AFFINE HECKE ALGEBRAS FOR LANGLANDS PARAMETERS

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Abstract. It is well-known that affine Hecke algebras are very useful to describe the smooth representations of any connected reductive p-adic group $\mathcal{G}$, in terms of the supercuspidal representations of its Levi subgroups. The goal of this paper is to create a similar role for affine Hecke algebras on the Galois side of the local Langlands correspondence.

To every Bernstein component of enhanced Langlands parameters for $\mathcal{G}$ we canonically associate an affine Hecke algebra (possibly extended with a finite R-group). We prove that the irreducible representations of this algebra are naturally in bijection with the members of the Bernstein component, and that the set of central characters of the algebra is naturally in bijection with the collection of cuspidal supports of these enhanced Langlands parameters. These bijections send tempered or (essentially) square-integrable representations to the expected kind of Langlands parameters.

Furthermore we check that for many reductive p-adic groups, if a Bernstein component $\mathfrak{B}$ for $\mathcal{G}$ corresponds to a Bernstein component $\mathfrak{B}^\vee$ of enhanced Langlands parameters via the local Langlands correspondence, then the affine Hecke algebra that we associate to $\mathfrak{B}^\vee$ is Morita equivalent with the Hecke algebra associated to $\mathfrak{B}$. This constitutes a generalization of Lusztig's work on unipotent representations. It might be useful to establish a local Langlands correspondence for more classes of irreducible smooth representations.

Contents

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Introduction

Let F be a non-archimedean local field and let $\mathcal{G}$ be a connected reductive algebraic group defined over F. The conjectural local Langlands correspondence (LLC) provides a bijection between the set of irreducible smooth $\mathcal{G}(F)$-representations $\mathrm{Irr}(\mathcal{G}(F))$ and the set of enhanced L-parameters $\Phi_e(\mathcal{G}(F))$, see [Bor, Vog, ABPS5].

Let $\mathfrak{s}$ be an inertial equivalence class for $\mathcal{G}(F)$ and let $\mathrm{Irr}(\mathcal{G}(F))^{\mathfrak{s}}$ be the associated Bernstein component. Similarly, inertial equivalence classes $\mathfrak{s}^\vee$ and Bernstein components $\Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee}$ for enhanced L-parameters were developed in [AMS1]. It can be expected that every $\mathfrak{s}$ corresponds to a unique $\mathfrak{s}^\vee$ (an "inertial Langlands correspondence"), such that the LLC restricts to a bijection

$$\mathrm{Irr}(\mathcal{G}(F))^{\mathfrak{s}} \longleftrightarrow \Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee}. \tag{1}$$

The left hand side can be identified with the space of irreducible representations of a direct summand $\mathcal{H}(\mathcal{G}(F))^{\mathfrak{s}}$ of the full Hecke algebra of $\mathcal{G}(F)$. It is known that in many cases $\mathcal{H}(\mathcal{G}(F))^{\mathfrak{s}}$ is Morita equivalent to an affine Hecke algebra, see [ABPS5, §2.4] and the references therein for an overview.

To improve our understanding of the LLC, we would like to canonically associate to $\mathfrak{s}^\vee$ an affine Hecke algebra $\mathcal{H}(\mathfrak{s}^\vee)$ whose irreducible representations are naturally parametrized by $\Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee}$. Then (1) could be written as

$$\mathrm{Irr}(\mathcal{G}(F))^{\mathfrak{s}} \cong \mathrm{Irr}\big(\mathcal{H}(\mathcal{G}(F))^{\mathfrak{s}}\big) \longleftrightarrow \mathrm{Irr}(\mathcal{H}(\mathfrak{s}^\vee)) \cong \Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee}, \tag{2}$$

and the LLC for this Bernstein component would become a comparison between two algebras of the same kind. If moreover $\mathcal{H}(\mathfrak{s}^\vee)$ were Morita equivalent to $\mathcal{H}(\mathcal{G}(F))^{\mathfrak{s}}$, then (1) could even be categorified to

$$\mathrm{Rep}(\mathcal{G}(F))^{\mathfrak{s}} \cong \mathrm{Mod}(\mathcal{H}(\mathfrak{s}^\vee)). \tag{3}$$

Such algebras $\mathcal{H}(\mathfrak{s}^\vee)$ would also be useful to establish the LLC in new cases. Suppose one would like to match $\mathfrak{s}^\vee$ (essentially a set of cuspidal enhanced Langlands parameters for a Levi subgroup $\mathcal{L}(F)$) with a yet unknown supercuspidal Bernstein block for $\mathcal{L}(F)$. Motivated by some examples, we increase the scope of (3) by considering it only for the full subcategories of finite length objects:

$$\mathrm{Rep}_{\mathrm{fl}}(\mathcal{G}(F))^{\mathfrak{s}} \cong \mathrm{Mod}_{\mathrm{fl}}(\mathcal{H}(\mathfrak{s}^\vee)). \tag{4}$$

One could compare $\mathcal{H}(\mathfrak{s}^\vee)$ with the algebras $\mathcal{H}(\mathcal{G}(F))^{\mathfrak{s}}$ for various $\mathfrak{s} = [\mathcal{L}(F), \sigma]$, and only the Bernstein components $\mathrm{Irr}(\mathcal{G}(F))^{\mathfrak{s}}$ for which (4) holds would be good candidates for the image of $\Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee}$ under the LLC. If one would know a lot about $\mathcal{H}(\mathfrak{s}^\vee)$, this could substantially reduce the number of possibilities for a LLC for both $\Phi_e(\mathcal{L}(F))^{\mathfrak{s}^\vee}$ and $\Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee}$.

This strategy was already employed by Lusztig, for unipotent representations [Lus5, Lus7]. Bernstein components of enhanced L-parameters had not yet been defined when the papers [Lus5, Lus7] were written, but the constructions in them can be interpreted in that way. Lusztig found a bijection between:

- the set of ("arithmetic") affine Hecke algebras associated to unipotent Bernstein blocks of adjoint, unramified groups;
- the set of ("geometric") affine Hecke algebras associated to unramified enhanced L-parameters for such groups.

However, the comparison of these two families of Hecke algebras is not enough to specify a canonical bijection between Bernstein components on the p-adic and the Galois sides. The problem is that one affine Hecke algebra can appear (up to isomorphism) several times on either side. This already happens in the unipotent case for exceptional groups, and the issue seems to be outside the scope of these techniques. In [Lus5, 6.6–6.8] Lusztig wrote down some remarks about this problem, but he does not work it out completely.

The main goal of this paper is the construction of an affine Hecke algebra for any Bernstein component of enhanced L-parameters, for any $\mathcal{G}$. But it quickly turns out that this is not exactly the right kind of algebra. Firstly, our geometric construction, which relies on [Lus2, AMS2], naturally includes some complex parameters $\mathbf{z}_i$, which we abbreviate to $\vec{\mathbf{z}}$. Secondly, an affine Hecke algebra with (indeterminate) parameters is still too simple. In general one must consider the crossed product of such an object with a twisted group algebra (of some finite "R-group"). We call this a twisted affine Hecke algebra, see Proposition 2.2 for a precise definition. Like for reductive groups, there are good notions of tempered representations and of (essentially) discrete series representations of such algebras (Definition 2.6).

Theorem 1. [see Theorem 3.18]

(*a*) *To every Bernstein component of enhanced L-parameters $\mathfrak{s}^\vee$ one can canonically associate a twisted affine Hecke algebra $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$.*

(*b*) *For every choice of parameters $z_i \in \mathbb{R}_{>0}$ there exists a natural bijection*

$$\Phi_{\mathrm{e}}(\mathcal{G}(F))^{\mathfrak{s}^\vee} \longleftrightarrow \mathrm{Irr}\big(\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})/(\{\mathbf{z}_i - z_i\}_i)\big)$$

(*c*) *For every choice of parameters $z_i \in \mathbb{R}_{\geq 1}$ the bijection from part (b) matches enhanced bounded L-parameters with tempered irreducible representations.*

(*d*) *Suppose that $\Phi_{\mathrm{e}}(\mathcal{G}(F))^{\mathfrak{s}^\vee}$ contains enhanced discrete L-parameters, and that $z_i \in \mathbb{R}_{>1}$ for all i. Then the bijection from part (b) matches enhanced discrete L-parameters with irreducible essentially discrete series representations.*

(*e*) *The bijection in part (b) is equivariant with respect to the canonical actions of the group of unramified characters of $\mathcal{G}(F)$.*

This can be regarded as a far-reaching generalization of parts of [Lus5, Lus7]: we allow any reductive group over a non-archimedean local field, and all enhanced L-parameters for that group. We check (see Section 5) that in several cases where the LLC is known, indeed

$$\mathcal{H}(\mathcal{G}(F))^{\mathfrak{s}} \quad \text{is Morita equivalent to} \quad \mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})/(\{\mathbf{z}_i - z_i\}_i) \tag{5}$$

for suitable $z_i \in \mathbb{R}_{>1}$, obtaining (3). Notice that on the p-adic side the parameters z_i are determined by $\mathcal{H}(\mathcal{G}(F))^{\mathfrak{s}}$, whereas on the Galois side we specify them manually. In fact, in all our examples we can take $z_i = q_F^{1/2}$. That is a good sign, which indicates that in general $z_i = q_F^{1/2}$ could be the best specialization of the parameters to compare with an affine Hecke algebra coming from a p-adic group.

Yet in general the categorification (3) is asking for too much. We discovered that for inner twists of $\mathrm{SL}_n(F)$ (5) does not always hold. Rather, these algebras are equivalent in a weaker sense: the category of finite length modules of $\mathcal{H}(\mathcal{G}(F))^{\mathfrak{s}}$ (i.e. the finite length objects in $\mathrm{Rep}(\mathcal{G}(F))^{\mathfrak{s}}$) is equivalent to the category of finite dimensional representations of $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}}) / \big(\{ \mathbf{z}_i - q_F^{1/2} \}_i \big)$.

Let us describe the contents of the paper more concretely. Our starting point is a triple $(G, M, q\mathcal{E})$ where

- G is a possibly disconnected complex reductive group,
- M is a quasi-Levi subgroup of G (the G-centralizer of the connected centre of a Levi subgroup of G°),
- $q\mathcal{E}$ is a M-equivariant cuspidal local system on a unipotent orbit $\mathcal{C}_v^M$ in M.

To these data we attach a twisted affine Hecke algebra $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$. This algebra can be specialized by setting $\vec{\mathbf{z}}$ equal to some $\vec{z} \in (\mathbb{C}^\times)^d$. Of particular interest is the specialization at $\vec{z} = \vec{1}$:

$$\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}}) / (\{ \mathbf{z}_i - 1 \}_i) = \mathcal{O}(T) \rtimes \mathbb{C}[W_{q\mathcal{E}}, \natural],$$

where $T = \mathrm{Z}(M)^\circ$, while the subgroup $W_{q\mathcal{E}} \subset \mathrm{N}_G(M)/M$ and the 2-cocycle $\natural \colon W_{q\mathcal{E}}^2 \to \mathbb{C}^\times$ also come from the data.

The goal of Section 2 is to understand and parametrize representations of the algebra $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$. We follow a strategy similar to that in [Lus3]. The centre naturally contains $\mathcal{O}(T)^{W_{q\mathcal{E}}} = \mathcal{O}(T/W_{q\mathcal{E}})$, so we can study $\mathrm{Mod}(\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}}))$ via localization at suitable subsets of $T/W_{q\mathcal{E}}$. In Paragraph 2.1 we reduce to representations with $\mathcal{O}(T)^{W_{q\mathcal{E}}}$-character in $W_{q\mathcal{E}} T_{\mathrm{rs}}$, where T_{rs} denotes the maximal real split subtorus of T. This involves replacing $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$ by an algebra of the same kind, but for a smaller G.

In Paragraph 2.2 we reduce further, to representations of a (twisted) graded Hecke algebra $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}})$. We defined and studied such algebras in our previous paper [AMS2]. But there we only considered the case with a single parameter $\mathbf{r}$, here we need $\vec{\mathbf{r}} = (\mathbf{r}_1, \ldots, \mathbf{r}_d)$. The generalization of the results of [AMS2] to a multi-parameter setting is carried out in Section 1. With that at hand we can use the construction of "standard" $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}})$-modules and the classification of irreducible $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}})$-modules from [AMS2] to achieve the same for $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$. For the parametrization we use triples (s, u, ρ) where:

- $s \in G^\circ$ is semisimple,
- $u \in \mathrm{Z}_G(s)^\circ$ is unipotent,
- $\rho \in \mathrm{Irr}\big(\pi_0(\mathrm{Z}_G(s, u))\big)$ such that the quasi-cuspidal support of (u, ρ), as defined in [AMS1, §5], is G-conjugate to $(M, \mathcal{C}_v^M, q\mathcal{E})$.

Theorem 2. [see Theorem 2.13]

(*a*) *Let $\vec{z} \in \mathbb{R}_{\geq 0}^d$. There exists a canonical bijection, say $(s, u, \rho) \mapsto \bar{M}_{s,u,\rho,\vec{z}}$, between:*
  - *G-conjugacy classes of triples (s, u, ρ) as above,*
  - $\mathrm{Irr}\big(\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}}) / (\{ \mathbf{z}_i - z_i \}_i)\big)$.

(*b*) *Let $\vec{z} \in \mathbb{R}_{\geq 1}^d$. The module $\bar{M}_{s,u,\rho,\vec{z}}$ is tempered if and only if s is contained in a compact subgroup of G°.*

(c) *Let $\vec{z} \in \mathbb{R}^d_{>1}$. The module $\bar{M}_{s,u,\rho,\vec{z}}$ is essentially discrete series if and only if u is distinguished unipotent in G° (i.e. does not lie in a proper Levi subgroup).*

In the case $M = T$, $\mathcal{C}_v^M = \{1\}$ and $q\mathcal{E}$ trivial, the irreducible representations in $\mathcal{H}(G^\circ, T, q\mathcal{E} = \text{triv})$ were already classified in the landmark paper [KaLu], in terms of similar triples. In Paragraph 2.3 we check that the parametrization from Theorem 2 agrees with the Kazhdan–Lusztig parametrization for these algebras.

Remarkably, our analysis also reveals that [KaLu] does not agree with the classification of irreducible representations in [Lus5]. To be precise, the difference consists of a twist with a version of the Iwahori–Matsumoto involution. Since [KaLu] is widely regarded (see for example [Ree, Vog]) as the correct local Langlands correspondence for Iwahori-spherical representations, this entails that the parametrizations obtained by Lusztig in [Lus5, Lus7] can be improved by composition with a suitable involution. In the special case $G = \mathrm{Sp}_{2n}(\mathbb{C})$, that already transpired from work of Mœglin and Waldspurger [Wal].

