = \alpha^*(f \circ \sigma) = f \circ \sigma \circ \alpha = (f_*)_m(\sigma \circ \alpha) = (f_*)_m(\alpha^*(\sigma)) \ .$$

Hence $f_*$ is a morphism of simplicial sets. □

In order to define a notion of homotopy, we introduce simplicial sets that play the role of the point and the interval in simplicial sets. Both arise as special cases of the following more general notion.

**Definition 5.6.** For $k \geq 0$, let $\underline{\Delta}^k$ be the simplicial set with

$$(\underline{\Delta}^k)_m = \mathrm{Hom}_{\Delta}([m],[k]) = \{\gamma\colon [m] \to [k] \mid \gamma \text{ is order preserving}\}\ .$$

The structure map induced by $\alpha\colon [m] \to [n]$ is

$$\alpha^*\colon (\underline{\Delta}^k)_n \to (\underline{\Delta}^k)_m, \qquad (\gamma\colon [n] \to [k]) \mapsto \gamma \circ \alpha\ .$$

The compatibility condition for the structure maps is a direct consequence of the definition.

The simplicial set $\underline{\Delta}^0$ plays the role of the one point space. Since there is only one (necessarily constant) map $[m] \to [0]$, the set of $m$-simplices consists only of this single element for every $m$, and all structure maps between the sets $(\underline{\Delta}^0)_m$ have to be constant.

The simplicial set $\underline{\Delta}^1$ plays the role of the interval. One can check that its set of $m$-simplices $(\underline{\Delta}^1)_m$ has $m+2$ elements. We note that for $i = 0$ or $i = 1$, the maps

$$j_i\colon [0] \to [1], \qquad 0 \mapsto i$$

induce maps $\underline{\Delta}^0 \to \underline{\Delta}^1$ sending $\alpha\colon [m] \to [0]$ in $\underline{\Delta}^0$ to the $m$-simplex $j_i \circ \alpha\colon [m] \to [1]$ of $\underline{\Delta}^1$. We think of these two maps $j_0$ and $j_1$ as the "inclusions of end points".

The last bit that is needed to define a homotopy of maps of simplicial set is the notion of the product of simplicial sets.

**Definition 5.7.** For simplicial sets $K$ and $L$, their product is the simplicial set $K \times L$ with $(K \times L)_m = K_m \times L_m$ and structure maps $\alpha^* = \alpha^* \times \alpha^*\colon K_n \times L_n \to K_m \times L_m$.

We note that if $h\colon K \to L$ and $h'\colon K' \to L'$ are maps of simplicial sets, there is an induced map $h \times h'\colon K \times K' \to L \times L'$ on the product. Since there is a canonical bijection from a set $K_m$ to its product $K_m \times \{*\}$ with a one element set, there is a canonical isomorphism of simplicial sets $K \to K \times \underline{\Delta}^0$.

**Definition 5.8.** Let $h_0, h_1\colon K \to L$ be morphisms of simplicial sets. A *simplicial homotopy* from $h_0$ to $h_1$ is a morphism of simplicial sets $F\colon K \times \underline{\Delta}^1 \to L$ such that each $h_i$ is given by

$$\begin{tikzcd} K \arrow[r, "\cong"] & K \times \underline{\Delta}^0 \arrow[r, "\mathrm{id} \times j_i"] & K \times \underline{\Delta}^1 \arrow[r, "F"] & L \end{tikzcd}\ .$$

Although this will not play a role in what follows, we point out that "simplicial homotopy" is not an equivalence relation because it is in general neither symmetric nor transitive.

**Proposition 5.9.** *If $f_0, f_1\colon X \to Y$ are homotopic continuous maps, then there is a simplicial homotopy from $(f_0)_*\colon \mathcal{S}(X) \to \mathcal{S}(Y)$ to $(f_1)_*\colon \mathcal{S}(X) \to \mathcal{S}(Y)$. In other words, $\mathcal{S}(-)\colon \mathrm{Top} \to \mathrm{sSet}$ is a homotopy preserving functor.*

*Proof.* Let $H\colon X\times[0,1]\to Y$ be a homotopy from $f_0$ to $f_1$. As in Lecture 3, we consider the homeomorphism $\iota\colon \Delta^1\to[0,1], (1-t,t)\mapsto t$ that identifies the 1-simplex with the interval. Moreover, we note that there is a canonical map of simplicial sets

$$u\colon \underline{\Delta}^1\to\mathcal{S}(\Delta^1), \qquad (\alpha\colon[m]\to[1])\mapsto(\alpha_*\colon\Delta^m\to\Delta^1)\,.$$

Using these maps, we define $F$ to be the composite

$$\mathcal{S}(X)\times\underline{\Delta}^1\xrightarrow{\mathrm{id}\times u}\mathcal{S}(X)\times\mathcal{S}(\Delta^1)\xrightarrow{\mathrm{id}\times\iota_*}\mathcal{S}(X)\times\mathcal{S}([0,1])\xrightarrow{\cong}\mathcal{S}(X\times[0,1])\xrightarrow{H_*}\mathcal{S}(Y)\,.$$

Here the third map is the isomorphism of simplicial sets that results from the observation that a continuous map $\sigma\colon\Delta^m\to X\times[0,1]$ corresponds to a pair of continuous maps $\Delta^m\to X$ and $\Delta^m\to[0,1]$. For $i=0,1$, the composite of

$$\underline{\Delta}^0\xrightarrow{j_i}\underline{\Delta}^1\xrightarrow{u}\mathcal{S}(\Delta^1)\xrightarrow{\iota_*}\mathcal{S}([0,1])$$

sends the constant map with value 0 in $(\underline{\Delta}^0)_m$ to the map $\Delta^m\to[0,1]$ with constant value $i$. This implies that the map $F\circ j_i$ sends the singular $m$-simplex $\alpha\colon\Delta^m\to X$ to the map

$$\Delta^m\to X\times[0,1],\quad t\mapsto H(\alpha(t),i).$$

Therefore, $F\circ j_i$ coincides with $(H|_{X\times\{i\}})_*\colon\mathcal{S}(X)_m\to\mathcal{S}(Y)_m$ and $F$ is indeed a simplicial homotopy from $(f_0)_*$ to $(f_1)_*$. □

**Remark 5.10.** In the language and categories and functors, the morphism $u$ in the previous proof arises from an application of the *Yoneda lemma*: The Yoneda lemma provides a bijection between the elements of $\mathcal{S}(\Delta^1)([1])=\mathcal{S}(\Delta^1)_1$ and the set of natural transformations from the representable functor $\underline{\Delta}^1=\mathrm{Hom}_\Delta(-,[1])\colon\Delta^{\mathrm{op}}\to\mathrm{Set}$ to the functor $\mathcal{S}(\Delta^1)\colon\Delta^{\mathrm{op}}\to\mathrm{Set}$. Under the bijection of the Yoneda lemma, $u$ corresponds to the identity $\Delta^1\to\Delta^1$.

For the next steps in the proof, we need a notion of homotopies for chain maps:

**Definition 5.11.** Let $f_0,f_1\colon C\to D$ be chain maps. A chain homotopy from $f_0$ to $f_1$ is a sequence of group homomorphisms $P_n\colon C_n\to D_{n+1}$ such that

$$\partial^D_{n+1}\circ P_n+P_{n-1}\circ\partial^C_n=(f_1)_n-(f_0)_n\,.$$

We will sometimes just write $\partial\circ P_n+P_{n-1}\circ\partial$ for the left hand side of this equation.

The following statement shows that this notion of homotopy enables us to perform the last step of the strategy outlined in the beginning.

**Proposition 5.12.** *If $P$ is a chain homotopy from $f_0\colon C\to D$ to $f_1\colon C\to D$, then $(f_0)_*=(f_1)_*$ as maps $H_n(C)\to H_n(D)$.*

*Proof.* Let $x\in C_n$ represent $[x]\in H_n(C)$. Then $x\in\ker\partial^C_n$, and hence

$$f_1(x)-f_0(x)=(\partial^D_{n+1}\circ P_n)(x)+(P_{n-1}\circ\partial^C_n)(x)=\partial^D_{n+1}(P_n(x))\,.$$

Hence $f_0(x)$ and $f_1(x)$ represent the same class in $H_n(D)=\ker\partial^D_n/\operatorname{im}\partial^D_{n+1}$. □

The construction of the singular chain complex $C(X;A)$ from $\mathcal{S}(X)$ generalizes to a functor $C(-;A)\colon \mathrm{sSet} \to \mathrm{Ch}$. Indeed, if $K$ is a simplicial set and $A$ is an abelian group, then we set

$$C(K;A)_n = A[K_n]\ .$$

The map $\delta_i\colon [n-1] \to [n]$ induces a map of sets $(\delta_i)^*\colon K_n \to K_{n-1}$, and we also denote the induced map on $A$-linearizations by $(\delta_i)^*\colon A[K_n] \to A[K_{n-1}]$. Using the latter maps, we set

$$\partial_n = \sum_{i=0}^{n}(-1)^i(\delta_i)^*\ .$$

The same arguments as used in the case of the singular chain complex show that $C(K;A)$ is a chain complex that depends functorially on $K$.

**Proposition 5.13.** *Let $h_0, h_1\colon K \to L$ be morphisms of simplicial sets. If there exists a simplicial homotopy from $h_0$ to $h_1$, then the induced maps $(h_0)_*, (h_1)_*\colon C(K;A) \to C(L;A)$ are chain homotopic. In other words, the functor $C(-;A)\colon \mathrm{sSet} \to \mathrm{Ch}$ preserves homotopy.*

*Proof.* Let $F\colon K \times \underline{\Delta}^1 \to L$ be the simplicial homotopy from $h_0$ to $h_1$. In order to construct a chain homotopy from $F$, we define for each integer $i$ with $-1 \le i \le n$ an order preserving map $\alpha_i\colon [n+1] \to [1]$ by requiring that $\alpha_i(j) = 0$ if $j \le i$ and that $\alpha_i(j) = 1$ if $j > i$. So $\alpha_i$ is the unique order preserving map that sends $i+1$ elements of $[n+1]$ to zero.

We now use $F$ and the maps $\alpha_i$ to define a map of sets

$$F((\sigma_i)^*(-),\alpha_i)\colon K_n \to L_{n+1}, \qquad x \mapsto F_{n+1}((\sigma_i)^*(x),\alpha_i)\ .$$

Here $\sigma_i\colon [n+1] \to [n]$ is again the unique order preserving surjective map hitting $i$ twice.

Writing also $F((\sigma_i)^*(-),\alpha_i)$ for the induced map of $A$-linearizations, we define

$$P_n\colon A[K_n] \to A[L_{n+1}], \qquad P_n = \sum_{i=0}^{n}(-1)^i F((\sigma_i)^*(-),\alpha_i)\ .$$

We claim that $P_n$ is the desired chain homotopy. To verify this, we consider

$$(\partial_{n+1} \circ P_n)(x) = \sum_{j=0}^{n+1}(-1)^j(\delta_j)^*(P_n(x)) = \sum_{j=0}^{n+1}\sum_{i=0}^{n}(-1)^{i+j}(\delta_j)^*(F_{n+1}((\sigma_i)^*(x),\alpha_i)) \tag{5.1}$$

where $x$ is now an element of $A[K_n]$. Since $F$ is a morphism of simplicial sets, it follows that $(\delta_j)^*(F_{n+1}((\sigma_i)^*(-),\alpha_i))$ and $F_n((\delta_j)^*((\sigma_i)^*(-)),(\delta_j)^*(\alpha_i))$ coincide as maps $K_n \to L_n$ and therefore also induce the same maps on $A$-linearizations.

We will analyze the double sum (5.1) by first restricting our attention to the summands with $j = i$ and $j = i+1$ and addressing the remaining summands later. For the first case, we note that we have $\delta_i^*(\alpha_i) = \alpha_{i-1}$ as maps $[n] \to [1]$ since precomposing with $\delta_i$ means we have one element with value 0 less. We also have $\delta_{i+1}^*(\alpha_i) = \alpha_i$. Moreover, inspecting the definition of the $\delta_i$ and the $\sigma_i$ we see that

$$\sigma_i \circ \delta_i = \mathrm{id}_{[n]} = \sigma_i \circ \delta_{i+1}\ .$$

Hence the sum over all summands in (5.1) with $j = i$ and $j = i + 1$ simplifies as follows:

$$\begin{aligned}
&\sum_{i=0}^{n}(-1)^{i+i}F_n((\delta_i)^*((\sigma_i)^*(x)),(\delta_i)^*(\alpha_i)) + \sum_{i=0}^{n}(-1)^{i+i+1}F_n((\delta_{i+1})^*((\sigma_i)^*(x)),(\delta_{i+1})^*(\alpha_i))\\
=&\sum_{i=0}^{n}(-1)^{i+i}F_n((\sigma_i\circ\delta_i)^*(x),\alpha_{i-1}) + \sum_{i=0}^{n}(-1)^{i+i+1}F_n((\sigma_i\circ\delta_{i+1})^*(x)),\alpha_i))\\
=&F_n(x,\alpha_{-1}) - F_n(x,\alpha_n) = ((h_1)_* - (h_0)_*)(x)\ .
\end{aligned}$$

To simplify the sum over the remaining summands in (5.1), we first note that we have

$$(\delta_j)^*\circ(\sigma_i)^* = (\sigma_i\circ\delta_j)^* = (\delta_j\circ\sigma_{i-1})^* = (\sigma_{i-1})^*\circ(\delta_j)^* \qquad \text{for } j < i$$

and

$$(\delta_j)^*\circ(\sigma_i)^* = (\sigma_i\circ\delta_j)^* = (\delta_{j-1}\circ\sigma_i)^* = (\sigma_i)^*\circ(\delta_{j-1})^* \qquad \text{for } j > i+1.$$

We now use these relations to observe that the remaining summands simplify in the following way (where we set $i' = i - 1$ and $j' = j - 1$ in the second line):

$$\begin{aligned}
&\sum_{j<i}(-1)^{i+j}F_n(((\sigma_{i-1})^*\circ(\delta_j)^*)(x)),\alpha_{i-1}) + \sum_{j>i+1}(-1)^{i+j}F_n(((\sigma_i)^*\circ(\delta_{j-1})^*)(x)),\alpha_i)\\
=&-\sum_{j\leq i'}(-1)^{i'+j}F_n(((\sigma_{i'})^*\circ(\delta_j)^*)(x)),\alpha_{i'}) - \sum_{j'\geq i+1}(-1)^{i+j'}F_n(((\sigma_i)^*\circ(\delta_{j'})^*)(x)),\alpha_i)\\
=&-\sum_{j=0}^{n}(-1)^j\sum_{i=0}^{n-1}(-1)^iF_n(((\sigma_i)^*\circ(\delta_j)^*)(x)),\alpha_i)\\
=&-P_{n-1}(\partial_n(x))
\end{aligned}$$

Altogether, this shows that $\partial_{n+1}\circ P_n = (h_1)_* - (h_0)_* - P_{n-1}\circ\partial_n$. □

**Theorem 5.14.** *If $f_0, f_1\colon X \to Y$ are homotopic continuous maps and $A$ is an abelian group, then $(f_0)_*$ and $(f_1)_*$ coincide as maps $H_n(X;A) \to H_n(Y;A)$.*

*Proof.* This follows by combining Propositions 5.9, 5.12 and 5.13. □

# Lecture 6: The mapping degree and a step towards excision

The aim of this lecture is to study the mapping degree and to reduce the proof of the Excision Theorem to a statement about "small simplices".

## Reduced homology

It is sometimes convenient to work with the following variation of singular homology.

**Definition 6.1.** Let $X$ be a non-empty space and let $A$ be an abelian group. Then the *reduced homology* of $X$ with coefficients in $A$ is defined by

$$\widetilde{H}_n(X;A) = \begin{cases} \ker(H_0(X;A) \to H_0(*;A)) & \text{if } n = 0 \\ H_n(X;A) & \text{if } n > 0 \end{cases}$$

where $H_0(X;A) \to H_0(*;A)$ is the map induced by the map $X \to *$ to the one-point space.

We note that by definition, the reduced homology $\widetilde{H}(X;A)$ only differs from $H_*(X;A)$ in degree 0. Any continuous map $f\colon X \to Y$ of non-empty spaces induces a group homomorphism $f_*\colon \widetilde{H}_n(X;A) \to \widetilde{H}_n(Y;A)$, and we have $(g \circ f)_* = g_* \circ f_*$ for composable maps.

We now give an alternative description of $\widetilde{H}(X;A)$ that uses a slightly more general notion of chain complexes than the one we considered before, namely chain complexes that are also allowed to have an abelian group in degree $-1$. For such a chain complex $C$ indexed by integers $n \geq -1$, we have an additional differential $\partial_0\colon C_0 \to C_{-1}$ satisfying $\partial_0 \circ \partial_1 = 0$ and set $H_0(C) = (\ker \partial_0)/(\operatorname{im} \partial_1)$.

We let $\widetilde{C}(X;A)$ be the chain complex indexed by integers $n \geq -1$ that coincides with the singular chain complex $C(X;A)$ in non-negative degrees, but has the group $A$ in degree $-1$ and has as additional differential $\partial_0\colon C_0(X;A) \to A$ the map $C_0(X;A) \to C_0(*;A) \xrightarrow{\cong} A$ induced by $X \to *$. In other words, $\widetilde{C}(X;A)$ has the form

$$\dots \xrightarrow{\partial_{n+1}} C_n(X;A) \xrightarrow{\partial_n} \dots \xrightarrow{\partial_2} C_1(X;A) \xrightarrow{\partial_1} C_0(X;A) \xrightarrow{\partial_0} A \to 0\,.$$

We have $\partial_0 \circ \partial_1 = 0$ because this composite coincides with the composite of

$$C_1(X;A) \xrightarrow{(X \to *)_*} C_1(*;A) \xrightarrow{\partial_1} C_0(*;A)$$

by naturality, and we have shown earlier that $\partial_1\colon C_1(*;A) \to C_0(*;A)$ is the trivial map.

**Lemma 6.2.** *The reduced homology groups $\widetilde{H}_n(X;A)$ are isomorphic to the homology groups of the chain complex $\widetilde{C}(X;A)$.*

*Proof.* Let $D$ be the chain complex with $D_0 = A$, $D_n = 0$ for $n \geq 1$, and trivial differentials. Then the above map $\partial_0\colon C_0(X;A) \to A$ extends to a chain map $\varepsilon\colon C(X;A) \to D$ with $\epsilon_n = 0$ for $n \geq 1$. Setting $B_n = \ker \epsilon_n$ for $n \geq 0$ then defines a chain complex that coincides with $C(X;A)$ in positive levels and has $B_0 = \ker \partial_0$. Hence the homology groups of $B$ are isomorphic to the homology groups of $\widetilde{C}(X;A)$. Since $H_0(D) \cong A$ is the only non-trivial homology group of $D$, the long exact homology sequence of $0 \to B \to C(X;A) \to D \to 0$ gives isomorphisms $H_n(\widetilde{C}(X;A)) \cong \widetilde{H}_n(X;A)$ for $n \geq 1$ and a short exact sequence

$$0 \to H_0(\widetilde{C}(X;A)) \to H_0(X;A) \to A \to 0$$

that gives the desired isomorphism between $H_0(\widetilde{C}(X;A))$ and $\widetilde{H}_0(X;A)$ because the map $H_0(X;A) \to A$ induced by $C(X;A) \to D$ coincides with the composite of the isomorphism $H_0(*;A) \to A$ and the map $H_0(X;A) \to H_0(*;A)$ that we used to define $\widetilde{H}_0(X;A)$. □

If $(X, X')$ is a pair of spaces with $X' \neq \emptyset$, we set $\widetilde{H}_n(X, X';A) = H_n(X, X';A)$. Then $\widetilde{H}_n(X, X';A)$ is isomorphic to the $n$-th homology of the quotient complex $\widetilde{C}(X;A)/\widetilde{C}(X';A)$, and the short exact sequence of chain complexes

$$0 \to \widetilde{C}(X';A) \to \widetilde{C}(X;A) \to \widetilde{C}(X;A)/\widetilde{C}(X';A) \to 0$$

implies that there is a long exact sequence of reduced homology groups

$$\dots \to \widetilde{H}_n(X';A) \to \widetilde{H}_n(X;A) \to \widetilde{H}_n(X, X';A) \to \widetilde{H}_{n-1}(X';A) \to \dots .$$

The use of reduced homology groups enables us to phrase some of our earlier results without a case distinction that distinguishes homology in degree 0 from homology in positive degrees. For example, the homotopy invariance of singular homology and our computation of $H_n(*;A)$ implies the following statement.

**Corollary 6.3.** *If $X$ is a contractible space, then $\widetilde{H}_n(X;A) \cong 0$ for all $n \geq 0$.*

To illustrate how the use of reduced homology can simplify arguments in a more substantial example, we give a generalization of our earlier argument for the inductive computation of the homology groups of spheres. For this, we define the *suspension* of a topological space $X$ to be the quotient space $SX$ of the product $X \times [-1,1]$ by the equivalence relation generated by $(x,-1) \sim (x',-1)$ for all $x, x' \in X$ and $(x,1) \sim (x',1)$ for all $x, x' \in X$. In other words, $SX$ is obtained by taking a "cylinder" with base $X$ and collapsing both its top and its bottom to a point (see Figure 6.1).

**Proposition 6.4.** *Let $X$ be a non-empty space, let $A$ be an abelian group, and let $n \geq 1$ be an integer. There is a natural isomorphism $\widetilde{H}_n(SX;A) \cong \widetilde{H}_{n-1}(X;A)$.*

*Proof.* We write $\pi\colon X \times [-1,1] \to SX$ for the quotient map and set $U_{-1} = \pi(X \times [-1, \frac{1}{2}])$ and $U_1 = \pi(X \times [-\frac{1}{2}, 1])$. Then $U_1 \cup U_2 = X$ and both $U_{-1}$ and $U_1$ are closed since $\pi^{-1}(U_{-1})$ and $\pi^{-1}(U_1)$ are. Moreover, we choose a point $x_0 \in X$ and write $x_{-1} = \pi(x_0, -1)$ and $x_1 = \pi(x_0, 1)$. The map

$$H\colon (X \times [-1, \frac{1}{2}]) \times [0,1] \to (X \times [-1, \frac{1}{2}]), \quad ((x,s),t) \mapsto (x, (1-t)s - t)$$

is a homotopy from the identity to the map $(x,s) \mapsto (x,-1)$. It induces a homotopy $H'\colon U_{-1} \times [0,1] \to U_{-1}$ from the identity to the constant map with value $x_{-1}$ whose existence shows that $\{x_{-1}\} \to U_{-1}$ is a homotopy equivalence. (The implicit claim that $H'$ is

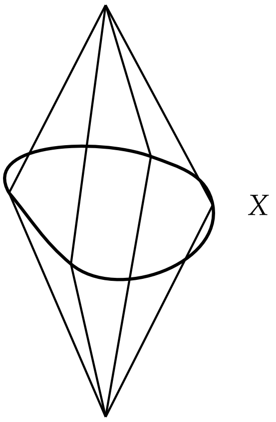

Figure 6.1: Suspension of $X$

continuous with respect to the product topology can for example be deduced from Proposition 8.14 below.) Analogously, the inclusion $\{x_1\} \to U_1$ is a homotopy equivalence, and $\pi$ restricts to a homotopy equivalence $X \times \{0\} \to U_{-1} \cap U_1$.

With this at hand, we get a chain of isomorphisms

$$\begin{aligned}\widetilde{H}_n(SX;A) &\cong \widetilde{H}_n(SX,\{x_1\};A) \cong \widetilde{H}_n(SX,U_1;A) \cong \widetilde{H}_n(U_{-1},U_{-1}\cap U_1;A)\\ &\cong \widetilde{H}_{n-1}(U_{-1}\cap U_1;A) \cong \widetilde{H}_{n-1}(X\times\{0\};A) \cong \widetilde{H}_{n-1}(X;A)\end{aligned}$$

where the first isomorphism comes from the long exact sequence of the pair $(SX,\{x_1\})$, the second is induced by the homotopy equivalence $\{x_1\} \to U_1$, the third is the excision theorem applied to $SX \setminus U_{-1} \subseteq U_1 \subseteq X$, the fourth comes from the long exact sequence of the pair $(U_{-1}, U_{-1} \cap U_1)$ and the contractibility of $U_{-1}$, the fifth comes from the homotopy equivalence $X \times \{0\} \to U_{-1} \cap U_1$, and the last one comes from the homeomorphism $X \to X \times \{0\}$. □

It is easy to see that the suspension of the $n-1$-sphere is homeomorphic to the $n$-sphere. Moreover, it follows from the definition of reduced homology that $\widetilde{H}_n(S^0;A) \cong 0$ if $n > 0$ and $\widetilde{H}_n(S^0;A) \cong A$ if $n = 0$ (see also the proof of Proposition 6.8 below). With these two observations, the last proposition does in particular allow to inductively compute $\widetilde{H}_n(S^m;A)$, and we can express the result without making a case distinction for the 0-sphere $S^0$:

**Corollary 6.5.** *Let $A$ be an abelian group and let $m, n \geq 0$ be non-negative integers. Then there are isomorphisms*

$$\widetilde{H}_n(S^m;A) \cong \begin{cases} A & \text{if } m = n \\ 0 & \text{if } m \neq n\,. \end{cases}$$

## The mapping degree

If $X$ is a non-empty space, we let $\widetilde{H}_n(X) = \widetilde{H}_n(X;\mathbb{Z})$ be the $n$-th homology with coefficients in $\mathbb{Z}$. We will now use the following elementary observation about abelian groups: If $A$ is an abelian group that is isomorphic to $\mathbb{Z}$, then every group homomorphism $f\colon A \to A$ is of the form $f(a) = k \cdot a$ for some $k \in \mathbb{Z}$.

**Definition 6.6.** Let $m \geq 0$ be a non-negative integer and let $f\colon S^m \to S^m$ be a continuous map. The *mapping degree* of $f$ is the unique integer $\deg(f) \in \mathbb{Z}$ with $\deg(f) \cdot a = f_*(a)$ for all $a \in \widetilde{H}_m(S^m)$.

We collect some immediate properties of the mapping degree:

**Lemma 6.7.** *(i) If $f$ and $f'$ are homotopic maps $S^m \to S^m$, then* $\deg(f) = \deg(f')$.

*(ii) If $f\colon S^m \to S^m$ and $g\colon S^m \to S^m$ are continuous maps, then* $\deg(g \circ f) = \deg(g) \cdot \deg(f)$ .

*(iii) If $f\colon S^m \to S^m$ is a homotopy equivalence, then* $\deg(f) \in \{1, -1\}$.

*Proof.* Part (i) follows from the homotopy invariance of singular homology. Part (ii) follows since

$$\deg(g \circ f) \cdot a = (g \circ f)_*(a) = g_*(f_*(a)) = g_*(\deg(f) \cdot a) = \deg(g) \cdot \deg(f) \cdot a \, .$$

If $f$ is a homotopy equivalence, part (i) and (ii) imply that multiplication by $\deg(f)$ is an isomorphism $\widetilde{H}_m(S^m) \to \widetilde{H}_m(S^m)$. Hence we must have $\deg(f) \in \{1, -1\}$. □

For $m \geq 0$, we now consider the continuous map

$$f_m\colon S^m \to S^m, \qquad (x_0, \dots, x_m) \mapsto (-x_0, x_1, \dots, x_m) \, .$$

Geometrically, $f_m$ is the reflection at the hyperplane in $\mathbb{R}^{m+1}$ given by the points $(x_0, \dots, x_m)$ with $x_0 = 0$.

**Proposition 6.8.** *We have* $\deg(f_m) = -1$.

*Proof.* We prove this statement by induction on $m$. For $m = 0$, we have $S^0 = \{-1, 1\} \subseteq \mathbb{R}^1$, and $f_0$ switches the two points. Earlier we have shown $H_0(S^0) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $H_0(*) \cong \mathbb{Z}$. Under these isomorphisms, the canonical map $H_0(S^0) \to H_0(*)$ corresponds to the map $(a, b) \mapsto a + b$. Hence $\widetilde{H}(S^0)$ is isomorphic to the subgroup $\{(a, -a) \mid a \in \mathbb{Z}\}$ of $\mathbb{Z} \oplus \mathbb{Z}$. Under the given isomorphisms, $f_0$ induces the map $(a, -a) \mapsto (-a, a)$. This shows that $\deg(f_0) = -1$.

In the inductive step, we let $m \geq 1$ and again write $D^m = \{(x_0, \dots, x_m) \in S^m \mid x_m \leq 0\}$ for the "southern hemisphere" in $S^m$ and view $S^{m-1} \subset D^m$ as the "equator". This leads to a commutative diagram

$$\begin{tikzcd}
\widetilde{H}_m(S^m) \arrow[r, "="] \arrow[d, "(f_m)_*"'] & H_m(S^m) \arrow[r, "\cong"] & H_m(D^m, S^{m-1}) \arrow[r, "\cong"] & \widetilde{H}_{m-1}(S^{m-1}) \arrow[d, "(f_{m-1})_*"] \\
\widetilde{H}_m(S^m) \arrow[r, "="] & H_m(S^m) \arrow[r, "\cong"] & H_m(D^m, S^{m-1}) \arrow[r, "\cong"] & \widetilde{H}_{m-1}(S^{m-1})
\end{tikzcd}$$

where the horizontal isomorphisms are the ones that were used in our calculation of the homology groups of spheres. The commutativity shows that $\deg(f_m) = \deg(f_{m-1})$. □

**Corollary 6.9.** *For any $i$ with $0 \leq i \leq m$, the reflection*

$$f_{(m,i)}\colon S^m \to S^m, \qquad (x_0, \dots, x_m) \mapsto (x_0, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_m)$$

*has degree* $-1$.

*Proof.* Let $h_i\colon \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ be the linear isomorphism with $h_i(e_0) = e_i, h_i(e_i) = e_0$ and $h_i(e_j) = e_j$ if $j \neq 0, i$. Since $h_i$ preserves the standard scalar product, it induces a homeomorphism $h_i\colon S^m \to S^m$. This map has the property $h_i^{-1} \circ f_{(m,i)} \circ h_i = f_m$. The calculation

$$\begin{aligned}-1 =&\deg(f_m) = \deg(h_i^{-1} \circ f_{(m,i)} \circ h_i) = \deg(h_i^{-1}) \cdot \deg(f_{(m,i)}) \cdot \deg(h_i)\\ =&\deg(f_{(m,i)}) \cdot \deg(h_i^{-1}) \cdot \deg(h_i) = \deg(f_{(m,i)}) \cdot \deg(h_i^{-1} \circ h_i) = \deg(f_{(m,i)})\end{aligned}$$

shows the claim. ☐

**Corollary 6.10.** *The antipodal map* $-1\colon S^m \to S^m, x \mapsto -x$ *has degree* $(-1)^{m+1}$.

*Proof.* The antipodal map is a composite of the $m+1$-reflections $f_{(m,0)}, \dots, f_{(m,m)}$ considered in the previous corollary. ☐

**Corollary 6.11.** *For even* $m$*, the antipodal map* $-1\colon S^m \to S^m$ *cannot be homotopic to the identity map* $S^m \to S^m$.

*Proof.* Homotopic maps have the same degree while the degrees of the two maps in question are $(-1)^{m+1} = -1$ and 1 if $m$ is even. ☐

**Lemma 6.12.** *Let* $f\colon S^m \to S^m$ *be a continuous map and let* $m$ *be even. Then there exists an* $x \in S^m$ *such that either* $f(x) = x$ *or* $f(x) = -x$.

*Proof.* Suppose that no such $x$ exists. Then for every $x$, both the line through $x$ and $f(x)$ and the line through $-x$ and $f(x)$ are well-defined and do not contain 0. Hence for every $t \in [0,1]$ and every $x \in S^m$, the vectors $t \cdot f(x) + (1-t) \cdot x$ and $-t \cdot x + (1-t) \cdot f(x)$ are non-zero in $\mathbb{R}^{m+1}$. This implies that we obtain continuous maps $F, G\colon S^m \times [0,1] \to S^m$ defined by

$$F(x,t) = \frac{t \cdot f(x) + (1-t) \cdot x}{\|t \cdot f(x) + (1-t) \cdot x\|} \quad \text{and} \quad G(x,t) = \frac{-t \cdot x + (1-t) \cdot f(x)}{\| -t \cdot x + (1-t) \cdot f(x)\|}\,.$$

The map $F$ is a homotopy from $\mathrm{id}_{S^m}$ to $f$, while $G$ is a homotopy from $f$ to $-1\colon S^m \to S^m$. This contradicts the previous lemma. ☐

**Definition 6.13.** A vector field on $S^m$ is a continuous map $F\colon S^m \to \mathbb{R}^{m+1}$ such that $F(x) \perp x$ for all $x \in S^m$.