Having obtained a good understanding of affine Hecke algebras attached to disconnected reductive groups, we turn to Langlands parameters. Let

$$\phi \colon \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathcal{G}$$

be a L-parameter and let ρ be an enhancement of ϕ. (See Section 3 for the precise notions.) Let $\mathcal{G}^\vee_{\mathrm{ad}}$ be the adjoint group of the complex dual group $\mathcal{G}^\vee$ and let $\mathcal{G}^\vee_{\mathrm{sc}}$ be the simply connected cover of $\mathcal{G}^\vee_{\mathrm{ad}}$. Let $\mathrm{Z}_{\mathcal{G}^\vee_{\mathrm{ad}}}(\phi(\mathbf{I}_F))$ be the centralizer of $\phi(\mathbf{I}_F)$ in $\mathcal{G}^\vee_{\mathrm{ad}}$, and let $J_\phi = \mathrm{Z}^1_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi(\mathbf{I}_F))$ denote its inverse image in $\mathcal{G}^\vee_{\mathrm{sc}}$. Similarly, we consider the group G_ϕ defined to be inverse image in $\mathcal{G}^\vee_{\mathrm{sc}}$ of the centralizer of $\phi(\mathbf{W}_F)$ in $\mathcal{G}^\vee_{\mathrm{ad}}$. We emphasize that the complex groups J_ϕ and G_ϕ can be disconnected – this is the main reason why we have to investigate Hecke algebras for disconnected reductive groups.

Recall that ϕ is determined up to $\mathcal{G}^\vee$-conjugacy by $\phi|_{\mathbf{W}_F}$ and the unipotent element $u_\phi = \phi\big(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\big)$. As the image of a Frobenius element is allowed to vary within one Bernstein component, $(\phi|_{\mathbf{I}_F}, u_\phi)$ contains almost all information about such a Bernstein component.

The cuspidal support of (u_ϕ, ρ) for $G = G_\phi$ is a triple $(M, \mathcal{C}_v^M, q\mathcal{E})$ as before. Thus we can associate to (ϕ, ρ) the twisted affine Hecke algebra $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$. This works quite well in several cases, but in general it is too simple, we encounter various technical difficulties. The main problem is that the torus $T = \mathrm{Z}(M)^\circ$ will not always match up with the torus from which the Bernstein component of $\Phi_{\mathrm{e}}(\mathcal{G}(F))$ containing (ϕ, ρ) is built.

Instead we consider the twisted graded Hecke algebra $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}})$, and we tensor it with the coordinate ring of a suitable vector space to compensate for the difference between $\mathcal{G}^\vee_{\mathrm{sc}}$ and $\mathcal{G}^\vee$. In Paragraph 3.1 we prove that the irreducible representations of the ensuing algebra are naturally parametrized by a subset of the Bernstein component $\Phi_{\mathrm{e}}(\mathcal{G}(F))^{\mathfrak{s}^\vee}$ containing (ϕ, ρ). In Paragraph 3.3 we glue families of such algebras together, to obtain the twisted affine Hecke algebras $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$ featuring in Theorem 1. This requires careful analysis of the involved tori and root systems, which we perform in Paragraph 3.2.

We discuss then, in Section 4, the relation of the above theory with the stable Bernstein center on the Galois side of the LLC. In Section 5 we explain and work out the examples of general linear, special linear and classical groups. It turns out that, for general linear groups (and their inner twists) and classical groups, the extended affine Hecke algebras for enhanced Langlands parameters (with a suitable specialization of the parameters) are Morita equivalent to those obtained from representations of reductive p-adic groups. In the case of inner twists of special linear groups we establish a slightly weaker result.

Let us compare our paper with similar work by other authors. Several mathematicians have noted that, when two Bernstein components give rise to isomorphic affine Hecke algebras, this often has to do with the centralizers of the corresponding Langlands parameters. It is known from the work of Bushnell–Kutzko (see in particular [BuKu2]) that every affine Hecke algebra associated to a semisimple type for $\mathrm{GL}_n(F)$ is isomorphic to the Iwahori–spherical Hecke algebra of some $\prod_i \mathrm{GL}_{n_i}(F_i)$, where $\sum_i n_i \leq n$ and F_i is a finite extension of the field F. A similar statement holds for Bernstein components in the principal series of F-split reductive groups [Roc, Lemma 9.3].

Dat [Dat, Corollary 1.1.4] has generalized this to groups of "GL-type", and in [Dat, Theorem 1.1.2] he proves that for such a group $\mathrm{Z}_{\mathcal{G}^\vee}(\phi(\mathbf{I}_F))$ determines $\prod_{\mathfrak{s}} \mathrm{Rep}(\mathcal{G}(F))^{\mathfrak{s}}$, where $\mathfrak{s}$ runs over all Bernstein components that correspond to extensions of $\phi|_{\mathbf{I}_F}$ to $\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C})$. In [Dat, §1.3] Dat discusses possible generalizations of these results to other reductive groups, but he did not fully handle the cases where $\mathrm{Z}_{\mathcal{G}^\vee}(\phi(\mathbf{I}_F))$ is disconnected. (It is always connected for groups of GL-type.) Theorem 1, in combination with the considerations about inner twists of $\mathrm{GL}_n(F)$ in Paragraph 5.1, provide explanations for all the equivalences between Hecke algebras and between categories found by Dat.

Heiermann [Hei2, §1] has associated affine Hecke algebras (possibly extended with a finite R-group) to certain collections of enhanced L-parameters for classical groups (essentially these sets constitute unions of Bernstein components). Unlike Lusztig he does not base this on geometric constructions in complex groups, rather on affine Hecke algebras previously found on the p-adic side in [Hei1]. In his setup (2) holds true by construction, but the Hecke algebras are only related to L-parameters via the LLC, so not in an explicit way.

In [Hei2, §2] it is shown that every Bernstein component of enhanced L-parameters for a classical group is in bijection with a Bernstein component of enhanced unramified L-parameters for a product of classical groups of smaller rank. (Some cases require extending the relevant notions to full orthogonal groups, which is straightforward.) So in the context of [Hei2] the data that we use for affine Hecke algebras are present, and the algebras appear as well (at least up to Morita equivalence), but the link between them is not yet explicit. In Paragraph 5.3 we discuss how our results clarify this.

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1. Twisted graded Hecke algebras

We will recall some aspects of the (twisted) graded Hecke algebras studied in [AMS2]. Let G be a complex reductive group, possibly disconnected. Let M be a quasi-Levi subgroup of G, that is, a group of the form $M = \mathrm{Z}_G(\mathrm{Z}(L)^\circ)$ where L is a Levi subgroup of G°. Notice that $M^\circ = L$ in this case.

We write $T = \mathrm{Z}(M)^\circ = \mathrm{Z}(M^\circ)^\circ$, a torus in G°. Let $P^\circ = M^\circ U$ be a parabolic subgroup of G° with Levi factor M° and unipotent radical U. We put $P = MU$. Let $\mathfrak{t}^*$ be the dual space of the Lie algebra $\mathfrak{t} = \mathrm{Lie}(T)$.

Let $v \in \mathfrak{m} = \mathrm{Lie}(M)$ be nilpotent, and denote its adjoint orbit by $\mathcal{C}_v^M$. Let $q\mathcal{E}$ be an irreducible M-equivariant cuspidal local system on $\mathcal{C}_v^M$. Then the stalk $q\epsilon = q\mathcal{E}|_v$ is an irreducible representation of $A_M(v) = \pi_0(\mathrm{Z}_M(v))$. Conversely, v and $q\epsilon$ determine $\mathcal{C}_v^M$ and $q\mathcal{E}$. By definition the cuspidality means that $\mathrm{Res}^{A_M(v)}_{A_{M^\circ}(v)} q\epsilon$ is a direct sum of irreducible cuspidal $A_{M^\circ}(v)$-representations. Let $\epsilon \in \mathrm{Irr}(A_{M^\circ}(v))$ be one of them, and let $\mathcal{E}$ be the corresponding M°-equivariant cuspidal local system on $\mathcal{C}_v^{M^\circ}$. Then $\mathcal{E}$ is a subsheaf of $q\mathcal{E}$. See [AMS1, §5] for more background.
The triple $(M, \mathcal{C}_v^M, q\mathcal{E})$ (or $(M, v, q\epsilon)$) is called a cuspidal quasi-support for G. We denote its G-conjugacy class by $[M, \mathcal{C}_v^M, q\mathcal{E}]_G$. To these data we associate the groups

$$\begin{aligned} &W_{q\mathcal{E}} = \mathrm{N}_G(q\mathcal{E})/M, \quad \text{where } \mathrm{N}_G(q\mathcal{E}) = \mathrm{Stab}_{\mathrm{N}_G(M)}(q\mathcal{E}),\\ &W_{q\mathcal{E}}^\circ = \mathrm{Stab}_{\mathrm{N}_{G^\circ M}(M)}(q\mathcal{E})/M = \mathrm{N}_{G^\circ M}(M)/M,\\ &W_{\mathcal{E}} = \mathrm{Stab}_{\mathrm{N}_{G^\circ}(M^\circ)}(\mathcal{E})/M^\circ = \mathrm{N}_{G^\circ}(M^\circ)/M^\circ,\\ &\mathfrak{R}_{q\mathcal{E}} = \mathrm{N}_G(P, q\mathcal{E})/M, \quad \text{where } \mathrm{N}_G(P, q\mathcal{E}) = \mathrm{N}_G(q\mathcal{E}) \cap \mathrm{N}_G(P). \end{aligned} \tag{1.1}$$

The group $W_{q\mathcal{E}}$ acts naturally on the set

$$R(G^\circ, T) := \{\alpha \in X^*(T) \setminus \{0\} : \alpha \text{ appears in the adjoint action of } T \text{ on } \mathfrak{g}\}.$$

By [Lus1, Theorem 9.2] (see also [AMS2, Lemma 2.1]) $R(G^\circ, T)$ is a root system with Weyl group $W_{\mathcal{E}} \cong W_{q\mathcal{E}}^\circ$. The group $\mathfrak{R}_{q\mathcal{E}}$ is the stabilizer of the set of positive roots determined by P and

$$W_{q\mathcal{E}} = W_{q\mathcal{E}}^\circ \rtimes \mathfrak{R}_{q\mathcal{E}}.$$

We choose semisimple subgroups $G_j \subset G^\circ$, normalized by $\mathrm{N}_G(q\mathcal{E})$, such that the derived group G°_{der} is the almost direct product of the G_j. In other words, every G_j is semisimple, normal in $G^\circ M$, normalized by $W_{q\mathcal{E}}$ (which makes sense because it is already normalized by M), and the multiplication map

$$m_{G^\circ} \colon \mathrm{Z}(G^\circ)^\circ \times G_1 \times \cdots \times G_d \to G^\circ \tag{1.2}$$

is a surjective group homomorphism with finite central kernel. The number d is not specified in advance, it indicates the number of independent variables in our upcoming Hecke algebras. Of course there are in general many ways to achieve (1.2). Two choices are always canonical:

(1.3)
- $G_1 = G^\circ_{\mathrm{der}}$, with $d = 1$;
- every G_j is of the form $N_1 N_2 \cdots N_k$, where $\{N_1, \ldots, N_k\}$ is a $\mathrm{N}_G(q\mathcal{E})$-orbit of simple normal subgroups of G°.

In any case, (1.2) gives a decomposition

$$\mathfrak{g} = \mathrm{Z}(\mathfrak{g}) \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_d \quad \text{where } \mathrm{Z}(\mathfrak{g}) = \mathrm{Lie}(\mathrm{Z}(G^\circ)), \mathfrak{g}_j = \mathrm{Lie}(G_j). \tag{1.4}$$

Each root system

$$R_j := R(G_j T, T) = R(G_j, G_j \cap T)$$

is a $W_{q\mathcal{E}}$-stable union of irreducible components of $R(G^\circ, T)$. Thus we obtain an orthogonal, $W_{q\mathcal{E}}$-stable decomposition

$$R(G^\circ, T) = R_1 \sqcup \cdots \sqcup R_d. \tag{1.5}$$

We let $\vec{\mathbf{r}} = (\mathbf{r}_1, \ldots, \mathbf{r}_d)$ be an array of variables, corresponding to (1.2) and (1.5) in the sense that $\mathbf{r}_j$ is relevant for G_j and R_j only. We abbreviate

$$\mathbb{C}[\vec{\mathbf{r}}] = \mathbb{C}[\mathbf{r}_1, \ldots, \mathbf{r}_d].$$

Let $\natural \colon (W_{q\mathcal{E}}/W_{q\mathcal{E}}^\circ)^2 \to \mathbb{C}^\times$ be a 2-cocycle. Recall that the twisted group algebra $\mathbb{C}[W_{q\mathcal{E}}, \natural]$ has a $\mathbb{C}$-basis $\{N_w : w \in W_{q\mathcal{E}}\}$ and multiplication rules

$$N_w \cdot N_{w'} = \natural(w, w') N_{ww'}.$$

In particular it contains the group algebra of $W_{q\mathcal{E}}^\circ$.

Let $c \colon R(G^\circ, T)_{\mathrm{red}} \to \mathbb{C}$ be a $W_{q\mathcal{E}}$-invariant function.

Proposition 1.1. *There exists a unique structure of an associative graded algebra on $\mathbb{C}[W_{q\mathcal{E}}, \natural] \otimes S(\mathfrak{t}^*) \otimes \mathbb{C}[\vec{\mathbf{r}}]$, such that:*

(i) the twisted group algebra $\mathbb{C}[W_{q\mathcal{E}}, \natural]$ is embedded as subalgebra in degree 0;
(ii) the algebra $S(\mathfrak{t}^*) \otimes \mathbb{C}[\vec{\mathbf{r}}]$ of polynomial functions on $\mathfrak{t} \oplus \mathbb{C}^d$ is embedded as a subalgebra, with twice the usual grading on $S(\mathfrak{t}^*)$ and each $\mathbf{r}_j$ in degree 2;
(iii) $\mathbb{C}[\vec{\mathbf{r}}]$ is central;
(iv) the braid relation $\xi N_{s_\alpha} - N_{s_\alpha} {}^{s_\alpha}\xi = c(\alpha)\mathbf{r}_j(\xi - {}^{s_\alpha}\xi)/\alpha$ holds for all $\xi \in S(\mathfrak{t}^*)$ and all simple roots $\alpha \in R_j$
(v) $N_\gamma \xi N_\gamma^{-1} = {}^\gamma\xi$ for all $\xi \in S(\mathfrak{t}^*)$ and $\gamma \in \mathfrak{R}_{q\mathcal{E}}$.

Proof. For $d = 1, G_1 = G^\circ_{\mathrm{der}}$ this is [AMS2, Proposition 2.2]. The general case can be shown in the same way. $\square$

We denote the algebra just constructed by $\mathbb{H}(\mathfrak{t}, W_{q\mathcal{E}}, c\vec{\mathbf{r}}, \natural)$. When $W_{q\mathcal{E}}^\circ = W_{q\mathcal{E}}$, there is no 2-cocycle, and write simply $\mathbb{H}(\mathfrak{t}, W_{q\mathcal{E}}^\circ, c\vec{\mathbf{r}})$. It is clear from the defining relations that

$$S(\mathfrak{t}^*)^{W_{q\mathcal{E}}} \otimes \mathbb{C}[\vec{\mathbf{r}}] = \mathcal{O}(\mathfrak{t} \times \mathbb{C}^d)^{W_{q\mathcal{E}}} \text{ is a central subalgebra of } \mathbb{H}(\mathfrak{t}, W_{q\mathcal{E}}, c\vec{\mathbf{r}}, \natural). \tag{1.6}$$

By a central character of an $\mathbb{H}(\mathfrak{t}, W_{q\mathcal{E}}, c\vec{\mathbf{r}}, \natural)$-module we shall mean an element of $\mathfrak{t}/W_{q\mathcal{E}} \times \mathbb{C}^d$ by which $\mathcal{O}(\mathfrak{t} \times \mathbb{C}^d)^{W_{q\mathcal{E}}}$ acts on that module. For $\zeta \in \mathfrak{t}^{W_{q\mathcal{E}}} = Z(\mathfrak{g})^{\mathfrak{R}_{q\mathcal{E}}}$ and $(\pi, V) \in \mathrm{Mod}(\mathbb{H}(\mathfrak{t}, W_{q\mathcal{E}}, c\vec{\mathbf{r}}, \natural))$ we define $(\zeta \otimes \pi, V) \in \mathrm{Mod}(\mathbb{H}(\mathfrak{t}, W_{q\mathcal{E}}, c\vec{\mathbf{r}}, \natural))$ by

$$(\zeta \otimes \pi)(f_1 f_2 N_w) = f_1(\zeta)\pi(f_1 f_2 N_w) \qquad f_1 \in S(\mathfrak{t}^*), f_2 \in \mathbb{C}[\vec{\mathbf{r}}], w \in W_{q\mathcal{E}}.$$

To the cuspidal quasi-support $[M, \mathcal{C}_v^M, q\mathcal{E}]_G$ we associated a particular 2-cocycle

$$\natural_{q\mathcal{E}} \colon (W_{q\mathcal{E}}/W_{q\mathcal{E}}^\circ)^2 \to \mathbb{C}^\times,$$

see [AMS1, Lemma 5.3]. The pair (M°, v) also gives rise to a $W_{q\mathcal{E}}$-invariant function $c \colon R(G^\circ, T)_{\mathrm{red}} \to \mathbb{Z}$, see [Lus2, Proposition 2.10] or [AMS2, (12)]. We denote the algebra $\mathbb{H}(\mathfrak{t}, W_{q\mathcal{E}}, c\vec{\mathbf{r}}, \natural_{q\mathcal{E}})$, with this particular c, by $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}})$.

In [AMS2] we only studied the case $d = 1, R_1 = R(G^\circ, T)$, and we denoted that algebra by $\mathbb{H}(G, M, q\mathcal{E})$. Fortunately the difference with $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}})$ is so small that almost all properties of $\mathbb{H}(G, M, q\mathcal{E})$ discussed in [AMS2] remain valid for $\mathbb{H}(\mathfrak{t}, W_{q\mathcal{E}}, c\vec{\mathbf{r}}, \natural_{q\mathcal{E}})$. We will proceed to make this precise.

Write $v = v_1 + \cdots + v_d$ with $v_j \in \mathfrak{g}_j = \mathrm{Lie}(G_j)$. Then

$$\mathcal{C}_v^{M^\circ} = \mathcal{C}_{v_1}^{M_1} + \cdots + \mathcal{C}_{v_d}^{M_d} \text{ , where } M_j = M^\circ \cap G_j.$$

The M°-action on $(\mathcal{C}_v^{M^\circ}, \mathcal{E})$ can be inflated to $Z(G^\circ)^\circ \times M_1 \times \cdots \times M_d$, and the pullback of $\mathcal{E}$ becomes trivial on $Z(G^\circ)^\circ$ and decomposes uniquely as

$$m_{G^\circ}^* \mathcal{E} = \mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_d \tag{1.7}$$

with $\mathcal{E}_j$ a M_j-equivariant cuspidal local system on $\mathcal{C}_{v_j}^{M_j}$. From Proposition 1.1 and [AMS2, Proposition 2.2] we see that

$$\mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}}) = \mathbb{H}(G_1, M_1, \mathcal{E}_1) \otimes \cdots \otimes \mathbb{H}(G_d, M_d, \mathcal{E}_d). \tag{1.8}$$

Furthermore the proof of [AMS2, Proposition 2.2] shows that

$$\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}}) = \mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}}) \rtimes \mathbb{C}[\mathfrak{R}_{q\mathcal{E}}, \natural_{q\mathcal{E}}]. \tag{1.9}$$

1.1. **Standard modules.**

To parametrize the irreducible representations of the above algebras we use some elements of the Lie algebras of the involved algebraic groups. Let $\sigma_0 \in \mathfrak{g}$ be semisimple and $y \in Z_{\mathfrak{g}}(\sigma_0)$ be nilpotent. We decompose them along (1.4):

$$\begin{aligned} \sigma_0 &= \sigma_z + \sigma_{0,1} + \cdots + \sigma_{0,d} \quad \text{with } \sigma_{0,j} \in \mathfrak{g}_j, \sigma_z \in Z(\mathfrak{g}), \\ y &= y_1 + \cdots + y_d \quad \text{with } y_j \in \mathfrak{g}_j. \end{aligned}$$

Choose algebraic homomorphisms $\gamma_j : SL_2(\mathbb{C}) \to Z_{G_j}(\sigma_{0,j})$ with $d\gamma_j \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right) = y_j$. Given $\vec{r} \in \mathbb{C}^d$, we write $\sigma_j = \sigma_{0,j} + d\gamma_j \left(\begin{smallmatrix} r_j & 0 \\ 0 & -r_j \end{smallmatrix} \right)$ and

$$\begin{aligned} d\vec{\gamma} \left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix} \right) &= d\gamma_1 \left(\begin{smallmatrix} r_1 & 0 \\ 0 & -r_1 \end{smallmatrix} \right) + \cdots + d\gamma_d \left(\begin{smallmatrix} r_d & 0 \\ 0 & -r_d \end{smallmatrix} \right), \\ \sigma &= \sigma_0 + d\vec{\gamma} \left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix} \right). \end{aligned} \tag{1.10}$$

Notice that $[\sigma, y_j] = [\sigma_j, y_j] = 2r_j y_j$. Let us recall the construction of the standard modules from [Lus2, AMS2]. We need the groups

$$\begin{aligned} M_j(y_j) &= \big\{ (g_j, \lambda_j) \in G_j \times \mathbb{C}^\times : \mathrm{Ad}(g_j) y_j = \lambda_j^2 y_j \big\}, \\ \vec{M}^\circ(y) &= \big\{ (g, \vec{\lambda}) \in G^\circ \times (\mathbb{C}^\times)^d : \mathrm{Ad}(g) y_j = \lambda_j^2 y_j \ \forall j = 1, \ldots, d \big\}, \\ \vec{M}(y) &= \big\{ (g, \vec{\lambda}) \in G^\circ N_G(q\mathcal{E}) \times (\mathbb{C}^\times)^d : \mathrm{Ad}(g) y_j = \lambda_j^2 y_j \ \forall j = 1, \ldots, d \big\}, \end{aligned}$$

and the varieties

$$\begin{aligned} \mathcal{P}_{y_j} &= \big\{ g(P^\circ \cap G_j) \in G_j/(P^\circ \cap G_j) : \mathrm{Ad}(g^{-1}) y_j \in \mathcal{C}_{v_j}^{M_j} + \mathrm{Lie}(U \cap G_j) \big\}, \\ \mathcal{P}_y^\circ &= \big\{ gP^\circ \in G^\circ/P^\circ : \mathrm{Ad}(g^{-1}) y \in \mathcal{C}_v^{M^\circ} + \mathrm{Lie}(U) \big\}, \\ \mathcal{P}_y &= \big\{ gP \in G^\circ N_G(q\mathcal{E})/P : \mathrm{Ad}(g^{-1}) y \in \mathcal{C}_v^{M} + \mathrm{Lie}(U) \big\}. \end{aligned}$$

The local systems $\mathcal{E}_j, \mathcal{E}$ and $q\mathcal{E}$ give rise to local systems $\dot{\mathcal{E}}_j, \dot{\mathcal{E}}$ and $\dot{q\mathcal{E}}$ on $\mathcal{P}_{y_j}, \mathcal{P}_y^\circ$ and $\mathcal{P}_y$, respectively. The groups $M_j(y_j), \vec{M}^\circ(y)$ and $\vec{M}(y)$ act naturally on, respectively, $(\mathcal{P}_{y_j}, \dot{\mathcal{E}}_j), (\mathcal{P}_y^\circ, \dot{\mathcal{E}})$ and $(\mathcal{P}_y, \dot{q\mathcal{E}})$. With the method from [Lus2] and [AMS2, §3.1] we can define an action of $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}}) \times \vec{M}(y)$ on the equivariant homology $H_*^{\vec{M}(y)^\circ}(\mathcal{P}_y, \dot{q\mathcal{E}})$, and similarly for $H_*^{\vec{M}^\circ(y)^\circ}(\mathcal{P}_y^\circ, \dot{\mathcal{E}})$ and $H_*^{M_j(y)^\circ}(\mathcal{P}_{y_j}, \dot{\mathcal{E}}_j)$. As in [Lus2] we build

$$E_{y_j, \sigma_j, r_j}^\circ = \mathbb{C}_{\sigma_j, r_j} \underset{H^*_{M_j(y_j)^\circ}(\{y_j\})}{\otimes} H_*^{M_j(y)^\circ}(\mathcal{P}_{y_j}, \dot{\mathcal{E}}_j).$$

Similarly we introduce

$$E^\circ_{y,\sigma,\vec{r}} = \mathbb{C}_{\sigma,\vec{r}} \underset{H^*_{\vec{M}^\circ(y)^\circ}(\{y\})}{\otimes} H_*^{\vec{M}^\circ(y)^\circ}(\mathcal{P}^\circ_y, \dot{\mathcal{E}}),$$

$$E_{y,\sigma,\vec{r}} = \mathbb{C}_{\sigma,\vec{r}} \underset{H^*_{\vec{M}(y)^\circ}(\{y\})}{\otimes} H_*^{\vec{M}(y)^\circ}(\mathcal{P}_y, \dot{q\mathcal{E}}).$$

By [AMS2, Theorem 3.2 and Lemma 3.6] these are modules over, respectively, $\mathbb{H}(G_j, M_j, \mathcal{E}_j) \times \pi_0(\mathrm{Z}_{G_j}(\sigma_{0,j}, y_j))$, $\mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}}) \times \pi_0(\mathrm{Z}_{G^\circ}(\sigma_0, y))$ and $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}}) \times \pi_0(\mathrm{Z}_{G^\circ \mathrm{N}_G(q\mathcal{E})}(\sigma_0, y))$. This last action is the reason to use $G^\circ \mathrm{N}_G(q\mathcal{E})$ instead of G in the definition of $\mathcal{P}_y$.

When $\mathcal{P}_y$ is nonempty, $\mathcal{P}^\circ_{\mathrm{Ad}(g)y}$ is nonempty for some $g \in G^\circ N_G(q\mathcal{E})$. As $\mathcal{P}_{\mathrm{Ad}(g)y} = g\mathcal{P}_y \cong \mathcal{P}_y$, it suffices to consider the cases where $\mathcal{P}^\circ_y$ is nonempty. Then $\mathcal{P}_y \cong \mathcal{P}^\circ_y \times \mathfrak{R}_{q\mathcal{E}}$ [AMS2, (17)] and (1.9) leads to a natural module isomorphism

$$E_{y,\sigma,\vec{r}} \cong \mathrm{ind}_{\mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}})}^{\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}})} E^\circ_{y,\sigma,\vec{r}}. \tag{1.11}$$

It can be proven in the same way as the analogous statement with only one variable $\mathbf{r}$, which is [AMS2, Lemma 3.3].

Lemma 1.2. *With the identifications (1.8) there is a natural isomorphism of $\mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}})$-modules*

$$E^\circ_{y,\sigma,\vec{r}} \cong \mathbb{C}_{\sigma_z} \otimes E^\circ_{y_1,\sigma_1,r_1} \otimes \cdots \otimes E^\circ_{y_d,\sigma_d,r_d},$$

which is equivariant for the actions of the appropriate subquotients of $\vec{M}^\circ(y)$.

Proof. From (1.2) and $Z(G^\circ)Z(G_j) \subset P^\circ$ we get natural isomorphisms

$$\mathcal{P}_{y_1} \times \cdots \times \mathcal{P}_{y_d} \to \mathcal{P}^\circ_y. \tag{1.12}$$

Looking at (1.7) and the construction of $\dot{\mathcal{E}}$ in [Lus2, §3.4], we deduce that

$$\dot{\mathcal{E}} \cong \dot{\mathcal{E}}_1 \otimes \cdots \otimes \dot{\mathcal{E}}_d \text{ as sheaves on } \mathcal{P}^\circ_y. \tag{1.13}$$

From (1.2) we also get a central extension

$$1 \to \ker m_{G^\circ} \to \mathrm{Z}(G^\circ)^\circ \times M_1(y_1) \times \cdots \times M_d(y_d) \to \vec{M}^\circ(y) \to 1. \tag{1.14}$$

Here $\ker m_{G^\circ}$ refers to the kernel of (1.2), a finite central subgroup which acts trivially on the sheaf $\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_d \cong m^*_{G^\circ}\mathcal{E}$. Restricting to connected components, we obtain a central extension of $\vec{M}^\circ(y)^\circ$ by

$$\tilde{M} := \mathrm{Z}(G^\circ)^\circ \times M_1(y_1)^\circ \times \cdots \times M_d(y_d)^\circ$$

In fact, equivariant (co)homology is inert under finite central extensions, for all groups and all varieties. We sketch how this can be deduced from [Lus2, §1]. By definition

$$H^*_{\vec{M}^\circ(y)^\circ}(\mathcal{P}^\circ_y, \dot{\mathcal{E}}) = H^*\big(\vec{M}^\circ(y)^\circ \backslash (\Gamma \times \mathcal{P}^\circ_y), {}_\Gamma\dot{\mathcal{E}}\big)$$

for a suitable (in particular free) $\vec{M}^\circ(y)^\circ$-variety Γ and a local system derived from $\dot{\mathcal{E}}$. On the right hand side we can replace $\vec{M}^\circ(y)^\circ$ by $\tilde{M}$ without changing anything. If $\tilde{\Gamma}$ is a suitable variety for $\tilde{M}$, then $\tilde{\Gamma} \times \Gamma$ is also one. (The freeness is preserved because (1.14) is an extension of finite index.) The argument in [Lus2, p. 149] shows that

$$H^*\big(\tilde{M} \backslash (\Gamma \times \mathcal{P}^\circ_y), {}_\Gamma\dot{\mathcal{E}}\big) \cong H^*\big(\tilde{M} \backslash (\tilde{\Gamma} \times \Gamma \times \mathcal{P}^\circ_y), {}_{\tilde{\Gamma}\times\Gamma}\dot{\mathcal{E}}\big) = H^*_{\tilde{M}}(\mathcal{P}^\circ_y, \dot{\mathcal{E}}).$$

In a similar way, using [Lus2, Lemma 1.2], one can prove that

$$H_*^{\vec{M}^\circ(y)^\circ}(\mathcal{P}_y^\circ, \dot{\mathcal{E}}) \cong H_*^{\tilde{M}}(\mathcal{P}_y^\circ, \dot{\mathcal{E}}). \tag{1.15}$$