**Theorem 6.14.** *If* $m$ *is even, there is no vector field on* $S^m$ *that is everywhere non-zero.*

*Proof.* Suppose there is a vector field $F$ on $S^m$ with $F(x) \neq 0$ for all $x \in S^m$. Consider the map

$$f\colon S^m \to S^m, \qquad x \mapsto \frac{F(x)}{\|F(x)\|}\,.$$

Then $f$ is continuous since $F$ is. Since $x \perp f(x)$ and both $x$ and $f(x)$ are always non-zero, we cannot have $f(x) = x$ or $f(x) = -x$. By the previous lemma, this is a contradiction. ☐

### Excision, Mayer–Vietoris sequences, and small simplices

We will now make a first step towards the proof of the Excision Theorem 4.17 and a related result, the Mayer–Vietoris sequence:

**Theorem 6.15.** *Let $A$ be an abelian group, let $X$ be a topological space and let $U, V \subseteq X$ be two subspaces with $X = \operatorname{interior}(U) \cup \operatorname{interior}(V)$. Moreover let $Y \subset X$ be another subspace and let $K \subseteq Y \cap U$ and $L \subseteq Y \cap V$ be subspaces with $Y = \operatorname{interior}(K) \cup \operatorname{interior}(L)$ (where the latter interiors are formed with respect to the subspace topology of $K$ and $L$ inherited from $Y$).*

*Then there is the following long exact sequence in homology*

$$\cdots \to H_n(U \cap V, K \cap L; A) \xrightarrow{(i^U_*, i^V_*)} H_n(U, K; A) \oplus H_n(V, L; A) \xrightarrow{j^U_* - j^V_*} H_n(X, Y; A)$$
$$\xrightarrow{\partial_n} H_{n-1}(U \cap V, K \cap L; A) \to \ldots$$

*where the indicated maps are induced by the four inclusion maps $i^U \colon (U \cap V, K \cap L) \to (U, K)$, $i^V \colon (U \cap V, K \cap L) \to (V, L)$, $j^U \colon (U, K) \to (X, Y)$, and $j^V \colon (V, L) \to (X, Y)$.*

For convenience we also state the simpler absolute version that is obtained by setting $Y = K = L = \emptyset$ in the above theorem:

**Corollary 6.16.** *Let $A$ be an abelian group, let $X$ be a topological space and let $U, V \subseteq X$ be two subspaces with $X = \operatorname{interior}(U) \cup \operatorname{interior}(V)$. Then there is the following long exact sequence in homology:*

$$\ldots \to H_n(U \cap V; A) \xrightarrow{(i^U_*, i^V_*)} H_n(U; A) \oplus H_n(V; A) \xrightarrow{j^U_* - j^V_*} H_n(X; A) \xrightarrow{\partial_n} H_{n-1}(U \cap V; A) \to \ldots$$

To prepare for the proof of Theorem 6.15, we fix a topological space $X$ and recall that a cover $\mathcal{O} = (O_i)_{i \in I}$ of $X$ is a collection of subsets $\mathcal{O}_i \subseteq X$ such that $\bigcup_{i \in I} O_i = X$. We say that the cover $\mathcal{O}$ is *admissible* if

$$X = \bigcup_{i \in I} \operatorname{interior}(O_i) .$$

Moreover, we say that a singular simplex $\sigma \colon \Delta^n \to X$ is *$\mathcal{O}$-small* if there exists an $i \in I$ with $\sigma(\Delta^n) \subseteq O_i$. We write

$$\mathcal{S}_{\mathcal{O}}(X)_n = \{\sigma \in \mathcal{S}(X)_n \mid \sigma \text{ is } \mathcal{O}\text{-small} \} .$$

For an order preserving map $\alpha \colon [m] \to [n]$ and $\sigma \in \mathcal{S}_{\mathcal{O}}(X)_n$, the composite $\sigma \circ \alpha_* \in \mathcal{S}(X)_m$ is also $\mathcal{O}$-small. This implies that the collection of sets $\mathcal{S}_{\mathcal{O}}(X)_n$ form a sub-simplicial set of $\mathcal{S}(X)$.

The following theorem is the key step towards the proof of Theorem 6.15 and the Excision Theorem 4.17.

**Theorem 6.17** (Small Simplices)**.** *Let $\mathcal{O}$ be an admissible cover of a topological space $X$ and let $A$ be an abelian group. Then the inclusion of simplicial sets $\mathcal{S}_{\mathcal{O}}(X) \to \mathcal{S}(X)$ induces an isomorphism*

$$H_n(C(\mathcal{S}_{\mathcal{O}}(X); A)) \to H_n(C(\mathcal{S}(X); A)) = H_n(X; A)$$

*for all $n$.*

The theorem states that we can use the simplicial set of $\mathcal{O}$-small simplices $\mathcal{S}_{\mathcal{O}}(X)$ (rather than the singular complex $\mathcal{S}(X)$ itself) to compute the singular homology groups of a topological space $X$.

*Proof of Theorem 6.15 assuming the "Small Simplices"-theorem.* We let $\mathcal{O}$ denote the admissible cover of $X$ given by $\{U, V\}$ and write $\mathcal{P}$ for the admissible cover of $Y$ given by $\{K, L\}$. Then the simplicial sets of $\mathcal{O}$- and $\mathcal{P}$-small simplices in $X$ and $Y$ have associated chain complexes $C_*(\mathcal{S}_{\mathcal{O}}(X); A)$ and $C_*(\mathcal{S}_{\mathcal{P}}(Y); A)$, and we let $C_*(\mathcal{S}_{\mathcal{O}}(X), \mathcal{S}_{\mathcal{P}}(Y); A)$ be the quotient of the canonical inclusion of chain complexes $C_*(\mathcal{S}_{\mathcal{P}}(Y); A) \to C_*(\mathcal{S}_{\mathcal{O}}(X); A)$.

The inclusions of the $\mathcal{O}$- and $\mathcal{P}$-small simplices into all simplices give rise to a diagram of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & C_*(\mathcal{S}_{\mathcal{P}}(Y); A) \arrow[r] \arrow[d] & C_*(\mathcal{S}_{\mathcal{O}}(X); A) \arrow[r] \arrow[d] & C_*(\mathcal{S}_{\mathcal{O}}(X), \mathcal{S}_{\mathcal{P}}(Y); A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_*(Y; A) \arrow[r] & C_*(X; A) \arrow[r] & C_*(X, Y; A) \arrow[r] & 0
\end{tikzcd}$$

where both rows are short exact sequences of chain complexes. Theorem 6.17 states that the first two vertical maps induce isomorphisms in homology. A Five Lemma argument implies the last vertical map also induces an isomorphism on homology groups.

Next we consider the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C_n(K \cap L; A) \arrow[r, "(i^U_*, i^V_*)"] \arrow[d] & C_n(K; A) \oplus C_n(L; A) \arrow[r, "j^U_* - j^V_*"] \arrow[d] & C_n(\mathcal{S}_{\mathcal{P}}(Y); A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(U \cap V; A) \arrow[r, "(i^U_*, i^V_*)"] \arrow[d] & C_n(U; A) \oplus C_n(V; A) \arrow[r, "j^U_* - j^V_*"] \arrow[d] & C_n(\mathcal{S}_{\mathcal{O}}(X); A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(U \cap V, K \cap L; A) \arrow[r, "(i^U_*, i^V_*)"] \arrow[d] & C_n(U, K; A) \oplus C_n(V, L; A) \arrow[r, "j^U_* - j^V_*"] \arrow[d] & C_n(\mathcal{S}_{\mathcal{O}}(X), \mathcal{S}_{\mathcal{P}}(Y); A) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The three columns are exact by construction. Next we claim that the two upper rows are exact. The argument is the same for both, so we focus on the second row. The injectivity of $(i^U_*, i^V_*)$ and surjectivity of $j^U_* - j^V_*$ are easy to see. An element $(x, y)$ is in the kernel of $j^U_* - j^V_*$ if and only if the singular chain $j^U_*(x) = j^V_*(y)$ is a formal sum of simplices with image in the intersection of $U$ and $V$, i.e., if $(x, y)$ is in the image of $(i^U_*, i^V_*)$. It is immediate that $(j^U_* - j^V_*) \circ (i^U_*, i^V_*) = 0$ holds also in the last row. To see exactness in the last row, we view all three rows as chain complexes and the vertical maps as a short exact sequence of chain complexes. Then in the associated long exact homology sequence, the homology terms of the upper to rows are zero and hence so are the homology terms of the last sequence. This implies the exactness of the last sequence.

The long exact homology sequence of the last sequence in the diagram and the isomorphism $H_*(C_*(\mathcal{S}_{\mathcal{O}}(X), \mathcal{S}_{\mathcal{P}}(Y); A)) \to H_*(X, Y; A)$ established above provide the desired long exact sequence. $\square$

The Excision Theorem 4.17 can be derived from Theorem 6.15 or deduced from the "Small Simplices"-Theorem. For convenience, we repeat the statement of Theorem 4.17:

**Theorem 6.18** (Excision)**.** *Let $(X, X')$ be a pair of spaces and let $Y \subset X'$ be a subspace such that* $\text{closure}(Y) \subseteq \text{interior}(X')$*. Then the inclusion $X \setminus Y \to X$ induces isomorphisms of relative homology groups*

$$H_n(X \setminus Y, X' \setminus Y; A) \to H_n(X, X'; A)$$

*for all $n \geq 0$ and all abelian coefficient groups $A$.*

*assuming the "Small Simplices"-Theorem.* The statement of the excision theorem provides us with a sequence of subspaces $Y \subseteq X' \subseteq X$ such that $\text{closure}(Y) \subseteq \text{interior}(X')$. We set $X'' = X \setminus Y$ and observe that since

$$X \setminus \text{interior}(X'') = \text{closure}(Y) \subseteq \text{interior}(X'),$$

the cover $\mathcal{O} = \{X', X''\}$ is admissible. With the notation $C_{\mathcal{O}}(X) = C(\mathcal{S}_{\mathcal{O}}(X); A)$ and $C(X) = C(\mathcal{S}(X); A)$, the above theorem implies that $H_n(C_{\mathcal{O}}(X)) \to H_n(C(X))$ is an isomorphism for all $n$.

The inclusion $X' \to X$ induces a levelwise injective map of simplicial sets $\mathcal{S}(X') \to \mathcal{S}_{\mathcal{O}}(X)$ which in turn induces a levelwise injective map of chain complexes $C(X') \to C_{\mathcal{O}}(X)$. The two out of three property for short exact sequences of chain complexes that we derived from the Five Lemma therefore implies that the induced map of quotient complexes

$$C_{\mathcal{O}}(X)/C(X') \to C(X)/C(X') \tag{6.1}$$

induces an isomorphism on all homology groups.

We now claim that the inclusion $X'' \to X$ induces an isomorphism of chain complexes

$$f \colon C(X'')/C(X' \cap X'') \to C_{\mathcal{O}}(X)/C(X')\,. \tag{6.2}$$

We choose $x \in \mathcal{S}_{\mathcal{O}}(X)_n$ and $a \in A$. If $x \in \mathcal{S}(X'')_n$, then $a \cdot x$ with $a \in A$ is in the image of $f$. If $x \in \mathcal{S}(X')_n$, then $a \cdot x$ represents 0 in the quotient and is therefore also contained in the image. Hence $f$ is surjective in degree $n$. If the composite of

$$C(X'') \to C_{\mathcal{O}}(X) \to C_{\mathcal{O}}(X)/C(X')$$

maps $a \cdot x$ to 0, then we must have $x \in \mathcal{S}(X')$. Hence $a \cdot x = 0$ in $C(X'')/C(X' \cap X'')$. This implies injectivity in degree $n$ and finishes the proof of the claim.

Since $X'' = X \setminus Y$ and $X' \cap X'' = X' \setminus Y$, the homology groups of the quotient complex $C(X'')/C(X' \cap X'')$ are the relative homology groups of the pair $(X \setminus Y, X' \setminus Y)$. Therefore, the homology isomorphism (6.1) and the isomorphism of chain complexes (6.2) imply the excision theorem. □

## The barycentric subdivision

We conclude this lecture with outlining the basic idea behind the proof of the "small simplices"-theorem. Consider vectors $v_0, \ldots, v_n \in \mathbb{R}^k$. The tuple $(v_0, \ldots, v_n)$ determines an *affine n-simplex*

$$\{t_1 \cdot v_0 + \cdots + t_n \cdot v_n \mid t_i \geq 0, \sum t_i = 1\} = \text{conv}(v_0, \ldots, v_n)$$

that we will simply refer to as $(v_0, \ldots, v_n)$.

Figure 6.2: Barycentric subdivision of affine 1- and 2-simplices

**Example 6.19.** The standard basis $(e_0, \dots, e_n)$ of $\mathbb{R}^{n+1}$ gives rise to the standard $n$-simplex $\Delta^n$.

The point

$$b = \sum_{i=0}^{n} \frac{1}{n+1} v_i$$

is called the barycenter of the affine $n$-simplex $(v_0, \dots, v_n)$. We can use the barycenter to decompose an affine $n$-simplex into a union of smaller $n$-simplices, called the *barycentric subdivision* of $(v_0, \dots, v_n)$. The cases of 1- and 2-simplices are illustrated in Figure 6.2.

The set $\beta_n(v_0, \dots, v_n)$ of $n$-simplices of the barycentric subdivision of $(v_0, \dots, v_n)$ can be inductively describe as follows. The set $\beta_0(v_0)$ of 0-simplices of the barycentric subdivision of the 0-simplex $(v_0)$ contains just $v_0$. Now assume that $\beta_{n-1}$ is defined for affine $n-1$-simplices. The boundary $\partial(v_0, \dots, v_n)$ of an affine $n$-simplex is the union of the affine $n-1$-simplices obtained by deleting one of the entries in the collection $(v_0, \dots, v_n)$. Forming the union of the sets of $n-1$-simplices in the barycentric subdivision of each of the $n-1$-dimensional faces in the boundary, we obtain the set of $n-1$-simplices of the barycentric subdivision $\beta_{n-1}(\partial(v_0, \dots, v_n))$ of the boundary. Now let $b$ be the barycenter of $(v_0, \dots, v_n)$. We define

$$\beta_n(v_0, \dots, v_n) = \{(w_0, \dots, w_{n-1}, b) \mid (w_0, \dots, w_{n-1}) \in \beta_{n-1}(\partial(v_0, \dots, v_n))\}\,.$$

This completes the inductive description of the barycentric subdivision.

The relevance of the barycentric subdivision for the theorem about "small simplices" is as follows: If $\sigma\colon \Delta^n \to X$ is a singular $n$-simplex and $c$ is an $n$-cycle having $a \cdot \sigma$ as a summand, then we will show that we can replace $a \cdot \sigma$ by

$$\sum_{(v_0,\dots,v_n) \in \beta_n(\Delta_n)} \pm a \cdot \sigma|_{\mathrm{conv}(v_0,\dots,v_n)}$$

in $c$ without changing the homology class of $c$. This will imply that every homology class can be represented as a sum of smaller and smaller simplices. After a finite number of iterations, this will provide a representative of a homology class which is small relative to a given admissible cover.

# Lecture 7: The proof of the "small simplices" theorem

We begin by recalling the barycentric subdivision that was introduced in the last lecture. Given vectors $v_0, \dots, v_n \in \mathbb{R}^k$, the affine $n$-simplex $(v_0, \dots, v_n)$ is the set

$$\{t_1 \cdot v_0 + \cdots + t_n \cdot v_n \mid t_i \geq 0, \sum t_i = 1\} = \operatorname{conv}(v_0, \dots, v_n) .$$

The *barycenter* of $(v_0, \dots, v_n)$ is the point

$$b = \sum_{i=0}^{n} \frac{1}{n+1} v_i .$$

The *barycentric subdivision* of $(v_0, \dots, v_n)$ is a subdivision of the affine $n$-simplex $(v_0, \dots, v_n)$ into a union of smaller affine $n$-simplices. The cases of 1- and 2-simplices are illustrated in Figure 7.1.

The set $\beta_n(v_0, \dots, v_n)$ of $n$-simplices of the barycentric subdivision of $(v_0, \dots, v_n)$ can be inductively described as follows. We set $\beta_0(v_0) = \{v_0\}$, that is, the 0-simplices in the barycentric subdivision of a point are just the point itself. If we assume that the set of $n-1$-simplices of the barycentric subdivision of an affine $n-1$-simplex is defined, we can apply this to all the $n-1$-dimensional simplices in the boundary of an affine $n$-simplex and obtain the set $\beta_{n-1}(\partial(v_0, \dots, v_n))$ of $n-1$-simplices in the boundary of $(v_0, \dots, v_n)$. Writing $b$ for the barycenter of $(v_0, \dots, v_n)$, we inductively define

$$\beta_n(v_0, \dots, v_n) = \{(w_0, \dots, w_{n-1}, b) \mid (w_0, \dots, w_{n-1}) \in \beta_{n-1}(\partial(v_0, \dots, v_n))\} .$$

This completes the inductive description of the barycentric subdivision. In the case of the standard $n$-simplex $\Delta^n$ spanned by $(e_0, \dots, e_n)$, we will write $\beta_n(\Delta^n)$ for $\beta_n(e_0, \dots, e_n)$.

Given vectors $w_0, \dots, w_n \in \Delta^n$, the affine simplex $(w_0, \dots, w_n)$ is a subspace of $\Delta^n$ since $\Delta^n$ is convex. Now let $X$ be a topological space and let $\sigma \colon \Delta^n \to X$ be a continuous map. For given $w_0, \dots, w_n \in \Delta^n$, we consider the map

$$\sigma_{(w_0,\dots,w_n)} \colon \Delta^n \to X, \qquad (t_0, \dots, t_n) \mapsto \sigma(t_0 w_0 + \cdots + t_n w_n) .$$

One can think of $\sigma_{(w_0,\dots,w_n)}$ as the restriction of $\sigma$ to the affine $n$-simplex $(w_0, \dots, w_n)$.

Our main tool for the proof of the "small simplices" theorem is the following statement. Loosely speaking, it states that every singular chain in $X$ can be replaced by a singular chain made of "smaller" simplices.

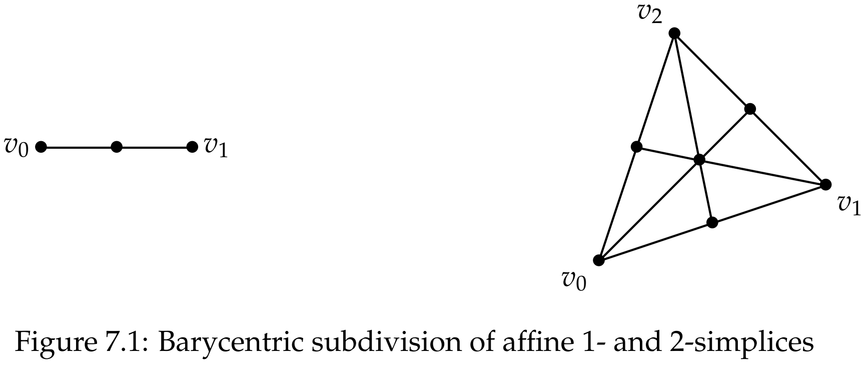

Figure 7.1: Barycentric subdivision of affine 1- and 2-simplices

**Proposition 7.1.** *Let $X$ be a topological space and let $A$ be an abelian group. Then there exists a chain map $u\colon C(X;A) \to C(X;A)$ and a chain homotopy from* id *to $u$ such that the following statements hold:*

*(i) The chain map $u$ and the chain homotopy $D$ are natural in $X$ in the sense explained below.*

*(ii) For a singular simplex $\sigma\colon \Delta^m \to X$ and $a \in A$, we have*

$$u(a \cdot \sigma) = \sum_{(w_0,\ldots,w_m)\in\beta_m(\Delta^m)} \pm a \cdot \sigma_{(w_0,\ldots,w_m)} .$$

*(iii) If $\mathcal{O} = (O_i)_{i\in I}$ is an admissible cover of $X$ and $\sigma$ is $\mathcal{O}$-small, then $u(a\cdot\sigma)$ and $D(a\cdot\sigma)$ are $\mathcal{O}$-small.*

In the statement of the proposition, the chains $u(a\cdot\sigma)$ and $D(a\cdot\sigma)$ are $\mathcal{O}$-small if they lie in the chain complex $C_{\mathcal{O}}(X;A)$ considered last time, that is, if they can be written as formal linear combinations of singular simplices $\sigma_j$ such that the image of each $\sigma_j$ is contained in one of the sets $O_i$ participating in the admissible cover.

The naturality requirement (i) means the following. If $f\colon X \to X'$ is a continuous map, $u\colon C(X;A) \to C(X;A)$ is the chain map for the space $X$ and $u'\colon C(X';A) \to C(X';A)$ is the chain map for $X'$, then the diagram of chain maps

$$\begin{tikzcd}
C_m(X;A) \arrow[r, "u_m"] \arrow[d, "f_*"'] & C_m(X;A) \arrow[d, "f_*"] \\
C_m(X';A) \arrow[r, "u'_m"] & C_m(X';A)
\end{tikzcd}$$

commutes. We also require that the analogous diagram for the chain homotopies commutes.

The following terminology will be used in the proof of the proposition.

**Definition 7.2.** A singular $m$-simplex $\sigma\colon \Delta^m \to \Delta^n$ in the topological space $\Delta^n$ is called an *affine singular simplex* if there exist $w_0,\ldots,w_m \in \Delta^n$ such that

$$\sigma(t_0,\ldots,t_m) = t_0 \cdot w_0 + \cdots + t_m \cdot w_m$$

for all $(t_0,\ldots,t_m) \in \Delta^m$. We write $(w_0,\ldots,w_m)$ for the affine singular $m$-simplex determined by $(w_0,\ldots,w_m)$.

Writing $(w_0,\ldots,\widehat{w}_i,\ldots,w_m)$ for $(w_1,\ldots,w_{i-1},w_{i+1},\ldots,w_m)$, the value of the boundary operator $\partial_m\colon C_m(\Delta^n;A) \to C_{m-1}(\Delta^n;A)$ takes the form

$$\partial_m(a\cdot(w_0,\ldots,w_m)) = \sum_{i=0}^{m}(-1)^i a\cdot(w_0,\ldots,\widehat{w}_i,\ldots,w_m)$$

since precomposing the map $\Delta^m \to \Delta^n$ determined by $(w_0,\ldots,w_m)$ with $\delta_i\colon \Delta^{m-1} \to \Delta^m$ gives the map $\Delta^{m-1} \to \Delta^n$ determined by $(w_0,\ldots,\widehat{w}_i,\ldots,w_m)$. In particular, this shows that the $A$-linearizations of the sets of affine singular simplices give rise to a subcomplex $C'(\Delta^n;A)$ of the singular chain complex $C(\Delta^n;A)$. The following lemma will be the key step in the proof of Proposition 7.1.

**Lemma 7.3.** *For all $n \geq 0$, there exist a chain map $u\colon C'(\Delta^n; A) \to C'(\Delta^n; A)$ and a chain homotopy $D$ from* id *to $u$ such that $u$ and $D$ are natural in affine linear maps and such that statements (ii) and (iii) of Proposition 7.1 are satisfied (with $X = \Delta^n$).*

Before proving the lemma, we show how it implies Proposition 7.1.

*Proof of Proposition 7.1, assuming Lemma 7.3.* For $\sigma\colon \Delta^n \to X$ and $a \in A$, we define

$$u(a \cdot \sigma) = \sigma_*(u(a \cdot \mathrm{id}_{\Delta^n})) \quad \text{and} \quad D(a \cdot \sigma) = \sigma_*(D(a \cdot \mathrm{id}_{\Delta^n}))\,.$$

Here the inner terms $u(a \cdot \mathrm{id}_{\Delta^n})$ and $D(a \cdot \mathrm{id}_{\Delta^n})$ are defined since the identity of $\Delta^n$ is an affine singular $n$-simplex. Since any $n$-chain in $C_n(X; A)$ is a sum of chains of the form $a \cdot \sigma$, requiring that $u$ and $D$ are group homomorphisms in every degree gives a definition on all chains.

Let $f\colon X \to Y$ be a continuous map. Since

$$f_*(u(a \cdot \sigma)) = f_*\sigma_*(u(a \cdot \mathrm{id}_{\Delta^n})) = u(a \cdot (f_*(\sigma)))\,,$$

the chain map $u$ is natural in $X$. An analogous statement holds for $D$. The property (ii) in the proposition is inherited from property (ii) in the lemma.

For property (iii), we notice that a singular $n$-simplex $\sigma\colon \Delta^n \to X$ is $\mathcal{O}$-small if and only if $\mathrm{id}_{\Delta^n}$ is small with respect to the admissible cover $(\sigma^{-1}(O_i))_{i \in I}$ of $\Delta^n$. If $a \cdot \sigma$ is $\mathcal{O}$-small, then $a \cdot \mathrm{id}_{\Delta^n}$ is $(\sigma^{-1}(O_i))_{i \in I}$ small. Hence part (iii) of the lemma implies that $D(a \cdot \mathrm{id}_{\Delta^n})$ is also $(\sigma^{-1}(O_i))_{i \in I}$ small. This in turn implies that $D(a \cdot \sigma) = \sigma_*(D(a \cdot \mathrm{id}_{\Delta^n}))$ is $\mathcal{O}$-small. To see that $u$ preserves $\mathcal{O}$-smallness, we can argue analogously or directly derive this property from (ii). □

Let $b = (\frac{1}{n+1}, \dots, \frac{1}{n+1})$ be the barycenter of $\Delta^n$. In the next proof, we will use the homomorphism $\beta\colon C'_m(\Delta^n; A) \to C'_{m+1}(\Delta^n; A)$ that is determined by

$$\beta(a \cdot (w_0, \dots, w_m)) = a \cdot (w_0, \dots, w_m, b)\,.$$

We note that it satisfies the formula

$$\partial \circ \beta = \beta \circ \partial + (-1)^{m+1}\mathrm{id} \tag{7.1}$$

since we have

$$\begin{aligned}\partial(\beta(a \cdot (w_0, \dots, w_m))) &= \sum_{i=0}^{m+1} (-1)^i a \cdot (w_0, \dots, \widehat{w}_i, \dots, w_m, b) \\ &= (\beta \circ \partial + (-1)^{m+1}\mathrm{id})(a \cdot (w_0, \dots, w_m))\end{aligned}$$

*Proof of Lemma 7.3.* We construct the required chain map $u$ and the chain homotopy $D$ by induction. For all $n \geq 0$, we define the first stages of $u$ and $D$ by $u_0 = \mathrm{id}$ and $D_0 = 0$.

Now we consider the case $m > 0$ and assume that $u_0, \dots, u_{m-1}$ and $D_0, \dots, D_{m-1}$ have been defined for all $n$ and satisfy all requirements for chain maps and chain homotopies that only involve the homomorphism up to $u_{m-1}$ and $D_{m-1}$. To define $u_m$ and $D_m$, we first focus on the special case $n = m$ and an affine singular simplex of the form $a \cdot \mathrm{id}_{\Delta^n}$. Writing $a \cdot \mathbf{1}$ for $a \cdot \mathrm{id}_{\Delta_n}$, we set

$$u_m(a \cdot \mathbf{1}) = (-1)^n \beta u \partial(a \cdot \mathbf{1}) \quad \text{and} \quad D_m(a \cdot \mathbf{1}) = (-1)^{n+1}\beta(a \cdot \mathbf{1} - D\partial(a \cdot \mathbf{1}))\,. \tag{7.2}$$

For general $n$ and $a \cdot \sigma \in C'(\Delta^n; A)$, we use the same naturality trick as in the proof of Proposition 7.1 to define $u_m(a \cdot \sigma)$ and $D_m(a \cdot \sigma)$. Then naturality holds again by construction, while (ii) and (iii) follow by induction from the definition.

The only thing that remains to show is that the maps $u_m$ and $D_m$ defined in (7.2) satisfy the formulas for a chain map and a chain homotopy in degree $m$. By naturality, it is enough to check that these formulas hold when we evaluate them on $a \cdot \mathbf{1}$ and $m = n$. Using (7.1), the calculation

$$\begin{aligned}\partial u(a \cdot \mathbf{1}) &= (-1)^n \partial \beta u \partial(a \cdot \mathbf{1}) \\ &= (-1)^n \beta \partial u \partial(a \cdot \mathbf{1}) + (-1)^n (-1)^{n-1+1} u \partial(a \cdot \mathbf{1}) \\ &= u \partial(a \cdot \mathbf{1})\end{aligned}$$

shows that $u_m$ provides the next stage of a chain map, and

$$\begin{aligned}(\partial D + D\partial)(a \cdot \mathbf{1}) &= \partial(-1)^{n+1}\beta(a \cdot \mathbf{1} - D\partial(a \cdot \mathbf{1})) + D\partial(a \cdot \mathbf{1}) \\ &= (-1)^{n+1}\beta\partial(a \cdot \mathbf{1}) + a \cdot \mathbf{1} - (-1)^{n+1}\beta\partial D\partial(a \cdot \mathbf{1}) - D\partial(a \cdot \mathbf{1}) + D\partial(a \cdot \mathbf{1}) \\ &= (-1)^{n+1}\beta\partial(a \cdot \mathbf{1}) + a \cdot \mathbf{1} - (-1)^{n+1}\beta(\mathrm{id} - u - D\partial)(\partial(a \cdot \mathbf{1})) \\ &= a \cdot \mathbf{1} - u(a \cdot \mathbf{1}) \\ &= (\mathrm{id} - u)(a \cdot \mathbf{1})\end{aligned}$$

shows that $D_m$ provides the next stage of a chain homotopy. □

In the next step, we convince ourselves that the barycentric subdivision does indeed make simplices smaller. For this purpose, we call the maximal distance between two points in the affine $n$-simplex $(v_0, \dots, v_n)$ the *diameter* of $(v_0, \dots, v_n)$. We note that the diameter is the maximal distance between two of the vectors $v_0, \dots, v_n$.

**Lemma 7.4.** *For any $(w_0, \dots, w_n) \in \beta_n(v_0, \dots, v_n)$, we have*

$$\operatorname{diameter}(w_0, \dots, w_n) \leq \frac{n}{n+1} \operatorname{diameter}(v_0, \dots, v_n) .$$

*Proof.* By construction of the barycentric subdivision, $(w_0, \dots, w_{n-1}) \in \beta_{n-1}(\partial(v_0, \dots, v_n))$ and the $w_n$ is the barycenter

$$b = \frac{1}{n+1} v_0 + \cdots + \frac{1}{n+1} v_n$$

of $(v_0, \dots, v_n)$. For any two vertices $w_i$ and $w_j$ of $(w_0, \dots, w_{n-1})$, we can find a $k$ with $w_i, w_j \in (v_0, \dots, \widehat{v}_k, \dots, v_n)$ and thus have

$$\|w_i - w_j\| \leq \frac{n-1}{(n-1)+1} \operatorname{diameter}(v_0, \dots, \widehat{v}_k, \dots, v_n) \leq \frac{n}{n+1} \operatorname{diameter}(v_0, \dots, v_n) .$$

It remains to establish the same estimate for the distance between $w_i$ and $b$. First we note that there is a $k$ with $\|w_i - b\| \leq \|v_k - b\|$ since the maximal distance between a point in $(v_0, \dots, v_n)$ and its barycenter $b$ is the distance between one of the $v_i$ and $b$. Hence we can conclude that

$$\begin{aligned}\|w_i - b\| \leq \|v_k - b\| = \|v_k - \sum_{i=0}^{n} \frac{1}{n+1} v_n\| &= \| \sum_{i \neq k} \frac{1}{n+1}(v_k - v_i)\| \\ &\leq \frac{n}{n+1} \max_i \|v_k - v_i\| \leq \frac{n}{n+1} \operatorname{diameter}(v_0, \dots, v_n) . \quad \square\end{aligned}$$

Now let $X$ be a topological space, let $A$ be an abelian group, and let $\mathcal{O} = (O_i)_{i\in I}$ be an admissible cover of $X$. Recall that $\mathcal{S}_{\mathcal{O}}(X)$ denotes the simplicial set of $\mathcal{O}$-small simplices and that $C_{\mathcal{O}}(X;A) = C(\mathcal{S}_{\mathcal{O}}(X);A)$ denotes the associated chain complex. We view $C_{\mathcal{O}}(X;A)$ as a subcomplex of the singular chain complex $C(\mathcal{S}(X);A)$.

**Lemma 7.5.** *For every $c \in C_n(X;A)$, there exists a $k \geq 0$ with $u^k(c) \in C_{\mathcal{O}}(X;A)$. Here $u^k = u \circ \cdots \circ u$ denotes the k-fold iteration of the chain map u from Proposition 7.1.*

*Proof.* The singular $n$-chain has the form $c = \sum_{j=0}^{l} a_j \cdot \sigma_j$. For each $j$, the family $(\sigma_j^{-1}(O_i))_{i\in I}$ is an admissible cover of $\Delta^n$. By the Lebesgue covering theorem (Exercise 12.5), we can choose an $\epsilon_j > 0$ such that for all $x \in \Delta^n$, there is an $i \in I$ with

$$\{y \in \Delta^n \mid \|x-y\| < \epsilon_j\} \subseteq \sigma_j^{-1}(O_i) \,.$$

Let $k_j$ be a positive integer with $\left(\frac{n}{n+1}\right)^{k_j} \leq \epsilon_j$. Then the previous lemma implies that for any affine $n$-simplex $(v_0, \ldots, v_n)$ in the $k_j$-fold iterated barycentric subdivision of $\Delta^n$, we have

$$\operatorname{conv}(v_0, \ldots, v_n) \subseteq \sigma_j^{-1}(O_i)$$

for some $i \in I$. This implies that $u^{k_j}(a \cdot \sigma_j)$ is $\mathcal{O}$-small. Setting $k = \max_j k_j$, we have $u^k(c) \in C_{\mathcal{O}}(X;A)$. □

**Theorem 7.6.** *Let X be a topological space, let A be an abelian group, and let $\mathcal{O} = (O_i)_{i\in I}$ be an admissible cover of X. Then the inclusion $C_{\mathcal{O}}(X;A) \to C(X;A)$ induces an isomorphism on homology groups.*

*Proof.* We use the chain map $u$ and the chain homotopy $D$ arising from Proposition 7.1. For $k \geq 1$, we set $P^k = \sum_{i=0}^{k-1} u^i D$. The calculation

$$\mathrm{id} - u^k = \sum_{i=0}^{k-1} u^i(\mathrm{id} - u) = \sum_{i=0}^{k-1} u^i(D\partial + \partial D) = \sum_{i=0}^{k-1} u^i D\partial + \partial u^i D = \partial P^k + P^k \partial$$

shows that $P^k$ is a chain homotopy from id to the $k$-fold iteration $u^k$ of $u$.