The upshot of (1.12), (1.13) and (1.15) is that we can factorize the entire setting along (1.8), which gives

$$H_*^{M_1(y)^\circ}(\mathcal{P}_{y_1}, \dot{\mathcal{E}}_1) \otimes \cdots \otimes H_*^{M_d(y)^\circ}(\mathcal{P}_{y_d}, \dot{\mathcal{E}}_d) \cong H_*^{\vec{M}^\circ(y)^\circ}(\mathcal{P}_y^\circ, \dot{\mathcal{E}}). \tag{1.16}$$

The equivariant cohomology of a point with respect to a connected group depends only on the Lie algebra [Lus2, §1.11], so (1.14) implies a natural isomorphism

$$H^*_{Z(G^\circ)^\circ}(\{1\}) \times H^*_{M_1(y_1)^\circ}(\{y_1\}) \times \cdots \times H^*_{M_d(y_d)^\circ}(\{y_d\}) \cong H^*_{\vec{M}^\circ(y)^\circ}(\{y\}).$$

Thus we can tensor both sides of (1.16) with $\mathbb{C}_{\sigma,\vec{r}}$ and preserve the isomorphism. □

Given $\rho_j \in \mathrm{Irr}\big(\pi_0(\mathrm{Z}_{G_j}(\sigma_{0,j}, y_j))\big)$, we can form the standard $\mathbb{H}(G_j, M_j, \mathcal{E}_j)$-module

$$E^\circ_{y_j,\sigma_j,r_j,\rho_j} := \mathrm{Hom}_{\pi_0(\mathrm{Z}_{G_j}(\sigma_{0,j},y_j))}(\rho_j, E^\circ_{y_j,\sigma_j,r_j}).$$

Similarly $\rho^\circ \in \mathrm{Irr}\big(\pi_0(\mathrm{Z}_{G^\circ}(\sigma_0, y))\big)$ and $\rho \in \mathrm{Irr}\big(\pi_0(\mathrm{Z}_{G^\circ \mathrm{N}_G(q\mathcal{E})}(\sigma_0, y))\big)$ give rise to

$$\begin{aligned} E^\circ_{y,\sigma,\vec{r},\rho^\circ} &:= \mathrm{Hom}_{\pi_0(\mathrm{Z}_{G^\circ}(\sigma_0,y))}(\rho^\circ, E^\circ_{y,\sigma,\vec{r}}), \\ E_{y,\sigma,\vec{r},\rho} &:= \mathrm{Hom}_{\pi_0(\mathrm{Z}_{G^\circ \mathrm{N}_G(q\mathcal{E})}(\sigma_0,y))}(\rho, E_{y,\sigma,\vec{r}}). \end{aligned} \tag{1.17}$$

We call these standard modules for respectively $\mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}})$ and $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}})$.

The canonical map (1.2) induces a surjection

$$\pi_0(\mathrm{Z}_{G_1}(\sigma_{0,1}, y_1)) \times \cdots \times \pi_0(\mathrm{Z}_{G_d}(\sigma_{0,d}, y_d)) \to \pi_0(\mathrm{Z}_{G^\circ}(\sigma_0, y)). \tag{1.18}$$

Lemma 1.3. *Let $\rho^\circ \in \mathrm{Irr}\big(\pi_0(\mathrm{Z}_{G^\circ}(\sigma_0, y))\big)$ and let $\bigotimes_{j=1}^d \rho_j$ be its inflation to $\prod_{j=1}^d \pi_0(\mathrm{Z}_{G_j}(\sigma_{0,j}, y_j))$ via (1.18). There is a natural isomorphism of $\mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}})$-modules*

$$E^\circ_{y,\sigma,\vec{r},\rho^\circ} \cong \mathbb{C}_{\sigma_z} \otimes E^\circ_{y_1,\sigma_1,r_1,\rho_1} \otimes \cdots \otimes E^\circ_{y_d,\sigma_d,r_d,\rho_d}.$$

Every $\bigotimes_{j=1}^d \rho_j \in \mathrm{Irr}\big(\prod_{j=1}^d \pi_0(\mathrm{Z}_{G_j}(\sigma_{0,j}, y_j))\big)$ for which $\bigotimes_{j=1}^d E^\circ_{y_j,\sigma_j,r_j,\rho_j}$ is nonzero comes from $\pi_0(\mathrm{Z}_{G^\circ}(\sigma_0, y))$ via (1.18).

Proof. The module isomorphism follows from the naturality and the equivariance in Lemma 1.2.

Suppose that $\bigotimes_{j=1}^d \rho_j \in \mathrm{Irr}\big(\prod_{j=1}^d \pi_0(\mathrm{Z}_{G_j}(\sigma_{0,j}, y_j))\big)$ appears in $\bigotimes_{j=1}^d E^\circ_{y_j,\sigma_j,r_j}$. By [AMS2, Proposition 3.7] the cuspidal support $\Psi_{\mathrm{Z}_G(\sigma_{0,j})}(y_j, \rho_j)$ is G_j-conjugate to $(M_j, \mathcal{C}^{M_j}_{y_j}, \mathcal{E}_j)$. In particular ρ_j has the same $\mathrm{Z}(G_j)$-character as $\mathcal{E}_j$, see [Lus1, Theorem 6.5.a]. Hence $\otimes_j \rho_j$ has the same central character as $m^*_{G_0}\mathcal{E}$. That central character factors through the multiplication map (1.2) whose kernel is central, so $\otimes_j \rho_j$ also factors through (1.2). That is, the map (1.18) induces a bijection between the relevant irreducible representations on both sides. □

For some choices of ρ the standard module $E_{y,\sigma,\vec{r},\rho}$ is zero. To avoid that, we consider triples (σ_0, y, ρ) with:

- $\sigma_0 \in \mathfrak{g}$ is semisimple,
- $y \in \mathrm{Z}_{\mathfrak{g}}(\sigma_0)$ is nilpotent,
- $\rho \in \mathrm{Irr}\big(\pi_0(\mathrm{Z}_G(\sigma_0, y))\big)$ is such that the cuspidal quasi-support $q\Psi_{\mathrm{Z}_G(\sigma_0)}(y, \rho)$ from [AMS1, §5] is G-conjugate to $(M, \mathcal{C}_v^M, q\mathcal{E})$.

Given in addition $\vec{r} \in \mathbb{C}^d$, we construct $\sigma = \sigma_0 + \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right) \in \mathfrak{g}$ as in (1.10). Although this depends on the choice of $\vec{\gamma}$, the conjugacy class of σ does not.

By definition

$$\mathbb{H}(G^\circ \mathrm{N}_G(q\mathcal{E}), M, q\mathcal{E}, \vec{\mathbf{r}}) = \mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}}),$$

but of course $\pi_0(\mathrm{Z}_{G^\circ \mathrm{N}_G(q\mathcal{E})}(\sigma_0, y))$ can be a proper subgroup of $\pi_0(\mathrm{Z}_G(\sigma_0, y))$. As shown in the proof of [AMS2, Lemma 3.21], the functor $\mathrm{ind}_{\pi_0(\mathrm{Z}_{G^\circ \mathrm{N}_G(q\mathcal{E})}(\sigma_0,y))}^{\pi_0(\mathrm{Z}_G(\sigma_0,y))}$ provides a bijection between the $\tilde{\rho}$ in the triples for $G^\circ \mathrm{N}_G(q\mathcal{E})$ and the ρ in the triples for G. For $\rho = \mathrm{ind}_{\pi_0(\mathrm{Z}_{G^\circ \mathrm{N}_G(q\mathcal{E})}(\sigma_0,y))}^{\pi_0(\mathrm{Z}_G(\sigma_0,y))} \tilde{\rho}$ we define, in terms of (1.17),

$$E_{y,\sigma,\vec{r},\rho} = E_{y,\sigma,\vec{r},\tilde{\rho}}. \tag{1.19}$$

We would like to exhibit the central characters of these standard $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}})$-modules. It has turned out that the treatment of this aspect in [AMS2] was flawed, we correct that here. We fix a homomorphism of algebraic groups

$$\gamma_v \colon \mathrm{SL}_2(\mathbb{C}) \to M \quad \text{with} \quad \mathrm{d}\gamma_v \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = v.$$

We write

$$\mathrm{d}\gamma_v \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = \sigma_v = \sigma_{v,1} + \cdots + \sigma_{v,d} \quad \text{where} \quad \sigma_{v,j} \in \mathrm{Lie}(M \cap G_j). \tag{1.20}$$

For $\vec{r} \in \mathbb{C}^d$ we put

$$\vec{r}\sigma_v = r_1 \sigma_{v,1} + \cdots + r_d \sigma_{v,d} \in \mathfrak{m}.$$

We record the linear bijection

$$\begin{array}{cccc} \Sigma_v \colon & \mathfrak{t} \oplus \mathbb{C}^d & \to & \mathfrak{t} \oplus \mathbb{C}^d(\sigma_v, 1) \\ & (\sigma_0, \vec{r}) & \mapsto & (\sigma_0 + \vec{r}\sigma_v, \vec{r}) \end{array}.$$

Here the target is a linear subspace of $\mathfrak{m} \oplus \mathbb{C}^d$ and the inverse map is

$$\Sigma_v^{-1} \colon (\sigma, \vec{r}) \mapsto (\sigma - \vec{r}\sigma_v, \vec{r}).$$

The next result is a correction of [AMS2, Proposition 3.5], which was based on a wrong interpretation of [Lus4, §8]. Our improvement consists mainly of adding $\Sigma_v^{\pm 1}$ at the right places.

Proposition 1.4. *Let $(y, \sigma, \vec{r})$ be as in (1.10) and assume that $\mathcal{P}_y^\circ$ is nonempty.*

(a) $(\mathrm{Ad}(\mathrm{N}_G(P, q\mathcal{E})G^\circ)\sigma - \vec{r}\sigma_v) \cap \mathfrak{t}$ is a single $W_{q\mathcal{E}}$-orbit in $\mathfrak{t}$.
(b) The $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}})$-module $E_{y,\sigma,\vec{r}}$ admits the central character $((\mathrm{Ad}(\mathrm{N}_G(P, q\mathcal{E})G^\circ)\sigma - \vec{r}\sigma_v) \cap \mathfrak{t}, \vec{r})$.
(c) The pair (y, σ) is G°-conjugate to one with σ_0 and $\vec{r}\sigma_v + \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} -\vec{r} & 0 \\ 0 & \vec{r} \end{smallmatrix}\right)$ in $\mathfrak{t}$.
(d) Suppose (y, σ) has the properties as in (c). Then σ_0, σ_v and $\mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ commute, and $\sigma_v + \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathfrak{t}_\mathbb{R}$.
(e) Suppose (y, σ) has the properties as in (c). Then the central character of $E_{y,\sigma,\vec{r},\rho}$ can be expressed as $W_{q\mathcal{E}}\big(\sigma_0 \pm (\vec{r}\sigma_v + \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} -\vec{r} & 0 \\ 0 & \vec{r} \end{smallmatrix}\right)), \vec{r}\big)$.

Proof. (a) By [Lus4, Theorem 8.11], $\mathbb{H}(G_j, M_j, \mathcal{E}_j, \mathbf{r}_j)$ is canonically isomorphic to the endomorphism algebra of a certain perverse sheaf K_j^*, in the $G_j \times \mathbb{C}^\times$-equivariant bounded derived category of constructible sheaves on $\mathfrak{g}_j$. According to [Lus4, §8.13.a], there exists a canonical surjection

$$H^*_{G_j \times \mathbb{C}^\times}(\text{point}) \cong \mathcal{O}(\mathfrak{g}_j \oplus \mathbb{C})^{G_j \times \mathbb{C}^\times} = \mathcal{O}(\mathfrak{g}_j)^{G_j} \otimes \mathbb{C}[\mathbf{r}_j] \to \mathrm{Z}(\mathrm{End}(K_j^*)). \tag{1.21}$$

By [AMS2, Lemma 2.3] the right hand side is

$$\text{(1.22)} \qquad \mathrm{Z}(\mathrm{End}(K_j^*)) \cong \mathrm{Z}(\mathbb{H}(G_j, M_j, \mathcal{E}_j, \mathbf{r}_j)) \cong \mathcal{O}(\mathfrak{t} \cap \mathfrak{g}_j)^{W_{\mathcal{E}_j}} \otimes \mathbb{C}[\mathbf{r}_j].$$

By [Lus4, §8.13.b], the composition of (1.21) and (1.22) corresponds to an injection like Σ_v, namely

$$\text{(1.23)} \qquad \begin{array}{ccc} (\mathfrak{t} \cap \mathfrak{g}_j)/W_{\mathcal{E}_j} \oplus \mathbb{C} & \to & \mathrm{Irr}\big(\mathcal{O}(\mathfrak{g}_j \oplus \mathbb{C})^{G_j \times \mathbb{C}^\times}\big) \\ (\sigma_{0,j}, r_j) & \mapsto & (\sigma_{0,j} + r_j \sigma_{v,j}, r_j) \end{array},$$

where the right hand side is the variety of semisimple adjoint orbits in $\mathfrak{g}_j \oplus \mathbb{C}$. Hence

$$(\mathrm{Ad}(G_j)\sigma_j - r_j \sigma_{v,j}) \cap \mathfrak{t} \cap \mathfrak{g}_j = \mathrm{Ad}(G_j)\sigma_{0,j} \cap \mathfrak{t} \cap \mathfrak{g}_j$$

is either empty or a single $W_{\mathcal{E}_j}$-orbit. We will see in the proof of part (b) that it is nonempty. Combining these statements for all $j = 1, \ldots, d$, we find that $(\mathrm{Ad}(G^\circ)\sigma - \vec{r}\sigma_v) \cap \mathfrak{t}$ is a single $W_{\mathcal{E}}$-orbit $W_{\mathcal{E}}\sigma' \subset \mathfrak{t}$. As M stabilizes $\mathfrak{m}$ and centralizes $\mathfrak{t}$:

$$\text{(1.24)} \qquad \begin{aligned} \mathrm{Ad}(G^\circ M)\sigma \cap (\mathfrak{t} + \vec{r}\sigma_v) &= \mathrm{Ad}(M)(W_{\mathcal{E}}\sigma' + \vec{r}\sigma_v) \cap (\mathfrak{t} + \vec{r}\sigma_v) \\ &= (W_{\mathcal{E}}\sigma' + \mathrm{Ad}(M)\vec{r}\sigma_v) \cap (\mathfrak{t} + \vec{r}\sigma_v). \end{aligned}$$

Here $\mathrm{Ad}(M)(\vec{r}\sigma_v)$ lies in the derived subalgebra of $\mathfrak{m}$, so the right hand side of (1.24) equals $W_{\mathcal{E}}\sigma' + \vec{r}\sigma_v$. In other words,

$$(\mathrm{Ad}(G^\circ M)\sigma - \vec{r}\sigma_v) \cap \mathfrak{t} = W_{\mathcal{E}}\sigma'.$$

As $\mathrm{N}_G(P, q\mathcal{E})G^\circ/G^\circ M \cong W_{q\mathcal{E}}/W_{\mathcal{E}}$, we can pass from $\mathrm{Ad}(G^\circ M)$-orbits to $\mathrm{Ad}(\mathrm{N}_G(P, q\mathcal{E})G^\circ)$-orbits in the required way.

(b) The assumption $\mathcal{P}_y^\circ \neq \emptyset$ implies that $H_*^{\vec{M}(y)^\circ}(\mathcal{P}_y^\circ, \dot{\mathcal{E}})$ is nonzero. By [Lus2, Proposition 8.6.c] and because the semisimple adjoint orbits in $\mathrm{Lie}(\vec{M}(y)^\circ)$ form an irreducible variety, $E^\circ_{y,\sigma,\vec{r}}$ is nonzero for all eligible $(\sigma, \vec{r})/\sim$.

The action of $\mathcal{O}(\mathfrak{t} \cap \mathfrak{g}_j)^{W_{\mathcal{E}_j}} \otimes \mathbb{C}[\mathbf{r}_j]$ on E_{y_j,σ_j,r_j} can be realized as

$$Z(\mathbb{H}(G_j, M_j, \mathcal{E}_j, \mathbf{r}_j)) \leftarrow H^*_{G_j \times \mathbb{C}^\times}(\text{point}) \to H^*_{M_j(y_j)^\circ}(y_j) \to H^*_{M_j(y_j)^\circ}(\mathcal{P}_{y_j})$$

and then the product in equivariant homology. By construction $H^*_{M_j(y_j)^\circ}(y_j)$ acts on E_{y_j,σ_j,r_j} via the character $(\sigma_j, r_j)/\sim$. Hence $H^*_{G_j \times \mathbb{C}^\times}(\text{point})$ acts via the character $\mathrm{Ad}(G_j \times \mathbb{C}^\times)(\sigma_j, r_j)$. In view of (1.21)–(1.23), $\mathrm{Z}(\mathbb{H}(G_j, M_j, \mathcal{E}_j, \mathbf{r}_j))$ acts via

$$((\mathrm{Ad}(G_j)\sigma_j - r_j\sigma_{v,j}) \cap \mathfrak{g}_j \cap \mathfrak{t}, r_j).$$

For all $j = 1, \ldots, d$ together, this shows that $\mathrm{Z}(\mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}})$ acts on $E^\circ_{y,\sigma,\vec{r}}$ as $((\mathrm{Ad}(G^\circ)\sigma - \vec{r}\sigma_v) \cap \mathfrak{t}, \vec{r})$. Now we use that $\mathrm{N}_G(P, q\mathcal{E})G^\circ/G^\circ \cong W_{q\mathcal{E}}/W_{\mathcal{E}}$ and

$$\mathrm{Z}(\mathbb{H}(G, M, q\mathcal{E}, \mathbf{r})) = \mathrm{Z}(\mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}}))^{W_{q\mathcal{E}}/W_{\mathcal{E}}},$$

and we conclude with (1.11).

(c) By part (b) with $r = 0$ we may assume that $\sigma_0 \in \mathfrak{t}$. Then $\exp(y)$ is contained in the reductive group $\mathrm{Z}_G(\sigma_0)^\circ$, so we can arrange that the image of $\vec{\gamma}$ lies in there. Applying part (b) to this group, we find $g \in \mathrm{Z}_G(\sigma_0)^\circ$ such that

$$\mathrm{Ad}(g)\sigma = \sigma_0 + \mathrm{Ad}(g)\mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right) \quad \text{lies in} \quad \mathfrak{t} + \vec{r}\sigma_v.$$

Then $\mathrm{Ad}(g)\mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right) - \vec{r}\sigma_v \in \mathfrak{t}$, so $(\mathrm{Ad}(g)y, \mathrm{Ad}(g)\sigma)$ has the required properties.

(d) The assumption and $\sigma_v \in \mathfrak{m}$ imply that $\mathrm{d}\gamma\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathfrak{m}$. As $\sigma_0 \in \mathfrak{t} = \mathrm{Z}(\mathfrak{m})$, it commutes with both σ_v and $\mathrm{d}\gamma \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The latter two differ by an element of $\mathfrak{t} = \mathrm{Z}(\mathfrak{m})$, so they commute as well. It follows that

$$\chi_{y,v} \colon z \mapsto \gamma \left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right) \gamma_v \left(\begin{smallmatrix} z^{-1} & 0 \\ 0 & z \end{smallmatrix}\right) \tag{1.25}$$

is an algebraic cocharacter of T. By definition of $\mathfrak{t}_\mathbb{R}$, the derivative

$$\mathrm{d}\chi_{y,v} \colon r \mapsto \mathrm{d}\gamma \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) - r\sigma_v$$

evaluates to an element of $X_*(T) \otimes_\mathbb{Z} \mathbb{R} = \mathfrak{t}_\mathbb{R}$ for every $r \in \mathbb{R}$.
(e) As in the proof of part (c), we may assume that $\mathrm{im}(\vec{\gamma}) \subset \mathrm{Z}_G(\sigma_0)^\circ$. Put $s_y = \gamma \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and consider the parameter $(\mathrm{Ad}(s_y)\sigma, \mathrm{Ad}(s_y)y, \vec{r})$. We have

$$\mathrm{Ad}(s_y)\sigma + \vec{r}s_v = \sigma_0 + \mathrm{d}\vec{\gamma} \left(\begin{smallmatrix} -\vec{r} & 0 \\ 0 & \vec{r} \end{smallmatrix}\right) - \vec{r}(-s_v) \in \mathfrak{t}.$$

Here $-s_v$ is the semisimple element in the $\mathfrak{sl}_2$-triple $\mathrm{d}\gamma_v \left(\begin{smallmatrix} 0 & 0 \\ -1 & 0 \end{smallmatrix}\right), -s_v, -v$, which is conjugate to $v, s_v, \mathrm{d}\gamma_v \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ by $\gamma_v \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in M$. Thus part (c) says that

$$W_{q\mathcal{E}} \big(\sigma_0 + \vec{r}\sigma_v + \mathrm{d}\vec{\gamma} \left(\begin{smallmatrix} -\vec{r} & 0 \\ 0 & \vec{r} \end{smallmatrix}\right), \vec{r} \big)$$

is also the central character of $E_{y,\sigma,\vec{r},\rho}$. □

1.2. **Irreducible modules.**

The standard modules and the irreducible modules which are annihilated by $\vec{\mathbf{r}}$ must be treated somewhat differently from the others. We need an improvement on the analysis in [AMS2, Lemma 3.9 and Proposition 3.10].

Theorem 1.5. *Let $E^\circ_{y,\sigma_0,0,\rho^\circ}$ be a nonzero standard $\mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}})$-module on which each $\mathbf{r}_j$ acts as zero.*

(a) *$E^\circ_{y,\sigma_0,0,\rho^\circ} = \mathrm{Hom}_{\pi_0(\mathrm{Z}_{G^\circ}(\sigma_0,y))}(\rho^\circ, H_*(\mathcal{P}^\circ_y, \dot{\mathcal{E}}))$ as graded $\mathbb{C}[W_\mathcal{E}]$-modules, where we use the homological grading from $H_*(\mathcal{P}^\circ_y, \dot{\mathcal{E}})$.*
(b) *For each $d \in \mathbb{Z}$,*
$$\bigoplus\nolimits_{n \geq d} \mathrm{Hom}_{\pi_0(\mathrm{Z}_{G^\circ}(\sigma_0,y))} \big(\rho^\circ, H_n(\mathcal{P}^\circ_y, \dot{\mathcal{E}}) \big)$$
is an $\mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}})$-submodule of $E^\circ_{y,\sigma_0,0,\rho^\circ}$.
(c) *$E^\circ_{y,\sigma_0,0,\rho^\circ}$ has a unique irreducible quotient isomorphic to the module $M^\circ_{y,\sigma_0,0,\rho^\circ}$ from* [AMS2, Proposition 3.8].

In the proof of [AMS2, Lemma 3.9] it was assumed incorrectly that $S(\mathfrak{t}^*)$ acts semisimply on $E^\circ_{y,\sigma_0,0,\rho^\circ}$, which lead to the wrong claim that it is always completely reducible as $\mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}})$-module.

Proof. (a) By [Lus4, 10.13], $E^\circ_{y,\sigma_0,0,\rho^\circ}$ can be identified with $H_*(\mathcal{P}^\circ_y, \dot{\mathcal{E}})$ as graded $W_\mathcal{E} \times \pi_0(\mathrm{Z}_{G^\circ}(\sigma_0, y))$-representations. In fact these finite groups both preserve the homological grading. Now apply $\mathrm{Hom}_{\pi_0(\mathrm{Z}_{G^\circ}(\sigma_0,y))}(\rho^\circ, ?)$ to both modules.
(b) First we assume that σ_0 is central in $\mathfrak{g}$. Write

$$\mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}}) = S(\mathrm{Z}(\mathfrak{g})^*) \otimes \mathbb{H}(G^\circ_{\mathrm{der}}, M \cap G^\circ_{\mathrm{der}}, \mathcal{E}, \vec{\mathbf{r}})$$

as in (1.8). Then $S(\mathrm{Z}(\mathfrak{g})^*)$ acts on $E^\circ_{y,\sigma_0,0,\rho^\circ}$ by the character σ_0 and the restriction of $E^\circ_{y,\sigma_0,0,\rho^\circ}$ to $\mathbb{H}(G^\circ_{\mathrm{der}}, M \cap G^\circ_{\mathrm{der}}, \mathcal{E}, \vec{\mathbf{r}})$ is

$$E^\circ_{y,0,0} = \mathbb{C}_{0,0} \underset{H^*_{\vec{M}^\circ(y)^\circ}(\{y\})}{\otimes} H_*^{\vec{M}^\circ(y)^\circ}(\mathcal{P}^\circ_y, \dot{\mathcal{E}}).$$

Here $H_*^{\vec{M}^\circ(y)^\circ}(\mathcal{P}_y^\circ, \dot{\mathcal{E}})$ is a graded $\mathbb{H}(G^\circ_{\text{der}}, M \cap G^\circ_{\text{der}}, \mathcal{E}, \vec{\mathbf{r}})$-module by [Lus2, Theorem 8.13], although typically not semisimple. As $\mathbb{C}_{0,0}$ is a graded $H^*_{\vec{M}^\circ(y)^\circ}(\{y\})$-module it follows that $E^\circ_{y,0,0}$ is a graded $\mathbb{H}(G^\circ_{\text{der}}, M \cap G^\circ_{\text{der}}, \mathcal{E}, \vec{\mathbf{r}})$-module. For $E^\circ_{y,\sigma_0,0}$ as $\mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}})$-module, we can still say that the action of $\mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}})$ only raises degrees, and that an element $x \in \mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}})$ of degree n can only raise the degree of an element of $E^\circ_{y,\sigma_0,0}$ by at most n.

Now we lift the condition on σ_0 and we consider the Levi subgroup $Q^\circ = \mathrm{Z}_{G^\circ}(\sigma_0)$ of G°. By Proposition 1.4.a, $\mathrm{Z}(\mathrm{Lie}(M^\circ))$ contains a G°-conjugate of σ_0. Upon replacing (y, σ_0) by a suitable G°-conjugate, we may assume that M° centralizes σ_0, so that $Q^\circ \supset M^\circ$. By [Lus6, Corollary 1.18], there is a natural isomorphism of $\mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}})$-modules

$$(1.26)\quad W_{\mathcal{E}} \ltimes S(\mathfrak{t}^*) \underset{W_{\mathcal{E}}^{Q^\circ} \ltimes S(\mathfrak{t}^*)}{\otimes} E^{Q^\circ}_{y,\sigma_0,0} = \mathbb{H}(G^\circ, M^\circ, \mathcal{E}) \underset{\mathbb{H}(Q^\circ, M^\circ, \mathcal{E})}{\otimes} E^{Q^\circ}_{y,\sigma_0,0} \longrightarrow E^\circ_{y,\sigma_0,0}.$$

We note that [Lus6, Corollary 1.18] is applicable because $r = 0$ and $\mathrm{ad}(\sigma_0)$ is an invertible linear transformation of $\mathrm{Lie}(U_{Q^\circ})$, where U_{Q° is the unipotent radical of a parabolic subgroup of G° with Levi factor Q°. The map (1.26) comes from a morphism $\mathcal{P}_y^{Q^\circ} \to \mathcal{P}_y^\circ$, which entails that it changes all homological degrees (see part a) by the same amount, namely $\dim \mathcal{P}_y^\circ - \dim \mathcal{P}_y^{Q^\circ}$. From (1.26) we deduce that, as $S(\mathfrak{t}^*)$-modules and as graded vector spaces,

$$E^\circ_{y,\sigma_0,0}[\dim \mathcal{P}_y^{Q^\circ} - \dim \mathcal{P}_y^\circ] = \bigoplus_{w \in W_{\mathcal{E}}/W_{\mathcal{E}}^{Q^\circ}} \mathbb{C}w \otimes_{\mathbb{C}} E^{Q^\circ}_{y,\sigma_0,0} = \bigoplus_{w \in W_{\mathcal{E}}/W_{\mathcal{E}}^{Q^\circ}} (w)^* E^{Q^\circ}_{y,\sigma_0,0},$$

where $[\cdots]$ denotes a degree shift.