To show that $H_n(C_{\mathcal{O}}(X;A)) \to H_n(C(X;A))$ is surjective, we consider a homology class $[c] \in H_n(C(X;A))$ represented by a cycle $c \in C_n(X;A)$. By the previous lemma, we can choose a $k$ with $u^k(c) \in C_{\mathcal{O}}(X;A)_n$. Then $u^k(c)$ is in the kernel of $\partial_n$ since $c$ is and since $u^k$ is a chain map. The calculation

$$c - u^k(c) = (\partial P^k + P^k \partial)(c) = \partial P^k(c)$$

shows that $[c]$ and $[u^k(c)]$ represent the same homology class. Hence the map in question is surjective.

For the injectivity, we consider a cycle $c \in C_{\mathcal{O}}(X;A)_n$ whose image in $C_n(X;A)$ represents 0 in homology. Then there exists a $b \in C_{n+1}(X;A)$ such that $\partial_{n+1}(b) = c$. We choose an $l \geq 1$ such that $u^l(b) \in C_{\mathcal{O}}(X;A)_{n+1}$. Then we have

$$c - \partial(u^l(b)) = \partial(b - u^l(b)) = \partial(P^l\partial + \partial P^l)(b) = \partial P^l(\partial b) = \partial P^l(c) \,.$$

Hence we can express $c$ as a sum

$$c = \partial P^l(c) + \partial u^l(b) \,.$$

Since both $u$ and $D$ preserve $\mathcal{O}$-smallness by part (iii) of Proposition 7.1, so does $P^l$, and it follows that $\partial P^l(c)$ is $\mathcal{O}$-small. The second summand $\partial u^l(b)$ is $\mathcal{O}$-small by construction. Hence $c$ is the boundary of a $\mathcal{O}$-small simplex and therefore represents the trivial homology class in $H_n(C_{\mathcal{O}}(X; A))$. $\square$

# Lecture 8: Cell attachments

The aim of this lecture is to explore the process of attaching cells to topological spaces.

## Review of some point-set topological results

Since there are different conventions in the literature about whether or not the notion of a compact space includes the Hausdorff property, we fix the convention that will be used in this lecture course.

**Definition 8.1.** Let $X$ be a topological space. We say that $X$ is *quasi-compact* if every open cover of $X$ has a finite sub-cover. We say that $X$ is *compact* if it is quasi-compact and satisfies the Hausdorff property.

We recall the following results.

(i) If $X$ is a Hausdorff space and $U \subseteq X$ is a compact subspace, then $U$ is a closed subspace of $X$.

(ii) If $X$ is compact and $U \subseteq X$ is closed, then $U$ is compact.

(iii) If $f\colon X \to Y$ is a continuous and bijective map, $X$ is quasi-compact, and $Y$ is Hausdorff, then $f$ is a homeomorphism.

Now we consider topological spaces and continuous maps

$$B \xleftarrow{i} A \xrightarrow{f} X.$$

**Definition 8.2.** The *pushout* $X \cup_A B$ of this diagram is the quotient space $X \amalg B/\sim$ of the disjoint union $X \amalg B$ modulo the equivalence relation $\sim$ that is generated by the identification $i(a) \sim f(a)$ for all $a \in A$.

Despite the fact that $i$ and $f$ are not indicated in the notation $X \cup_A B$, the space $X \cup_A B$ does of course depend on the choice of these maps (see e.g. Examples 8.5 (ii) and (iii) below).

The pushout comes with canonical continuous maps $X \to X \cup_A B$ and $B \to X \cup_A B$. It can be characterized by the following universal property. For every commutative diagram of continuous maps

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd} \tag{8.1}$$

there is a unique map $X \cup_A B \to Y$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d] \arrow[ddr] & \\ B \arrow[r] \arrow[drr] & X \cup_A B \arrow[dr] & \\ & & Y \end{tikzcd}$$

commutes.

**Definition 8.3.** We say that the square (8.1) is a pushout square if $X \cup_A B \to Y$ is a homeomorphism.

### Cell attachments

Let $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ be the $n$-dimensional disk, let $\partial D^n = S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ be its boundary, and let $i\colon \partial D^n \to D^n$ be the inclusion.

**Definition 8.4.** Let $X \to Y$ be a continuous map between topological spaces. We say that $Y$ arises from $X$ by attaching an $n$-cell along $f$ if there is a pushout square

$$\begin{tikzcd}
\partial D^n \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d] \\
D^n \arrow[r] & Y
\end{tikzcd}$$

that has the given map as the right hand vertical map. We refer to $f$ as the *attaching* map and usually do not mention the given map $X \to Y$ in our notation.

**Example 8.5.** (i) The $n$-sphere $S^n$ arises by attaching an $n$-cell to the 0-dimensional disk $D^0 = *$ along the unique map $\partial D^n \to D^0$. Equivalently, $S^n$ is homeomorphic to the quotient space $D^n/\partial D^n$.

(ii) For $f = \mathrm{id}\colon \partial D^n \to S^{n-1}$, the pushout $S^{n-1} \cup_{\partial D^n} D^n$ is homeomorphic to $D^n$.

(iii) For a constant map $f\colon \partial D^n \to S^{n-1}$, the pushout $S^{n-1} \cup_{\partial D^n} D^n$ is homeomorphic to $S^{n-1} \vee S^n$, the one-point union of an $n-1$-dimensional sphere and an $n$-dimensional sphere.

To construct interesting spaces, it can be useful to attach many $n$-cells at a time.

**Definition 8.6.** Let $X \to Y$ be a continuous map. We say that $Y$ arises from $X$ by attaching $n$-cells indexed by a set $J$ if there is a continuous map $f\colon J \times \partial D^n \to X$ and a pushout square

$$\begin{tikzcd}
J \times \partial D^n \arrow[r, "f"] \arrow[d, "\mathrm{id}_J \times i"'] & X \arrow[d] \\
J \times D^n \arrow[r] & Y
\end{tikzcd}$$

that has the given map as the right hand vertical map.

Here $J \times \partial D^n$ and $J \times D^n$ carry the product topology with respect to the discrete topology on $J$. Hence the map $f\colon J \times \partial D^n \to X$ is continuous if and only if the restrictions $f|_{\{j\}\times\partial D^n}$ are continuous for all $j \in J$. In the definition, we impose no condition on the size of the set $J$ since there are in fact interesting examples of cell attachments where $J$ is an (even uncountably) infinite set.

**Example 8.7.** The $n$-dimensional sphere $S^n$ arises from attaching two $n$-cells to $S^{n-1}$ along the attaching map $f\colon \{a, b\} \times \partial D^n \to S^{n-1}$ which is the identity when restricted to one of the copies of $\partial D^n$ in the source.

## Pushouts and homotopies

We consider continuous maps $f\colon A \to X$ and $i\colon A \to B$ and a commutative diagram of topological spaces and continuous maps

$$\begin{tikzcd}
A \times [0,1] \arrow[r, "f\times \mathrm{id}_{[0,1]}"] \arrow[d, "i\times \mathrm{id}_{[0,1]}"'] & X \times [0,1] \arrow[d, "F"] \\
B \times [0,1] \arrow[r, "G"] & Z\,.
\end{tikzcd}$$

This means that $F\colon X \times [0,1] \to Z$ and $G\colon B \times [0,1] \to Z$ are homotopies on the spaces $X$ and $B$ which are compatible, that is, which coincide when we restrict them to $A \times [0,1]$ along $f \times \mathrm{id}_{[0,1]}$ and $i \times \mathrm{id}_{[0,1]}$. In this situation, one can ask if these homotopies induce a homotopy $H\colon (X \cup_A B) \times [0,1] \to Z$ on the pushout $X \cup_A B$ that extends both $F$ and $G$. The universal property of the pushout provides the upper horizontal map $H'$ in the diagram

$$\begin{tikzcd}
(X \times [0,1]) \cup_{(A\times[0,1])} (B \times [0,1]) \arrow[r, "H'"] \arrow[d, "p"'] & Z\,. \\
(X \cup_A B) \times [0,1] \arrow[ur, dashed, "H"'] &
\end{tikzcd} \tag{8.2}$$

However, the desired homotopy would be the indicated dashed map $H$. The problem is that we need to know that the left hand vertical map $p$ in the diagram is a homeomorphism. Writing $Y$ for the pushout $X \cup_A B$, this map $p$ arises from the universal property of the pushout applied to in the diagram

$$\begin{tikzcd}
A \times [0,1] \arrow[r, "f\times \mathrm{id}_{[0,1]}"] \arrow[d, "i\times \mathrm{id}_{[0,1]}"'] & X \times [0,1] \arrow[d] \arrow[ddr, bend left] & \\
B \times [0,1] \arrow[r] \arrow[drr, bend right] & (X \times [0,1]) \cup_{(A\times[0,1])} (B \times [0,1]) \arrow[dr, "p"] & \\
 & & Y \times [0,1].
\end{tikzcd} \tag{8.3}$$

**Proposition 8.8.** *The map p in the diagram* (8.3) *is a homeomorphism. In other words, the product with the interval* $[0,1]$ *preserves pushout squares so that the outer diagram in* (8.3) *is a pushout.*

Before proving the proposition, we note that it solves our original problem.

**Corollary 8.9.** *The homotopy H in* (8.2) *extending F and G exists.*

*Proof.* Since $p$ is a homeomorphism, we can define $H$ to be the composite $H' \circ p^{-1}$. $\square$

**Example 8.10.** Let $f_0, f_1\colon X \to Z$ be continuous maps and let $A \subseteq X$ be a subspace such that $f_0$ and $f_1$ are the constant maps with value $z_0$ when restricted to $A$, that is, $f_0(A) = \{z_0\} = f_1(A)$. In this situation, $f_0$ and $f_1$ induce unique continuous maps $h_0, h_1\colon X/A \to Z$ such that the composite of $h_i$ with the canonical projection $X \to X/A$ is $f_i$.

Now let $F\colon X \times [0,1] \to Z$ be a homotopy from $f_0$ to $f_1$ and suppose that $F$ is a constant map with value $z_0$ when restricted to $A \times [0,1]$. Identifying $X/A$ with the pushout of $* \leftarrow A \to X$, we can apply the previous corollary to the homotopy $F$ and the homotopy $* \times [0,1] \to Z$ with constant value $z_0$ to get a homotopy $H\colon X/A \times [0,1] \to Z$ from $h_0$ to $h_1$.

The following terminology will be used in the proof of Proposition 8.8.

**Definition 8.11.** We say that a surjective continuous map $q\colon X \to Y$ of topological spaces is a *quotient map* if a subset $O \subseteq Y$ is open if and only if $q^{-1}(O) \subseteq X$ is open.

**Definition 8.12.** A topological space $X$ is *locally compact* if for every $x$ in $X$ and every neighborhood $U$ of $x$, there exists a compact neighborhood $V$ of $x$ with $V \subseteq U$.

**Example 8.13.** The space $\mathbb{R}^n$ and the interval $[0,1]$ are locally compact since in both cases, one can find a closed ball that is contained in a given open ball around a point.

**Proposition 8.14.** *Let $f\colon X \to Y$ be a quotient map and let $K$ be a locally compact space. Then $f \times \mathrm{id}_K\colon X \times K \to Y \times K$ is a quotient map.*

*Proof.* We have to show that given a subset $W \subseteq Y \times K$ such that $(f \times \mathrm{id}_K)^{-1}(W) \subseteq X \times K$ is open, $W$ is open in $Y \times K$. Choose a point $(y_0, k_0) \in W$. We will show the claim by constructing a neighborhood of $(y_0, k_0)$ in $Y \times K$ that is contained in $W$. We choose a point $x_0 \in f^{-1}(y_0)$ and define

$$K_0 = \{k \in K \mid (x_0, k) \in (f \times \mathrm{id}_K)^{-1}(W)\}\ .$$

Given any point $k \in K_0$, the definition of the product topology on $K \times X$ implies that there are open sets $O_K \subseteq K$ and $O_X \subseteq X$ with $(x_0, k) \in O_X \times O_K \subseteq (f \times \mathrm{id}_K)^{-1}(W)$. This implies that $O_K \subseteq K_0$ is an open neighborhood of $k$. We conclude that $K_0$ is open.

By our assumption that $K$ is locally compact, there exists a compact neighborhood $L$ of $k_0$ with $L \subseteq K_0$. We note that since $\{x_0\} \times L \subseteq (f \times \mathrm{id}_K)^{-1}(W)$, we have

$$\{y_0\} \times L = (f \times \mathrm{id}_K)(\{x_0\} \times L) \subseteq W\ .$$

Setting $U = \{y \in Y \mid \{y\} \times L \subseteq W\}$, this implies $(y_0, k_0) \in U \times L \subseteq W$. If we can show that $U$ is a neighborhood of $y_0$, then $U \times L$ will provide the desired neighborhood of $(y_0, k_0)$ in $W$. In fact, we will show that $U$ is an open subset of $Y$. Since $f$ is a quotient map, this is equivalent to showing that

$$f^{-1}(U) = \{x \in X \mid \{x\} \times L \subseteq (f \times \mathrm{id}_K)^{-1}(W)\}$$

is an open subset of $X$. Since $L$ is compact and $(f \times \mathrm{id}_K)^{-1}(W)$ is open, this follows from Exercise 6.1. $\square$

*Proof of Proposition 8.8.* Inspecting the equivalence relations defining the pushouts in the source and the target of $p$ it follows that $p$ is a bijection. Now consider the commutative square

$$\begin{tikzcd}
(X \times [0,1]) \amalg (B \times [0,1]) \arrow[r, "u"] \arrow[d, "h"', "\cong"] & (X \times [0,1]) \cup_{A \times [0,1]} (B \times [0,1]) \arrow[d, "p"] \\
(X \amalg B) \times [0,1] \arrow[r, "v"] & (X \cup_A B) \times [0,1].
\end{tikzcd}$$

Here $h$ is the canonical homeomorphism. It is in particular a quotient map. Since $X \amalg B \to X \cup_A B$ is a quotient map, Proposition 8.14 implies that $v$ is a quotient map. Because the composite of two quotient maps is a quotient map, it follows that $vh = pu$ is a quotient map. If $O \subseteq (X \cup_A B) \times [0,1]$ is a subset such that $p^{-1}(O)$ is open, then $u^{-1}p^{-1}(O)$ is open since $u$ is continuous. Since $u^{-1}p^{-1}(O) = (pu)^{-1}(O)$ and $vh = pu$ is a quotient map, $O$ is open. Hence $p$ is a homeomorphism. $\square$

Slightly generalizing the statement of Proposition 8.8, our arguments show that the product with a locally compact space $K$ preserves pushouts.

### Analysis of cell attachments

We continue to let $X$ be a topological space, $J$ be a set, and $f\colon J \times \partial D^n \to X$ be an attaching map. Moreover, we write

$$p\colon X \amalg J \times D^n \to X \cup_{J \times \partial D^n} J \times D^n$$

be the quotient map and let $\mathring{D}^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$ be the interior of the $n$-disk. We think of $\mathring{D}^n$ as an "open $n$-cell".

**Lemma 8.15.** *(i) The subspace $p(X) \subseteq X \cup_{J \times \partial D^n} J \times D^n$ is closed, and $p$ induces a homeomorphism $X \to p(X)$.*

*(ii) The subspace $p(J \times \mathring{D}^n) \subseteq X \cup_{J \times \partial D^n} J \times D^n$ is open, and $p$ induces a homeomorphism $J \times \mathring{D}^n \to p(J \times \mathring{D}^n)$.*

*Proof.* For part (i), we notice that $p|_X\colon X \to p(X)$ is continuous and surjective by construction. Since the boundary inclusion $\partial D^n \to D^n$ is injective, the equivalence relation defining the quotient $X \cup_{J \times \partial D^n} J \times D^n$ does not identify different points in $X$. This implies that $p|_X$ is also injective and hence bijective. To see that $p(X)$ is closed and that $(p|_X)^{-1}$ is continuous, it is sufficient to show that $p$ preserves closed sets. Let $A \subseteq X$ be a closed subset. By definition of the quotient topology, $p(A)$ is closed if and only if $p^{-1}(p(A)) = A \cup f^{-1}(A)$ is closed in $X \amalg J \times D^n$. Since $f^{-1}(A)$ is closed in $J \times \partial D^n$ by continuity of $f$ and $J \times \partial D^n$ is a closed subset of $J \times D^n$, we deduce that $f^{-1}(A)$ is closed in $J \times D^n$. Hence $A \cup f^{-1}(A)$ is closed in $X \amalg J \times D^n$.

For part (ii), we notice that $p|_{J \times \mathring{D}^n}\colon J \times \mathring{D}^n \to p(J \times \mathring{D}^n)$ is continuous and bijective since the equivalence relation defining the quotient $X \cup_{J \times \partial D^n} J \times D^n$ does not identify different interior points of the $n$-cells. It remains to show that the inverse of this map is continuous and that $p(J \times \mathring{D}^n)$ is open in $X \cup_{J \times \partial D^n} J \times D^n$. For this it is sufficient to check that if $O \subseteq J \times \mathring{D}^n$ is open, then $p(O)$ is open in $X \cup_{J \times \partial D^n} J \times D^n$. This follows from the definition of the quotient topology since we have $p^{-1}(p(O)) = O$. $\square$

The argument for the proof of the second part of the lemma can be generalized to produce more examples of open subsets of $X \cup_{J \times \partial D^n} J \times D^n$. For this purpose, we say that a subset $U \subseteq X \amalg J \times D^n$ is *saturated* if $p^{-1}(p(U)) = U$.

**Lemma 8.16.** *Let $U \subseteq X \amalg J \times D^n$ be a saturated subset.*

*(i) If $U$ is open in $X \amalg J \times D^n$, then $p(U)$ is open in $X \cup_{J \times \partial D^n} J \times D^n$.*

*(ii) If $U$ is closed in $X \amalg J \times D^n$, then $p(U)$ is closed in $X \cup_{J \times \partial D^n} J \times D^n$.*

*Proof.* In both cases, this follows directly from the definition of the quotient topology. □

**Corollary 8.17.** *For every $j \in J$, let $V_j \subseteq \{j\} \times D^n$ be an open subset with $\{j\} \times \partial D^n \subseteq V_j$. Then $V = p(X \cup \bigcup_{j \in J} V_j)$ is open in $X \cup_{J \times \partial D^n} J \times D^n$.*

*Proof.* The conditions imply that $X \cup \bigcup_{j \in J} V_j$ is saturated. Hence the previous lemma applies. □

We now give an example of a topological space which looks like the result of a cell attachment at the first glance, but which fails to be one because it has a different topology. For this purpose, we let

$$K_n = \{x \in \mathbb{R}^2 \mid \|x - (\frac{1}{n}, 0)\| = \frac{1}{n}\}$$

be the circle with radius $\frac{1}{n}$ around the point $(\frac{1}{n}, 0)$ in $\mathbb{R}^2$. We let $H = \bigcup_{n \in \mathbb{N}} K_n$ be the union of the circles $K_n$ and view $H$ as a topological space equipped with the subspace topology induced from $\mathbb{R}^2$. The space $H$ is known as the *Hawaiian earring*.

**Lemma 8.18.** *The space $H$ is not homeomorphic to $D^0 \cup_{\mathbb{N} \times \partial D^1} \mathbb{N} \times D^1$.*

*Proof.* Using the previous corollary, one can construct an open neighborhood

$$O \subset D^0 \cup_{\mathbb{N} \times \partial D^1} \mathbb{N} \times D^1$$

of the image of the unique point of $D^0$ such that none of the circles $K_n$ is completely contained in $O$. On the other hand, any neighborhood of $0 \in H$ does contain infinitely many of the circles $K_n$. □

# Lecture 9: CW-complexes

The aim of this lecture is to introduce an extremely useful class of topological spaces, the so-called *CW-complexes*. Before we state their definition, we will develop a few more properties of cell attachments that we will need for our study of CW-complexes.

## Compactness properties of cell attachments

We briefly recall the setup from the last lecture. Let $X$ be a topological space, let $J$ be a set, let $n \geq 0$ be an integer, and let $f\colon J \times \partial D^n \to X$ is a continuous map. In this situation, we can form the pushout $X \cup_{J \times \partial D^n} J \times D^n$. This provides a space that arises from $X$ by attaching $n$-cells, and we write

$$p\colon X \amalg J \times D^n \to X \cup_{J \times \partial D^n} J \times D^n$$

for the corresponding quotient map.

**Lemma 9.1.** *If $X$ is a Hausdorff space, then $X \cup_{J \times \partial D^n} J \times D^n$ is a Hausdorff space.*

*Proof.* For $x, y \in X \cup_{J \times \partial D^n} J \times D^n$, we have to find disjoint open subsets of $X \cup_{J \times \partial D^n} J \times D^n$ that contain $x$ and $y$. For this we will use that the underlying set of $X \cup_{J \times \partial D^n} J \times D^n$ is a disjoint union of $p(X)$ and $p(J \times \mathring{D}^n)$.

If we have $x, y \in p(J \times \mathring{D}^n)$, then the claim follows because $J \times \mathring{D}^n$ is a Hausdorff space and the restriction of $p$ is a homeomorphism $J \times \mathring{D}^n \to p(J \times \mathring{D}^n)$ onto an open subset of $X \cup_{J \times \partial D^n} J \times D^n$.

If we have $x \in p(X)$ and $y \in p(\{j\} \times \mathring{D}^n)$ for some $j \in J$, we choose disjoint open subsets $U_j, V_j \subset \{j\} \times D^n$ with $\{j\} \times \partial D^n \subseteq V_j$ and $y \in p(U_j)$. Then

$$p(X \cup V_j \cup \bigcup_{i \neq j} \{i\} \times D^n)$$

is an open subset of $X \cup_{J \times \partial D^n} J \times D^n$ since it is the image of a saturated open set. Since it contains $x$ and is disjoint from $p(U_j)$, we have also established the Hausdorff property in this case.

In the last case, we assume that $x, y \in X$. Let $O_x, O_y$ be disjoint open subsets of $X$ with $x \in O_x$ and $y \in O_y$. As a consequence of Exercise 6.3, there exist an open subset

$$O \subseteq X \cup_{J \times \partial D^n} J \times D^n$$

containing $X$ and a continuous map $r\colon O \to X$ with $r(z) = z$ for all $z \in X$. (For example, we may set $O = p(X \cup J \times (D^n \setminus \{0\}))$ and let $r$ be the map induced by the identity on $X$ and the retraction $D^n \setminus \{0\} \to D^n, v \mapsto \frac{v}{\|v\|}$ on each of the cells.) Then $r^{-1}(O_x)$ and $r^{-1}(O_y)$ are disjoint open subsets of $X \cup_{J \times \partial D^n} J \times D^n$ that contain $x$ and $y$. □

**Corollary 9.2.** *If $X$ is compact and $J$ is finite, then $X \cup_{J \times \partial D^n} J \times D^n$ is compact.*

*Proof.* Under these assumptions, $X \amalg J \times D^n$ is quasi-compact. Since the quotient map to $X \cup_{J \times \partial D^n} J \times D^n$ is surjective, it follows that $X \cup_{J \times \partial D^n} J \times D^n$ is quasi-compact. Together with the Hausdorff property established in the last lemma, this implies the claim. □

The following lemma does in particular imply that $X \cup_{J\times\partial D^n} J \times D^n$ cannot be compact if $J$ is not finite.

**Lemma 9.3.** *If $K \subseteq X \cup_{J\times\partial D^n} J \times D^n$ is compact, then the set*

$$\{j \in J \mid K \cap p(\{j\} \times \mathring{D}^n) \neq \emptyset\}$$

*is finite. That is, $K$ intersects with only finitely many open cells.*

*Proof.* Suppose the indicated set is not finite. Then there exists an infinite set $S \subseteq K$ such that any two distinct elements of $S$ are contained in different open $n$-cells. That is, for $s, s' \in S$ with $s \neq s'$ there exist $j, j' \in J$ with $j \neq j'$ and $s \in p(\{j\} \times \mathring{D}^n)$ and $s' \in p(\{j'\} \times \mathring{D}^n)$. Hence all points in $p^{-1}(S) \subseteq X \amalg J \times D^n$ lie in different copies of $D^n$. This implies that $p^{-1}(S)$ is closed, which by definition of the quotient topology shows that $S \subseteq X \cup_{J\times\partial D^n} J \times D^n$ is closed. Since the same argument implies that any subset of $S$ is closed in $X \cup_{J\times\partial D^n} J \times D^n$, it follows that the subspace topology of $S$ inherited from $X \cup_{J\times\partial D^n} J \times D^n$ is the discrete topology. Because $S$ is a closed subset of the compact set $K$, it is also compact. But a discrete compact space has to be finite. This gives the desired contradiction. □

## CW-complexes

We can now state one of the main definitions of this lecture course.

**Definition 9.4.** Let $A$ be a topological space. A *CW-complex* relative to $A$ is a topological space $X$ together with a sequence of subspaces

$$A = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X$$

such that:

(i) For every $n \geq 0$, the space $X_n$ arises from $X_{n-1}$ by attaching $n$-cells.

(ii) We have $X = \cup_{n\geq -1} X_n$, and a subset $O \subseteq X$ is an open subset of $X$ if and only if $O \cap X_n$ is an open subset of $X_n$ for all $n \geq -1$.

Spelling out definitions, property (i) states that there are pushout squares

$$\begin{tikzcd} J_n \times \partial D^n \arrow[r, "f"] \arrow[d, "\mathrm{id}_{J_n}\times i"'] & X_{n-1} \arrow[d] \\ J_n \times D^n \arrow[r] & X_n \end{tikzcd}$$

where the right hand vertical map is the inclusion map $X_{n-1} \to X_n$ and $J_n$ denotes the set of $n$-cells. By our earlier analysis, this implies that $X_n \setminus X_{n-1}$ is a union of open $n$-cells. We note that in the definition it is allowed that $J_n = \emptyset$ so that $X_n = X_{n-1}$.

For every $j \in J_n$, the pushout above implies that there is a *characteristic map* $\chi_j \colon D^n \to X_n$ inducing a homeomorphism from $\mathring{D}^n$ to the open $n$-cell in $X_n$ that is indexed by $j$. The choice of such a characteristic map is not part of the data. (To see that $\chi_j$ is not unique, we note that precomposing it with any homeomorphism $D^n \to D^n$ that fixes the boundary provides another choice for $\chi_j$.)

We also note the following consequence of part (ii) of the definition.

**Corollary 9.5.** *Let $(X, A)$ be a relative CW-complex. A map $f\colon X \to Y$ to a topological space $Y$ is continuous if and only if the restriction $f|_{X_n}\colon X_n \to Y$ is continuous for all $n \geq -1$.*

*Proof.* Given a subset $O \subseteq Y$, property (ii) implies that $f^{-1}(O) \subseteq X$ is open if and only if $f^{-1}(O) \cap X_n = (f|_{X_n})^{-1}(O)$ is open in $X_n$. □

We introduce some more terminology about CW-complexes.

- The subspace $X_n \subseteq X$ is called the $n$-skeleton of $(X, A)$.
- An *absolute CW-complex* $X$ is a CW-complex relative to the empty space $\emptyset$.
- A relative CW-complex $(X, A)$ is *finite dimensional* if $X = X_n$ for some $n \geq 0$.
- A relative CW-complex $(X, A)$ is *finite* if it is finite dimensional and the indexing set of $n$-cells $J_n$ is finite for each $n$. In other words, $(X, A)$ is finite if and only if all of $X$ is built from $A$ by attaching finitely many cells of possibly different dimensions.

The notion of finiteness is well-defined because the above pushout square shows that there is a bijection $J_n \to \pi_0(X_n \setminus X_{n-1})$. Hence despite the fact that $J_n$ is not part of the data of a CW-complex, its cardinality can be read off from $X$ and its filtration.

**Example 9.6.** We discuss some simple examples of absolute CW-complexes.

(i) The $n$-dimensional sphere $S^n$ is homeomorphic to the cell complex $D^0 \cup_{\partial D^n} D^n$ with the unique map $\partial D^n \to D^0$ as the attaching map. So $S^n$ can be viewed as a CW-complex with one 0-cell and one $n$-cell.

(ii) Drawing $k$ points on a circle provides an example for a CW-structure for $S^1$ with the $k$ points as 0-cells and the $k$ line segments between them as 1-cells.

(iii) The surface of a cube admits a CW-structure where the 0-cells are the 8 vertices, the 1-cells are the 12 edges, and the 2-cells are the 6 faces.

(iv) The tetrahedron (which is $\partial\Delta^3$ in our earlier notation) has a CW-structure with 0-cells the 4 vertices, with 1-cells the 6 edges, and with 2-cells the 4 faces.

(v) The real projective space $\mathbb{R}P^n$ from Exercise 7.2 is a CW-complex having a single $i$-cell for every $i$ with $0 \leq i \leq n$.

Useful examples of relative CW-complexes $(X, A)$ with $A$ non-empty will appear in later lectures where we use this notion for inductive arguments.

**Definition 9.7.** The Euler characteristic of a finite (absolute) CW-complex $X$ is defined to be

$$\chi(X) = \sum_{n \geq 0} (-1)^n \cdot (\text{Number of } n\text{-cells of } X) \ .$$

The Euler characteristic is $1 + (-1)^n$ in Example 9.6(i), it is $k - k = 0$ in Example 9.6(ii), it is $8 - 12 + 6 = 2$ in Example 9.6(iii), and it is $4 - 6 + 4 = 2$ in Example 9.6(iv). The Euler characteristic of the CW-complex $\mathbb{R}P^n$ in Example 9.6(v) is 0 if $n$ is odd and 1 if $n$ is even. So for all the examples that are homotopy equivalent to $S^2$, the Euler characteristic is 2. This is a shadow of a much more general result: In one of the next lectures we will show that two finite CW-complexes have the same Euler characteristic if they are homotopy equivalent. This result may be surprising since the homotopy equivalence is not required to be compatible with the CW-complex structure. In particular, this implies that two finite CW-structures on the same space lead to the same Euler characteristic. Another special case of this result is Euler's polyhedron formula, stating that for a convex polyhedron with $V$ vertices, $E$ edges, and $F$ faces, we have

$$V - E + F = 2\,.$$

## Compactness and CW-complexes

**Lemma 9.8.** *Let $(X, A)$ be a finite CW-complex with $A$ compact. Then $X$ is compact.*

*Proof.* Since $X = X_n$ for some $n \geq 0$, this follows from Corollary 9.2 by induction. □

**Lemma 9.9.** *Let $(X, A)$ be a relative CW-complex. If $A$ is a Hausdorff space, then so is $X$.*

*Proof.* By Lemma 9.1, $X_n$ is Hausdorff for all $n$. Consider $x, y \in X$ with $x \neq y$. Choose $n \geq 0$ with $x, y \in X_n$, and choose $O_n, P_n \subseteq X_n$ open and disjoint with $x \in O_n$ and $y \in P_n$. As in Lemma 9.1, we use Exercise 6.3 to choose an open subset $V_n \subseteq X_{n+1}$ that contains $X_n$ and a continuous map $r_{n+1}\colon V_n \to X_n$ such that $r_{n+1}(z) = z$ for all $z \in X_n$. We set $O_{n+1} = r_{n+1}^{-1}(O_n)$ and $P_{n+1} = r_{n+1}^{-1}(P_n)$. Proceeding by induction, this provides a sequence of spaces $O_{n+2}, P_{n+2}, O_{n+3}, P_{n+3}, \ldots$ with analogous properties. We set

$$O = \bigcup_{i\geq 0} O_{n+i} \quad \text{and} \quad P = \bigcup_{i\geq 0} P_{n+i}\,.$$

Then $O \cap X_m$ is open in $X_m$ by construction, and so $O$ is open in $X$. Analogously, $P$ is open in $X$. Since the preimage of disjoint sets is disjoint, $P$ and $O$ are disjoint. Lastly, $x \in O$ and $y \in P$ hold by construction. □

If $(X, A)$ is a relative CW-complex, we will in the sequel assume $A$ to be Hausdorff unless otherwise stated.

**Definition 9.10.** Let $(X, A)$ be a relative CW-complex. A relative CW-complex $(Y, A)$ is a *subcomplex* of $(X, A)$ if $Y \subseteq X$ is a subspace such that $Y_n = Y \cap X_n$ for all $n \geq -1$ and $Y_n \setminus Y_{n-1}$ is a union of open $n$-cells in $X_n \setminus X_{n-1}$ for all $n \geq 0$.

**Lemma 9.11.** *Let $(X, A)$ be a relative CW-complex, and let $K \subseteq X$ be a compact subspace such that $K \subseteq X_n$ holds for some $n \geq -1$. Then $K$ is contained in a finite subcomplex of $(X, A)$.*

*Proof.* We prove this by induction over $n$. If $n = -1$, there is nothing to show. So assume that the statement holds for compact subspaces of $X_{n-1}$. We know from Lemma 9.3 that $K$ intersects nontrivially with only finitely many open $n$-cells in $X_n \setminus X_{n-1}$. Let $J'_n \subseteq J_n$ be the indexing set of these $n$-cells. Choosing characteristic maps $\chi_j\colon D^n \to X_n$ for the $n$-cells, we first note that $\chi_j(D^n) \subseteq X_n$ is compact since $D^n$ is compact, $\chi_j$ is continuous, and $X_n$ is a Hausdorff space. Because $J'_n$ is finite, it follows that

$$\bigcup_{j\in J'_n} \chi_j(D^n)$$

is a compact subset of $X_n$. Hence

$$X_{n-1} \cap \left( K \cup \bigcup_{j \in J'_n} \chi_j(D^n) \right)$$

is a compact subset of $X_{n-1}$ and thus contained in a finite subcomplex $(Y, A)$ of $(X, A)$. Then

$$Y \cup_{J'_n \times \partial D^n} J'_n \times D^n$$

(or rather its image under the canonical map to $X$) is the desired finite subcomplex of $X$ containing $K$. □

**Lemma 9.12.** *Let $(X, A)$ be a relative CW-complex and let $K \subseteq X$ be compact. Then there is an $n \geq -1$ with $K \subseteq X_n$.*

*Proof.* We prove this statement by contradiction and assume that there is no such $n$. Then we can find a sequence of non-negative integers $(n_i)_{i \geq 0}$ and a sequence of points $(x_i)_{i \geq 0}$ in $K$ such that $x_0 \in X_{n_0}$ and such that for all $i \geq 1$, we have $n_i > n_{i-1}$ and $x_i \in X_{n_i} \setminus X_{n_{i-1}}$.