We denote the ideal generated by the $\mathbf{r}_j$ by $(\vec{\mathbf{r}})$. As $(\vec{\mathbf{r}})$ is divided out in (1.26), the action of $w \in W_{\mathcal{E}}$ on $S(\mathfrak{t}^*)$ reduces to the usual action, induced from the action of $W_{\mathcal{E}}$ on $\mathfrak{t}$. Thus the property established above in the case σ_0 central remains valid here: the action of an element $x \in S(\mathfrak{t}^*)$ of degree n on $E^\circ_{y,\sigma_0,0}$ can only raise degrees, and raises them by at most n. Since

$$\mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}})/(\vec{\mathbf{r}}) = S(\mathfrak{t}^*) \otimes \mathbb{C}[W_{\mathcal{E}}]$$

as vector spaces and $\mathbb{C}[W_{\mathcal{E}}]$ preserves the degrees (because it sits in degree 0, see also part a), the degree properties of $E^\circ_{y,\sigma_0,0}$ also hold when we regard it as $\mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}})$-module. In particular, for any $d \in \mathbb{Z}$ the sum of the subspaces in degrees $\geq d$ is a submodule. The action of $\pi_0(\mathrm{Z}_{G^\circ}(\sigma_0, y))$ preserves the degrees, so this remains valid for

$$\mathrm{Hom}_{\pi_0(\mathrm{Z}_{G^\circ}(\sigma_0,y))}(\rho^\circ, E^\circ_{y,\sigma_0,0}) = E^\circ_{y,\sigma_0,0,\rho^\circ}.$$

Combine that with part (a) to obtain the stated form.
(c) With parts (a,b) instead of [AMS2, Lemma 3.9], the proof of [AMS2, Proposition 3.10] still works. It shows that $E^\circ_{y,\sigma_0,0,\rho^\circ}$ has a unique irreducible subquotient isomorphic to $M^\circ_{y,\sigma_0,0,\rho^\circ}$ and that the part of $E^\circ_{y,\sigma_0,0,\rho^\circ}$ in one specific homological degree projects bijectively onto this subquotient. More precisely, the argument [AMS2, (40)–(42)] shows that this subquotient appears as

$$\mathrm{Hom}_{\pi_0(\mathrm{Z}_{G^\circ}(\sigma_0,y))}(\rho^\circ, H_d(\mathcal{P}_y^\circ, \dot{\mathcal{E}})) \subset E^\circ_{y,\sigma_0,0,\rho^\circ},$$

where d is the minimal degree for which this space is nonzero. By part (b),

$$E^{\circ,>d}_{y,\sigma_0,0,\rho^\circ} := \bigoplus\nolimits_{n>d} \mathrm{Hom}_{\pi_0(\mathrm{Z}_{G^\circ}(\sigma_0,y))}\big(\rho^\circ, H_n(\mathcal{P}^\circ_y,\dot{\mathcal{E}})\big)$$

is a $\mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}})$-submodule. Further

$$E^{\circ,d}_{y,\sigma_0,0,\rho^\circ} := E^{\circ}_{y,\sigma_0,0,\rho^\circ} / E^{\circ,>d}_{y,\sigma_0,0,\rho^\circ} \cong \mathrm{Hom}_{\pi_0(\mathrm{Z}_{G^\circ}(\sigma_0,y))}(\rho^\circ, H_d(\mathcal{P}^\circ_y,\dot{\mathcal{E}}))$$

as $\mathbb{C}[W_\mathcal{E}]$-modules. The proof of part (b) can be modified to study the $\mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}})$-module $E^{\circ,d}_{y,\sigma_0,0,\rho^\circ}$. In the case that σ_0 is central, it shows that $E^{\circ,d}_{y,\sigma_0,0,\rho^\circ}$ is a semisimple module, on which the $\mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}})$-action factors through

$$\begin{array}{llll} \mathrm{ev}_{\sigma_0,0}\colon & \mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}})/(\vec{\mathbf{r}}) & \to & \mathbb{C}[W_\mathcal{E}] \\ & fx & \mapsto & f(\sigma_0)x \qquad f \in S(\mathfrak{t}^*), x \in \mathbb{C}[W_\mathcal{E}]. \end{array}$$

When σ_0 is not central, the functor $\mathrm{ind}_{W_\mathcal{E}^{Q^\circ} \ltimes S(\mathfrak{t}^*)}^{W_\mathcal{E} \ltimes S(\mathfrak{t}^*)}$ from (1.26) preserves irreducibility of modules that admit the $S(\mathfrak{t}^* \oplus \mathbb{C})$-character $(\sigma_0, 0)$, because $W_\mathcal{E}^{Q^\circ} = (W_\mathcal{E})_{\sigma_0}$. Hence $\mathrm{ind}_{W_\mathcal{E}^{Q^\circ} \ltimes S(\mathfrak{t}^*)}^{W_\mathcal{E} \ltimes S(\mathfrak{t}^*)}$ preserves the semisimplicity of $E^{Q^\circ}_{y,\sigma_0,0,\rho^\circ}$.

This shows that the distinguished irreducible subquotient of $E^{\circ}_{y,\sigma_0,0,\rho^\circ}$ is a direct summand of $E^{\circ,d}_{y,\sigma_0,0,\rho^\circ}$. Since $E^{\circ,d}_{y,\sigma_0,0,\rho^\circ}$ is a quotient of $E^{\circ}_{y,\sigma_0,0,\rho^\circ}$, so is our distinguished subquotient. $\square$

The next result generalizes [AMS2, Theorem 3.20] to several variables r_j. We define $\mathrm{Irr}_{\vec{r}}(\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}}))$ as the set of equivalence classes of those irreducible representations of $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}})$ on which each $\mathbf{r}_j$ acts as r_j.

Theorem 1.6. *Fix $\vec{r} \in \mathbb{C}^d$. The standard $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}})$-module $E_{y,\sigma,\vec{r},\rho}$ is nonzero if and only if $q\Psi_{\mathrm{Z}_G(\sigma_0)}(y, \rho) = (M, \mathcal{C}^M_v, q\mathcal{E})$ up to G-conjugacy. In that case it has a distinguished irreducible quotient $M_{y,\sigma,\vec{r},\rho}$, which appears with multiplicity one in $E_{y,\sigma,\vec{r},\rho}$.*

The map $M_{y,\sigma,\vec{r},\rho} \longleftrightarrow (\sigma_0, y, \rho)$ sets up a canonical bijection between $\mathrm{Irr}_{\vec{r}}(\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}}))$ and G-conjugacy classes of triples as after Lemma 1.3.

Proof. For $\mathbb{H}(G_j, M_j, \mathcal{E}_j)$ this is [AMS2, Proposition 3.7 and Theorem 3.11], where we use Theorem 1.5 to replace the input from the flawed [AMS2, Lemma 3.9]. With (1.8) and Lemma 1.3 we can generalize that to $\mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}})$. The method to go from there to $\mathbb{H}(G^\circ \mathrm{N}_G(q\mathcal{E}), M, q\mathcal{E}, \vec{\mathbf{r}})$ is exactly the same as in [AMS2, §3.3–3.4] (for $\mathbb{H}(G^\circ, M^\circ, \mathcal{E})$ and $\mathbb{H}(G^\circ \mathrm{N}_G(q\mathcal{E}), M, q\mathcal{E})$). That is, the proof of [AMS2, Theorem 3.20] applies and establishes the theorem for $\mathbb{H}(G^\circ \mathrm{N}_G(q\mathcal{E}), M, q\mathcal{E}, \vec{\mathbf{r}})$. In view of (1.19) we can replace $G^\circ \mathrm{N}_G(q\mathcal{E})$ by G. $\square$

The irreducible module $M_{y,\sigma,\vec{r},\rho}$ has the same central character as the standard module $E_{y,\sigma,\vec{r},\rho}$ of which it is a quotient. It can be found in Proposition 1.4.

The above modules are compatible with parabolic induction, in a suitable sense and under a certain condition. Let $Q \subset G$ be an algebraic subgroup containing M, such that Q° is a Levi subgroup of G°. Let $y, \sigma, \vec{r}, \rho$ be as in Theorem 1.6, with $\sigma, y \in \mathfrak{q} = \mathrm{Lie}(Q)$. By [Ree, §3.2] the natural map

$$\pi_0(\mathrm{Z}_Q(\sigma, y)) = \pi_0(\mathrm{Z}_{Q \cap \mathrm{Z}_G(\sigma_0)}(y)) \to \pi_0(\mathrm{Z}_{\mathrm{Z}_G(\sigma_0)}(y)) = \pi_0(\mathrm{Z}_G(\sigma, y)) \tag{1.27}$$

is injective, so we can consider the left hand side as a subgroup of the right hand side. Let $\rho^Q \in \mathrm{Irr}\big(\pi_0(\mathrm{Z}_Q(\sigma, y))\big)$ be such that $q\Psi_{\mathrm{Z}_Q(\sigma_0)}(y, \rho^Q) = (M, \mathcal{C}_v^M, q\mathcal{E})$. Then $E_{y,\sigma,r,\rho}, M_{y,\sigma,r,\rho}, E^Q_{y,\sigma,r,\rho^Q}$ and M^Q_{y,σ,r,ρ^Q} are defined.

Further, PQ° is a parabolic subgroup of G° with Q° as Levi factor. The unipotent radical $\mathcal{R}_u(PQ^\circ)$ is normalized by Q°, so its Lie algebra $\mathfrak{u}_Q = \mathrm{Lie}(\mathcal{R}_u(PQ^\circ))$ is stable under the adjoint actions of Q° and $\mathfrak{q}$. By (1.4) $\mathfrak{u}_Q$ decomposes as the direct sum of the subspaces $\mathfrak{u}_{Q,j} = \mathfrak{u}_Q \cap \mathfrak{g}_j$. In particular $\mathrm{ad}(y_j)$ acts on $\mathfrak{u}_{Q,j}$. We denote the cokernel of $\mathrm{ad}(y_j) \colon \mathfrak{u}_{Q,j} \to \mathfrak{u}_{Q,j}$ by ${}_y\mathfrak{u}_{Q,j}$. From $[\sigma_j, y_j] = 2r_j y_j$ we see that $\mathrm{ad}(\sigma_j)$ descends to a linear map ${}_y\mathfrak{u}_{Q,j} \to {}_y\mathfrak{u}_{Q,j}$.

Following Lusztig [Lus6, §1.16], we define

$$\begin{array}{llll} \epsilon_{y,j} : & \mathrm{Lie}(M^Q(y)^\circ) & \to & \mathbb{C} \\ & (\sigma, r) & \mapsto & \det(\mathrm{ad}(\sigma_j) - 2r_j : {}_y\mathfrak{u}_{Q,j} \to {}_y\mathfrak{u}_{Q,j}) \end{array}.$$

All parameters for which parabolic induction could behave problematically are zeros of a function $\epsilon_{y,j}$.

Proposition 1.7. *Let $y, \sigma, \vec{r}, \rho$ be as in Theorem 1.6, and assume that $\epsilon_{y,j}(\sigma, r) \neq 0$ for each $j = 1, \ldots, d$.*

(*a*) *There is a natural isomorphism of $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}})$-modules*

$$\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}}) \underset{\mathbb{H}(Q,M,q\mathcal{E},\vec{\mathbf{r}})}{\otimes} E^Q_{y,\sigma,\vec{r},\rho^Q} \cong \bigoplus\nolimits_\rho \mathrm{Hom}_{\pi_0(\mathrm{Z}_Q(\sigma,y))}(\rho^Q, \rho) \otimes E_{y,\sigma,r,\rho},$$

where the sum runs over all $\rho \in \mathrm{Irr}\big(\pi_0(\mathrm{Z}_G(\sigma, y))\big)$ with $q\Psi_{\mathrm{Z}_G(\sigma_0)}(y, \rho) = (M, \mathcal{C}_v^M, q\mathcal{E})$.

(*b*) *For $\vec{r} = \vec{0}$ part (a) contains an isomorphism of $S(\mathfrak{t}^*) \rtimes \mathbb{C}[W_{q\mathcal{E}}, \natural_{q\mathcal{E}}]$-modules*

$$\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}}) \underset{\mathbb{H}(Q,M,q\mathcal{E},\vec{\mathbf{r}})}{\otimes} M^Q_{y,\sigma,\vec{0},\rho^Q} \cong \bigoplus\nolimits_\rho \mathrm{Hom}_{\pi_0(\mathrm{Z}_Q(\sigma,y))}(\rho^Q, \rho) \otimes M_{y,\sigma,\vec{0},\rho}.$$

(*c*) *The multiplicity of $M_{y,\sigma,\vec{r},\rho}$ in $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}}) \underset{\mathbb{H}(Q,M,q\mathcal{E},\vec{\mathbf{r}})}{\otimes} E^Q_{y,\sigma,\vec{r},\rho^Q}$ is $[\rho^Q : \rho]_{\pi_0(\mathrm{Z}_Q(\sigma,y))}$. It already appears that many times as a quotient, via $E^Q_{y,\sigma,\vec{r},\rho^Q} \to M^Q_{y,\sigma,\vec{r},\rho^Q}$. More precisely, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathbb{H}(Q,M,q\mathcal{E},\vec{\mathbf{r}})}(M^Q_{y,\sigma,\vec{r},\rho^Q}, M_{y,\sigma,\vec{r},\rho}) \cong \mathrm{Hom}_{\pi_0(\mathrm{Z}_Q(\sigma,y))}(\rho^Q, \rho)^*.$$

Proof. For twisted graded Hecke algebras with only one parameter $\mathbf{r}$ this is [AMS2, Proposition 3.22], as corrected in [AMS2, Theorem A.1] and in the version with quasi-Levi subgroups as discussed on [AMS2, p. 47]. Using Theorem 1.6, the proof of that result also works in the present setting. □

For an improved parametrization we use the Iwahori–Matsumoto involution, whose definition we will now generalize to $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}})$. Extend the sign representation of the Weyl group $W^\circ_{q\mathcal{E}}$ to a character of $W_{q\mathcal{E}}$ which is trivial on $\mathfrak{R}_{q\mathcal{E}}$. Then we define

$$\mathrm{IM}(N_w) = \mathrm{sign}(w) N_w,\ \mathrm{IM}(\mathbf{r}_j) = \mathbf{r}_j,\ \mathrm{IM}(\xi) = -\xi\ (\xi \in \mathfrak{t}^*). \tag{1.28}$$

Notice that IM is canonically determined by G, P, M and $q\mathcal{E}$, precisely the data that are needed to define $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}})$. Twisting representations with this involution is useful in relation with the properties temperedness and discrete series (with which Theorem 1.6 is incompatible), see [AMS2, §3.5].

Proposition 1.8. (*a*) *Fix* $\vec{r} \in \mathbb{C}^d$. *There exists a canonical bijection*

$$(\sigma_0, y, \rho) \longleftrightarrow \mathrm{IM}^* M_{y, \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right) - \sigma_0, \vec{r}, \rho}$$

between conjugacy classes triples as in Theorem 1.6 and $\mathrm{Irr}_{\vec{r}}(\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}}))$.

(*b*) *Suppose that* $\Re(\vec{r}) \in \mathbb{R}^d_{\geq 0}$. *Then* $\mathrm{IM}^* M_{y, \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right) - \sigma_0, \vec{r}, \rho}$ *is tempered if and only if* $\sigma_0 \in i\mathfrak{t}_{\mathbb{R}} = i\mathbb{R} \otimes_{\mathbb{Z}} X_*(T)$.

(*c*) *Suppose that* $\Re(\vec{r}) \in \mathbb{R}^d_{>0}$. *Then* $\mathrm{IM}^* M_{y, \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right) - \sigma_0, \vec{r}, \rho}$ *is essentially discrete series if and only if y is distinguished in* $\mathfrak{g}$. *In this case* $\sigma_0 \in Z(\mathfrak{g})$.

(*d*) *Let* $\zeta \in \mathfrak{g}^G = \mathrm{Z}(\mathfrak{g})^{G/G^\circ}$. *Then part (a) maps* $(\zeta + \sigma_0, y, \rho)$ *to* $\zeta \otimes \mathrm{IM}^* M_{y, \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right) - \sigma_0, \vec{r}, \rho}$.

(*e*) *Suppose that* $\Re(\vec{r}) \in \mathbb{R}^d_{>0}$ *and that* $\sigma_0 \in i\mathfrak{t}_{\mathbb{R}} + \mathrm{Z}(\mathfrak{g})$. *Then*

$$\mathrm{IM}^* M_{y, \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right) - \sigma_0, \vec{r}, \rho} = \mathrm{IM}^* E_{y, \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right) - \sigma_0, \vec{r}, \rho}.$$

(*f*) *Suppose that* $\sigma_0, \sigma_v + \mathrm{d}\gamma \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathfrak{t}$ *(which can always be arranged by Proposition 1.4.c). Both* $\mathrm{IM}^* M_{y, \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right) - \sigma_0, \vec{r}, \rho}$ *and* $\mathrm{IM}^* E_{y, \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right) - \sigma_0, \vec{r}, \rho}$ *admit the central character* $W_{q\mathcal{E}}\big(\sigma_0 \pm (\vec{r}\sigma_v + \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} -\vec{r} & 0 \\ 0 & \vec{r} \end{smallmatrix}\right)), \vec{r}\big)$.