We consider the set $S = \{x_1, x_2, \dots\} \subseteq K$. For every $i \geq 0$, we have $S \cap X_i \subseteq S \cap X_{n_i} = \{x_1, \dots, x_i\}$ so that each $S \cap X_i$ is a finite set. Since $X_i$ is a Hausdorff space, each one point subset and hence also each finite subset of $X_i$ is closed. So $S \cap X_i$ is closed for all $i \geq 0$. By the definition of the topology of $X$, this implies that $S$ is a closed subset of $X$. Since the same argument shows that any subset of $S$ is closed in $X$, it follows that the subspace topology on $S$ inherited from $X$ is the discrete topology. Using that $S \subseteq K$ holds by construction, $K$ being compact and $S$ being closed imply that $S$ is compact. Since a discrete compact space has to be finite, we arrive at a contradiction. □

**Theorem 9.13.** *Let $(X, A)$ be a relative CW-complex and let $K \subseteq X$ be compact. Then $K$ is contained in a finite subcomplex of $(X, A)$.*

*Proof.* This follows by combining Lemma 9.11 and Lemma 9.12. □

**Corollary 9.14.** *Let $(X, A)$ be a relative CW-complex. Then the closure of every $n$-cell is contained in a finite subcomplex.*

*Proof.* We choose a characteristic map $\chi_j \colon D^n \to X$ for the $n$-cell in question. As noted earlier, its image $\chi_j(D^n)$ in $X$ is compact since $D^n$ is compact and $X$ is a Hausdorff space. So $\chi_j(D^n)$ is contained in a finite subcomplex by the previous theorem. Since the compactness also implies that $\chi_j(D^n) \subseteq X$ is closed and $\chi_j(D^n)$ contains the open $n$-cell $\chi_j(\mathring{D}^n)$, the claim follows. (In fact, it is easy to show that $\chi_j(D^n)$ equals the closure of $\chi_j(\mathring{D}^n)$.) □

# Lecture 10: Cellular homology

In the last lecture we defined CW-complexes. By definition, a CW-complex $(X, A)$ is a topological space $X$ together with a filtration $A = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \dots$ with $X = \bigcup_{n \geq -1} X_n$ such that $X_n$ arises from $X_{n-1}$ by attaching $n$-cells and such that a subset of $X$ is open if and only if for all $n \geq -1$, its intersection with $X_n$ is open in $X_n$. We also showed that if $K \subseteq X$ is a compact subset, then $K$ is contained in a finite subcomplex $(Y, A)$ of $(X, A)$. For later use, we note that the latter result has the following consequence about singular chains:

**Corollary 10.1.** *Let $X$ be a CW-complex, let $A$ be an abelian group, and let $c \in C_n(X; A)$ be a singular $n$-chain. Then there exists a $p \geq 0$ such that $c$ is in the image of the map $C_n(X_p; A) \to C_n(X; A)$ induced by the inclusion $X_p \subseteq X$.*

*Proof.* Since $c$ is a formal linear combination of singular $n$-simplices, it is sufficient to show that every continuous map $\sigma\colon \Delta^n \to X$ factors through the inclusion $X_p \subseteq X$ for some $p \geq 0$. Since $\sigma(\Delta^n) \subseteq X$ is quasi-compact and $X$ is a Hausdorff space, $\sigma(\Delta^n)$ is compact and therefore contained in a finite subcomplex $Y$ of $X$. Letting $p$ be the maximal dimension of the cells in $Y$, it follows that $Y$ and hence $\sigma(\Delta^n)$ is contained in $X_p$. □

To prepare for the subsequent constructions, introduce the long exact homology sequence of a triple of spaces.

**Proposition 10.2.** *Let $Y'' \subseteq Y' \subseteq Y$ be spaces and let $A$ be an abelian group. Then there is a long exact sequence*

$$\dots \to H_n(Y', Y''; A) \to H_n(Y, Y''; A) \to H_n(Y, Y'; A) \to H_{n-1}(Y', Y''; A) \to \dots$$

*where*

- *the first map is induced by the inclusion $(Y', Y'') \subseteq (Y, Y'')$,*
- *the second map is induced by the identity of $Y$ viewed as a map $(Y, Y'') \subseteq (Y, Y')$, and*
- *the third map is the composite of the connecting map $H_n(Y, Y'; A) \to H_{n-1}(Y'; A)$ of the long exact sequence for $(Y, Y')$ and the map $H_{n-1}(Y'; A) \to H_{n-1}(Y', Y''; A)$ from the long exact sequence of $(Y', Y'')$.*

*Proof.* We write $C = C(Y; A)$, $C' = C(Y'; A)$ and $C'' = C(Y''; A)$. The inclusions and quotient maps induce a commuatative diagram of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & C' \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & C/C' \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & C'/C'' \arrow[r] & C/C'' \arrow[r] & C/C' \arrow[r] & 0
\end{tikzcd}$$

where the lines are exact since there are canonical isomorphisms of abelian groups $C_n/C'_n \cong (C_n/C''_n)/(C'_n/C''_n)$. The long exact sequence of the tripel is the long exact homology sequence that arises from the lower short exact sequence of chain complexes. The connecting homomorphism $H_n(C/C') \to H_{n-1}(C'/C'')$ has the factorization described in the proposition because by naturality of long exact homology sequences, it is the composite of the connecting homomorphism $H_n(C/C') \to H_{n-1}(C')$ of the upper sequence and the homomorphism $H_{n-1}(C') \to H_{n-1}(C'/C'')$ induced by the vertical map on the left. □

## The cellular chain complex

We now use the long exact sequence of a triple to construct a chain complex associated with a CW-complex.

**Definition 10.3.** Let $X$ be an absolute CW-complex and let $A$ be an abelian group. The *cellular chain complex* of $X$ with coefficients in $A$ has $\widetilde{C}_n(X;A) = H_n(X_n, X_{n-1};A)$ as its group of $n$-chains. The differential

$$\widetilde{\partial}_n\colon \widetilde{C}_n(X;A) \to \widetilde{C}_{n-1}(X;A)$$

is the connecting homomorphism $H_n(X_n, X_{n-1};A) \to H_{n-1}(X_{n-1}, X_{n-2};A)$ from the long exact sequence of the triple $X_{n-2} \subseteq X_{n-1} \subseteq X_n$.

The next lemma shows that this indeed defines a chain complex.

**Lemma 10.4.** *The differential of the cellular chain complex satisfies* $\widetilde{\partial}_{n-1} \circ \widetilde{\partial}_n = 0$.

*Proof.* Writing out the definitions, $\widetilde{\partial}_{n-1} \circ \widetilde{\partial}_n$ is the composite of

$$\begin{aligned} H_n(X_n, X_{n-1};A) \to H_{n-1}(X_{n-1};A) \to H_{n-1}(X_{n-1}, X_{n-2};A)& \\ \to H_{n-2}(X_{n-2};A) \to H_{n-2}(X_{n-2}, X_{n-3};A)&\ . \end{aligned}$$

The composite of the two middle maps is 0 since they are subsequent maps in the long exact sequence of the pair $(X_{n-1}, X_{n-2})$. □

We now let $X$ be an absolute CW-complex, write $J_n$ for its set of $n$-cells and choose characteristic maps $\chi_j\colon D^n \to X_n$ for all $j \in J_n$ so that there are pushout squares

$$\begin{tikzcd} J_n \times \partial D^n \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ J_n \times D^n \arrow[r, "\amalg \chi_j"] & X_n\ . \end{tikzcd}$$

We note that we can view the characteristic maps as maps of pairs $\chi_j\colon (D^n, \partial D^n) \to (X_n, X_{n-1})$.

**Lemma 10.5.** *The characteristic maps induce isomorphisms*

$$\textstyle\bigoplus_{J_n} H_n(D^n, \partial D^n;A) \to H_n(X_n, X_{n-1};A)$$

*for all* $n \geq 0$.

*Proof.* The characteristic maps induce the horizontal maps in the following commutative diagram:

$$\begin{tikzcd} \bigoplus_{J_n} H_n(D^n, \partial D^n;A) \arrow[r] \arrow[d, "\cong"'] & H_n(X_n, X_{n-1};A) \arrow[d, "\cong"] \\ \bigoplus_{J_n} H_n(D^n, D^n \setminus \{0\};A) \arrow[r] & H_n(X_n, X_n \setminus \bigcup_{j\in J_n} \chi_j(0);A) \\ \bigoplus_{J_n} H_n(\mathring{D}^n, \mathring{D}^n \setminus \{0\};A) \arrow[r] \arrow[u, "\cong"] & H_n\Big(X_n \setminus X_{n-1}, X_n \setminus \Big(X_{n-1} \cup \bigcup_{j\in J_n} \chi_j(0)\Big);A\Big) \arrow[u, "\cong"'] \end{tikzcd}$$

The vertical maps are induced by the inclusions. They are isomorphisms by excision and homotopy invariance. The lower horizontal map is an isomorphism since it is induced by a homeomorphism. Hence the upper horizontal map is an isomorphism. □

**Corollary 10.6.** *If $X$ is an absolute CW-complex with set of $n$-cells $J_n$, then the group of cellular $n$-chains $\widetilde{C}_n(X;A)$ is isomorphic to $A[J_n]$.*

*Proof.* This follows from the last lemma and our earlier computation $H_n(D^n, \partial D^n; A) \cong A$. □

The cellular chain complex is useful because of the following result.

**Theorem 10.7.** *Let $X$ be an absolute CW-complex and let $A$ be an abelian group. Then there is an isomorphism*

$$H_n(X;A) \cong H_n(\widetilde{C}(X;A))$$

*identifying the homology groups of the cellular chain complex with the singular homology groups of $X$.*

Together with the previous corollary, this result implies that the homology groups of a CW-complex can be computed from the chain complex $\widetilde{C}(X;A)$ that often has considerably more accessible chain groups than the singular chain complex. To really make use of this in examples, we also need to get hands on the differential of the cellular chain complex. This will be done in the next lecture.

The theorem has the following powerful implications about the Euler characteristic

$$\chi(X) = \sum_{n \geq 0} (-1)^n \cdot |J_n|$$

of a finite CW-complex $X$ with set of $n$-cells $J_n$.

**Corollary 10.8.** *Let $X$ be a finite CW-complex and let $k$ be a field. Then the homology groups $H_n(X;k)$ are finite dimensional $k$-vector spaces, $H_n(X;k)$ is only non-trivial for finitely many $n$, and*

$$\chi(X) = \sum_{n \geq 0} (-1)^n \cdot \dim_k H_n(X;k) \ .$$

*Proof.* By Theorem 10.7, $H_n(X;k)$ is isomorphic to the homology group of a chain complex of finite dimensional $k$-vector spaces. Since the passage to sub vector spaces and quotient factor spaces preserves the property of being finite dimensional, $H_n(X;k)$ is finite dimensional. The assumption that $X$ is finite also implies that $H_n(X;k)$ is trivial if $n$ exceeds the maximal dimension of a cell in $X$. The description of the Euler characteristic follows by applying Exercise 2.4 to the cellular chain complex of $X$. □

**Theorem 10.9.** *If $X$ and $Y$ are homotopy equivalent finite CW-complexes, then $\chi(X) = \chi(Y)$.*

*Proof.* This is an immediate consequence of the last corollary and the homotopy invariance of singular homology. □

## The proof of Theorem 10.7

Throughout this section, we let $X$ be an absolute CW-complex and $A$ be an abelian group. The following lemma will be repeatedly used in the proof of the theorem.

**Lemma 10.10.** *Let $n > m \geq -1$ be integers. Then $H_k(X_n, X_m; A) = 0$ for all $k$ with $k > n$ or $k \leq m$.*

*Proof.* We prove the statement by induction over $n - m$. The case $n - m = 1$ results from Lemma 10.5. For the inductive step, we consider the triple $X_m \subseteq X_{n-1} \subseteq X_n$ and the following part of its long exact homology sequence:

$$H_k(X_{n-1}, X_m; A) \to H_k(X_n, X_m; A) \to H_k(X_n, X_{n-1}; A)$$

The induction hypothesis implies that the first and last term are trivial. Exactness implies that the middle term is trivial, too. $\square$

**Corollary 10.11.** *The inclusions $X_{n+1} \to X_{n+2} \to \dots$ induce a sequence of isomorphisms*

$$H_n(X_{n+1}; A) \xrightarrow{\cong} H_n(X_{n+2}; A) \xrightarrow{\cong} \dots .$$

*Proof.* Let $q \geq 1$ and consider the exact sequence

$$H_{n+1}(X_{n+q+1}, X_{n+q}; A) \to H_n(X_{n+q}; A) \to H_n(X_{n+q+1}; A) \to H_n(X_{n+q+1}, X_{n+q}; A)$$

arising from the long exact sequence of $(X_{n+q+1}, X_{n+q})$. By the last lemma, the first and the last term are trivial. So the middle map is an isomorphism. $\square$

**Proposition 10.12.** *The inclusion $X_{n+1} \to X$ induces an isomorphism $H_n(X_{n+1}; A) \to H_n(X; A)$ .*

*Proof.* For surjectivity, consider a cycle $c \in C_n(X; A)$ representing $[c] \in H_n(X; A)$. By Corollary 10.1, there is a $p \geq 0$ such that $c$ is in the image of $C_n(X_p; A) \to C_n(X; A)$, and we may assume that $p > n$. Then $[c]$ is in the image of $H_n(X_p; A) \to H_n(X; A)$. By the previous corollary, it is also in the image of $H_n(X_{n+1}; A) \to H_n(X; A)$.

For injectivity, we consider a cycle $c \in C_n(X_{n+1}; A)$ that represents the trivial homology class in $H_n(X; A)$. By definition, this means that there is a $b \in C_{n+1}(X; A)$ with $\partial(b) = c$. Arguing as above, there is a $p > n$ such that $b$ is in the image of $C_{n+1}(X_p; A) \to C_{n+1}(X; A)$. Hence $c$ represents the trivial class in $H_n(X_p; A)$. By the previous corollary, it also represents the trivial class in $H_n(X_{n+1}; A)$. $\square$

**Lemma 10.13.** *The inclusions $(X_{n+1}, X_i) \subseteq (X_{n+1}, X_{i+1})$ induce a chain of isomorphisms*

$$H_n(X_{n+1}; A) \cong H_n(X_{n+1}, X_{-1}; A) \xrightarrow{\cong} H_n(X_{n+1}, X_0; A) \xrightarrow{\cong} \dots \xrightarrow{\cong} H_n(X_{n+1}, X_{n-2}; A)$$

*Proof.* The long exact sequence of the triple $X_{q-1} \subseteq X_q \subseteq X_{n+1}$ provides an exact sequence

$$H_n(X_q, X_{q-1}; A) \to H_n(X_{n+1}, X_{q-1}; A) \to H_n(X_{n+1}, X_q; A) \to H_{n-1}(X_q, X_{q-1}; A)$$

By Lemma 10.5, the first and the last term are trivial if $n - 1 > q$, i.e., if $q \leq n - 2$. Exactness provides the desired isomorphism. $\square$

The following corollary summarizes what we have proved so far:

**Corollary 10.14.** *The inclusions induce isomorphisms*

$$H_n(X; A) \xleftarrow{\cong} H_n(X_{n+1}; A) \xrightarrow{\cong} H_n(X_{n+1}, X_{n-2}; A) .$$

Hence it remains to construct an isomorphism between $H_n(X_{n+1}, X_{n-2}; A)$ and the $n$-th homology group of the cellular chain complex.

*Proof of Theorem 10.7.* The long exact sequence of the triple $X_{n-2} \subseteq X_{n-1} \subseteq X_n$ provides the upper exact sequence in the following commutative diagram:

$$\begin{tikzcd}
H_n(X_{n-1}, X_{n-2}; A) \arrow[r] \arrow[d, "\cong"] & H_n(X_n, X_{n-2}; A) \arrow[r, "\iota"] \arrow[d, "\cong"] & H_n(X_n, X_{n-1}; A) \arrow[r] \arrow[d, "="] & H_{n-1}(X_{n-1}, X_{n-2}; A) \arrow[d, "="] \\
0 \arrow[r] & \ker \widetilde{\partial}_n \arrow[r] & \widetilde{C}_n(X; A) \arrow[r, "\widetilde{\partial}_n"] & \widetilde{C}_{n-1}(X; A)
\end{tikzcd}$$

Here the two vertical arrows on the right hand side are identity maps by the definition of the cellular chain complex, and the left hand vertical arrow is an isomorphism by Lemma 10.5. Hence exactness implies that the map $\iota\colon H_n(X_n, X_{n-2}; A) \to H_n(X_n, X_{n-1}; A)$ is isomorphic to the inclusion $\ker \widetilde{\partial}_n \to \widetilde{C}_n(X; A)$.

For the next step, we consider the long exact sequence for $X_{n-2} \subseteq X_n \subseteq X_{n+1}$. It provides the upper exact sequence in the diagram

$$\begin{tikzcd}
H_{n+1}(X_{n+1}, X_n; A) \arrow[r, "\rho"] & H_n(X_n, X_{n-2}; A) \arrow[r] \arrow[d, "\iota"] & H_n(X_{n+1}, X_{n-2}; A) \arrow[r] & H_n(X_{n+1}, X_n; A) \arrow[d, "\cong"] \\
& H_n(X_n, X_{n-1}; A) & & 0
\end{tikzcd} \tag{10.1}$$

The vertical isomorphism on the right hand side again arises from Lemma 10.5. Writing out the definitions, the composite $\iota \circ \rho$ in (10.1) is the composite of the maps

$$H_{n+1}(X_{n+1}, X_n; A) \to H_n(X_n; A) \to H_n(X_n, X_{n-2}; A) \to H_n(X_n, X_{n-1}; A)$$

arising from the long exact sequences of the relevant pairs. This composite coincides with $\widetilde{\partial}_{n+1}\colon \widetilde{C}_{n+1}(X; A) \to \widetilde{C}_n(X; A)$ . Hence the above identification of $\ker \widetilde{\partial}_n$ and the exactness of the upper sequence in (10.1) show that $(\ker \widetilde{\partial}_n)/(\operatorname{im} \widetilde{\partial}_{n+1})$ is isomorphic to $H_n(X_{n+1}, X_{n-2}; A)$ . Together with the previous corollary, this shows the claim. □

### The homology of the complex projective spaces

We now give an example for a computation of cellular homology groups that is particularly easy because it will only involve trivial differentials $\widetilde{\partial}_n$.

**Definition 10.15.** Let $n \geq 1$ be an integer. The complex projective space $\mathbb{C}P^n$ is the quotient space of $\mathbb{C}^{n+1} \setminus \{0\}$ by the equivalence relation $\sim$ defined by $x \sim \lambda x$ for $x \in \mathbb{C}^{n+1} \setminus \{0\}$ and $\lambda \in \mathbb{C} \setminus \{0\}$.

We can view $\mathbb{C}P^{n-1}$ as a subspace of $\mathbb{C}P^n$ via the map $i_n\colon \mathbb{C}P^{n-1} \to \mathbb{C}P^n$ induced by

$$\mathbb{C}^n \to \mathbb{C}^{n+1}, \quad (z_1, \dots, z_n) \mapsto (z_1, \dots, z_n, 0) .$$

Analogous to the case of the real projective space, there is a pushout

$$\begin{tikzcd}
\partial D^{2n} \arrow[r, "\chi_{2n}|_{\partial D^{2n}}"] \arrow[d, "\mathrm{incl}"'] & \mathbb{C}P^{n-1} \arrow[d, "i_n"] \\
D^{2n} \arrow[r, "\chi_{2n}"] & \mathbb{C}P^n
\end{tikzcd}$$

that exhibits $\mathbb{C}P^n$ as a space arising from $\mathbb{C}P^{n-1}$ by attaching a $2n$-cell. To see this, one can view $D^{2n}$ as $\{(z_1,\dots,z_n) \in \mathbb{C}^n \mid |z_1|^2+\cdots+|z_n|^2 \leq 1\}$ and use the characteristic map

$$\chi_{2n}\colon D^{2n} \to \mathbb{C}P^n, \quad (z_1,\dots,z_n) \mapsto \left(z_1,\dots,z_n,\sqrt{1-|z_1|^2-\cdots-|z_n|^2}\right) .$$

Hence $\mathbb{C}P^k$ is a CW-complex with one cell in every even dimension between 0 and $2k$. Its cellular chain complex $\widetilde{C}(\mathbb{C}P^k;A)$ has the form

$$0 \to A \to 0 \to A \to \cdots \to A \to 0 \to A .$$

This implies that all cellular differentials have to be trivial. From this we deduce the following result.

**Theorem 10.16.**

$$H_n(\mathbb{C}P^k;A) = \begin{cases} A & \textit{if } n \textit{ is even and } 0 \leq n \leq 2k, \\ 0 & \textit{else} \end{cases}$$

# Lecture 11: Computations with cellular homology

Last time we introduced the cellular chain complex $\widetilde{C}(X;A)$ of a CW-complex $X$ with coefficients in an abelian group $A$. By definition, its group of $n$-chains $\widetilde{C}_n(X;A)$ is the relative homology group $H_n(X_n, X_{n-1};A)$. The differential $\widetilde{\partial}_n \colon \widetilde{C}_n(X;A) \to \widetilde{C}_{n-1}(X;A)$ is the connecting homomorphism in the long exact sequence of the triple $(X_n, X_{n-1}, X_{n-2})$. Our next aim is to examine this differential in more detail so that we are able to perform computations of cellular homology groups also in cases where the $\widetilde{\partial}_n$ are non-trivial. By the main theorem of the last lecture, this is an approach to calculate singular homology groups.

## The differential of the cellular chain complex

We will now consider an absolute CW-complex $X$, focus on homology with coefficients in $A = \mathbb{Z}$, and omit the coefficient group from the notation. As usual, we write $J_n$ for the set of $n$-cells in $X$ . Moreover, we choose a characteristic map $\chi_\alpha \colon D^n \to X_n$ for every $n$-cell $\alpha \in J_n$. Lastly, we choose a generator $\mathbf{1}_n \in H_n(D^n, \partial D^n)$ for all $n \geq 0$. (In other words, we choose an isomorphism $\iota_n \colon \mathbb{Z} \to H_n(D^n, \partial D^n)$ and set $\mathbf{1}_n = \iota_n(1)$ ). Then Lemma 10.5 implies:

**Corollary 11.1.** *The elements $e^n_\alpha = (\chi_\alpha)_*(\mathbf{1}_n) \in H_n(X_n, X_{n-1})$ form a $\mathbb{Z}$-basis of $H_n(X_n, X_{n-1})$.*

In other words, any element $a \in H_n(X_n, X_{n-1})$ can be written in a unique way as a linear combination of finitely many $e^n_\alpha$ with coefficients in $\mathbb{Z}$. Using the basis in degree $n-1$, we can write $\widetilde{\partial}_n(e^n_\alpha)$ as a linear combination

$$\widetilde{\partial}_n(e^n_\alpha) = \sum_{\beta \in J_{n-1}} d_{\alpha\beta} \cdot e^{n-1}_\beta$$

for suitable coefficients $d_{\alpha\beta} \in \mathbb{Z}$, and these coefficients determine $\widetilde{\partial}_n$. Our aim is to express the coefficients $d_{\alpha\beta}$ using topological structure of the CW-complex $X$.

The first step is to give a different characterization of the image of $e^n_\alpha$ under the differential $\widetilde{\partial}_n \colon H_n(X_n, X_{n-1}) \to H_{n-1}(X_{n-1}, X_{n-2})$. For this, we write $\widetilde{\delta}_n \colon H_n(D^n, \partial D^n) \to \widetilde{H}_{n-1}(\partial D^n)$ for the connecting homomorphism in the long exact sequence of the reduced homology groups of the pair $(D^n, \partial D^n)$. It sends our chosen generator $\mathbf{1}_n \in H_n(D^n, \partial D^n)$ to a generator $\widetilde{\delta}_n(\mathbf{1}_n)$ of $\widetilde{H}_{n-1}(\partial D^n)$.

**Lemma 11.2.** *The element $\widetilde{\partial}_n(e^n_\alpha) \in H_{n-1}(X_{n-1}, X_{n-2})$ equals the image of $\widetilde{\delta}_n(\mathbf{1}_n)$ under the composite*

$$\widetilde{H}_{n-1}(\partial D^n) \to \widetilde{H}_{n-1}(X_{n-1}) \to H_{n-1}(X_{n-1}, X_{n-2})$$

*where the first map is induced by $\chi_\alpha|_{\partial D^n} \colon \partial D^n \to X_{n-1}$ and the second map is from the long exact sequence of reduced homology groups of the pair $(X_{n-1}, X_{n-2})$.*

*Proof.* The argument that we used in Proposition 10.2 to identify the connecting homomorphism in the long exact sequence of a tripel also applies when we use the chain complexes that give rise to reduced singular homology (rather than ordinary singular homology). Hence the differential of the cellular chain complex can also be described as the composite

$$H_n(X_n, X_{n-1}) \to \widetilde{H}_{n-1}(X_{n-1}) \to H_{n-1}(X_{n-1}, X_{n-2})$$

resulting from the long exact reduced homology sequences of $(X_n, X_{n-1})$ and $(X_{n-1}, X_{n-2})$. By naturality, precomposing the first map with $(\chi_\alpha)_*$ is the same as first using the connecting map $H_n(D^n, \partial D^n) \to \widetilde{H}_{n-1}(\partial D^n)$ of the pair $(D^n, \partial D^n)$ and then applying the map induced by $\chi_\alpha$. □

Now let $b = \sum_{\gamma \in J_{n-1}} b_\gamma e_\gamma^{n-1} \in H_{n-1}(X_{n-1}, X_{n-2})$ be an element that has coefficients $b_\gamma$ with respect to our chosen basis. Our aim is to determine the coefficient $b_\beta$ from $b$ for a fixed $\beta \in J_{n-1}$.

To do so, we consider the following commutative diagram of continuous maps explained below:

$$\begin{tikzcd}
X_{n-1}/X_{n-2} \arrow[dr, "q_\beta"'] & (J_{n-1} \times D^{n-1})/(J_{n-1} \times \partial D^{n-1}) \arrow[l, "\cong"'] \arrow[d, "\pi_\beta"] & D^{n-1}/\partial D^{n-1} \arrow[l, "\iota_\gamma"'] \arrow[dl, "g_{\gamma\beta}"] \\
& D^{n-1}/\partial D^{n-1} &
\end{tikzcd} \tag{11.1}$$

The upper left hand map is induced by the disjoint union of the characteristic maps of the $n-1$-cells. It is bijective since $X_{n-1} \setminus X_{n-2}$ is the union of the open $n-1$-cells, and it is a homeomorphism because the characteristic maps are closed maps. The vertical map $\pi_\beta$ is induced by the identity on the copy of $D^{n-1}$ indexed by $\beta$ and the constant map to the basepoint on all other copies of $D^{n-1}$ (where the collapsed space $\partial D^{n-1}$ serves as the basepoint). The map $q_\beta$ is defined to be the composite of the inverse of the upper left hand homeomorphism and the map $\pi_\beta$. The map $\iota_\gamma$ is the inclusion of the summand indexed by $\gamma \in J_{n-1}$. Consequently, the composite $g_{\gamma\beta} = \pi_\beta \iota_\gamma$ is the identity if $\beta = \gamma$ and the constant map to the basepoint otherwise.

Next we note that for a pair of spaces $(Y, Y')$, there is a natural map

$$p \colon H_k(Y, Y') \to \widetilde{H}_k(Y/Y')$$

from its relative homology to the reduced homology of the quotient. It is given as the composite of the map $H_k(Y, Y') \to H_k(Y/Y', Y'/Y')$ induced by the quotient map and the inverse of the isomorphism $\widetilde{H}_k(Y/Y') \to H_k(Y/Y', Y'/Y')$ that results from the long exact reduced homology sequence of the pair $(Y/Y', Y'/Y')$ and the fact that $H_l(Y'/Y', Y'/Y') \cong 0$ for all $l$.

For the pair $(D^{n-1}, \partial D^{n-1})$, this natural map $p$ provides an isomorphism

$$H_{n-1}(D^{n-1}, \partial D^{n-1}) \to \widetilde{H}_{n-1}(D^{n-1}/\partial D^{n-1}).$$

The fact that it is an isomorphism is a consequence of Exercise 6.4, and can be seen by using the commutative diagram

$$\begin{tikzcd}
H_k(D^m, \partial D^m) \arrow[r, "\cong"] \arrow[d] & H_k(D^m, U) \arrow[d] & H_k(\mathring{D}^m, \mathring{D}^m \setminus \{0\}) \arrow[l, "\cong"'] \arrow[d, "\cong"] \\
H_k(D^m/\partial D^m, \partial D^m/\partial D^m) \arrow[r, "\cong"] & H_k(D^m/\partial D^m, U/\partial D^m) & H_k((D^m/\partial D^m) \setminus P, U/(\partial D^m) \setminus P) \arrow[l, "\cong"']
\end{tikzcd}$$

in which $U = D^m \setminus \{0\}$, $P = \partial D^m/\partial D^m$, the left hand upper horizontal map is an isomorphism by homotopy invariance, the left hand lower horizontal map is an isomorphism by homotopy invariance and Corollary 8.9, the right hand horizontal maps are isomorphisms by excision, and the left hand vertical map is an isomorphism because it is induced by a homeomorphism.

(It also follows from Exercises 6.3 and 6.4 that the map $p$ for the pair $(X_{n-1}, X_{n-2})$ is an isomorphism, but we will not make use of this here.)

As the final ingredient for the next proposition, we claim that we can choose a homeomorphism $h_{n-1}\colon D^{n-1}/\partial D^{n-1} \to \partial D^n$ such that the composite

$$H_{n-1}(D^{n-1}, \partial D^{n-1}) \to \widetilde{H}_{n-1}(D^{n-1}/\partial D^{n-1}) \xrightarrow{(h_{n-1})_*} \widetilde{H}_{n-1}(\partial D^n)$$

sends our chosen generator $\mathbf{1}_{n-1}$ to the image $\widetilde{\delta}_n(\mathbf{1}_n)$ of $\mathbf{1}_n$ under $\widetilde{\delta}_n\colon H_n(D^n, \partial D^n) \to \widetilde{H}_{n-1}(\partial D^n)$. To verify this claim, we choose any homeomorphism $D^{n-1}/\partial D^{n-1} \to \partial D^n$. If it satisfies the condition on the generators, we take it as $h_{n-1}$. If not, we compose it with a map of degree $-1$ to obtain $h_{n-1}$.

Altogether, we obtain a commutative diagram where the left and square is induced by the maps $p$ discussed above, the middle triangle is obtained by applying reduced homology to the outer triangle in (11.1), and $h_{n-1}$ is the map discussed in the previous paragraph.

$$\begin{tikzcd}
H_{n-1}(D^{n-1}, \partial D^{n-1}) \arrow[r, "\cong", "p"'] \arrow[d, "(\chi_\gamma)_*"'] & \widetilde{H}_{n-1}(D^{n-1}/\partial D^{n-1}) \arrow[r, "(g_{\gamma\beta})_*"] \arrow[d, "(\chi_\gamma)_*"'] & \widetilde{H}_{n-1}(D^{n-1}/\partial D^{n-1}) \arrow[r, "(h_{n-1})_*"] & \widetilde{H}_{n-1}(\partial D^n) . \\
H_{n-1}(X_{n-1}, X_{n-2}) \arrow[r, "p"'] & \widetilde{H}_{n-1}(X_{n-1}/X_{n-2}) \arrow[ru, "(q_\beta)_*"'] & &
\end{tikzcd}$$

When we write an element in $H_{n-1}(X_{n-1}, X_{n-2})$ as a linear combination of the basis vectors $e_\gamma^{n-1}$, we can now use this diagram to read off the coefficient of $e_\beta^{n-1}$ :

**Proposition 11.3.** *Let $b = \sum_{\gamma \in J_{n-1}} b_\gamma e_\gamma^{n-1}$ be an element in $H_{n-1}(X_{n-1}, X_{n-2})$. Then the bottom composite $H_{n-1}(X_{n-1}, X_{n-2}) \to \widetilde{H}_{n-1}(\partial D^n)$ in the previous diagram sends $b$ to $b_\beta \cdot \widetilde{\delta}_n(\mathbf{1}_n)$.*

*Proof.* The previous commutative diagram implies that if $\gamma \neq \beta$, then the bottom composite sends $b_\gamma \cdot e_\gamma^{n-1} = b_\gamma \cdot (\chi_\gamma)_*(\mathbf{1}_{n-1})$ to 0 since $g_{\gamma\beta}$ is the constant map and thus induces the trivial map on reduced homology. Similarly, $b_\beta \cdot e_\beta^{n-1}$ is sent to $b_\beta \cdot \widetilde{\delta}_n(\mathbf{1}_n)$ since $g_{\beta\beta}$ is the identity and $(h_{n-1})_*$ sends $\mathbf{1}_{n-1}$ to $\widetilde{\delta}_n(\mathbf{1}_n)$. The general case follows by additivity. □

As a final ingredient, we write $q\colon X_{n-1} \to X_{n-1}/X_{n-2}$ for the quotient map and consider the composite

$$\partial D^n \xrightarrow{\chi_\alpha|_{\partial D^n}} X_{n-1} \xrightarrow{q} X_{n-1}/X_{n-2} \xrightarrow{q_\beta} D^{n-1}/\partial D^{n-1} \xrightarrow{h_{n-1}} \partial D^n .$$

We also recall that the degree of a continuous map $f\colon \partial D^n \to \partial D^n$ is the integer $\deg(f)$ with $\deg(f) \cdot a = f_*(a)$ for all $a \in \widetilde{H}_{n-1}(\partial D^n)$.