Proof. Part (a) follows immediately from Theorem 1.6. Parts (b) and (c) are consequences of [AMS2, §3.5], see in particular (82) and (83) therein.
(d) From (1.17) and Lemma 1.3 we see that

$$E_{y, \sigma' - \zeta, \vec{r}, \rho} = -\zeta \otimes E_{y, \sigma', \vec{r}, \rho}$$

whenever both sides are defined. By Theorem 1.6 the analogous equation for $M_{y, \sigma', \vec{r}, \rho}$ holds. Apply this with $\sigma' = \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right) - \sigma_0$ and use that IM^* turns $-\zeta$ into ζ.
(e) Notice that $\sigma_0 - \sigma_z \in i\mathfrak{t}_{\mathbb{R}}$. We may assume that $\mathcal{P}_y^\circ$ is nonempty, so that (1.11) holds. Write $\rho = \tau^* \ltimes \rho^\circ$ as in [AMS2, Lemma 3.13]. By Lemma 1.3 and [Lus6, Theorem 1.21](for the simple factors of G°_{der})

$$\begin{aligned} M^\circ_{y, \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right) - \sigma_0, \vec{r}, \rho^\circ} &= M^{G^\circ_{\mathrm{der}}}_{y, \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right) + (\sigma_z - \sigma_0), \vec{r}, \rho^\circ} \otimes \mathbb{C}_{-\sigma_z} \\ &= E^{G^\circ_{\mathrm{der}}}_{y, \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right) + (\sigma_z - \sigma_0), \vec{r}, \rho^\circ} \otimes \mathbb{C}_{-\sigma_z} = E^\circ_{y, \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right) - \sigma_0, \vec{r}, \rho^\circ}. \end{aligned}$$

By [AMS2, Lemma 3.16], which uses (1.11), we may identify

$$M_{y, \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right) - \sigma_0, \vec{r}, \rho} = \tau \ltimes M^\circ_{y, \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right) - \sigma_0, \vec{r}, \rho^\circ}.$$

Similarly [AMS2, Lemma 3.18] gives an isomorphism of $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}})$-modules

$$E_{y, \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right) - \sigma_0, \vec{r}, \rho} = \tau \ltimes E^\circ_{y, \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right) - \sigma_0, \vec{r}, \rho^\circ}.$$

Applying IM^* to both these modules, we obtain the desired statement.
(f) Since the first module is a quotient of the second, it suffices to consider the latter. From (1.28) we see that the effect of IM^* on central characters is $W_{q\mathcal{E}}(\sigma, \vec{r}) \mapsto W_{q\mathcal{E}}(-\sigma, \vec{r})$. Combine that with Proposition 1.4.e. □

2. Twisted affine Hecke algebras

We would like to push the results of [AMS2] and the previous section to affine Hecke algebras, because these appear more directly in the representation theory of reductive p-adic groups. This can be achieved with Lusztig's reduction theorems [Lus3]. The first reduces to representations with a "real" central character (to be made precise later), and the second reduction theorem relates representations of affine Hecke algebras with representations of graded Hecke algebras.

Our goal is a little more specific though, we want to consider not just one (twisted) graded Hecke algebra, but a family of those, parametrized by a torus. We want to find a (twisted) affine Hecke algebra which contains all members of this family as some kind of specialization. Let us mention here that, although we phrase this section with quasi-Levi subgroups and cuspidal quasi-supports, all the results are equally valid for Levi subgroups and cuspidal supports.

Let G be a possibly disconnected complex reductive group and let $(M, \mathcal{C}_v^M, q\mathcal{E})$ be a cuspidal quasi-support for G. For any $t \in T = Z(M)^\circ$ the reductive group $G_t = \mathrm{Z}_G(t)$ contains M, and we can consider the twisted graded Hecke algebra

$$\mathbb{H}(G_t, M, q\mathcal{E}, \vec{\mathbf{r}}) = \mathbb{H}(\mathfrak{t}, \mathrm{N}_{G_t}(q\mathcal{E})/M, c_t\vec{\mathbf{r}}, \natural_{q\mathcal{E},t}).$$

Here $\vec{\mathbf{r}} = (\mathbf{r}_1, \ldots, \mathbf{r}_d)$ refers to the almost direct factorization of G_t° induced by (1.4). Let us investigate how this family of algebras depends on t. For any $t \in T$, the 2-cocycle $\natural_{q\mathcal{E},t}$ of $\mathrm{N}_{G_t}(q\mathcal{E})/M$ is just the restriction of $\natural_{q\mathcal{E}} \colon W_{q\mathcal{E}}^2 \to \mathbb{C}^\times$. This can be seen from [Lus1, §3] and the proofs of [AMS1, Proposition 4.5 and Lemma 5.4]. More concretely, the perverse sheaves $(\mathrm{pr}_1)_! \dot{q\mathcal{E}}$ and $(\mathrm{pr}_1)_! \dot{q\mathcal{E}}^*$ on $\mathrm{Lie}(G)$ from [AMS2, (90)] and [Lus2, §3.4] extend the perverse sheaves $q\pi_*(\widetilde{q\mathcal{E}})$ and $q\pi_*(\widetilde{q\mathcal{E}}^*)$ on $\mathrm{Lie}(G)_{\mathrm{RS}}$ (see [AMS2, SS2] for the definition) from [AMS1, §5]. The latter naturally contain the corresponding objects $q\pi_{t,*}(\widetilde{q\mathcal{E}})$ and $q\pi_{t,*}(\widetilde{q\mathcal{E}}^*)$ for G_t. We denote the category of G-equivariant perverse sheaves on a G-variety X by $\mathcal{P}_G(X)$. The algebra

$$\mathbb{C}[\mathrm{N}_{G_t}(q\mathcal{E})/M, \natural_{q\mathcal{E},t}] \cong \mathrm{End}_{\mathcal{P}_{G_t}\mathrm{Lie}(G_t)_{\mathrm{RS}}}\big(q\pi_{t,*}(\widetilde{q\mathcal{E}})\big)$$

from [AMS1, Proposition 4.5 and Lemma 5.4] is canonically embedded in

$$\mathbb{C}[W_{q\mathcal{E}}, \natural_{q\mathcal{E}}] \cong \mathrm{End}_{\mathcal{P}_G\mathrm{Lie}(G)_{\mathrm{RS}}}\big(q\pi_*(\widetilde{q\mathcal{E}})\big).$$

We will simply write $W_{q\mathcal{E},t}$ for $\mathrm{N}_{G_t}(q\mathcal{E})/M$, and $\natural_{q\mathcal{E}}$ for $\natural_{q\mathcal{E},t}$.

On the other hand, the parameter function $c_t \colon R(\mathrm{Z}_G(t)^\circ, T)_{\mathrm{red}} \to \mathbb{C}$ could depend on t, we have to specify which t we use for a given root α. Recall that $c_t(\alpha)$ was defined in [Lus2, §2]. For any root $\alpha \in R(G^\circ, T)$:

$$\mathfrak{g}_\alpha \subset \mathrm{Lie}(G_t) \iff \alpha(t) = 1.$$

From [Lus2, Proposition 2.2] we know that $R(G^\circ, T)$ is a root system, so $R(G^\circ, T) \cap \mathbb{R}\alpha \subset \{\alpha, 2\alpha, -\alpha, -2\alpha\}$ for every nondivisible root α.

Proposition 2.1. [Lus2, Propositions 2.8, 2.10 and 2.12]
Let $y \in \mathfrak{m}$ be an element of the nilpotent orbit defined by the cuspidal quasi-support $(M, \mathcal{C}_v^M, q\mathcal{E})$.

(a) *Suppose that $R(G^\circ, T) \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$. Then $c_t(\alpha)$ satisfies*

$$0 = \mathrm{ad}(y)^{c_t(\alpha)-1} \colon \mathfrak{g}_\alpha \to \mathfrak{g}_\alpha \quad \textit{and} \quad 0 \neq \mathrm{ad}(y)^{c_t(\alpha)-2} \colon \mathfrak{g}_\alpha \to \mathfrak{g}_\alpha. \tag{2.1}$$

This condition is independent of t, as long as $\mathfrak{g}_\alpha \subset \mathrm{Lie}(G_t)$. So we can unambiguously write $c(\alpha)$ for $c_t(\alpha)$ in this case. Moreover $c(\alpha) \in \mathbb{N}$ is even.

(b) *Suppose that $R(G^\circ, T) \cap \mathbb{R}\alpha = \{\alpha, 2\alpha, -\alpha, -2\alpha\}$.*

When $\alpha(t) = 1$, $\{\alpha, 2\alpha\} \subset R(Z_G(t)^\circ, T)$. Then $c_t(\alpha)$ is again given by (2.1), and it is odd. We write $c(\alpha) = c_t(\alpha)$ for such a $t \in T$. Furthermore $c_t(2\alpha)$ is given by (2.1) with 2α instead of α, and it equals 2.

When $\alpha(t) = -1$, still $2\alpha \in R(Z_G(t)^\circ, T)$, and $c_t(2\alpha)$ is given by (2.1) with 2α instead of α. It equals 2, and we write $c(2\alpha) = 2$.

With the conventions from Proposition 2.1, c_t is always the restriction of $c\colon R(G^\circ, T) \to \mathbb{C}$ to $R(Z_G(t)^\circ, T)_{\mathrm{red}}$.

Now we construct the algebras that we need.

Proposition 2.2. *Consider the following data:*

- *a root datum $\mathcal{R} = (R, X, R^\vee, Y)$ with a basis of R and an orthogonal decomposition $R = \bigsqcup_{j=1}^d R_j$;*
- *an array of invertible variables $\vec{\mathbf{z}} = (\mathbf{z}_1, \ldots, \mathbf{z}_d)$,*
- *a finite group Γ acting on $\mathcal{R}$, stabilizing each R_j and the set of simple roots;*
- *$W(R) \rtimes \Gamma$-invariant functions $\lambda\colon R_{\mathrm{red}} \to \mathbb{Z}_{\geq 0}$ and $\lambda^*\colon \{\alpha \in R_{\mathrm{red}} : \alpha^\vee \in 2Y\} \to \mathbb{Z}_{\geq 0}$;*
- *a 2-cocycle $\natural\colon \Gamma^2 \to \mathbb{C}^\times$;*
- *for every $\gamma \in \Gamma$ an element $t_\gamma \in \mathrm{Hom}(X, \mathbb{C}^\times)$ such that $\alpha(t_\gamma) = 1$ for all $\alpha \in R$ and such that $\gamma \cdot \theta_x := x(t_\gamma)\theta_{\gamma x}$ defines an action of Γ on $\mathbb{C}[X] = \mathrm{span}\{\theta_x : x \in X\}$ by algebra automorphisms.*

The vector space

$$\mathbb{C}[X] \otimes \mathbb{C}[\vec{\mathbf{z}}, \vec{\mathbf{z}}^{-1}] \otimes \mathbb{C}[W(R)] \otimes \mathbb{C}[\Gamma, \natural]$$

admits a unique algebra structure such that:

(i) *$\mathbb{C}[X], \mathbb{C}[\vec{\mathbf{z}}, \vec{\mathbf{z}}^{-1}]$ and $\mathbb{C}[\Gamma, \natural]$ are embedded as subalgebras.*

(ii) *$\mathbb{C}[\vec{\mathbf{z}}, \vec{\mathbf{z}}^{-1}] = \mathbb{C}[\mathbf{z}_1, \mathbf{z}_1^{-1}, \ldots, \mathbf{z}_d, \mathbf{z}_d^{-1}]$ is central.*

(iii) *The span of $W(R)$ is the Iwahori–Hecke algebra $\mathcal{H}(W(R), \vec{\mathbf{z}}^{2\lambda})$ of $W(R)$ with parameters $\vec{\mathbf{z}}^{2\lambda(\alpha)}$. That is, it has a basis $\{N_w : w \in W(R)\}$ such that*

$$\begin{array}{rcll} N_w N_v & = & N_{wv} & \text{if } \ell(w) + \ell(v) = \ell(wv), \\ (N_{s_\alpha} + \mathbf{z}_j^{-\lambda(\alpha)})(N_{s_\alpha} - \mathbf{z}_j^{\lambda(\alpha)}) & = & 0 & \text{if } \alpha \in R_j \text{ is a simple root.} \end{array}$$

(iv) *For $\gamma \in \Gamma, w \in W(R)$ and $x \in X$:*

$$N_\gamma N_w \theta_x N_\gamma^{-1} = N_{\gamma w \gamma^{-1}} x(t_\gamma) \theta_{\gamma(x)}.$$

(v) *For a simple root $\alpha \in R_j$ and $x \in X$:*

$$\theta_x N_{s_\alpha} - N_{s_\alpha} \theta_{s_\alpha(x)} =$$
$$\begin{cases} \big(\mathbf{z}_j^{\lambda(\alpha)} - \mathbf{z}_j^{-\lambda(\alpha)}\big)(\theta_x - \theta_{s_\alpha(x)})/(\theta_0 - \theta_{-\alpha}) & \alpha^\vee \notin 2Y \\ \big(\mathbf{z}_j^{\lambda(\alpha)} - \mathbf{z}_j^{-\lambda(\alpha)} + \theta_{-\alpha}(\mathbf{z}_j^{\lambda^*(\alpha)} - \mathbf{z}_j^{-\lambda^*(\alpha)})\big)(\theta_x - \theta_{s_\alpha(x)})/(\theta_0 - \theta_{-2\alpha}) & \alpha^\vee \in 2Y \end{cases}.$$

Proof. In the case $\Gamma = 1$, the existence and uniqueness of such an algebra is well-known. It follows for instance from [Lus3, §3], once we identify T_{s_α} from [Lus3] with $\mathbf{z}_j^{\lambda(\alpha)} N_{s_\alpha}$. It is called an affine Hecke algebra and denoted by $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \vec{\mathbf{z}})$.