**Theorem 11.4.** *The coefficient $d_{\alpha\beta}$ in $\widetilde{\partial}_n(e_\alpha^n) = \sum_{\beta \in J_{n-1}} d_{\alpha\beta} \cdot e_\beta^{n-1}$ is given by*

$$d_{\alpha\beta} = \deg(h_{n-1} \circ q_\beta \circ q \circ \chi_\alpha|_{\partial D^n}) ,$$

*the degree of the map $\partial D^n \to \partial D^n$ discussed before the theorem.*

*Proof.* By combining Lemma 11.2 and Proposition 11.3, we see that the composite

$$\begin{aligned}\widetilde{H}_{n-1}(\partial D^n) \xrightarrow{(\chi_\alpha)_*} \widetilde{H}_{n-1}(X_{n-1}) \to H_{n-1}(X_{n-1}, X_{n-2}) \xrightarrow{p} \widetilde{H}_{n-1}(X_{n-1}/X_{n-2})& \\ \xrightarrow{(q_\beta)_*} \widetilde{H}_{n-1}(D^{n-1}/\partial D^{n-1}) \xrightarrow{(h_{n-1})_*} \widetilde{H}_{n-1}(\partial D^n)&\end{aligned}$$

sends $\widetilde{\delta}_n(\mathbf{1}_n)$ to $d_{\alpha\beta} \cdot \widetilde{\delta}_n(\mathbf{1}_n)$. The commutative square

$$\begin{tikzcd}
\widetilde{H}_{n-1}(X_{n-1}) \arrow[rr, "q_*"] \arrow[d] & & \widetilde{H}_{n-1}(X_{n-1}/X_{n-2}) \arrow[d, "\cong"] \\
H_{n-1}(X_{n-1}, X_{n-2}) \arrow[rr] \arrow[urr, "p"] & & H_{n-1}(X_{n-1}/X_{n-2}, X_{n-2}/X_{n-2})
\end{tikzcd}$$

shows that the composite of the second and the third map in that chain equals the map induced by $q$. Since $\widetilde{\delta}_n(\mathbf{1}_n)$ is a generator of $\widetilde{H}_{n-1}(\partial D^n) \cong \mathbb{Z}$, this implies the statement about mapping degrees. □

In words, this means that we get the coefficients that determine the differential in the cellular complex as follows. We first take the attaching map $\partial D^n \to X_{n-1}$ of the $n$-cell $\alpha$, then collapse the $(n-2)$-skeleton $X_{n-2}$, then collapse all $(n-1)$-cells except $\beta$, and lastly identify the resulting space $D^{n-1}/\partial D^{n-1}$ with $\partial D^n$ again using $h_{n-1}$. The degree of this map then gives the coefficient $d_{\alpha\beta}$.

**Remark 11.5.** The homeomorphism $h_{n-1}$ chosen above also exists when $n = 1$. To see this, recall that $\partial D^0 = \emptyset$ by our earlier convention. Collapsing an empty subspace has the effect of forming the disjoint union with a one-point space. The latter statement can be viewed as a convention, or deduced from defining $L/K$ to be the pushout of $\{*\} \leftarrow K \to L$ for a general pair of spaces $K \subseteq L$. In particular, there exists a homeomorphism $h_0 \colon D^0/\partial D^0 \to \partial D^1$ since both spaces are discrete spaces with two points.

**Remark 11.6.** Suppose that $h'_{n-1} \colon D^{n-1}/\partial D^{n-1} \to \partial D^n$ is a homeomorphism whose induced map in homology differs from the induced map of $h_{n-1}$ by $-1$. Then the degree of

$$h'_{n-1} \circ q_\beta \circ q \circ \chi_\alpha|_{\partial D^n}$$

differs from the degree of

$$h_{n-1} \circ q_\beta \circ q \circ \chi_\alpha|_{\partial D^n}$$

by the sign $-1$. This implies that if we ignore the compatibility with the generators of the groups $H_n(D^n, \partial D^n)$ when choosing $h_{n-1}$, we will still determine $\widetilde{\partial}$ up to a sign. This is sufficient for the computation of cellular homology groups because changing the sign of a differential in a chain complex does not change its kernel and image.

So in examples, we will just choose some homeomorphism $D^{n-1}/\partial D^{n-1} \to \partial D^n$ and keep in mind that we determine $\widetilde{\partial}_n$ only up to a sign. One can actually avoid this ambiguity by inductively making coherent choices for the generators of $H_n(D^n, \partial D^n)$ and the homeomorphisms $D^{n-1}/\partial D^{n-1} \to \partial D^n$. However, this has the disadvantage that in examples, we then have to make sure that generators of $H_n(D^n, \partial D^n)$ and the homeomorphisms $D^{n-1}/\partial D^{n-1} \to \partial D^n$ we are working with lead to the same signs as the ones chosen for the general theory.

## The homology of the torus

The torus $T$ can be described as the quotient space of the square $[0,1] \times [0,1]$ by the equivalence relation generated by the identifications $(s,0) \sim (s,1)$ for all $s \in [0,1]$ and $(0,t) \sim (1,t)$ for all $t \in [0,1]$. This is displayed in Figure (11.1). In the notation used in the figure, we glue each of the two edges $v_0, v_1$ and $w_0, w_1$ together using the orientation indicated by the arrows. Gluing only the first two edges together turns the square into a cylinder, and gluing together the second pair turns the cylinder into a torus.

The square $[0,1] \times [0,1]$ has a CW-structure with four 0-cells, four 1-cells and one 2-cell. Under the identifications that are made when passing to the torus, this gives rise to a CW-structure on the torus with one 0-cell $p$, two 1-cells $v$ and $w$, and one 2-cell $u$. Hence the cellular chain complex $\widetilde{C}(T;\mathbb{Z})$ has the form

$$0 \to \mathbb{Z}\{e^2_u\} \to \mathbb{Z}\{e^1_v\} \oplus \mathbb{Z}\{e^1_w\} \to \mathbb{Z}\{e^0_p\} .$$

Since the 1-sphere $S^1$ has a CW-structure with one 0-cell and one 1-cell and we already know that $H_k(S^1;\mathbb{Z}) \cong \mathbb{Z}$ for $k = 0, 1$, the differential $\widetilde{\partial}_1$ in $\widetilde{C}(S^1;\mathbb{Z})$ must be zero. The inclusion of one of the edges of the square $[0,1] \times [0,1]$ induces a map $S^1 \to T$ that maps the 1-cell of $S^1$ to one of the 1-cells of $T$ and the 0-cell of $S^1$ to the 0-cell of $T$. The naturality of the long exact sequence of triples provides a map of cellular chain complexes, and the observation about the $\widetilde{\partial}_1$ of $S^1$ implies that the $\widetilde{\partial}_1$ of $T$ must be trivial on the generators $v$ and $w$. To compute $\widetilde{\partial}_2$, we use the last theorem. It implies that when we write $\widetilde{\partial}_2(e^2_u) = a_v \cdot e^1_v + a_w \cdot e^1_w$, then the coefficients $a_v$ and $a_w$ are given by mapping degrees. For $a_v$, the relevant map is given by mapping a circle to $\partial([0,1] \times [0,1])$ by first running through $v$, then through $w$, then through $v^{-1}$, and then through $w^{-1}$. Composing with the map that collapses the cell $w$ to a point, we are left with a closed path that first runs through $v$ and then through $v^{-1}$. Since this is a contractible closed path, $a_v = 0$. The same argument shows $a_w = 0$. Hence all differentials in the cellular chain complex are trivial, and it follows that

$$H_k(T;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } k = 1 \\ \mathbb{Z} & \text{if } k = 2 \\ 0 & \text{if } k > 2 \end{cases}$$

## Computing mapping degrees

We now develop a new tool for computing mapping degrees. We work in the following setup: Let $n \geq 1$, let $f\colon S^n \to S^n$ be a continuous map, and let $v \in S^n$ be a point such that $f^{-1}(v) = \{u_1, \ldots, u_m\}$ is a finite set. Let $U_1, \ldots, U_m$ be disjoint open neighborhoods of the points $u_1, \ldots, u_m$ in the preimage of $v$, and let $V$ be an open neighborhood of $v$ such that $f(U_i) \subseteq V$ holds for all $i$ with $1 \leq i \leq m$. One can think of these neighborhoods as "small" open balls around the points $u_1, \ldots, u_m$ and $v$.

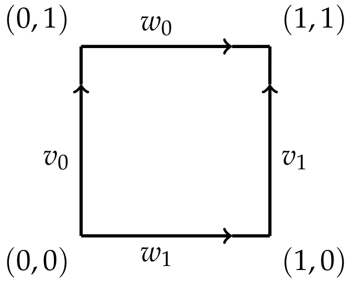

Figure 11.1: The torus as a quotient of a square

Now we consider the following commutative diagrams whose maps are explained below:

$$\begin{tikzcd}
 & H_n(U_i, U_i \setminus \{u_i\}; \mathbb{Z}) \arrow[r, "f_*"] \arrow[dl, "\cong"'] \arrow[d, "k_i"] & H_n(V, V \setminus \{v\}; \mathbb{Z}) \arrow[d, "\cong"] \\
H_n(S^n, S^n \setminus \{u_i\}; \mathbb{Z}) & H_n(S^n, S^n \setminus f^{-1}(v); \mathbb{Z}) \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n \setminus \{v\}; \mathbb{Z}) \\
 & H_n(S^n; \mathbb{Z}) \arrow[r, "f_*"] \arrow[u, "j"'] \arrow[ul, "\cong"] & H_n(S^n; \mathbb{Z}) \arrow[u, "\cong"']
\end{tikzcd}$$

Here

- $k_i$ is induced by the inclusion $(U, U \setminus \{u_i\}) \subseteq (S^n, S^n \setminus f^{-1}(v))$,
- $p_i$ is induced by the inclusion $(S^n, S^n \setminus f^{-1}(v)) \subseteq (S^n, S^n \setminus \{u_i\})$,
- $j$ is the map in the long exact sequence of the pair $(S^n, S^n \setminus f^{-1}(v))$,
- the two upper isomorphisms result from applying the excision theorem,
- the two lower isomorphisms result from homotopy invariance, and
- all morphisms decorated with $f_*$ are induced by $f$.

Now we choose a generator $\mathbf{1} \in H_n(S^n; \mathbb{Z})$. In other words, we choose an isomorphism $\mathbb{Z} \to H_n(S^n; \mathbb{Z})$ and let $\mathbf{1}$ be the image of $1 \in \mathbb{Z}$ under the chosen isomorphism. Via the isomorphisms on the two sides of the diagram, the generator $\mathbf{1} \in H_n(S^n; \mathbb{Z})$ determines generators in $H_n(U_i, U_i \setminus \{u_i\}; \mathbb{Z})$ and $H_n(V, V \setminus \{v\}; \mathbb{Z})$ that we also denote by $\mathbf{1}$.

**Definition 11.7.** In the above situation, the local degree of $f$ at $u_i$ is the integer $\mathrm{deg} f|_{u_i}$ for which $f_* \colon H_n(U_i, U_i \setminus \{u_i\}; \mathbb{Z}) \to H_n(V, V \setminus \{v\}; \mathbb{Z})$ satisfies $f_*(\mathbf{1}) = (\mathrm{deg} f|_{u_i}) \cdot \mathbf{1}$ .

If we work with the other generator $-\mathbf{1}$ of $H_n(S^n; \mathbb{Z})$, then the generators on both sides of $f_*$ change their sign. This implies that the local degree does not depend on the choice of the generator of $H_n(S^n; \mathbb{Z})$.

**Proposition 11.8.** *In the above situation,* $\mathrm{deg} f = \sum_{i=1}^{m} \mathrm{deg} f|_{u_i}$ .

*Proof.* By excision, the inclusions $k_i$ induce an isomorphism

$$\bigoplus_{1\leq i\leq m} H_n(U_i, U_i \setminus \{u_i\}; \mathbb{Z}) \to H_n(S^n, S^n \setminus f^{-1}(v); \mathbb{Z}) \,.$$

Hence the commutativity of the upper left hand triangle implies that under this isomorphism, $p_i$ corresponds to the projection to the $i$-th summand. So by the commutativity of the lower left hand triangle, it follows that $j(\mathbf{1}) = \sum_{i=1}^m k_i(\mathbf{1})$. The commutativity of the upper square on the right hand side and the definition of the local mapping degree imply that the middle instance of $f_*$ takes $k_i(\mathbf{1})$ to $(\deg f|_{u_i}) \cdot \mathbf{1}$. The commutativity of the lower square on the right hand side then implies that the lower instance of $f_*$ takes $\mathbf{1}$ to $(\sum_{i=1}^m \deg f|_{u_i}) \cdot \mathbf{1}$ . □

## The homology of the real projective spaces

Recall from Exercise 7.2 that the real projective space $\mathbb{R}P^n$ is the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ modulo the equivalence relation $\sim$ defined by $x \sim \lambda x$ for $x \in \mathbb{R}^{n+1} \setminus \{0\}$ and $\lambda \in \mathbb{R} \setminus \{0\}$. We view $\mathbb{R}P^{n-1}$ as a subspace of $\mathbb{R}P^n$ via the inclusion map

$$\mathbb{R}P^{n-1} \to \mathbb{R}P^n \quad [x_0, \ldots, x_{n-1}] \mapsto [x_0, \ldots, x_{n-1}, 0]$$

induced by the linear map $\mathbb{R}^n \to \mathbb{R}^{n+1}$ that adds the entry 0 at the end of a vector. Moreover, we consider the continuous map

$$\chi_n \colon D^n \to \mathbb{R}P^n, \quad (x_1, \ldots, x_n) \mapsto [(x_1, \ldots, x_n, \sqrt{1 - x_1^2 - \cdots - x_n^2})] \,.$$

From this we obtain a commutative square

$$\begin{tikzcd}
\partial D^n \arrow[r, "\chi_n|_{\partial D^n}"] \arrow[d, "\mathrm{incl}"'] & \mathbb{R}P^{n-1} \arrow[d, "i_n"] \\
D^n \arrow[r, "\chi_n"] & \mathbb{R}P^n \,.
\end{tikzcd}$$

The fact that this square is a pushout implies that $\mathbb{R}P^n$ arises from $\mathbb{R}P^{n-1}$ by attaching a single $n$-cell and that $\mathbb{R}P^k$ is a CW-complex with a single $i$-cell in dimensions $0, \ldots, k$. (To see that it is pushout, one first verifies that the induced map $D^n \cup_{\partial D^n} \mathbb{R}P^{n-1} \to \mathbb{R}P^n$ is a bijection and then checks that its source is quasi-compact and that its target is a Hausdorff space.)

We now use the characteristic maps $\chi_n$ specified above to compute the differential in the cellular chain complex of $\mathbb{R}P^n$ up to signs. As a first step, we note that $\chi_n$ induces a homeomorphism

$$D^n/\partial D^n \to \mathbb{R}P^n/\mathbb{R}P^{n-1}, \quad [(x_1, \ldots, x_n)] \mapsto \left[[(x_1, \ldots, x_n, \sqrt{1 - x_1^2 - \cdots - x_n^2})]\right] \,. \tag{11.2}$$

(The outer brackets in the target indicate equivalence classes in the quotient $\mathbb{R}P^n/\mathbb{R}P^{n-1}$.) Now we let

$$U_1 = \{(x_1, \ldots, x_{n+1}) \in \partial D^{n+1} \,|\, x_{n+1} > 0\} \quad \text{and} \quad U_2 = \{(x_1, \ldots, x_{n+1}) \in \partial D^{n+1} \,|\, x_{n+1} < 0\}$$

be the northern and southern hemi-sphere in $\partial D^{n+1}$, write $u_1 = (0, \dots, 0, 1) \in U_1$ for the north pole and $u_2 = (0, \dots, 0, -1) \in U_2$ for the south pole, and set

$$V' = \mathring{D}^n \subset D^n/\partial D^n \quad \text{and} \quad v' = (0, \dots, 0) \in V' .$$

Moreover, we define $f$ to be the composite of

$$\partial D^{n+1} \xrightarrow{\chi|_{\partial D^{n+1}}} \mathbb{R}P^n \to \mathbb{R}P^n/\mathbb{R}P^{n-1} \overset{\cong}{\to} D^n/\partial D^n .$$

where the first map is the attaching map for the $n+1$-cell of $\mathbb{R}^{n+1}$, the second map is the quotient map, and the last map is the inverse of the homeomorphism (11.2). Restricting $f$ to $U_1$ and $U_2$ in the source and to $V'$ in the target provides homeomorphisms

$$\begin{aligned} f|_{U_1} &: U_1 \to V', (x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_n) \quad \text{and} \\ f|_{U_2} &: U_2 \to V', (x_1, \dots, x_{n+1}) \mapsto (-x_1, \dots, -x_n) . \end{aligned}$$

Here the minus signs in the second map arise because we have to switch to a representative of $(x_1, \dots, x_{n+1})$ in $\mathbb{R}P^n$ with positive last coordinate before applying the inverse of (11.2). We also note that $f|_{U_i}(u_i) = v'$.

Since the $f|_{U_i}$ are homeomorphisms, they induce isomorphisms

$$H_n(U_i, U_i \setminus \{u_i\}; \mathbb{Z}) \to H_n(V', V' \setminus \{v'\}; \mathbb{Z})$$

and hence send generators to generators. Therefore we can choose a homeomorphism

$$h_n \colon D^n/\partial D^n \to \partial D^{n+1}$$

such that the composite $h_n \circ f \colon \partial D^{n+1} \to \partial D^{n+1}$ has local degree 1 at the point $u_1$ (where we set $V = h_n(V')$ and $v = h_n(v')$ to be in the setup of Proposition 11.8). The antipodal map $-1 \colon \partial D^{n+1} \to \partial D^{n+1}$ restricts to a homeomorphism $\tau \colon U_1 \to U_2$ satisfying $f|_{U_1} = f|_{U_2} \circ \tau$. Since we know that the antipodal map $-1 \colon \partial D^{n+1} \to \partial D^{n+1}$ has degree $(-1)^{n+1}$, this implies that the local degree of $h_n \circ f$ at $u_2$ is $(-1)^{n+1}$. By Proposition 11.8, it follows that

$$\deg(h_n \circ f) = 1 + (-1)^{n+1}.$$

Hence we have shown that $\widetilde{\partial}_{n+1}$ is multiplication by $1 + (-1)^{n+1}$. So the cellular chain complex $\widetilde{C}(\mathbb{R}P^k; \mathbb{Z})$ of $\mathbb{R}P^k$ has (up to the signs of the differentials) the form

$$\overset{k+1}{0} \to \overset{k}{\mathbb{Z}} \xrightarrow{\cdot 2} \dots \xrightarrow{0} \overset{2}{\mathbb{Z}} \xrightarrow{\cdot 2} \overset{1}{\mathbb{Z}} \xrightarrow{0} \overset{0}{\mathbb{Z}}$$

if $k$ is even and the form

$$\overset{k+1}{0} \to \overset{k}{\mathbb{Z}} \xrightarrow{0} \dots \xrightarrow{0} \overset{2}{\mathbb{Z}} \xrightarrow{\cdot 2} \overset{1}{\mathbb{Z}} \xrightarrow{0} \overset{0}{\mathbb{Z}}$$

if $k$ is odd. (Here the numbers over the groups indicate the levels in the chain complexes.) Since multiplication by 2 is injective and has the even integers as its image, this implies the following result.

**Theorem 11.9.** *The homology groups of* $\mathbb{R}P^k$ *with coefficients in* $\mathbb{Z}$ *are given by*

$$H_n(\mathbb{R}P^k; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0 \text{ or if } n = k \text{ and } k \text{ is odd,} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n \text{ is odd and } 0 < n < k, \\ 0 & \text{else.} \end{cases}$$

# Lecture 12: The homotopy extension property

In the last lectures we considered the cellular chain complex $\widetilde{C}(X;A)$ where $\widetilde{C}_n(X;A)$ is the relative homology group $H_n(X_n, X_{n-1};A)$. We showed that the homology groups of $\widetilde{C}(X;A)$ are isomorphic to the singular homology groups of $X$ with coefficients in $A$.

When it comes to the maps of homology groups induced by a continuous map $f\colon X \to Y$, there is an important difference between cellular and singular homology. While we showed that $f$ induces group homomorphisms $C_n(X;A) \to C_n(Y;A)$ on the singular $n$-chains and $H_n(X;A) \to H_n(Y;A)$ on the singular homology groups, the analogous statement for cellular homology does not hold for general $f$. The problem is that $f$ may not restrict to a map of pairs $(X_n, X_{n-1}) \to (Y_n, Y_{n-1})$. This issue is one (but by far not the most important) motivation for the following terminology.

**Definition 12.1.** Let $f\colon (X,A) \to (Y,B)$ a map of relative CW-complexes, i.e., a continuous map $f\colon X \to Y$ with $f(A) \subseteq B$. Then $f$ is called *cellular* if $f(X_n) \subseteq Y_n$ for $n \geq -1$.

In other words, a cellular map preserves skeleta of CW-complexes. If $f$ is cellular, then $f(X_n) \subseteq Y_n$ and $f(X_{n-1}) \subseteq Y_{n-1}$ imply that there is an induced map $H_n(X_n, X_{n-1};A) \to H_n(Y_n, Y_{n-1};A)$ and therefore also an induced map on cellular homology groups.

In this and the next lecture, we will prove the following *Cellular Approximation Theorem* that provides us with a large supply of cellular maps.

**Theorem 12.2.** *Every map $f\colon (X,A) \to (Y,B)$ of relative CW-complexes is homotopic relative to $A$ to a cellular map. That is, there exists a continuous map $H\colon X \times [0,1] \to Y$ such that $H|_{X\times\{0\}} = f$, such that $H|_{X\times\{1\}}$ is cellular, and such that $H(x,t) = f(x)$ for all $x \in A$ and $t \in [0,1]$.*

The following statement is an important consequence (or rather a special case) of the theorem: For $m < n$, every continuous map $S^m \to S^n$ between spheres is homotopic to a constant map. To see this, we recall that there are homeomorphisms $S^m \cong D^0 \cup_{\partial D^m} D^m$ and $S^n \cong D^0 \cup_{\partial D^n} D^n$ that exhibit $S^m$ and $S^n$ as CW-complexes with one 0-cell and one $m$- or $n$-cell. If $m < n$, the $m$-skeleton of $D^0 \cup_{\partial D^n} D^n$ consists of a point, and any cellular map has to be constant. So the theorem implies the claim. Later we will interpret this statement about homotopy classes of maps between spheres as a vanishing of higher homotopy groups.

## The homotopy extension property

For the proof of the Cellular Approximation Theorem we will need more tools to construct homotopies. The following terminology will be central for this.

**Definition 12.3.** Let $X$ be a topological space and let $A \subset X$ be a subspace. Then the pair $(X,A)$ has the *homotopy extension property* (HEP) if the following holds: Given a topological space $Z$, a continuous map $f\colon X \to Z$ and a homotopy $F\colon A \times [0,1] \to Z$ with $F|_{A\times\{0\}} = f|_A$, there exists a homotopy $H\colon X \times [0,1] \to Z$ with $H|_{X\times\{0\}} = f$ and $H|_{A\times[0,1]} = F$ .

In other words, given a continuous map $f\colon X \to Z$ and a homotopy $F\colon A \times [0,1] \to Z$ starting from the restriction of $f$ to $A$, there is an extension of $F$ to a homotopy defined on $X \times [0,1]$ that starts from $f$.

We write $X \cup_A (A \times [0,1])$ for the pushout of the diagram

$$X \leftarrow A \rightarrow (A \times [0,1])$$

given by the inclusion $A \to X$ and the map $A \to (A \times [0,1]), a \mapsto (a,0)$. We note that by the universal property of the pushout, the continuous maps

$$X \to X \times [0,1],\ x \mapsto (x,0) \quad \text{and} \quad A \times [0,1] \to X \times [0,1],\ (a,t) \mapsto (a,t)$$

induce a continuous map $X \cup_A (A \times [0,1]) \to X \times [0,1]$.

**Lemma 12.4.** *The pair $(X, A)$ has the homotopy extension property if and only if every continuous map $X \cup_A (A \times [0,1]) \to Z$ extends to a continuous map $X \times [0,1] \to Z$.*

*Proof.* Since a continuous map $X \cup_A (A \times [0,1]) \to Z$ is uniquely specified by continuous maps $X \to Z$ and $A \times [0,1] \to Z$ that coincide when restricted to $A$, this follows directly from the definition. □

One can express the condition of the lemma in terms of a commutative diagram. That is, we require that for every continuous map $X \cup_A (A \times [0,1]) \to Z$ there exists a dashed arrow such that the following diagram is commutative:

$$\begin{tikzcd}
X \cup_A (A \times [0,1]) \arrow[r] \arrow[d] & Z \\
X \times [0,1] \arrow[ur, dashed] &
\end{tikzcd}$$

Here the vertical map is again the canonical map induced by the inclusions.

Our next aim is to give a description of the homotopy extension property that does not make use of the "test map" $X \cup_A (A \times [0,1]) \to Z$.

**Definition 12.5.** Let $Y$ be a topological space. A subspace $B \subseteq Y$ is a *retract* if there exists a continuous map $r \colon Y \to B$ such that $r(b) = b$ for all $b \in B$.

**Lemma 12.6.** *Let $(Y, B)$ be a pair of spaces. The following are equivalent.*

*(i) Every continuous map $B \to Z$ extends to a continuous map $Y \to Z$.*

*(ii) $B$ is a retract of $Y$.*

*Proof.* To see that (i) implies (ii), we extend $\mathrm{id}_B$ to $Y \to B$ and note that the latter extension provides the desired retraction. To see that (ii) implies (i), we choose a retraction $Y \to B$ and compose it with $B \to Z$ to get the desired extension $Y \to Z$. □

To use this lemma for a characterization of the homotopy extension property, we recall a point set topological result.

**Lemma 12.7.** *Let $T$ be a topological space and let $U, V \subseteq T$ be closed subspaces. Then the inclusion maps $U \to T$ and $V \to T$ induce a homeomorphism $U \cup_{U \cap V} V \to U \cup V$ from the pushout of the diagram $U \leftarrow U \cap V \to V$ to the subspace $U \cup V$ of $T$.*

*Proof.* The universal property of the pushout provides a map $f\colon U \cup_{U\cap V} V \to U \cup V$ that is continuous. It follows from the construction that $f$ is bijective. To see that $f^{-1}$ is also continuous, it is sufficient to show that $f$ preserves closed sets. If $C \subseteq U \cup_{U\cap V} V$ is closed, then its preimage under the quotient map $q\colon U \amalg V \to U \cup_{U\cap V} V$ is closed by the definition of the topology on the pushout. This preimage is $(U \cap f(C)) \amalg (V \cap f(C))$. Since $U$ and $V$ are closed in $T$, this implies that $U \cap f(C)$ and $V \cap f(C)$ are closed in $T$. Hence $f(C) = (U \cap f(C)) \cup (V \cap f(C))$ is closed in $U \cup V$. ☐

The statement of the lemma does not hold if we drop the assumption that $U$ and $V$ are closed. To see this, take $T = [0,2]$, $U = [0,1)$, $V = [1,2]$. Then $U \cup V = [0,2]$, while the pushout of $U \leftarrow U \cap V \to V$ is the disjoint union $[0,1) \amalg [1,2]$.

**Corollary 12.8.** *If $A \subset X$ is a closed subspace, then the canonical map*

$$j\colon X \cup_A (A \times [0,1]) \to X \times \{0\} \cup A \times [0,1]$$

*from the pushout to the subspace $X \cup A \times [0,1]$ of $X \times [0,1]$ is a homeomorphism.*

*Proof.* Both $X \times \{0\}$ and $A \times [0,1]$ are closed subspaces of $X \times [0,1]$. Using the canonical homeomorphisms $A \to A \times \{0\}$ and $X \to X \times \{0\}$ and the last lemma, the claim follows. ☐

**Corollary 12.9.** *Let $A \subset X$ be a closed subspace. Then the pair $(X, A)$ has the HEP if and only if the subspace $X \times \{0\} \cup A \times [0,1]$ of $X \times [0,1]$ is a retract.*

*Proof.* Via the homeomorphism from the last lemma, extending a map

$$X \cup_A (A \times [0,1]) \to Z$$

to $X \times [0,1]$ is equivalent to extending a map defined on the subspace $X \times \{0\} \cup A \times [0,1]$ of $X \times [0,1]$ to a map defined on $X \times [0,1]$. This is equivalent to $X \times \{0\} \cup A \times [0,1]$ being a retract of $X \times [0,1]$ by Lemma 12.6. ☐

## The homotopy extension property for relative CW-complexes

Our next aim is to show that every relative CW-complex $(X, A)$ has the homotopy extension property. The proof breaks up into various steps.

**Proposition 12.10.** *For $m \geq 0$, the pair $(D^m, \partial D^m)$ has the homotopy extension property.*

*Proof.* We need to define a retraction

$$r\colon D^m \times [0,1] \to D^m \times \{0\} \cup \partial D^m \times [0,1]\,.$$

We can interpret $D^m \times [0,1]$ as a cylinder with bottom $D^m \times \{0\}$ and side $\partial D^m \times [0,1]$. Viewing $D^m \times [0,1]$ as a subspace of $\mathbb{R}^{m+1}$, we define $r(x,t)$ to be the intersection of the line through $(0,2)$ and $(x,t)$ with $D^m \times \{0\} \cup \partial D^m \times [0,1]$, see Figure 12.1. This defines a continuous map that fixes the points in $D^m \times \{0\} \cup \partial D^m \times [0,1]$. Although it may not make things clearer, one can check in terms of elements that $r$ is given by

$$r(x,t) = \begin{cases} (\frac{2}{2-t}x, 0) & \text{if } \|x\| \leq 1 - \frac{t}{2} \\ \frac{1}{\|x\|}(x, t + 2(\|x\| - 1)) & \text{if } \|x\| \geq 1 - \frac{t}{2}. \end{cases}$$

☐

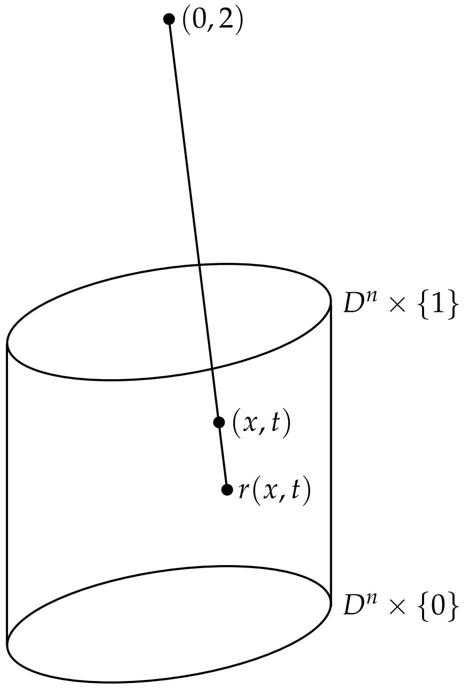

Figure 12.1: Construction of the retraction in the proof of Proposition 12.10

**Proposition 12.11.** *Suppose that $X$ arises from $X'$ by attaching $n$-cells. Then $(X, X')$ has the homotopy extension property.*

*Proof.* Let $J$ be the set of $n$-cells so that $X = X' \cup_{J\times\partial D^m} J \times D^m$ (where we as usual drop the attaching map from the notation). The canonical map $X' \to X$ allows us to view $X'$ as a closed subspace of $X$. By the last proposition, there is a retraction

$$r\colon J \times D^m \times [0,1] \to J \times D^m \times \{0\} \cup J \times \partial D^m \times [0,1]\,.$$

Using it, we obtain the following sequence of maps explained below:

$$\begin{aligned}
& X \times [0,1] \\
\xrightarrow{\cong}\; & X' \times [0,1] \cup_{J\times\partial D^m\times[0,1]} (J \times D^m \times [0,1]) \\
\xrightarrow{\mathrm{id}\cup r}\; & X' \times [0,1] \cup_{J\times\partial D^m\times[0,1]} (J \times D^m \times \{0\} \cup J \times \partial D^m \times [0,1]) \\
\xrightarrow{\cong}\; & X' \times [0,1] \cup_{J\times\partial D^m\times[0,1]} (J \times \partial D^m \times [0,1] \cup_{J\times\partial D^m\times\{0\}} J \times D^m \times \{0\}) \\
\xrightarrow{\cong}\; & (X' \times [0,1] \cup_{J\times\partial D^m\times[0,1]} J \times \partial D^m \times [0,1]) \cup_{J\times\partial D^m\times\{0\}} J \times D^m \times \{0\} \\
\xrightarrow{\cong}\; & X' \times [0,1] \cup_{J\times\partial D^m\times\{0\}} (J \times D^m \times \{0\}) \\
\xrightarrow{\cong}\; & (X' \times [0,1] \cup_{X'\times\{0\}} X' \times \{0\}) \cup_{J\times\partial D^m\times\{0\}} (J \times D^m \times \{0\}) \\
\xrightarrow{\cong}\; & X' \times [0,1] \cup X \times \{0\}
\end{aligned}$$

The first map is the homeomorphism resulting from Corollary 8.9. The second map is induced by the retraction $r$ and the identity on $X' \times [0,1]$. It is well-defined since $r|_{J \times \partial D^m \times [0,1]}$ is the identity on $J \times \partial D^m \times [0,1]$. The third map is the homeomorphism resulting from applying Lemma 12.7 to the subspaces $J \times \partial D^m \times [0,1]$ and $J \times D^m \times \{0\}$ of $J \times D^m \times [0,1]$. The fourth map is the homeomorphism that results from commuting pushouts. The fifth map is induced by the canonical homeomorphism $X' \times [0,1] \cup_{J \times \partial D^m \times [0,1]} J \times \partial D^m \times [0,1] \to X' \times [0,1]$. The sixth map arises from another application of Lemma 12.7. The last map arises from commuting pushouts once more and yet another application of Lemma 12.7.

The composite of these maps is the desired retraction.

□

In order to establish the homotopy extension property for relative CW-complexes, we need an additional lemma about continuous maps from products of CW-complexes with locally compact spaces.

**Lemma 12.12.** *Let $(X, A)$ be a relative CW-complex, let $Z$ be a topological space, let $K$ be a locally compact space, and let $f\colon X \times K \to Z$ be a map of underlying sets. Then $f$ is continuous if and only if the restrictions $f|_{X_m \times K}\colon X_m \times K \to Z$ are continuous for all $m \geq -1$.*

*Proof.* By property (ii) in the definition of CW-complexes, the map

$$p\colon \coprod_{m \geq -1} X_m \to X$$

from the disjoint union of the skeleta induced by the inclusions $X_m \to X$ is a quotient map. Proposition 8.14 implies that the composite $q$ of

$$\coprod_{m \geq -1} (X_m \times K) \xrightarrow{\cong} \left( \coprod_{m \geq -1} X_m \right) \times K \xrightarrow{p \times \mathrm{id}} X \times K$$

is also a quotient map. Hence $f$ is continuous if and only if $f \circ q$ is continuous. By the definition of the topology on the disjoint union, the latter condition is equivalent to

$$f|_{X_m \times K}\colon X_m \times K \to Z$$

being continuous for all $m \geq -1$. □

**Theorem 12.13.** *Let $(X, A)$ be a relative CW-complex. Then $(X, A)$ has the homotopy extension property.*

*Proof.* We need to construct a retraction

$$r\colon X \times [0,1] \to X \times \{0\} \cup A \times [0,1] \,. \tag{12.1}$$

To do so, we begin by inductively constructing a sequence of retractions

$$r_m\colon X_m \times [0,1] \to X_m \times \{0\} \cup A \times [0,1]$$

with the property $r_m|_{X_{m-1} \times [0,1]} = r_{m-1}$. The first of these retractions is

$$r_{-1} = \mathrm{id}_{A \times [0,1]}\colon A \times [0,1] \to A \times [0,1] \,.$$

Now we assume that $r_{m-1}$ is already constructed. Then we define $r_m$ to be the composite of

$$X_m \times [0,1] \xrightarrow{r'_m} X_m \times \{0\} \cup X_{m-1} \times [0,1] \xrightarrow{\mathrm{id} \cup r_{m-1}} X_m \times \{0\} \cup A \times [0,1]$$

where $r'_m$ is a retraction arising from Proposition 12.11. The second map $\mathrm{id}_{X_m \times \{0\}} \cup r_{m-1}$ is well defined and continuous since $r_{m-1}|_{X_{m-1}} = \mathrm{id}_{X_{m-1}}$ and both $X_m \times \{0\}$ and $X_{m-1} \times [0,1]$ are closed subspaces of $X_m \times [0,1]$. By construction, $r_m$ is indeed a retraction that satisfies $r_m|_{X_{m-1} \times [0,1]} = r_{m-1}$.

Now we define the desired retraction (12.1) by setting $r(x,t) = r_n(x,t)$ if $x \in X_n$. This is a well-defined map of sets since condition $r_m|_{X_{m-1} \times [0,1]} = r_{m-1}$ ensures that the restrictions $r_n|_{X_m \times [0,1]}$ coincide with $r_m$ if $n > m$. Since $r|_{X_m \times [0,1]} = r_m$ is continuous by construction, the last lemma implies that $r$ is continuous. □

To conclude our discussion of the homotopy extension property, we discuss a pair of spaces that fails to have this property.

**Example 12.14.** Let $A = [-1,0]$ and $X = [-1,0] \cup \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$ with the subspace topology coming from the inclusions $A \subset X \subset \mathbb{R}$. Any homotopy $H \colon X \times [0,1] \to X$ with $H|_{X \times \{0\}} = \mathrm{id}_X$ satisfies $H(\frac{1}{n}, t) = \frac{1}{n}$ for all $t$ since each of the points $\frac{1}{n}$ lies in a separate path component. By continuity, this implies that $H(0,t) = 0$ for all $t \in [0,1]$. Now consider the homotopy $F \colon A \times [0,1] \to X, (t,s) \mapsto (1-s) \cdot t + s \cdot (-1)$ from the inclusion of $A$ in $X$ to the constant map with value $-1$. By the above observation, the homotopy $F$ and $\mathrm{id}_X$ do not extend to a homotopy defined on $X \times [0,1]$.

### A reduction step in the proof of the cellular approximation theorem

Next time we will prove the following lemma which states that if a map of relative CW-complexes is already cellular up to level $m-1$, then its restriction to the $m$-skeleton is homotopic (relative to the $m-1$-skeleton) to a cellular map.

**Lemma 12.15.** *Let $f \colon (X,A) \to (Y,B)$ be a map of relative CW-complexes such that $f(X_i) \subseteq Y_i$ for $i \leq m-1$. Then $f|_{X_m}$ is homotopic relative to $X_{m-1}$ to a map with image in $Y_m$.*

We will now show how this lemma and the homotopy extension property for relative CW-complexes imply the cellular approximation theorem. As an intermediate step, we first deduce the following corollary which already implies the cellular approximation theorem for finite dimensional CW-complexes.

**Corollary 12.16.** *Let $f \colon (X,A) \to (Y,B)$ be a map of relative CW-complexes. Then there exists a sequence of continuous maps $f_m \colon (X,A) \to (Y,B)$ for $m \geq -1$ and homotopies $H_m \colon X \times [0,1] \to Y$ for $m \geq 0$ such that $f_{-1} = f$ and the following conditions hold:*

*(i) $f_m(X_i) \subseteq Y_i$ for $i \leq m$.*

*(ii) $H_m$ is a homotopy from $f_{m-1}$ to $f_m$ relative to $X_{m-1}$.*

*Proof.* Suppose that $(f_0, H_0), \dots, (f_{m-1}, H_{m-1})$ are constructed. Then the previous lemma provides a homotopy $H'_m \colon X_m \times [0,1] \to Y$ relative to $X_{m-1} \times [0,1]$ with $H'_m|_{X_m \times \{0\}} = f_{m-1}|_{X_m}$ and $H'_m(X_m \times \{1\}) \subseteq Y_m$. Applying the homotopy extension property provides a homotopy $H_m \colon X \times [0,1] \to Y$ which extends $f_{m-1}$ and $H'_m$. Setting $f_m = H_m|_{X \times \{1\}}$ finishes the construction of $(f_m, H_m)$. ☐

*Proof of Theorem 12.2 (assuming Lemma 12.15).* Let $f \colon (X, A) \to (Y, B)$ be a map of relative CW-complexes, and let $f_m$ and $H_m$ be the maps and homotopies arising from last corollary. We use them to define the following map of sets $H \colon X \times [0,1] \to Y$. For $(x,t) \in X \times [0,1]$ with $\frac{m}{m+1} \leq t \leq \frac{m+1}{m+2}$, we define

$$H(x,t) = H_m(x, (m+1) \cdot (m+2) \cdot \left(t - \frac{m}{m+1}\right)) \,.$$

That is, up to re-parametrization we use the homotopy $H_0$ on $[0, \frac{1}{2}]$, the homotopy $H_1$ on $[\frac{1}{2}, \frac{2}{3}]$, the homotopy $H_2$ on $[\frac{2}{3}, \frac{3}{4}]$, and so on. To define $H$ for a point $(x, 1)$, we choose an $m$ with $x \in X_m$ and set $H(x,1) = f_m(x)$. This is well defined since for $n > m$, the homotopies $H_n$ are defined relative to $X_m \times [0,1]$ so that

$$f_m(x) = H_{m+1}(x,t) = f_{m+1}(x) = \dots = H_n(x,t) = f_n(x) = \dots$$

for all $t \in [0,1]$.

We note that $H|_{X_m \times [0,1]}$ is continuous since it is a re-parametrized concatenation of the homotopies $H_0|_{X_m \times [0,1]}, \dots, H_m|_{X_m \times [0,1]}$. Hence Lemma 12.12 implies that $H$ is continuous. ☐

# Lecture 13: Cellular approximation

The aim of this lecture is to complete the proof of the following statement that we refer to as the "Cellular Approximation Theorem".

**Theorem 13.1.** *Every map $f\colon (X,A) \to (Y,B)$ of relative CW-complexes is homotopic relative to $A$ to a cellular map. That is, there exists a continuous map $H\colon X \times [0,1] \to Y$ such that $H|_{X\times\{0\}} = f$, such that $H|_{X\times\{1\}}$ is cellular, and such that $H(x,t) = f(x)$ for all $x \in A$ and $t \in [0,1]$.*

Last time we reduced the proof of this theorem to showing that if $f$ preserves the skeleta up to level $m-1$, then $f|_{X_m}$ is homotopic relative to $X_{m-1}$ to a map with image in $Y_m$. In order to prove the latter statement, we will now formulate and prove the above theorem in the very special case where both CW-complexes only have one cell relative to their $-1$-skeleta.

**Proposition 13.2.** *Let $m < n$ and let $f\colon (A \cup_{\partial D^m} D^m, A) \to (B \cup_{\partial D^n} D^n, B)$ be a map of relative CW-complexes. Then $f$ is homotopic relative to $A$ to a map with image in $B$.*

We will prove the proposition by induction over $n$. When $n = 0$, there is nothing to show, and consequently our argument below will not rely on an induction hypothesis when $n = 1$.

The next lemmas will be used in the proof.

**Lemma 13.3.** *Let $q > 0$, and let $h\colon \partial D^q \to Z$ be a continuous map. Then $h$ is homotopic to a constant map if and only if $h$ extends to a map $D^q \to Z$.*

*Proof.* A homotopy $H\colon \partial D^q \times [0,1] \to Z$ with $H|_{\partial D^q\times\{0\}} = h$ factors as the composite of the quotient map

$$p\colon \partial D^q \times [0,1] \to (\partial D^q \times [0,1])/(\partial D^q \times \{1\})$$

and a continuous map $(\partial D^q \times [0,1])/(\partial D^q \times \{1\}) \to Z$ if and only if $H|_{\partial D^q\times\{1\}}$ is a constant map. So giving a homotopy from $h$ to a constant map is equivalent to giving a continuous map $(\partial D^q \times [0,1])/(\partial D^q \times \{1\}) \to Z$ whose composite with

$$\partial D^q \to \partial D^q \times [0,1]/(\partial D^q \times \{1\}), \quad x \mapsto p(x,0)$$

is $h$. Since there exists a homeomorphism $D^q \to \partial D^q \times [0,1]/(\partial D^q \times \{1\})$ sending $x \in \partial D^q$ to $p(x,0)$, the statement of the lemma follows. □

**Lemma 13.4.** *Suppose that Proposition 13.2 holds for $n-1$. Let $q < n$ and let $(a,b) \subset \mathbb{R}$ be an open interval. Then every continuous map $h\colon \partial D^q \to S^{n-1} \times (a,b)$ extends to a map on $D^q$.*

*Proof.* Let $c$ be a point in $(a,b)$ and consider the homotopy

$$H\colon S^{n-1} \times (a,b) \times [0,1] \to S^{n-1} \times (a,b), \quad (x,s,t) \mapsto (x, (1-t)\cdot s + t \cdot c)$$

that is the identity on $S^{n-1}$ and contracts the interval $(a,b)$ to the point $c$. Then

$$\partial D^q \times [0,1] \to S^{n-1} \times (a,b), \quad (x,t) \mapsto H(h(x),t)$$

is a homotopy from $h$ to a map with image in $S^{n-1} \times \{c\}$. Since $\partial D^q \cong \{*\} \cup_{\partial D^{q-1}} D^{q-1}$ and $S^{n-1} \cong \{*\} \cup_{\partial D^{n-1}} D^{n-1}$, the $n-1$-case of the proposition implies that the latter map is homotopic to a constant map. By the previous lemma, $h$ extends to $D^q$. □

The following lemma is the main step in the proof of Proposition 13.2.

**Lemma 13.5.** *Let $m < n$ and suppose that Proposition 13.2 holds for $n-1$. Then every continuous map $g\colon (D^m, \partial D^m) \to (B \cup_{\partial D^n} D^n, B)$ is homotopic relative $\partial D^m$ to a map $g'$ that does not have all of $\mathring{D}^n$ in its image.*

*Proof.* We let $p\colon B \amalg D^n \to B \cup_{\partial D^n} D^n$ be the quotient map resulting from a choice of a characteristic map for the $n$-cell, set

$$U' = \{x \in D^n \mid \|x\| < \frac{2}{3}\} \qquad \text{and} \qquad V' = \{x \in D^n \mid \|x\| > \frac{1}{3}\},$$

and set $U = p(U')$ and $V = p(B \cup V')$. By construction, $\{U, V\}$ is an open cover of $B \cup_{\partial D^n} D^n$. We will construct the desired map $g'$ so that its image is contained in $V$. This implies in particular that the center $p(0) \in B \cup_{\partial D^n} D^n$ of the $n$-cell will not be in the image of $g'$.

The quotient map $p$ induces a homeomorphism $U' \cap V' \to U \cap V$. Since

$$U' \cap V' \cong S^{n-1} \times (\frac{1}{3}, \frac{2}{3}),$$

the previous lemma implies that a continuous map $\partial D^q \to U \cap V$ extends to $D^q$ whenever $q < n$.

Since the $m$-dimensional disk $D^m$ is homeomorphic to the $m$-dimensional cube $[0,1]^m$, we may view $g$ as map $g\colon ([0,1]^m, \partial[0,1]^m) \to (B \cup_{\partial D^n} D^n, B)$. We now apply the Lebesgue covering theorem to the open cover $\{g^{-1}(U), g^{-1}(V)\}$ of $[0,1]^m$. It implies that there is a natural number $k$ such that each subcube of $[0,1]^m$ with side length $\frac{1}{k}$ is mapped to $U$ or $V$.

For $0 \leq i \leq m$, we define $C_i$ be the set of $i$-dimensional subcubes of $[0,1]^m$ that have side length $\frac{1}{k}$ and all of whose coordinates are multiples of $\frac{1}{k}$. Moreover, we define

$$P_i = \{c \in C_i \mid c \subseteq g^{-1}(V)\} \quad \text{and} \quad Q_i = \{c \in C_i \mid c \notin P_i\}\ .$$

We think of the subcubes in $P_i$ as the "good" ones since their image under $g$ is already contained in $V$. The remaining cubes are the "bad" ones, and the value of the desired map $g'$ will differ from the value of $g$ on those subcubes. We note that every bad cube $c$ satisfies $g(c) \subseteq U$.

In order to construct $g'$, we define a filtration of the $m$-cube $[0,1]^m$ by setting

$$K_{-1} = \bigcup_{\substack{0 \leq i \leq m \\ c \in P_i}} c \quad \text{and} \quad K_i = K_{i-1} \cup \bigcup_{c \in Q_i} c \quad \text{for } 1 \leq i \leq m\ .$$

That is, $K_{-1}$ is the union of all good subcubes (of all dimensions), and $K_i$ arises from $K_{i-1}$ by adding the bad cubes of dimension $i$. In particular, this means that $K_m = [0,1]^m$.

Now we inductively construct maps $g_i\colon K_i \to B \cup_{\partial D^n} D^n$ such that $g_i|_{K_{i-1}} = g_{i-1}$ and that $g_i(c) \subseteq U \cap V$ for all $c \in Q_i$. The construction starts by setting $g_{-1} = g|_{K_{-1}}$ which has the desired property by the construction of $K_{-1}$. To define $g_0$, we just choose points in $U \cap V$ as the values of $g_0$ on the bad cubes of dimension 0. For the induction step, we can assume that $g_{i-1}$ is already constructed. Now consider $c \in Q_i$. Let $c'$ be an $i-1$-dimensional subcube of the boundary $\partial c$ of $c$. If $c'$ is good, then $g_{i-1}(c') = g(c') \subseteq V$ since $g_{i-1}$ and $g$ coincide on good subcubes and $g_{i-1}(c') = g(c') \subseteq g(c) \subseteq U$ since $c$ is a bad subcube. If $c'$ is bad, then

$g_{i-1}(c') \subseteq U \cap V$ by our inductive assumption. Together, this implies that $g_{i-1}(\partial c) \subseteq U \cap V$. The observation made at the beginning of the proof implies that $g_{i-1}|_{\partial c}\colon \partial c \to U \cap V$ extends to a map defined on the subcube $c$. Pasting together $g_{i-1}$ and choices of such extensions for all $c \in Q_i$ defines the desired map $g_i\colon K_i \to B \cup_{\partial D^n} D^n$. This finishes the inductive construction of the $g_i$.

We now set $g' = g_m$. Then $g'(D^m) \subseteq V$, and it remains to show that $g'$ is homotopic to our original map $g$. For this, we note that $g|_{K_{-1}} = g'|_{K_{-1}}$ and let

$$Q = \bigcup_{\substack{0 \leq i \leq m \\ c \in Q_i}} c$$

be the union of all bad cubes. Then $g'(Q), g(Q) \subseteq U$. Using the inverse of the homeomorphism $U' \to U$ induced by the quotient map $p$, we may view the restrictions $g|_Q$ and $g'|_Q$ as maps $Q \to U'$ and use the homotopy $(x,t) \mapsto t \cdot g(x) + (1-t) \cdot g'(x)$ to define a homotopy between these two maps that is constant on all points where $g$ and $g'$ do already coincide. Composing this homotopy with the homeomorphism $U' \to U$ and the inclusion $U \to B \cup_{\partial D^n} D^n$ provides a homotopy between $g|_Q$ and $g'|_Q$ as maps $Q \to B \cup_{\partial D^n} D^n$ that is constant on all points where $g$ and $g'$ do already coincide. In particular, the homotopy is constant on $K_{-1} \cap Q$. Extending by the constant homotopy $K_{-1} \times [0,1] \to B \cup_{\partial D^n} D^n$, $(x,t) \mapsto g(x)$ gives the desired homotopy between $g$ and $g'$. □

*Proof of Proposition 13.2.* Let $\chi\colon D^m \to A \cup_{\partial D^m} D^m$ be a characteristic map and let

$$p\colon B \amalg D^n \to B \cup_{\partial D^n} D^n$$

be the quotient map. As usual, we use $p$ to identify $B$ with the subspace $p(B)$. By the last lemma, the composite $g = f \circ \chi$ is homotopic relative to $\partial D^m$ to a map $g'$ that does not hit all points of $\mathring{D}^n$ in its image. We may assume that $p(0)$ is not in the image. Extending the homotopy

$$D^n \setminus \{0\} \times [0,1] \to D^n \quad (s,t) \mapsto (1-t) \cdot s + t \cdot \frac{s}{\|s\|}$$

by the constant homotopy on the identity of $B$ provides a homotopy

$$F\colon (B \cup_{\partial D^n} D^n \setminus p(0)) \times [0,1] \to B \cup_{\partial D^n} D^n$$

from the inclusion $B \cup_{\partial D^n} D^n \setminus p(0) \to B \cup_{\partial D^n} D^n$ to a map with image in $B$. Then $(x,t) \mapsto F(g'(x),t)$ gives a homotopy from $g'$ to a map $g''$ with value in $B$. In the last step, we concatenate the homotopies from $g$ to $g'$ and from $g'$ to $g''$ to get a homotopy from $g$ to $g''$ and glue it together with the constant homotopy $A \times [0,1] \to B, (x,t) \mapsto f(x)$ to get the desired homotopy from $f$ to a cellular map. (We note that the above argument used Corollary 8.9 about gluing homotopies on cell attachments twice.) □

Now we are ready to prove the following statement that we used in the last lecture for the proof of the Cellular Approximation Theorem.

**Lemma 13.6.** *Let $f\colon (X,A) \to (Y,B)$ be a map of relative CW-complexes such that $f(X_i) \subseteq Y_i$ for $i \leq m-1$. Then $f|_{X_m}$ is homotopic relative to $X_{m-1}$ to a map with image in $Y_m$.*

*Proof.* Let $J_m$ be the set of $m$-cells for $(X, A)$ and let $\chi_j \colon D^m \to X_m, j \in J_m$ be characteristic maps for the $m$-cells. We view the pair of spaces $(Y, Y_m)$ as a relative CW-complex without cells of dimension less or equal than $m$.

Consider the $m$-cell of $(X, A)$ indexed by $j \in J_m$. Since $f(\chi_j(D^m)) \subseteq Y$ is compact, it lies in a finite subcomplex of $(Y, Y_m)$. We denote this finite subcomplex by $Y^j$ and view the composite $f \circ \chi_j$ as a map

$$g_j \colon (D^m, \partial D^m) \to (Y^j, Y_m) .$$

If $Y^j = Y_m$, we are done. Otherwise, we choose a cell of maximal dimension $n$ in $(Y^j, Y_m)$. Let $(\overline{Y}^j, Y_m)$ be the finite subcomplex of $(Y, Y_m)$ which has all cells of $Y^j$ apart from the chosen one of maximal dimension. Then $Y^j$ arises from $\overline{Y}^j$ by attaching a single $n$-cell, and we can also view $(Y^j, \overline{Y}^j)$ as a relative CW-complex with a single cell of dimension $n$. Since $n > m$, we can apply the above proposition to $g_j \colon (D^m, \partial D^m) \to (Y^j, \overline{Y}^j)$ to see that $g_j$ is homotopic to a map with image in $\overline{Y}^j$. Repeating this procedure for all of the finitely many cells of $(\overline{Y}^j, Y_m)$, we see that $g_j$ is homotopic to a map with image in $Y_m$.

Applying this argument for all $j \in J_m$ leads to a homotopy $G \colon J_m \times D^m \times [0, 1] \to Y$ relative to $J_m \times \partial D^m$ with $G|_{\{j\} \times D^m \times \{0\}} = g_j$ and $G(J_m \times D^m \times \{1\}) \subseteq Y_m$. Now we let $F \colon X_{m-1} \times [0, 1] \to Y$ be the constant homotopy given by $F(x, t) = f(x)$. Since $G$ and $F$ coincide when restricted to $J_m \times \partial D^m \times [0, 1]$, they induce a continuous map

$$X_{m-1} \times [0, 1] \cup_{J_m \times \partial D^m \times [0,1]} J_m \times D^m \times [0, 1] \to Y .$$

Composing this map with the inverse of the homeomorphism

$$(X_{m-1} \cup_{J_m \times \partial D^m} J_m \times D^m) \times [0, 1] \to X_{m-1} \times [0, 1] \cup_{J_m \times \partial D^m \times [0,1]} J_m \times D^m \times [0, 1]$$

arising from Corollary 8.9 provides the desired homotopy from $f|_{X_m}$ to a map with image in $Y_m$. □

## Products of CW-complexes

We conclude this lecture by giving brief explanation of how to obtain CW-structures on products. Let $X$ and $X'$ be CW-complexes with sets of $n$-cells $J_n$ and $J'_n$. For $j \in J_n$ and $j' \in J'_n$, we let $\chi_j \colon D^n \to X_n$ and $\chi'_{j'} \colon D^n \to X'_n$ be characteristic maps for the cells. We set

$$\widehat{X} = \coprod_{n \geq 0} J_n \times D^n \quad \text{and} \quad \widehat{X}' = \coprod_{n \geq 0} J'_n \times D^n$$

and note that the chosen characteristic maps induce continuous maps

$$q \colon \widehat{X} \to X \quad \text{and} \quad q' \colon \widehat{X}' \to X' .$$

It follows from Exercise 7.3 that $U \subseteq X$ is closed if and only if $q^{-1}(U) \subseteq \widehat{X}$ is closed. Hence $q$ and $q'$ are quotient maps.

Now we consider the product of these quotient maps

$$q \times q' \colon \widehat{X} \times \widehat{X}' \to X \times X' . \tag{13.1}$$

This is a continuous map. However, for general $X$ and $X'$ it may itself not be a quotient map. So if $U \subseteq X \times X'$ is open, then $(q \times q')^{-1}(U) \subseteq \widehat{X} \times \widehat{X}'$ is open by continuity, but it is in general not true that if $(q \times q')^{-1}(U) \subseteq \widehat{X} \times \widehat{X}'$ is open, then $U \subseteq X \times X'$ is open.

We will see below that the condition that $q \times q'$ is a quotient map is desirable because it ensures that $X \times X'$ inherits a CW-structure from $X$ and $X'$. To analyze how the CW-structures are inherited, we equip the set $X \times X'$ with the quotient topology that is induced from the surjective map $q \times q'$ from (13.1) and write $X \widehat{\times} X'$ for the resulting topological space. That is, $X \widehat{\times} X'$ is the topological space with underlying set $X \times X'$, and $U \subseteq X \widehat{\times} X'$ is open if and only if $(q \times q')^{-1}(U) \subseteq \widehat{X} \times \widehat{X}'$ is open. Since the map (13.1) is continuous, the identity $X \widehat{\times} X' \to X \times X'$ is a continuous map. We will now show that the space $X \widehat{\times} X'$ always inherits a CW-structure from $X$ and $X'$ and that the identity $X \widehat{\times} X' \to X \times X'$ is a homeomorphism if $X'$ is locally compact.

**Proposition 13.7.** *Let $X$ and $X'$ be CW-complexes. Then the space $X \widehat{\times} X'$ defined above inherits a CW-structure from $X$ and $X'$ with set of $n$-cells $\widehat{J}_n = \bigcup_{p+q=n} J_p \times J'_q$ and with $n$-skeleton*

$$(X \widehat{\times} X')_n = \bigcup_{p+q=n} X_p \times X'_q \subseteq X \widehat{\times} X' \,.$$

*Proof.* We set

$$(\widehat{X} \times \widehat{X}')_n = (q \times q')^{-1}((X \widehat{\times} X')_n) \subseteq \widehat{X} \times \widehat{X}' \cong \coprod_{p \geq 0, q \geq 0} J_p \times J'_q \times D^p \times D^q \,.$$

Then $(\widehat{X} \times \widehat{X}')_n$ is closed in $\widehat{X} \times \widehat{X}'$ since it is the disjoint union over all $\{j\} \times \{j'\} \times D^p \times D^q$ with $(j, j') \in J_p \times J'_q$ and $p + q \leq n$. This implies that the $n$-skeleton $(X \widehat{\times} X')_n$ is a closed subset of $X \widehat{\times} X'$ and that the topology on $(X \widehat{\times} X')_n$ as a subspace of $X \widehat{\times} X'$ coincides with the quotient topology induced by the restriction of $q \times q'$ to a map

$$(q \times q')_n \colon (\widehat{X} \times \widehat{X}')_n \to (X \widehat{\times} X')_n \,.$$

The definition of the topology of a disjoint union implies that a subset $V \subseteq \widehat{X} \times \widehat{X}'$ is closed if and only if for all $n \geq 0$, the intersection $V \cap (\widehat{X} \times \widehat{X}')_n$ is a closed subset of $(\widehat{X} \times \widehat{X}')_n$. So if $U \subseteq X \widehat{\times} X'$ is a subset such that $U \cap (X \widehat{\times} X')_n \subseteq (X \widehat{\times} X')_n$ is closed for all $n$, then

$$(q \times q')_n^{-1}(U) = (q \times q')^{-1}(U) \cap (\widehat{X} \times \widehat{X}')_n \subseteq (\widehat{X} \times \widehat{X}')_n$$

is closed for all $n$. Hence $(q \times q')^{-1}(U)$ is closed in $\widehat{X} \times \widehat{X}'$, and $U$ is closed in $X \widehat{\times} X'$. This implies that our filtration of $X \widehat{\times} X'$ by the $n$-skeleta satisfies the "weak topology" requirement in the definition of CW-complexes (which can both be formulated in terms of open and closed sets).

It remains to prove that $(X \widehat{\times} X')_n$ arises from $(X \widehat{\times} X')_{n-1}$ by attaching $n$-cells. For this we note that the quotient map $(q \times q')_n$ defines an equivalence relation $\sim_n$ on $(\widehat{X} \times \widehat{X}')_n$ where $x \sim_n x'$ if and only if $(q \times q')_n(x) = (q \times q')_n(x')$. In other words, we can identify $(X \widehat{\times} X')_n$ with the quotient of $(\widehat{X} \times \widehat{X}')_n$ by the equivalence relation $\sim_n$. We can rewrite $(\widehat{X} \times \widehat{X}')_n$ as

$$(\widehat{X} \times \widehat{X}')_n = (\widehat{X} \times \widehat{X}')_{n-1} \coprod \coprod_{p+q=n} J_p \times J'_q \times D^p \times D^q \,.$$

Using this decomposition, the quotient by the equivalence relation $\sim_n$ can be obtained by first forming the quotient of $(\widehat{X} \times \widehat{X}')_{n-1}$ by the equivalence relation $\sim_{n-1}$ and then taking the quotient by the equivalence relation $\approx_n$ generated by all identifications that involve points in $J_p \times J'_q \times D^p \times D^q$ for $p+q=n$. The quotient of $(\widehat{X} \times \widehat{X}')_{n-1}$ by the equivalence relation $\sim_{n-1}$ is $(X\widehat{\times}X')_{n-1}$, and the quotient of the equivalence relation $\approx_n$ on the disjoint union

$$(X\widehat{\times}X')_{n-1} \coprod \coprod_{p+q=n} J_p \times J'_q \times D^p \times D^q$$

coincides with the pushout of the diagram

$$\bigcup_{p+q=n} J_p \times J'_q \times D^p \times D^q \leftarrow \bigcup_{p+q=n} J_p \times J'_q \times (\partial D^p \times D^q \cup D^p \times \partial D^q) \to (X\widehat{\times}X')_{n-1}.$$

Since the pair $(D^p \times D^q, \partial D^p \times D^q \cup D^p \times \partial D^q)$ is homeomorphic to $(D^{p+q}, \partial D^{p+q})$, this shows that $(X\widehat{\times}X')_n$ arises from $(X\widehat{\times}X')_{n-1}$ by attaching $n$-cells indexed by $\widehat{J}_n$. □

**Lemma 13.8.** *Let $X$ and $X'$ be CW-complexes such that $X'$ is a locally compact space. Then the topology on $X\widehat{\times}X'$ coincides with the product topology (or, equivalently, the identity $X\widehat{\times}X' \to X \times X'$ is a homeomorphism).*

*Proof.* We factor the map $q \times q' \colon \widehat{X} \times \widehat{X}' \to X\widehat{\times}X'$ considered above as

$$\widehat{X} \times \widehat{X}' \xrightarrow{\mathrm{id}\times q'} \widehat{X} \times X' \xrightarrow{q\times\mathrm{id}} X\widehat{\times}X'$$

and apply Proposition 8.14 twice: The first map is a quotient map since $\widehat{X}$ is locally compact. The second map is a quotient map since $X'$ is locally compact. Hence $q \times q'$ is also a quotient map with respect to the product topology. So the quotient topology used to define $X\widehat{\times}X'$ and the product topology coincide. □

One can construct examples of CW-complexes $X$ and $X'$ that fail to be locally compact and that have the property that the identity $X\widehat{\times}X' \to X \times X'$ is not an homeomorphism. We will not discuss this here.

**Corollary 13.9.** *If $X$ and $X'$ are CW-complexes with $X'$ finite, then the product space $X \times X'$ inherits a CW-structure.*

*Proof.* If $X'$ is a finite CW-complex, then $X'$ is compact and therefore locally compact. Hence the above lemma applies. □

We can view the interval $[0,1]$ as a CW-complex with two 0-cells and one 1-cell. If $X$ is a CW-complex, the above implies that the product space $X \times [0,1]$ inherits a CW-structure. Each $n$-cell of $X$ gives rise to two $n$-cells and one $n+1$-cell of $X \times [0,1]$.

# Lecture 14: Higher homotopy groups

We choose a basepoint $s_0 \in S^n$ (where $n \geq 0$) and keep this choice for the rest of this lecture.

**Definition 14.1.** Let $X$ be a topological space with basepoint $x_0 \in X$ and let $n \geq 0$ be a natural number. Then we write

$$\pi_n(X, x_0) = [(S^n, s_0), (X, x_0)]_*$$

for the set of basepoint preserving homotopy classes of based continuous maps $S^n \to X$. We view $\pi_n(X, x_0)$ as a based set whose preferred element is the class of the constant map with value $x_0$.

- If $n = 0$, then $\pi_0(X, x_0)$ is the set of path components of $X$, viewed as a based set with the path component of the chosen point $x_0$ as preferred element.
- If $n = 1$, then $\pi_1(X, x_0)$ is the (underlying set of) the fundamental group of $X$ with basepoint $x_0$.
- If $n \geq 2$, then one can show that $\pi_n(X, x_0)$ admits an abelian group structure, and $\pi_n(X, x_0)$ is called the $n$-th homotopy group of $X$.

**Example 14.2.** By the cellular approximation theorem, $\pi_n(S^m, s_0)$ consists of only one element if $n < m$. In contrast, $\pi_n(S^m, s_0)$ is (highly) non-trivial for general $n > m$.

We also note that a basepoint preserving continuous map $f\colon (X, x_0) \to (Y, y_0)$ induces a well defined map of based sets

$$f_*\colon \pi_n(X, x_0) \to \pi_n(Y, y_0), \quad [\alpha] \mapsto [f \circ \alpha] .$$

Somewhat analogous to the situation of homology groups, it is also useful to consider a relative version of higher homotopy groups.

**Definition 14.3.** Let $(X, A)$ be a pair of spaces and let $x_0 \in A$ be a basepoint. For $n \geq 1$, we let $\pi_n(X, A, x_0)$ be the set of homotopy classes of continuous maps

$$\alpha\colon (D^n, S^{n-1}, s_0) \to (X, A, x_0)$$

that preserve the subspace $S^{n-1}$ and the basepoint $s_0$.

In more detail, we view $S^{n-1}$ as $\partial D^n \subset D^n$ an assume that $s_0 \in S^{n-1} \subset D^n$. Then a map $\alpha\colon (D^n, S^{n-1}, s_0) \to (X, A, x_0)$ is a continuous map $\alpha\colon D^n \to X$ with $\alpha(S^{n-1}) \subseteq A$ and $\alpha(s_0) = x_0$, and the homotopies $H\colon D^n \times [0,1] \to X$ between maps of this form are required to satisfy $H(S^{n-1} \times [0,1]) \subseteq A$ and $H(\{s_0\} \times [0,1]) = \{x_0\}$.

We view $\pi_n(X, A, x_0)$ as a based set with preferred element the homotopy class $[\mathrm{const}_{x_0}]$ represented by the constant map with value $x_0$.

## The definition of the group structures

To describe the group structure on $\pi_n(X, x_0)$ for $n \geq 1$, we let $I^n = [0,1]^n$ be the $n$-dimensional cube. Since $I^n/\partial I^n$ is homeomorphic to $S^n$, we can represent elements in $\pi_n(X, x_0)$ by homotopy classes of maps $(I^n, \partial I^n) \to (X, x_0)$.

Now consider maps of pairs $\alpha, \beta\colon (I^n, \partial I^n) \to (X, x_0)$. For every $i$ with $1 \leq i \leq n$, we define a map $\alpha +_i \beta\colon (I^n, \partial I^n) \to (X, x_0)$ by setting

$$(\alpha +_i \beta)(t_1, \ldots, t_n) = \begin{cases} \alpha(t_1, \ldots, t_{i-1}, 2t_i, t_{i+1}, \ldots, t_n) & \text{if } 0 \leq t_i \leq \frac{1}{2} \\ \beta(t_1, \ldots, t_{i-1}, 2t_i - 1, t_{i+1}, \ldots, t_n) & \text{if } \frac{1}{2} \leq t_i \leq 1\,. \end{cases}$$

Moreover, we set $[\alpha] +_i [\beta] = [\alpha +_i \beta]$. The same argument that shows that composition of loops provides a well-defined group structure on the fundamental group also shows that each $+_i$ provides a well-defined group structure on $\pi_n(X, x_0)$. For $n = 1$, this group structure is in fact just the usual group structure on the fundamental group $\pi_1(X, x_0)$.

In the case $n \geq 2$, it remains to compare the possibly different group structures $+_i$ for varying $i$. The main tool for this will be the following lemma (which is known as the *Eckmann–Hilton argument*).

**Lemma 14.4.** *Let $M$ be a set with two binary operations $*$ and $\circ$ that both admit a two-sided unit element and satisfy $(a \circ b) * (c \circ d) = (a * c) \circ (b * d)$ for all $a, b, c, d \in M$. Then the two operations coincide, and they are associative and commutative.*

*Proof.* Writing $1_*$ for the unit of $*$ and $1_\circ$ for the unit of $\circ$, we first note that

$$1_* = 1_* * 1_* = (1_* \circ 1_\circ) * (1_\circ \circ 1_*) = (1_* * 1_\circ) \circ (1_\circ * 1_*) = 1_\circ \circ 1_\circ = 1_\circ.$$

Hence we can write 1 for the common unit $1_* = 1_\circ$ of $*$ and $\circ$. For all $a, b \in M$, we have

$$\begin{aligned} a * b &= (a \circ 1) * (1 \circ b) = (a * 1) \circ (1 * b) = a \circ b \quad \text{and} \\ a * b &= (1 \circ a) * (b \circ 1) = (1 * b) \circ (a * 1) = b \circ a\,. \end{aligned}$$

This shows the two products coinicde and are commutative. For all $a, b, c \in M$, we have

$$(a * b) * c = (a * b) * (1 * c) = (a * 1) * (b * c) = a * (b * c)$$

and thus associativity. ☐

**Corollary 14.5.** *Let $n \geq 2$. Then the above group structures $+_i$ on $\pi_n(X, x_0)$ agree for all $1 \leq i \leq n$, and they are abelian.*

*Proof.* For $i \neq j$, the group structures $+_i$ and $+_j$ satisfy the assumptions of the last lemma. For $n = 2$, the compatibility is displayed in Figure 14.1. (In fact, we could have even used the last lemma to establish associativity, rather than arguing that we can re-use the proof for associativity of the group structure on the fundamental group.) ☐

The common group structure arising from the last corollary is referred to as "the" group structure on the higher homotopy group $\pi_n(X, x_0)$.

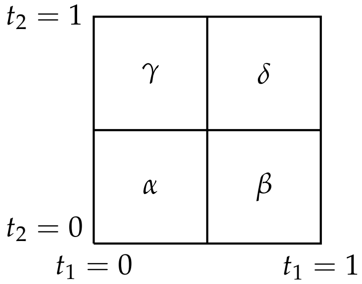

Figure 14.1: Compatibility of group structures $+_1$ and $+_2$ on $\pi_2(X, x_0)$

To define a group structure on $\pi_n(X, A, x_0)$ when $n \geq 2$, we consider the subspaces of the cube given by

$$I^{n-1} = \{(t_1, \dots, t_n) \in I^n \mid t_n = 0\} \quad \text{and}$$
$$J^{n-1} = \{(t_1, \dots, t_n) \in I^n \mid t_n = 1 \text{ or there exists } 1 \leq i \leq n-1 \text{ with } t_i = 0 \text{ or } t_i = 1\} \, .$$

We think of $I^{n-1}$ as the front face of $I^n$ and of $J^{n-1}$ as the union of all other faces of $I^n$. In particular, we note that $I^{n-1} \cup J^{n-1} = \partial I^n$ and $I^{n-1} \cap J^{n-1} = \partial I^{n-1}$. Since there is a homeomorphism $I^n / J^{n-1} \to D^n$ that restricts to a homeomorphism $\partial I^n / J^{n-1} \to S^{n-1}$, there is a bijection

$$\pi_n(X, A, x_0) \xrightarrow{\cong} [(I^n, \partial I^n, J^{n-1}), (X, A, x_0)] \, .$$

Using the latter description, we see that the maps $+_1, \dots, +_{n-1}$ defined above define group structures on $\pi_n(X, A, x_0)$ if $n \geq 2$. If $n \geq 3$, they coincide and define a unique abelian group law on $\pi_n(X, A, x_0)$.

## The long exact sequence of homotopy groups

We will now develop a relationship between higher homotopy groups and their relative version.

**Lemma 14.6.** *For every $n \geq 1$, there is a bijection $\pi_n(X, x_0) \xrightarrow{\cong} \pi_n(X, \{x_0\}, x_0)$. It is an isomorphism of groups if $n \geq 2$.*

*Proof.* From our earlier description

$$\pi_n(X, x_0) = [(I^n, \partial I^n), (X, x_0)] \quad \text{and} \quad \pi_n(X, \{x_0\}, x_0) \cong [(I^n, \partial I^n, J^{n-1}), (X, \{x_0\}, x_0)]$$

it is clear that on both sides, an element is represented by a map $I^n \to X$ sending $\partial I^n$ to $x_0$. This is compatible with passage to homotopy classes and the group structures. □

**Remark 14.7.** Alternatively, one may describe the map in the lemma by first observing that by the universal property of the quotient space $D^n / S^{n-1}$, continuous maps $\alpha \colon (D^n, S^{n-1}, s_0) \to (X, \{x_0\}, x_0)$ correspond to continuous and basepoint preserving maps $\overline{\alpha} \colon D^n / S^{n-1} \to X$. Next we choose a homeomorphism $\phi \colon S^n \to D^n / S^{n-1}$ that sends the chosen basepoint of $S^n$ to the image of the basepoint of $D^n$ in the quotient. By composition with $\phi$, basepoint preserving maps $D^n / S^{n-1} \to X$ correspond to basepoint preserving maps $S^n \to X$.

We now consider the following sequence of maps:

$$\begin{aligned} \cdots \to \pi_{n+1}(X,x_0) \xrightarrow{j} \pi_{n+1}(X,A,x_0) \xrightarrow{p} \pi_n(A,x_0) \\ \xrightarrow{i} \pi_n(X,x_0) \xrightarrow{j} \pi_n(X,A,x_0) \to \ldots \xrightarrow{i} \pi_0(X,x_0) \end{aligned} \quad (14.1)$$

The individual maps are defined as follows:

- The map $i\colon \pi_n(A,x_0) \to \pi_n(X,x_0)$ sends the element $[\alpha\colon (S^n,s_0) \to (A,x_0)]$ to the element $[\mathrm{incl} \circ \alpha\colon (S^n,s_0) \to (X,x_0)]$. It is a group homomorphism if $n \geq 1$.
- The map $j\colon \pi_n(X,x_0) \to \pi_n(X,A,x_0)$ is the composite of the isomorphism $\pi_n(X,x_0) \xrightarrow{\cong} \pi_n(X,\{x_0\},x_0)$ constructed in Lemma 14.6 with the map $\pi_n(X,x_0,x_0) \to \pi_n(X,A,x_0)$ obtained by viewing a map to $\{x_0\}$ as a map to $A$. By the lemma, it is a group homomorphism if $n \geq 2$.
- The map $p\colon \pi_n(X,A,x_0) \to \pi_{n-1}(A,x_0)$ sends the element $[\alpha\colon (D^n,S^{n-1},s_0) \to (X,A,x_0)]$ to the element $[\alpha|_{\partial D^n}\colon (S^{n-1},s_0) \to (A,x_0)]$. To see that it is a group homomorphism if $n \geq 2$, we represent an element in the source by a map $\alpha\colon (I^n,\partial I^n,J^{n-1}) \to (X,A,x_0)$ and note that its image is represented by $\alpha|_{I^{n-1}}\colon (I^{n-1},\partial I^{n-1}) \to (A,x_0)$. Inspecting the definition of the group structures, we note that it is a group homomorphism $n \geq 2$.

Next we recall the following notion from Exercise 12.2.

**Definition 14.8.** Let $A_1, A_2$ and $A_3$ be pointed sets with preferred elements $a_i \in A_i$. Let

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$$

be a sequence of maps of sets with $f_1(a_1) = a_2$ and $f_2(a_2) = a_3$. We say that this sequence is *exact* if $f_1(A_1) = f_2^{-1}(a_3)$.

If the maps $f_1$ and $f_2$ are group homomorphisms and the preferred elements are the neutral elements of the group structures, then this notion of exactness coincides with the notion of exactness considered earlier.

**Proposition 14.9.** *The sequence* (14.1) *is a long exact sequence.*

For the proof of the proposition, we need the following lemma.

**Lemma 14.10.** *Let $\alpha\colon (D^n,S^{n-1},s_0) \to (X,A,x_0)$ be a representative of $[\alpha] \in \pi_n(X,A,x_0)$. Then $[\alpha]$ coincides with $[\mathrm{const}_{x_0}]$ if and only if $\alpha$ is homotopic relative to $\partial D^n$ to a map with image in $A$.*

*Proof.* If $[\alpha] = [\mathrm{const}_{x_0}]$ in $\pi_n(X,A,x_0)$, then there exists a homotopy $F\colon D^n \times [0,1] \to X$ with

$$F(D^n \times \{0\}) = \{x_0\}, \quad F|_{D^n \times \{1\}} = \alpha, \quad F(\{s_0\} \times [0,1]) = \{x_0\}, \text{ and } \quad F(\partial D^n \times [0,1]) \subseteq A\,.$$

The problem is that this homotopy may not be a homotopy relative to $\partial D^n$. That is, for $x \in \partial D^n$ the value of $F(x,t) \in A$ may vary in $t$. To build a homotopy $H\colon D^n \times [0,1] \to X$ relative to $\partial D^n$, we view $D^n \times [0,1]$ as a cylinder in $\mathbb{R}^n \times \mathbb{R}$ and perform the following construction that is outlined in the case $n = 2$ in Figure 14.2. For any point $x$ in $D^n$, we consider the line in

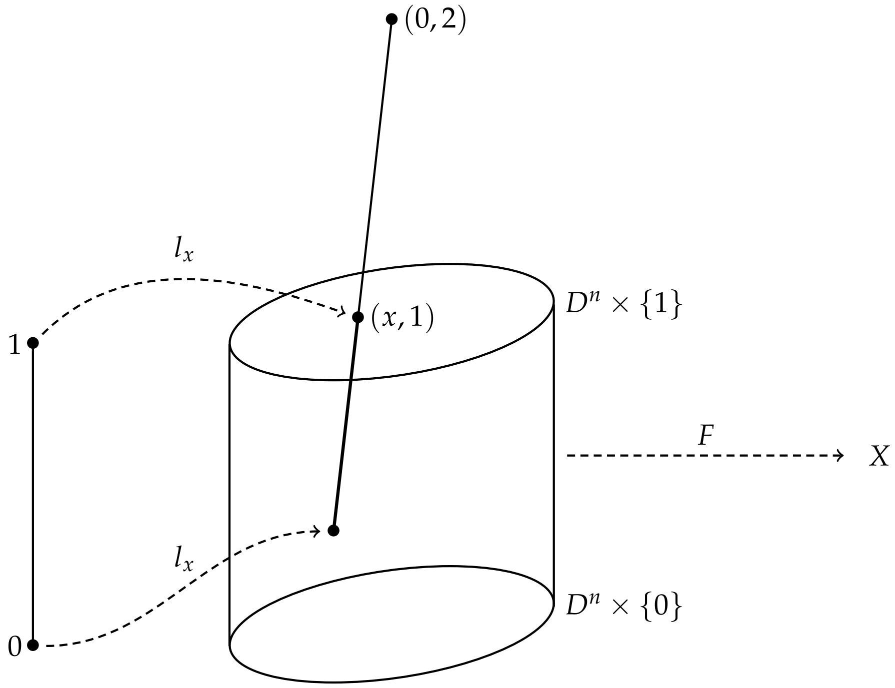

Figure 14.2: Construction of the homotopy $H$ in the proof of Lemma 14.10

$\mathbb{R}^{n+1}$ from $(0,2)$ to $(x,1)$ and let $l_x\colon [0,1] \to D^n \times [0,1]$ be the affine linear map that sends the interval to the intersection of this line with the $D^n \times [0,1]$. In particular, this means that $l_x(1) = (x,1)$, that $l_x(0) \in \partial D^n \times [0,1] \cup D \times \{0\}$, and that $l_x(t) \in \partial D^n \times \{1\}$ if $x \in \partial D^n$ and $t \in [0,1]$ is arbitrary. Since $l_x(t)$ is continuous in $x$ and $t$, we can define a new homotopy by $H(x,t) = F(l_x(t))$. Then $H(x,1) = F(x,1) = \alpha(x)$ for all $x \in D^n$, $H(x,0) \in A$ for all $x \in D^n$, and $H(x,t) = \alpha(x)$ for all $x \in \partial D^n$ and $t \in [0,1]$. Hence $H$ is a homotopy relative to $\partial D^n$ with $H|_{D^n \times \{1\}} = \alpha$ and $H(D^n \times \{0\}) \subseteq A$.

For the other direction, it is sufficient to show that any map $\alpha'\colon (D^n, S^{n-1}, s_0) \to (X, A, x_0)$ with $\alpha'(D^n) \subseteq A$ represents the class of $\mathrm{const}_{x_0}$ in $\pi_n(X, A, x_0)$. This follows by composing $\alpha'$ with a homotopy that contracts $D^n$ to the point $s_0$. ☐

*Proof of Proposition 14.9.* To show exactness at $\pi_n(X, x_0)$, we use a homeomorphism $D^n/\partial D^n \to S^n$ to view maps of pairs $\alpha\colon (D^n, \partial D^n) \to (X, x_0)$ as representatives of elements in $\pi_n(X, x_0)$. For a map $\alpha\colon (D^n, \partial D^n) \to (X, x_0)$, we have $[\alpha] \in i(\pi_n(X, x_0))$ if and only if $\alpha$ is homotopic relative to $\partial D^n$ to a map with image in $A$. By Lemma 14.10, the latter condition is satisfied if and only if $\alpha$ represents the class of the constant map in $\pi_n(X, A, x_0)$.

To show exactness at $\pi_n(A, x_0)$, we note that our earlier observation that a map $S^n \to X$ is homotopic to a constant map if and only if it extends to a map on $D^{n+1}$ also applies to basepoint preserving maps. Hence for $\alpha\colon (S^n, s_0) \to (A, x_0)$, it follows that $i([\alpha])$ represents the class of a constant maps if and only if $[\alpha]$ is in the image of $p\colon \pi_{n+1}(X, A, x_0) \xrightarrow{p} \pi_n(A, x_0)$.

For exactness at $\pi_{n+1}(X, A, x_0)$, we first note that

$$p(j([\alpha\colon (D^{n+1}, \partial D^{n+1}) \to (X, x_0)])) = [\alpha|_{\partial D^{n+1}}] = [\mathrm{const}_{x_0}] .$$

Hence $j(\pi_{n+1}(X, x_0)) \subseteq p^{-1}([\mathrm{const}_{x_0}])$. If $p([\alpha\colon (D^{n+1}, \partial D^{n+1}, s_0) \to (X, A, x_0)])$ is the class of the constant map, then $\alpha|_{\partial D^{n+1}}$ is homotopic relative to $s_0$ to a constant map. The homotopy extension property for the pair $(D^{n+1}, \partial D^{n+1})$ implies that this homotopy extends to a homotopy from $\alpha$ to a map $\alpha'$ with $\alpha'(\partial D^{n+1}) = \{x_0\}$. Hence $[\alpha]$ is in the image of $j\colon \pi_{n+1}(X, x_0) \to \pi_{n+1}(X, A, x_0)$. □

### The choice of basepoints

We now discuss how the higher homotopy groups $\pi_n(X, x_0)$ depend on the choice of the basepoint $x_0$.

**Proposition 14.11.** *Let X be a topological space and let n $\geq 1$ be an integer. Every path $w\colon [0,1] \to X$ induces a group isomorphism $w_*\colon \pi_n(X, w(1)) \to \pi_n(X, w(0))$ such that the following holds:*

*(i) If w and w′ are paths that are homotopic relative to their start and end points, then $w_* = w'_*$.*

*(ii) If w is constant, then $w_* = \mathrm{id}$.*

*(iii) If w and v are composable paths, then $(w * v)_* = w_* \circ v_*$. (Here $w * v$ denotes the composition of the two paths.)*

*(iv) If $f\colon X \to Y$ is continuous, then $f_* \circ w_* = (f \circ w)_* \circ f_*$. In other words, the square*

$$\begin{tikzcd}
\pi_n(X, w(1)) \arrow[r, "f_*"] \arrow[d, "w_*"'] & \pi_n(Y, f(w(1))) \arrow[d, "(f\circ w)_*"] \\
\pi_n(X, w(0)) \arrow[r, "f_*"] & \pi_n(Y, f(w(0))).
\end{tikzcd}$$

*commutes.*

*Proof.* Let $\alpha\colon (D^n, \partial D^n) \to (X, w(1))$ represent $[\alpha] \in \pi_n(X, w(1))$. Then we define $w_*([\alpha])$ to be the class represented by the composite

$$(D^n, \partial D^n) \xrightarrow{\cong} (D^n \cup_{\partial D^n \times \{1\}} \partial D^n \times [0,1], \partial D^n \times \{0\}) \xrightarrow{\alpha \cup \mathrm{pr} \circ w} (X, w(0)) \,.$$

Here the first homeomorphism is analogous to the one used in the proof of Lemma 14.10. The map $\alpha \cup \mathrm{pr} \circ w$ arises by applying the universal property of the pushout to $\alpha\colon D^n \to X$ and $\partial D^n \times [0,1] \xrightarrow{\mathrm{pr}} [0,1] \xrightarrow{w} X$. Arguments that are analogous to those establishing the well-defined group structure on the fundamental group show that properties (i) - (iv) hold. One also checks that $w_*$ is a homomorphism of groups. The claim that the map $w_*$ is a bijection follows since by properties (i) - (iii), the inverse path $w^{-1}\colon [0,1] \to X, t \mapsto w(1-t)$ provides an inverse $(w^{-1})_*$ to $w_*$. □

In the special case where $w(0) = w(1)$, the proposition provides an action of the fundamental group $\pi_1(X, x_0)$ on the set $\pi_n(X, x_0)$. This is the conjugation action if $n = 1$.

# Lecture 15: The Whitehead theorem

Let $X$ be a topological space with basepoint $x_0$. In the last lecture we have seen that

$$\pi_n(X, x_0) = [(S^n, s_0), (X, x_0)]_*$$

has a group structure if $n \geq 1$ that is abelian if $n \geq 2$. We also studied how $\pi_n(X, x_0)$ depends on the choice of the basepoint $x_0$: If $n \geq 1$, every path $w\colon [0,1] \to X$ induces a group isomorphism $w_*\colon \pi_n(X, w(1)) \to \pi_n(X, w(0))$ that only depends on the homotopy class of $w$ relative to the start and end point of $w$. Moreover, $w_*$ is compatible with composition and the maps of homotopy groups induced by continuous maps $f\colon X \to Y$.

**Corollary 15.1.** *Let $f\colon X \to Y$ be a continuous map, let $x_0 \in X$ be a point, and let $n \geq 1$. If $f_*\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is bijective (or surjective), then $f_*\colon \pi_n(X, x_1) \to \pi_n(Y, f(x_1))$ is bijective (or surjective) for any point $x_1$ in the same path component of $X$ as $x_0$.*

*Proof.* We choose a path $w$ from $x_0$ to $x_1$ and apply part (iv) of Proposition 14.11. □

We will also need the following compatibility of the bijections induced by paths and homotopies between continuous maps.

**Lemma 15.2.** *Let $f, g\colon X \to Y$ be continuous maps, let $H\colon X \times [0,1] \to Y$ be a homotopy from $f$ to $g$, and let $x_0 \in X$ be a basepoint. Let $w = H|_{\{x_0\}\times[0,1]}\colon [0,1] \to Y$ be the path in $Y$ from $f(x_0)$ to $g(x_0)$ that results from $H$. Then $f_*$ and $w_* \circ g_*$ coincide as maps $\pi_n(X, x_0) \to \pi_n(Y, f(x_0))$, i.e., the following diagram is commutative:*

$$\begin{tikzcd}
\pi_n(X, x_0) \arrow[r, "g_*"] \arrow[dr, "f_*"'] & \pi_n(Y, g(x_0)) \arrow[d, "w_*"] \\
& \pi_n(Y, f(x_0)).
\end{tikzcd} \tag{15.1}$$

*Proof.* Let $\alpha\colon (D^n, \partial D^n) \to (X, x_0)$ represent $[\alpha] \in \pi_n(X, x_0)$. Consider the homotopy

$$F\colon D^n \times [0,1] \to Y, \quad (x, t) \mapsto H(\alpha(x), t)\ .$$

Then $F|_{D^n\times\{1\}} = g \circ \alpha$, $F|_{D^n\times\{0\}} = f \circ \alpha$, and $F(x, t) = w(t)$ for all $x \in \partial D^n$ and $t \in [0,1]$. By an argument analogous to the one in the proof of Lemma 14.10, we can convert $F$ into a homotopy relative $\partial D^n$ from $f \circ \alpha$ to a representative of $w_*[g \circ \alpha]$. This implies that $[f \circ \alpha] = w_*[g \circ \alpha]$. □

**Definition 15.3.** A continuous map $f\colon X \to Y$ is a *weak homotopy equivalence* if it induces bijections $f_*\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ for all $n \geq 0$ and all basepoints $x_0 \in X$.

**Corollary 15.4.** *Every homotopy equivalence $f\colon X \to Y$ is a weak homotopy equivalence.*

*Proof.* It is easy to check from the definition that $f$ induces an isomorphism on path components. Let $g$ be a homotopy inverse of $f$, let $H$ be a homotopy from $g \circ f$ to $\mathrm{id}_X$ and let $x_0$ be a point in $X$. Set $w = H_{\{x_0\}\times[0,1]}\colon [0,1] \to X$. Then by Lemma 15.2, the diagram

$$\begin{tikzcd}
\pi_n(X, x_0) \arrow[r, "f_*"] \arrow[dr, "w_*"'] & \pi_n(Y, f(x_0)) \arrow[d, "g_*"] \\
& \pi_n(X, g(f(x_0)))
\end{tikzcd}$$

commutes. Since $w_*$ is a bijection, it follows that $f_*$ is injective and that $g_*$ is surjective. By Corollary 15.1, $g_*\colon \pi_n(Y, y_0) \to \pi_n(X, g(y_0))$ is surjective for any other point $y_0$ in the path component of $f(x_0)$. Since $f$ induces a bijection on path components, $g_*$ is surjective for any choice of the basepoint $y_0$. Reversing the roles of $f$ and $g$, it follows that $f_*$ is surjective for any choice of basepoint. Hence $f_*$ is bijective for all basepoints and all $n \geq 1$. ☐

In this lecture, we will show the "Whitehead Theorem". It states that if $X$ and $Y$ are CW-complexes, then the converse of the corollary is true. That is, every weak homotopy equivalence between CW-complexes is a homotopy equivalence.

Before we start to work towards the proof of this statement, we note that the property of being a weak homotopy equivalence is preserved under homotopy.

**Lemma 15.5.** *If $f\colon X \to Y$ is a weak homotopy equivalence and $f'\colon X \to Y$ is homotopic to $f$, then $f'$ is also a weak homotopy equivalence.*

*Proof.* Since homotopic maps induce the same maps on path components, the $n = 0$ case is clear. For the corresponding statement about $\pi_n$ with $n \geq 1$, we let $H$ be a homotopy from $f$ to $f'$. Moreover, let $x_0 \in X$ be a point and let $w = H(x_0, -)$ be the path from $f(x_0)$ to $f'(x_0)$ determined by $H$. By Lemma 15.2, the map $f_*\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is the composite of $f'_*\colon \pi_n(X, x_0) \to \pi_n(Y, f'(x_0))$ and the action of $w$. Since $w$ acts through an isomorphism and $f_*$ is an isomorphism, it follows that $f'_*$ is an isomorphism. ☐

## The Whitehead theorem for relative CW-complexes

We begin by using the long exact sequence of homotopy groups to express the property of being a weak homotopy equivalence in terms of relative homotopy groups.

**Lemma 15.6.** *Let $(X, A)$ be a pair of spaces such that the inclusion map $i\colon A \to X$ induces a bijective map $\pi_0(A) \to \pi_0(X)$ on path components. Then $i$ is a weak homotopy equivalence if and only if $\pi_n(X, A, x_0)$ is trivial for all $n \geq 1$ and all $x_0 \in A$.*

*Proof.* We argue with the long exact sequence of homotopy groups

$$\cdots \to \pi_{n+1}(X, x_0) \xrightarrow{j} \pi_{n+1}(X, A, x_0) \xrightarrow{p} \pi_n(A, x_0)$$
$$\xrightarrow{i} \pi_n(X, x_0) \xrightarrow{j} \pi_n(X, A, x_0) \to \ldots \xrightarrow{i} \pi_0(X, x_0)$$

and notice that this situation is very much analogous to the corresponding statement about homology isomorphisms and relative homology groups that we proved in Corollary 4.15. The argument based on Examples 4.4 and 4.5 used there is not affected by the fact that $\pi_2(X, A, x_0)$, $\pi_1(A, x_0)$ and $\pi_1(X, x_0)$ are not necessarily abelian. Apart from that, we are are lacking a group structure on $\pi_1(X, A, x_0)$. However, this is no problem since also for exact sequences of based sets it is true that $A_1 \to A_2 \to 0$ is exact if and only if $A_1 \to A_2$ is surjective. With our assumption that $i$ induces an isomorphism of path components, this shows that $\pi_1(A, x_0) \to \pi_1(X, x_0)$ is surjective if and only if the set $\pi_1(X, A, x_0)$ contains just one element. ☐

**Definition 15.7.** Let $(X,A)$ be a pair of spaces. Then $(X,A)$ is called *n-connected* if for every $m \leq n$ and every map of pairs $g\colon (D^m, \partial D^m) \to (X,A)$, there is a homotopy relative to $\partial D^m$ from $g$ to a map with image in $A$. That is, there exists a homotopy $H\colon D^m \times [0,1] \to X$ such that $H|_{D^m \times \{0\}} = g$, $H(x,t) = g(x)$ for all $x \in \partial D^m$ and $t \in [0,1]$, and $H(x,1) \in A$ for all $x \in D^m$.

For example, since $(D^0, \partial D^0) = (\{*\}, \emptyset)$, it follows directly from the definition that $(X,A)$ is 0-connected if and only if the map $\pi_0(A) \to \pi_0(X)$ of path components induced by the inclusion is surjective.

**Corollary 15.8.** *Let $(X,A)$ be a pair of spaces. If the inclusion $A \to X$ is a weak homotopy equivalence, then $(X,A)$ is n-connected for every $n$.*

*Proof.* The case $n = 0$ follows from the comment before the corollary. If $n \geq 1$, then $\pi_n(X,A,x_0)$ is trivial for all choices of $x_0 \in X$ by Lemma 15.6, and thus every map $g\colon (D^n, \partial D^n) \to (X,A)$ is homotopic relative $\partial D^n$ to a map with image in $A$ by Lemma 14.10. ☐

We recall the following notion which already appeared on the exercise sheets.

**Definition 15.9.** Let $X$ be a topological space and let $A \subseteq X$ be a subspace. Then $A$ is called a *deformation retract* of $X$ if there exists a continuous map $H\colon X \times [0,1] \to X$ such that $H(x,0) = x$ for all $x \in X$, $H(x,t) = x$ for all $x \in A$ and all $t \in [0,1]$, and $H(x,1) \in A$ for all $x \in X$. This means that $\mathrm{incl}_A\colon A \to X$ and $r\colon X \to A, x \mapsto H(x,1)$ are continuous maps such that $r \circ \mathrm{incl}_A = \mathrm{id}_A$ and such that $\mathrm{incl}_A \circ r$ is homotopic relative $A$ to $\mathrm{id}_X$. In particular, both $\mathrm{incl}_A$ and $r$ are homotopy equivalences which are homotopy inverse to each other.

**Lemma 15.10.** *If $A$ is a deformation retract of $X$, then $(X,A)$ is n-connected for every $n$.*

*Proof.* Let $H\colon X \times [0,1] \to X$ be a homotopy relative $A$ from $\mathrm{id}_X$ to a map with image in $A$. Given $g\colon (D^m, \partial D^m) \to (X,A)$, the map

$$D^m \times [0,1] \to X, \quad (x,t) \mapsto H(g(x),t)$$

is the desired homotopy relative $\partial D^m$ from $g$ to a map with image in $A$. Alternatively, one can use that the inclusion $A \to X$ is a homotopy equivalence and apply Corollaries 15.4 and 15.8. ☐

Our next aim is to show that for relative CW-complexes, the converse of the previous lemma holds. The next lemma is a first step towards this result.

**Lemma 15.11.** *Let $(X,A)$ be a relative CW-complex, let $(Y,B)$ be an n-connected pair of spaces, and let $f\colon (X,A) \to (Y,B)$ be a map of pairs. Then there is a map of pairs $f_n\colon (X,A) \to (Y,B)$ with $f_n(X_n) \subseteq B$ such that $f$ and $f_n$ are homotopic relative to $A$.*

*Proof.* We set $f_{-1} = f$ and inductively construct maps $f_m\colon (X,A) \to (Y,B)$ for $0 \leq m \leq n$ such that $f_m(X_m) \subseteq B$ and that $f_m$ is homotopic to $f_{m-1}$ relative to $X_{m-1}$.

For the inductive step, assume that $f_{m-1}$ is already constructed. Let $J_m$ be the set of $m$-cells for $X$ and let $\chi_j\colon (D^m, \partial D^m) \to (X_m, X_{m-1})$ be a characteristic map for the $m$-cell $j \in J_m$. By assumption, the composite $f_{m-1} \circ \chi_j \colon (D^m, \partial D^m) \to (Y, B)$ is homotopic relative $\partial D^m$ to a map with image in $B$. Taking these homotopies for all $j \in J_m$ together provides a homotopy

$$F \colon J_m \times D^m \times [0,1] \to Y$$

with $F|_{\{j\} \times D^m \times \{0\}} = f_{m-1} \circ \chi_j$, with $F(j, x, t) = (f_{m-1} \circ \chi_j)(x)$ if $x \in \partial D^m$ and $t \in [0,1]$, and $F(J_m \times D^m \times \{1\}) \subseteq B$. We can glue this homotopy together with the constant homotopy $X_{m-1} \times [0,1] \to Y, (x,t) \mapsto f_{m-1}(x)$ to get a homotopy

$$G \colon X_m \times [0,1] \to Y$$

relative to $X_{m-1}$ from $f_{m-1}|X_m$ to a map with image in $B$. Applying the homotopy extension property to $G$ and $f_{m-1}$ provides a homotopy $H \colon X \times [0,1] \to Y$. Setting $f_m = H|_{X \times \{1\}}$, we see that $f_m(X_m) \subseteq B$ and that $f_{m-1}$ and $f_m$ are homotopic relative $X_{m-1}$ via $H$. □

**Proposition 15.12.** *Let $(Y, B)$ be a pair of spaces such that the inclusion $B \to Y$ is $n$-connected for every $n$ and let $(X, A)$ be a relative CW-complex. Then any map of pairs $f \colon (X, A) \to (Y, B)$ is homotopic relative to $A$ to a map with image in $B$.*

*Proof.* The last lemma provides a sequence of maps $f_m \colon (X, A) \to (Y, B)$ for $m \geq -1$ with $f_{-1} = f$ and homotopies $H_m \colon X \times [0,1] \to Y$ from $f_{m-1}$ to $f_m$ relative to $X_{m-1}$ such that $f_m(X_m) \subseteq B$. As in the proof of the cellular approximation theorem, we can use these data to construct a homotopy $H$ from $f$ to a map with image in $B$. □

We can now show a first version of the Whitehead theorem for relative CW-complexes.

**Theorem 15.13.** *Let $(X, A)$ be a relative CW-complex. If the inclusion $A \to X$ is a weak homotopy equivalence, then it is also a homotopy equivalence.*

*Proof.* We apply the previous proposition to $\mathrm{id}_X \colon (X, A) \to (X, A)$. The resulting homotopy $H$ relative to $A$ from $\mathrm{id}_X$ to a map with image in $A$ exhibits $A$ as a deformation retract of $X$. In particular, the inclusion $A \to X$ is a homotopy equivalence. □

## Maps between CW-complexes

We now construct an extremely useful factorization of a continuous map into a composite of an inclusion followed by a homotopy equivalence.

**Definition 15.14.** Let $f \colon X \to Y$ be a continuous map between topological spaces. The mapping cylinder $M(f)$ of $f$ is defined to be the pushout $(X \times [0,1]) \cup_X Y$ of the diagram

$$X \times [0,1] \xleftarrow{\mathrm{incl}_1} X \xrightarrow{f} Y\,.$$

Here $\mathrm{incl}_1$ is defined by $\mathrm{incl}_1(x) = (x, 1)$.

Figure 15.1 displays the mapping cylinder in the case where $X = S^1$ and $f$ is injective. This picture is supposed to motivate the name "mapping cylinder".

We write $q \colon X \times [0,1] \amalg Y \to M(f)$ for the quotient map resulting from the definition of the pushout. There are several maps relating $X$, $Y$, and the mapping cylinder $M(f)$:

- We let $j_X \colon X \to M(f)$ be the map $x \mapsto q(x,0)$. It is continuous since it is the composite of the continuous maps $q$ and $\mathrm{incl}_0 \colon X \to X \times [0,1], x \mapsto (x,0)$. Moreover, the image $j_X(X) \subset M(f)$ is a closed subspace since its preimage under $q$ is closed, and $j_X$ induces a homeomorphism $X \to j_X(X)$ to this subspace. We therefore say that $j_X$ is a *closed inclusion* and sometimes identify $X$ with the subspace $j_X(X)$ of $M(f)$.
- The map $j_Y \colon Y \to M(f), y \mapsto q(y)$ is also a closed inclusion that allows to identify $Y$ with the subspace $j_Y(Y) \subseteq M(f)$.
- The continuous maps $\mathrm{pr} \colon X \times [0,1] \to X, (x,t) \to x$ and $\mathrm{id}_Y \colon Y \to Y$ have the property $\mathrm{id}_Y \circ f = f = f \circ \mathrm{pr} \circ \mathrm{incl}_1$. By the universal property of the pushout, $\mathrm{id}_Y$ and $f \circ \mathrm{pr}$ induce a unique continuous map $p \colon M(f) \to Y$. This map satisfies $p \circ j_X = f$ and $p \circ j_Y = \mathrm{id}_Y$.

The following commutative diagram (where $j_X = (q|_{X\times[0,1]}) \circ \mathrm{incl}_0$) summarizes the situation:

$$\begin{tikzcd}
 & X \arrow[r, "f"] \arrow[d, "\mathrm{incl}_1"'] & Y \arrow[d, "j_Y"'] \arrow[ddr, bend left, "\mathrm{id}_Y"] & \\
X \arrow[r, "\mathrm{incl}_0"] \arrow[drrr, bend right, "f"'] & X \times [0,1] \arrow[r, "q|_{X\times[0,1]}"] \arrow[drr, "f \circ \mathrm{pr}"] & M(f) \arrow[dr, "p"] & \\
 & & & Y
\end{tikzcd}$$

We now continue to analyze the mapping cylinder. Consider the homotopy

$$F \colon (X \times [0,1]) \times [0,1] \to X \times [0,1], \quad (x,s,t) \mapsto (x, (1-t)\cdot s + t \cdot 1)$$

that contracts $X \times [0,1]$ to $X \times \{1\}$. Together with the constant homotopy on $\mathrm{id}_Y$, the homotopy $F$ induces a homotopy

$$H \colon M(f) \times [0,1] \to M(f)$$

with $H|_{M(f)\times\{0\}} = \mathrm{id}_{M(f)}$, $H|_{M(f)\times\{1\}} = j_Y \circ p$ and $H(j_Y(y),t) = j_Y(y)$ for all $y \in Y$ and $t \in [0,1]$. So via the identification of $Y$ with $j_Y(Y)$, we can view $Y$ as a deformation retract of $M(f)$. In particular, $p$ is a homotopy equivalence.

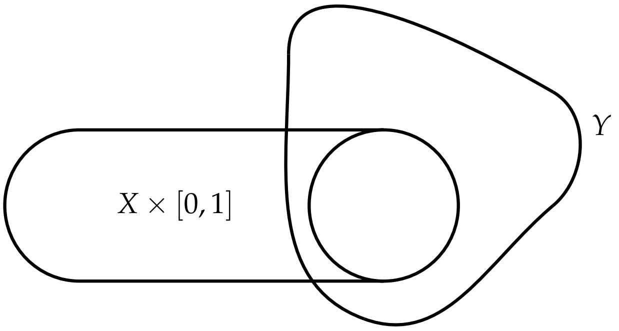

Figure 15.1: The mapping cylinder of $f \colon X \to Y$

Altogether, we have shown the following result:

**Proposition 15.15.** *Every continuous map $f\colon X \to Y$ factors as a composite*

$$X \xrightarrow{j_X} M(f) \xrightarrow{p} Y$$

*of a closed inclusion and a homotopy equivalence.*

By Exercise 11.2, the mapping cylinder has the additional property that the pair $(M(f), X)$ has the homotopy extension property.

Now let us assume in addition that $X$ and $Y$ are CW-complexes and that $f\colon X \to Y$ is a cellular map. Then $X \times [0,1]$ is a CW-complex by the results of the last lecture, and $X \times \{0\}$ can be viewed as a subcomplex of $X \times [0,1]$. Viewing $f$ as a cellular map $X \times \{0\} \to Y$, Exercise 11.3 shows that the mapping cylinder $M(f)$ becomes a CW-complex. In this situation, $j_X\colon X \to M(f)$ is a cellular inclusion, that is, $j_X$ induces a homeomorphism $X \to j_X(X)$ and $j_X(X) \subseteq M(f)$ is a subcomplex.

**Proposition 15.16.** *Every continuous map of CW-complexes $f\colon X \to Y$ is homotopic to the composite of a cellular inclusion and a homotopy equivalence.*

*Proof.* By the Cellular Approximation Theorem, $f$ is homotopic to a cellular map $f'$. Applying the last proposition and the discussion before the theorem to $f'$, the desired factorization is given by

$$X \xrightarrow{j_X} M(f') \xrightarrow{p} Y \qquad \square\,.$$

## The general Whitehead theorem

The following result is known as the *Whitehead Theorem*.

**Theorem 15.17.** *If $f\colon X \to Y$ is a weak homotopy equivalence between CW-complexes, then $f$ is a homotopy equivalence.*

*Proof.* We apply the cellular approximation theorem, let $f'$ be a cellular map that is homotopic to $f$, and consider the factorization

$$X \xrightarrow{j_X} M(f') \xrightarrow{p} Y$$

into a cellular inclusion $j_X$ and a homotopy equivalence $p$ resulting from Proposition 15.16. Since $f$ and $f'$ are homotopic, Lemma 15.5 implies that $f'$ is a weak equivalence. The map $p$ is a weak homotopy equivalence by Corollary 15.4. Since $p \circ j_X = f'$, it follows that $j_X$ is also a weak homotopy equivalence. Since $j_X$ is homeomorphic to the inclusion map in the relative CW-complex $(M(f'), j_X(X))$, Theorem 15.13 shows that $j_X$ is a homotopy equivalence. This in turn implies that $f' = p \circ j_X$ and $f$ are homotopy equivalences. $\square$

**Remark 15.18.** It is in general not true that two CW-complexes $X$ and $Y$ for which there are bijections between the sets $\pi_n(X, x_0)$ and $\pi_n(Y, y_0)$ are homotopy equivalent. In other words, it is an essential assumption of the Whitehead Theorem that there is a continuous map inducing such bijections.

In fact, the spaces $\mathbb{R}P^2$ and $S^2 \times \mathbb{R}P^\infty$ are not homotopy equivalent but have isomorphic homotopy groups in all degrees. However, we do not have developed all tools to show this.

# Lecture 16: CW-approximation and the Hurewicz-Theorem

In this last lecture we discuss two important results about homotopy and homology groups without proof.

## CW-approximation

We have seen that not all topological spaces can be equipped with the structure of a CW-complex. However, one may wonder if a general topological space is at least homotopy equivalent to a CW-complex. Also this turns out to be wrong: the Hawaiian earring discussed in the exercises is an example for a space that it not even homotopy equivalent to a CW-complex. An even weaker statement is to ask whether a given topological space is related to a CW-complex by a weak homotopy equivalence, that is, a map that induces isomorphisms on all homotopy groups with respect to all basepoints. It turns out that such a *CW-approximation* always exists:

**Theorem 16.1.** *For every topological space $X$, there is a CW-complex* $\mathrm{cw}(X)$ *and a weak homotopy equivalence* $\mathrm{cw}(X) \to X$.

We will not prove this theorem, but outline one possible construction. For this we recall that a simplicial set is a contravariant functor $K\colon \Delta \to \mathrm{Set}$ and that the standard $n$-simplices form a (covariant) functor $\Delta \to \mathrm{Top}$, $[n] \mapsto \Delta^n$.

**Definition 16.2.** Let $K$ be a simplicial set. Its *geometric realization* is the quotient space $|K|$ of the disjoint union $\coprod_{n\geq 0} K_n \times \Delta^n$ modulo the equivalence relation $\sim$ generated by $(\alpha^*(k), t) \sim (k, \alpha_*(t))$ for all $\alpha\colon [m] \to [n]$ in $\Delta$, all $k \in K_n$, and all $t \in \Delta^m$.

One can show that for all simplicial sets $K$, the geometric realization $|K|$ is a CW-complex. This may be plausible since $|K|$ is built by gluing simplices along their boundaries, which is analogous to glueing disks together along their boundaries. In fact one can define a filtration of the simplicial set $K$ by sub-simplicial sets that gives rise to the filtration of the CW-complex $|K|$ by its skeleta.

When the simplicial set $K$ is $\mathcal{S}(X)$, the singular complex of a topological space $X$, then there is a canonical map $\eta_X\colon |\mathcal{S}(X)| \to X$ that is induced by

$$\amalg_{n\geq 0}\mathcal{S}(X)_n \times \Delta^n, \quad (\sigma\colon \Delta^n \to X, t \in \Delta^n) \mapsto \sigma(t)\ .$$

To see that this map induces a well-defined map on the quotient space $|\mathcal{S}(X)|$, we have to check that $(\alpha^*(\sigma), t)$ and $(\sigma, \alpha_*(t))$ have the same image under this map. And indeed, we have $(\alpha^*(\sigma))(t) = (\sigma \circ \alpha_*)(t) = \sigma(\alpha_*(t))$.

One way to prove Theorem 16.1 is now to show that $\eta_X\colon |\mathcal{S}(X)| \to X$ is a weak homotopy equivalence. The map $\eta_X$ even has the additional feature that it is natural in the space $X$. That is, continuous maps on the CW-approximation in a way compatible with composition. (In more category theoretic terms, the geometric realization is left adjoint to singular complex functor, and the map $\eta_X\colon |\mathcal{S}(X)| \to X$ is the adjunction counit.)

The CW-approximation $\eta_X\colon |\mathcal{S}(X)| \to X$ is also interesting when $X$ is already a CW-complex. In this case, the Whitehead theorem implies that $\eta_X\colon |\mathcal{S}(X)| \to X$ is even a homotopy equivalence. So up to homotopy equivalence, every CW-complex $X$ can be recovered from its singular complex $\mathcal{S}(X)$, which just consists of "discrete data"! This means that if one studies CW-complexes $X$ up to homotopy equivalence or general topological spaces $X$ up to weak homotopy equivalence, then all the relevant information is encoded in the singular complex $\mathcal{S}(X)$.

**Remark 16.3.** Another way to construct CW-approximations is to attach cells corresponding to the generators and relations in the homotopy groups of a given space $X$.

## The Hurewicz-Theorem

In Lecture 3 we showed that if $X$ is a path connected space with basepoint $x_0$, then there is a group homomorphism $\phi\colon \pi_1(X, x_0) \to H_1(X; \mathbb{Z})$ which induces an isomorphism

$$\phi^{\mathrm{ab}}\colon \pi_1(X, x_0)^{\mathrm{ab}} \to H_1(X; \mathbb{Z})$$

from the abelianization of the fundamental group to the first homology group of $X$ with coefficients in $\mathbb{Z}$. It is a natural question to ask if there is an analogous isomorphism if $n \geq 2$. At least in one respect, the situation is better for higher homotopy groups than for the fundamental group: since $\pi_n(X, x_0)$ is already abelian if $n \geq 2$, an abelianization is not necessary. However, the most naive generalization of the isomorphism between the abelianization of the fundamental group and the first homology group fails badly: There are (interesting) path connected based spaces $(X, x_0)$ with $\pi_n(X, x_0) \cong 0$ and $H_n(X; \mathbb{Z}) \neq 0$.

**Remark 16.4.** Spaces with trivial higher homotopy groups and non-trivial homology groups are in fact not at all rare. The Kan–Thurston theorem states that for every path connected space $X$ one can find a space $X'$ and a continuous map $f\colon X' \to X$ such that

- $f$ induces isomorphisms $f_*\colon H_n(X'; A) \to H_n(X; A)$ for all $n \geq 0$ and all abelian groups $A$,
- $f$ induces a surjective map on fundamental groups, and
- $\pi_n(X', x_0) \cong 0$ for all $n \geq 2$.

This implies that even path connected spaces $X$ with $\pi_n(X; x_0) \cong 0$ for $n \geq 2$ can have homology groups that are as complicated as those of any topological space!

Nonetheless it turns out that there is a useful result about the relation between higher homotopy groups and homology groups. We now prepare to formulate its relative version. For this we recall that $H_n(D^n, \partial D^n; \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ and choose a generator $[e^n] \in H_n(D^n, \partial D^n; \mathbb{Z})$, i.e., an element $[e^n]$ such that $\mathbb{Z} \to H_n(D^n, \partial D^n; \mathbb{Z}), k \mapsto k[e^n]$ is an isomorphism. Now let $(X, A)$ be a pair of spaces and let $x_0 \in A$ be a basepoint. Then we can consider the map

$$h_n\colon \pi_n(X, A, x_0) \to H_n(X, A; \mathbb{Z}), \qquad [\alpha\colon (D^n, \partial D^n, s_0) \to (X, A, x_0)]_* \mapsto \alpha_*([e^n])$$

where $\alpha_*\colon H_n(D^n, \partial D^n; \mathbb{Z}) \to H_n(X, A; \mathbb{Z})$ is the map induced by $\alpha$. We call $h_n$ the (relative) Hurewicz map. It is is well-defined since if $\alpha$ and $\alpha'$ are two representatives of an element in

$\pi_n(X, A, x_0)$, then the maps $H_n(D^n, \partial D^n; \mathbb{Z}) \to H_n(X, A; \mathbb{Z})$ induced by $\alpha$ and $\alpha'$ coincide by the homotopy invariance of homology. In particular, $\alpha_*([e^n]) = \alpha'_*([e^n])$ in $H_n(X, A)$.

There is a version $h_n\colon \pi_n(X, x_0) \to H_n(X; \mathbb{Z})$ of the Hurewicz map for absolute homology and homotopy groups. Its definition uses a generator of $H_n(S^n; \mathbb{Z})$ instead of a generator for $H_n(D^n, \partial D^n; \mathbb{Z})$.

Both for the relative and the absolute version of the Hurewicz map, the homotopy group in the source depend on the choice of a basepoint $x_0$, while the homology groups in the target don't depend on $x_0$. This turns out to be a problem even if $X$ is path connected: We have seen earlier that the fundamental group $\pi_1(X, x_0)$ acts on $\pi_n(X, x_0)$. For $[w] \in \pi_1(X, x_0)$ and $[\alpha] \in \pi_n(X, x_0)$, one can check from our definition of $w_*[\alpha]$ that the representative we constructed is homotopic to $\alpha$ by a homotopy that may not preserve basepoints. This is enough to imply that $h_n$ sends $w_*[\alpha]$ and $[\alpha]$ to the same element of $H_n(X; \mathbb{Z})$. As a consequence, we note that $h_n$ cannot be injective if the fundamental group $\pi_1(X, x_0)$ acts on $\pi_n(X, x_0)$ in a non-trivial way. (When we studied the isomorphism $\pi_1(X, x_0)^{\mathrm{ab}} \to H_1(X; \mathbb{Z})$, this issue did not become explicit since the $\pi_1(X, x_0)$-action on $\pi_1(X, x_0)$ was the conjugation action that becomes trivial after abelianization.)

In view of these complications, we now restrict our attention to a class of spaces where these problems do not arise.

**Definition 16.5.** A topological space $X$ is simply connected if it is path connected and the fundamental group $\pi_1(X, x_0)$ is trivial for one (and hence any) choice of basepoint.

The following statement is known as the Hurewicz-Theorem.

**Theorem 16.6.** *Let $(X, A)$ be a pair of spaces with $A$ and $X$ simply connected, and let $n \geq 2$. Suppose that $\pi_i(X, A, x_0)$ is trivial for all $2 \leq i < n$ and some (and hence any) basepoint $x_0 \in A$. Then $H_i(X, A; \mathbb{Z}) \cong 0$ for $i < n$, and $h_n\colon \pi_n(X, A, x_0) \to H_n(X, A; \mathbb{Z})$ is an isomorphism of groups.*

There is an analogous version that states that $h_n\colon \pi_n(X, x_0) \to H_n(X; \mathbb{Z})$ is an isomorphism if $X$ is simply connected and $\pi_i(X, x_0) \cong 0$ if $i < n$. So for a simply connected space, the first non-trivial homotopy group coincides with the first non-trivial homology group with $\mathbb{Z}$-coefficients.

There is also a more general version of the relative Hurewicz-Theorem for pairs of spaces with non-trivial fundamental groups that takes the action of the fundamental group on $\pi_n(X, A, x_0)$ into account.

### Consequences of the Hurewicz-Theorem

The Hurewicz-Theorem has important and powerful applications. For example, we may apply it to the simply connected space $S^n$ where $n \geq 2$ to obtain the following result:

**Corollary 16.7.** *If $n \geq 2$, then $\pi_n(S^n, s_0)$ is isomorphic to $\mathbb{Z}$.*

*Proof.* We showed earlier that $\pi_i(S^n, s_0) \cong 0$ if $i < n$ and that $H_n(S^n; \mathbb{Z}) \cong \mathbb{Z}$. The claim follows from the absolute version of the Hurewicz-Theorem or the relative one with $A = \{s_0\}$. □

**Remark 16.8.** It is not entirely honest to present the calculation of $\pi_n(S^n, s_0)$ as a consequence of the Hurewicz-Theorem: In common proofs of the theorem, one uses this result as an input or establishes it simultaneously with the statement of the Hurewicz-Theorem, rather than deriving it from the Hurewicz-Theorem.

The Hurewicz-Theorem also leads to the following remarkable variant of the Whitehead-Theorem (which also goes under the name Whitehead-Theorem).

**Theorem 16.9.** *Let $f\colon X \to Y$ be a continuous map between simply connected CW-complexes that induces isomorphisms $f_*\colon H_n(X;\mathbb{Z}) \to H_n(Y;\mathbb{Z})$ for all $n \geq 2$. Then $f$ is a homotopy equivalence.*

*Proof.* As a first step, we note that since $X$ and $Y$ are simply connected, the groups $H_n(X;\mathbb{Z})$ and $H_n(Y;\mathbb{Z})$ are trivial for $n = 0$ and $n = 1$ by the results from Lecture 2. Hence $f_*\colon H_n(X;\mathbb{Z}) \to H_n(Y;\mathbb{Z})$ is an isomorphism for all $n$. Arguing with cellular approximation and the mapping cylinder as in the proof of the Whitehead theorem, we may assume that $(Y, X)$ is a relative CW-complex. Now the long exact sequence in homology implies that $H_n(Y, X;\mathbb{Z}) \cong 0$ for all $n \geq 0$. Hence the Hurewicz-Theorem implies that $\pi_n(Y, X, x_0) \cong 0$ for all $n \geq 1$. By the long exact sequence of homotopy groups, this implies that $f\colon X \to Y$ is a weak homotopy equivalence. Thus it follows from the Whitehead-Theorem that $f$ is a homotopy equivalence. $\square$

## The Hurewicz homomorphism is a group homomorphism

The only part of the Hurewicz-Theorem that we will actually prove in this lecture is the statement that the Hurewicz map $h_n$ in the theorem is indeed a group homomorphism. Also this requires some work since the group structures on the two sides are defined in completely different ways.

To prepare for this, we first prove a lemma about the reduced homology of one-point unions. We recall that if $Y_1$ and $Y_2$ are spaces with basepoints $y_1 \in Y_1$ and $y_2 \in Y_2$, then their one-point union $Y_1 \vee Y_2$ is the quotient space of the disjoint union $Y_1 \amalg Y_2$ obtained by identifying the two basepoints $y_1$ and $y_2$. In this situation, we write $p_j\colon Y_1 \vee Y_2 \to Y_j$ for the map that is the identity on the $j$th summand and the constant map to the basepoint on the other summand (where $j = 1$ or $j = 2$). Moreover, $i_k\colon Y_k \to Y_1 \vee Y_2$ is the inclusion of the $k$th summand (where $k = 1$ or $k = 2$). In reduced homology, these maps induce group homomorphisms

$$(p_1)_* \oplus (p_2)_*\colon \widetilde{H}_n(Y_1 \vee Y_2; A) \to \widetilde{H}_n(Y_1; A) \oplus \widetilde{H}_n(Y_2; A), \quad [a] \mapsto ((p_1)_*([a]), (p_2)_*([a]))$$

and

$$(i_1)_* + (i_2)_*\colon \widetilde{H}_n(Y_1; A) \oplus \widetilde{H}_n(Y_2; A) \to \widetilde{H}_n(Y_1 \vee Y_2; A), \quad ([b_1], [b_2]) \mapsto (i_1)_*([b_1]) + (i_2)_*([b_2])$$

where as usual $A$ is the coefficient group.

**Lemma 16.10.** *Suppose that the basepoint $y_1 \in Y_1$ admits an open neighborhood $U_1$ such that the inclusion $\{y_1\}$ is a deformation retract of $U_1$. Then the maps $p = (p_1)_* \oplus (p_2)_*$ and $i = (i_1)_* + (i_2)_*$ are inverse isomorphisms between $\widetilde{H}_n(Y_1 \vee Y_2; A)$ and $\widetilde{H}_n(Y_1; A) \oplus \widetilde{H}_n(Y_2; A)$.*

*Proof.* We first note that the composite $p_j \circ i_k$ equals the identity of $j = k$ and the constant map to the basepoint if $j \neq k$. Then the map $(p_j \circ i_k)_*\colon \widetilde{H}_n(Y_k; A) \to \widetilde{H}_n(Y_j; A)$ is the identity if $j = k$ and the trivial map if $j \neq k$, and we see that

$$\begin{aligned}(p \circ i)([b_1], [b_2]) &= \left((p_1)_* \left((i_1)_*([b_1]) + (i_2)_*([b_2])\right), (p_2)_* \left((i_1)_*([b_1]) + (i_2)_*([b_2])\right)\right) \\ &= ([b_1], [b_2]).\end{aligned}$$

This shows $p \circ i = \mathrm{id}$. We now claim that it is sufficient to show that $i$ is surjective. Assuming this, any $[a] \in \widetilde{H}_n(Y_1 \vee Y_2; A)$ can be written as $[a] = i([b_1], [b_2])$, and together with $p \circ i = \mathrm{id}$ this implies

$$(i \circ p)([a]) = (i \circ p \circ i)([b_1], [b_2]) = i([b_1], [b_2]) = [a]$$

so that we indeed also have $i \circ p = \mathrm{id}$.

To verify the claim about $i$, we consider the long exact reduced homology sequence of the pair $(Y_1 \vee Y_2, Y_1)$. The fact that the composite $p_1 \circ i_1$ is the identity implies that the map $(i_1)_* \colon \widetilde{H}_k(Y_1; A) \to \widetilde{H}_k(Y_1 \vee Y_2; A)$ in this sequence is injective for every $k$. By exactness, this implies that the boundary operator has trivial image in every degree. Hence we get a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{H}_n(Y_1; A) \arrow[r, "(i_1)_*"] & \widetilde{H}_n(Y_1 \vee Y_2; A) \arrow[r, "j"] & \widetilde{H}_n(Y_1 \vee Y_2, Y_1; A) \arrow[r] & 0\,. \end{tikzcd}$$

Since long exact sequences are natural, $j \circ (i_2)_*$ can be written as the composite of

$$\widetilde{H}_n(Y_2; A) \to \widetilde{H}_n(Y_2, \{y_2\}; A) \to \widetilde{H}_n(U_1 \vee Y_2, U_1; A) \to \widetilde{H}_n(Y_1 \vee Y_2, Y_1; A) \tag{16.1}$$

where the first map is from the long exact sequence of $(Y_2, \{y_2\})$ and the last two maps are induced by the inclusions. The first map in (16.1) is an isomorphism by the long exact reduced homology sequence of the pair $(Y_2, \{y_2\})$, the second is an isomorphism by homotopy invariance and the assumption that $\{y_1\} \to U_1$ is a homotopy equivalence, and the last one is an isomorphism by excision. Hence $j \circ (i_2)_* \colon \widetilde{H}_n(Y_2; A) \to \widetilde{H}_n(Y_2 \vee Y_1, Y_1; A)$ is an isomorphism. We obtain the following diagram of horizontal short exact sequences where the outer vertical maps are isomorphisms, $\mathrm{incl}_1$ is the inclusion of the first summand, and $\mathrm{proj}_2$ is the projection to the second summand:

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{H}_n(Y_1; A) \arrow[r, "\mathrm{incl}_1"] \arrow[d, "\mathrm{id}"] & \widetilde{H}_n(Y_1; A) \oplus \widetilde{H}_n(Y_2; A) \arrow[r, "\mathrm{proj}_2"] \arrow[d, "i"] & \widetilde{H}_n(Y_2; A) \arrow[r] \arrow[d, "j \circ (i_2)_*"] & 0 \\ 0 \arrow[r] & \widetilde{H}_n(Y_1; A) \arrow[r, "(i_1)_*"] & \widetilde{H}_n(Y_1 \vee Y_2; A) \arrow[r, "j"] & \widetilde{H}_n(Y_2; A) \arrow[r] & 0\,. \end{tikzcd}$$

The five lemma implies that $i$ is an isomorphism and particularly a surjection. ☐

We now return to the Hurewicz map. For simplicity, we focus on the absolute case and assume $n \geq 1$ so that we can work with reduced homology (which coincides with the absolute homology in degrees above 0). In this setup, the Hurewicz map has the form

$$h_n \colon \pi_n(X, x_0) \to \widetilde{H}_n(X; \mathbb{Z}), \qquad [\alpha \colon (I^n, \partial I^n) \to (X, x_0)]_* \mapsto \alpha_*([e^n])$$

where $[e^n]$ denotes a generator of $\widetilde{H}_n(I^n/\partial I^n; \mathbb{Z})$. Here we also write $\alpha \colon I^n/\partial I^n \to X$ for the map of based spaces induced by the map of pairs $\alpha \colon (I^n, \partial I^n) \to (X, \{x_0\})$ so that $\alpha_* \colon \widetilde{H}_n(I^n/\partial I^n; \mathbb{Z}) \to \widetilde{H}_n(X; \mathbb{Z})$ does indeed take $[e^n]$ to an element of $\widetilde{H}_n(X; \mathbb{Z})$.

We recall that the group structure on $\pi_n(X, x_0)$ can be defined as follows: If $\alpha, \beta \colon (I^n, \partial I^n) \to (X, x_0)$ represent elements $[\alpha], [\beta] \in \pi_n(X, x_0)$, then $[\alpha] + [\beta] = [\alpha +_1 \beta]$ where $\alpha +_1 \beta$ is the map obtained by running with "double speed" in the first variable and concatenating $\alpha$ and $\beta$.

We will now give a different description of this group structure that is of independent interest. For this we again view $I^n/\partial I^n$ as a based space and consider the "pinch map"

$$\gamma\colon I^n/\partial I^n \to I^n/\partial I^n \vee I^n/\partial I^n,$$

$$\gamma([t_1,\ldots,t_n]) = \begin{cases} (2t_1,t_2,\ldots,t_n) \text{ in the first summand} & \text{if } 0 \le t_1 \le \frac{1}{2} \\ (2t_1-1,t_2,\ldots,t_n) \text{ in the second summand} & \text{if } \frac{1}{2} \le t_1 \le 1\,. \end{cases}$$

It is well defined since it sends all points with $t_1 = \frac{1}{2}$ to the basepoint in $I^n/\partial I^n$ (which is the image of $\partial I^n \amalg \partial I^n$ under the map $I^n \amalg I^n \to I^n/\partial I^n \amalg I^n/\partial I^n \to I^n/\partial I^n \vee I^n/\partial I^n$).

We have $\alpha +_1 \beta = (\alpha \vee \beta) \circ \gamma$ where $\alpha \vee \beta\colon I^n/\partial I^n \vee I^n/\partial I^n \to X$ is the map which is $\alpha$ on the first summand and $\beta$ on the second summand. So in particular, the composite $(\alpha \vee \beta) \circ \gamma$ represents the sum of $[\alpha]$ and $[\beta]$ in $\pi_n(X,x_0)$.

We now again write

$$p_j\colon I^n/\partial I^n \vee I^n/\partial I^n \to I^n/\partial I^n \quad \text{and} \quad i_k\colon I^n/\partial I^n \to I^n/\partial I^n \vee I^n/\partial I^n$$

for the maps defined above and note that both $p_1 \circ \gamma$ and $p_2 \circ \gamma$ are homotopic to the identity.

With this preparation, we can now prove the additivity of the Hurewicz map.

**Proposition 16.11.** *The Hurewicz map $h_n\colon \pi_n(X,x_0) \to \widetilde{H}_n(X;\mathbb{Z})$ is a group homomorphism.*

*Proof.* Let $[e^n]$ be a generator of $\widetilde{H}_n(I^n/\partial I^n;\mathbb{Z})$. Since $p_j \circ \gamma$ is homotopic to id for $j = 1,2$, we have

$$((p_1)_* \oplus (p_2)_*)(\gamma_*([e^n])) = ((p_1 \circ \gamma)_*([e^n]),(p_2 \circ \gamma)_*([e^n])) = ([e^n],[e^n])\,.$$

Since $(i_1)_* + (i_2)_*$ is inverse to $(p_1)_* \oplus (p_2)_*$ by the previous lemma, applying $(i_1)_* + (i_2)_*$ to this equation gives

$$\gamma_*([e^n]) = (i_1)_*([e^n]) + (i_2)_*([e^n])\,.$$

Using this, we now get

$$\begin{aligned} h_n([\alpha]+[\beta]) &= h_n((\alpha \vee \beta)\circ\gamma) = ((\alpha\vee\beta)\circ\gamma)_*([e^n]) = (\alpha\vee\beta)_*\gamma_*([e^n]) \\ &= (\alpha\vee\beta)_*((i_1)_*([e^n]) + (i_2)_*([e^n])) = ((\alpha\vee\beta)\circ i_1)_*([e^n]) + ((\alpha\vee\beta)\circ i_2)_*([e^n]) \\ &= \alpha_*([e^n]) + \beta_*([e^n]) = h_n([\alpha]) + h_n([\beta]). \end{aligned}$$

□

**Higher homotopy groups** One can show that the group $\pi_n(S^n,s_0)$ is isomorphic to $\mathbb{Z}$ if $n \ge 1$. In general, $\pi_n(S^m,s_0)$ is non-trivial if $n > m$. The easiest example of a non-trivial element in such a higher homotopy group already lies in $\pi_3(S^2,s_0)$: The map

$$\mathbb{C}^2 \to \mathbb{C}\times\mathbb{R}, \qquad (z_1,z_2) \mapsto (2z_1\overline{z_2}, |z_1|^2 - |z_2|^2)$$

restricts to a map $\eta\colon S^3 \to S^2$ on the vectors of length 1. This map $\eta$ is known as the Hopf map, and it is not homotopic to a constant map. (In fact, the class of $\eta$ is a generator of $\pi_3(S^2,s_0) \cong \mathbb{Z}$.)

*Freudenthal's Suspension Theorem* states that reduced suspension induces isomorphisms

$$\pi_{n+k}(S^n,s_0) \xrightarrow{\cong} \pi_{n+1+k}(S^{n+1},s_0)$$

if $n \geq 1$ and $n > k + 1$. So for $k = 1$, suspension induces the following sequence of maps:

$$\pi_3(S^2, s_0) \to \pi_4(S^3, s_0) \overset{\cong}{\to} \pi_5(S^4, s_0) \overset{\cong}{\to} \pi_6(S^5, s_0) \overset{\cong}{\to} \dots$$

The groups $\pi_{n+1}(S^n, s_0)$ with $n \geq 3$ in this sequence are isomorphic to $\mathbb{Z}/2$.

For general $k \geq 0$, the group $\pi_{n+k}(S^n, s_0)$ for $n$ large enough so that the Freudenthal suspension theorem applies is known as the $k$-th stable homotopy group of spheres. This is a finite abelian group if $k \geq 1$. The computation of stable homotopy groups of spheres is a difficult (and in general open) problem.