Consider the linear bijection

$$\begin{array}{cccc} A_\gamma : & \mathcal{H}(\mathcal{R},\lambda,\lambda^*,\vec{\mathbf{z}}) & \to & \mathcal{H}(\mathcal{R},\lambda,\lambda^*,\vec{\mathbf{z}}) \\ & N_w \theta_x & \mapsto & N_{\gamma w \gamma^{-1}} x(t_\gamma) \theta_{\gamma(x)} \end{array} .$$

By assumption $A_\gamma |_{\mathbb{C}[X]}$ is an algebra homomorphism, while $A_\gamma |_{\mathcal{H}(W(R),\vec{\mathbf{z}}^{2\lambda})}$ is an algebra homomorphism because Γ stabilizes λ and the set of simple roots. As $s_\alpha(x)(t_\gamma) = x(t_\gamma)$,

$$A_\gamma \big(\theta_x N_{s_\alpha} - N_{s_\alpha} \theta_{s_\alpha(x)} \big) = x(t_\gamma) \big(\theta_{\gamma x} N_{s_{\gamma\alpha}} - N_{s_{\gamma\alpha}} \theta_{s_{\gamma\alpha}(x)} \big).$$

Similarly one computes, using $\gamma(R_j) = R_j$:

$$A_\gamma \Big(\big(\mathbf{z}_j^{\lambda(\alpha)} - \mathbf{z}_j^{-\lambda(\alpha)} \big) \frac{\theta_x - \theta_{s_\alpha(x)}}{\theta_0 - \theta_{-\alpha}} \Big) = x(t_\gamma) \big(\mathbf{z}_j^{\lambda(\gamma\alpha)} - \mathbf{z}_j^{-\lambda(\gamma\alpha)} \big) \frac{\theta_{\gamma x} - \theta_{s_{\gamma\alpha}(x)}}{\theta_0 - \theta_{-\gamma\alpha}}.$$

An analogous formula holds in the case $\alpha^\vee \in 2Y$ of (v), because Γ stabilizes λ^*. This shows that A_γ respects all the multiplication rules of $\mathcal{H}(\mathcal{R},\lambda,\lambda^*,\vec{\mathbf{z}})$, and hence is an algebra automorphism. The map

$$\Gamma \to \mathrm{Aut}(\mathcal{H}(\mathcal{R},\lambda,\lambda^*,\vec{\mathbf{z}})) : \gamma \mapsto A_\gamma$$

is a group homomorphism, because it so when we restrict to $\mathbb{C}[X]$ or to $\mathcal{H}(W(R),\vec{\mathbf{z}}^{2\lambda})$. Pick a central extension Γ^+ of Γ and a central idempotent $p_\natural$ such that $\mathbb{C}[\Gamma,\natural] \cong p_\natural \mathbb{C}[\Gamma^+]$. Now the same argument as in the proof of [AMS2, Proposition 2.2] shows that the algebra

$$\mathbb{C}[\Gamma,\natural] \ltimes \mathcal{H}(\mathcal{R},\lambda,\lambda^*,\vec{\mathbf{z}}) \cong p_\natural \mathbb{C}[\Gamma^+] \ltimes \mathcal{H}(\mathcal{R},\lambda,\lambda^*,\vec{\mathbf{z}}) \subset \Gamma^+ \ltimes \mathcal{H}(\mathcal{R},\lambda,\lambda^*,\vec{\mathbf{z}}) \tag{2.2}$$

has the required properties. □

Since (2.2) is built from an affine Hecke algebra $\mathcal{H}(\mathcal{R},\lambda,\lambda^*,\vec{\mathbf{z}})$ and a twisted group algebra $\mathbb{C}[\Gamma,\natural_{q\mathcal{E}}]$, we refer to it as a twisted affine Hecke algebra. When $\mathfrak{R}_{q\mathcal{E}} = 1$, specializations of $\mathcal{H}(\mathcal{R},\lambda,\lambda^*,\vec{\mathbf{z}})$ at $\vec{\mathbf{r}} = \vec{r} \in \mathbb{R}^d_{>0}$ figure for example in [Opd1]. In relation with p-adic groups one should think of the variables $\vec{\mathbf{z}}$ as as $(q_j^{1/2})_{j=1}^d$, where q_j is the cardinality of some finite field.

In this section we focus on the following instances on Proposition 2.2. We take the root datum

$$\mathcal{R}(G^\circ,T) = (R(G^\circ,T), X^*(T), R(G^\circ,T)^\vee, X_*(T))$$

with simple roots determined by P and the decomposition of $R(G^\circ,T)$ corresponding to the decomposition (1.4) of $\mathfrak{g}$. Then $W(R) = W^\circ_{q\mathcal{E}}$ and we may identify

$$\mathbb{C}[X] \otimes \mathbb{C}[\vec{\mathbf{z}},\vec{\mathbf{z}}^{-1}] = \mathcal{O}(T \times (\mathbb{C}^\times)^d).$$

We take $\Gamma = \mathfrak{R}_{q\mathcal{E}}$ and $t_\gamma = 1$ for all γ. We define, for $\alpha \in R(G^\circ,T)_{\mathrm{red}}$:

$$\begin{array}{lcll} \lambda(\alpha) & = & c(\alpha)/2 & 2\alpha \notin R(G^\circ,T) \\ \lambda^*(\alpha) & = & c(\alpha)/2 & 2\alpha \notin R(G^\circ,T), \alpha^\vee \in 2X_*(T) \\ \lambda(\alpha) & = & c(\alpha)/2 + c(2\alpha)/4 & 2\alpha \in R(G^\circ,T) \\ \lambda^*(\alpha) & = & c(\alpha)/2 - c(2\alpha)/4 & 2\alpha \in R(G^\circ,T). \end{array} \tag{2.3}$$

By Proposition 2.1 $\lambda(\alpha) \in \mathbb{Z}_{\geq 0}$ in all cases. For $\natural = \natural_{q\mathcal{E}}$ [AMS2, (91)] says that

$$\mathbb{C}[\mathfrak{R}_{q\mathcal{E}},\natural_{q\mathcal{E}}] \cong \mathrm{End}^+_{\mathcal{P}_G \mathrm{Lie}(G)_{\mathrm{RS}}} \big(q\pi_*(\widetilde{q\mathcal{E}}) \big).$$

We denote the algebra constructed in Proposition 2.1, with these extra data, by $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$. When $d = 1$ we simply write $\mathcal{H}(G, M, q\mathcal{E})$. We record that

$$\mathcal{H}(G, M, q\mathcal{E}) = \mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})/(\{\mathbf{z}_i - \mathbf{z}_j : 1 \leq i, j \leq d\}). \tag{2.4}$$

The same argument as for [AMS2, Lemma 2.8] shows that

$$\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}}) = \mathcal{H}(\mathcal{R}(G^\circ, T), \lambda, \lambda^*, \vec{\mathbf{z}}) \rtimes \mathrm{End}^+_{\mathcal{P}_G \mathrm{Lie}(G)_{\mathrm{RS}}}\big(q\pi_*(\widetilde{q\mathcal{E}})\big). \tag{2.5}$$

If we are in one of the cases (1.3), then with this interpretation $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$ depends canonically on $(G, M, q\mathcal{E})$. In general the algebra $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$ is not entirely canonical, since it involves the choice of a decomposition (1.4).

Lemma 2.3. *$\mathcal{O}(T \times (\mathbb{C}^\times)^d)^{W_{q\mathcal{E}}} = \mathcal{O}(T)^{W_{q\mathcal{E}}} \otimes \mathbb{C}[\vec{\mathbf{z}}, \vec{\mathbf{z}}^{-1}]$ is a central subalgebra of $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$. It equals $\mathrm{Z}(\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}}))$ if $W_{q\mathcal{E}}$ acts faithfully on T.*

Proof. The case $W_{q\mathcal{E}} = 1, d = 1$ is [Lus3, Proposition 3.11]. The general case follows readily from that, as observed in [Sol3, §1.2]. □

For $\zeta \in \mathrm{Z}(G) \cap G^\circ$ and $(\pi, V) \in \mathrm{Mod}(\mathcal{H}(G, M, q_{\mathcal{E}}, \vec{\mathbf{z}}))$ we define $(\zeta \otimes \pi, V) \in \mathrm{Mod}(\mathcal{H}(G, M, q_{\mathcal{E}}, \vec{\mathbf{z}}))$ by

$$(\zeta \otimes \pi)(f_1 f_2 N_w) = f_1(\zeta)\pi(f_1 f_2 N_w) \qquad f_1 \in \mathcal{O}(T), f_2 \in \mathbb{C}[\vec{\mathbf{z}}, \vec{\mathbf{z}}^{-1}], w \in W_{q\mathcal{E}}.$$

2.1. Reduction to real central character.

Let $T = T_{\mathrm{un}} \times T_{\mathrm{rs}}$ be the polar decomposition of the complex torus T, in a unitary and a real split part:

$$\begin{aligned} T_{\mathrm{un}} &= \mathrm{Hom}(X^*(T), S^1) = \exp(i\mathfrak{t}_{\mathbb{R}}), \\ T_{\mathrm{rs}} &= \mathrm{Hom}(X^*(T), \mathbb{R}_{>0}) = \exp(\mathfrak{t}_{\mathbb{R}}). \end{aligned} \tag{2.6}$$

We write the polar decomposition of an arbitrary element $t \in T$ as

$$t = (t\,|t|^{-1})\,|t| \in T_{\mathrm{un}} \times T_{\mathrm{rs}}.$$

By Lemma 2.3 every irreducible representation of $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$ admits a $\mathcal{O}(T \times (\mathbb{C}^\times)^d)^{W_{q\mathcal{E}}}$-character, an element of $T/W_{q\mathcal{E}} \times (\mathbb{C}^\times)^d$. We will refer to this as the central character. Following [BaMo, Definition 2.2] we say that a central character $(W_{\mathcal{E}} t, \vec{z})$ is "real" if $\vec{z} \in \mathbb{R}^d_{>0}$ and the unitary part $t\,|t|^{-1}$ is fixed by $W^\circ_{\mathcal{E}}$.

For $t \in T$ we define $\tilde{Z}_G(t)$ to be the subgroup of G generated by $\mathrm{Z}_G(t)$ and the root subgroups for $\alpha \in R(G^\circ, T)$ with $\alpha^\vee \in 2X_*(T)$ and $\alpha(t) = -1$. Thus $R(\tilde{Z}_G(t)^\circ, T)$ consists of the roots $\alpha \in R(G^\circ, T)$ with $s_\alpha(t) = t$. All roots $\alpha \in R(G^\circ, T)$ with $s_\alpha \in \tilde{Z}_G(t)$ have root spaces in $\tilde{Z}_G(t)^\circ$, so $\lambda(\alpha)$ and $\lambda^*(\alpha)$ are the same for G° and for $\tilde{Z}_G(t)$. The analogue of $\mathfrak{R}_{q\mathcal{E}}$ for $\tilde{Z}_G(t)$ is $\mathfrak{R}_{q\mathcal{E},t}$, the stabilizer of $R(\tilde{Z}_G(t)^\circ, T) \cap R(P, T)$ in $W_{q\mathcal{E},t}$.

Our first reduction theorem will relate modules of $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$ and of $\mathcal{H}(\tilde{Z}_G(t), M, q\mathcal{E}, \vec{\mathbf{z}})$. Assuming that every $\mathbf{z}_j$ acts via a positive real number, we end up with representations admitting a real central character. To describe the effect on $\mathcal{O}(T \times (\mathbb{C}^\times)^d)$-weights, we need some preparations. Consider the set

$$W^t_{\mathcal{E}} = \big\{w \in W_{\mathcal{E}} : w\big(R(\tilde{Z}_G(t)^\circ, T) \cap R(P, T)\big) \subset R(P, T)\big\}.$$

Recall that the parabolic subgroup $P \subset G^\circ$ determines a set of simple reflections and a length function on the Weyl group $W^\circ_{q\mathcal{E}} = W_{\mathcal{E}}$. We use this to define two cones in $\mathfrak{t}_\mathbb{R} = X_*(T) \otimes_\mathbb{Z} \mathbb{R}$:

$$\mathfrak{t}_\mathbb{R}^+ := \{x \in \mathfrak{t}_\mathbb{R} : \langle x , \alpha \rangle \geq 0 \; \forall \alpha \in R(P,T)\},$$
$$\mathfrak{t}_\mathbb{R}^- := \big\{ \textstyle\sum_{\alpha \in R(P,T)} x_\alpha \alpha^\vee : x_\alpha \leq 0 \big\}.$$

Lemma 2.4. (*a*) *$W_\mathcal{E}^t$ is the unique set of shortest length representatives for $W_\mathcal{E} / W(\tilde{Z_G}(t)^\circ, T)$ in $W_\mathcal{E}$.*

(*b*) *$\bigcup_{w \in W_\mathcal{E}^t} w^{-1} \mathfrak{t}_\mathbb{R}^+$ equals $\mathfrak{t}_\mathbb{R}^{+,t}$, the analogue of $\mathfrak{t}_\mathbb{R}^+$ for the group $\tilde{Z_G}(t)^\circ$. The same holds for $\mathfrak{t}_\mathbb{R}^{*,+}$.*

(*c*) *$\{x \in \mathfrak{t}_\mathbb{R} : W_\mathcal{E}^t x \subset \mathfrak{t}_\mathbb{R}^-\}$ equals $\mathfrak{t}_\mathbb{R}^{-,t}$, the analogue of $\mathfrak{t}_\mathbb{R}^-$ for $\tilde{Z_G}(t)^\circ$.*

Proof. (a) This well-known when $R(\tilde{Z_G}(t)^\circ, T)$ is parabolic subsystem [Hum, Proposition 1.10.c and §1.6], and the general case is mentioned in [Lus7, proof of Lemma 3.4]. For completeness, we provide a proof.

For any $w \in W_\mathcal{E}$, $w^{-1} W(P,T) \cap R(\tilde{Z_G}(t)^\circ, T)$ is a positive system in $R(\tilde{Z_G}(t)^\circ, T)$. By [Hum, Theorem 1.8] there exists a unique $v \in W(\tilde{Z_G}(t)^\circ, T)$ such that

$$v^{-1} \big(w^{-1} W(P,T) \cap R(\tilde{Z_G}(t)^\circ, T) \big) = W(P,T) \cap R(\tilde{Z_G}(t)^\circ, T).$$

This is also the unique $v \in W(\tilde{Z_G}(t)^\circ, T)$ such that

$$wv \big(R(\tilde{Z_G}(t)^\circ, T) \cap R(P,T) \big) \subset R(P,T).$$

Hence $W_\mathcal{E}^t$ is a set of representatives for $W_\mathcal{E} / W(\tilde{Z_G}(t)^\circ, T)$.

Consider $w \in W_\mathcal{E}$ of minimal length in $w W(\tilde{Z_G}(t)^\circ, T)$. By [Hum, Proposition 5.7], $w(\alpha) \in R(P,T)$ for all $\alpha \in R(\tilde{Z_G}(t)^\circ, T) \cap R(P,T)$, so $w \in W_\mathcal{E}^t$. We deduce that every left coset of $W(\tilde{Z_G}(t)^\circ, T)$ contains a unique element of minimal length, namely its representative in $W_\mathcal{E}^t$.

(b) Suppose that $x \in \mathfrak{t}_\mathbb{R}^+$ and $\alpha \in R(\tilde{Z_G}(t)^\circ, T) \cap R(P,T)$. For all $w \in W_\mathcal{E}^t$ we have $w\alpha \in R(P,T)$, so

$$\langle \alpha , w^{-1} x \rangle = \langle w \alpha , x \rangle \geq 0.$$

Hence $\bigcup_{w \in W_\mathcal{E}^t} w^{-1} \mathfrak{t}_\mathbb{R}^+ \subset \mathfrak{t}_\mathbb{R}^{+,t}$. Let S be a sphere in $\mathfrak{t}_\mathbb{R}$ centred in 0. Then

$$\mathrm{vol}(S) / \mathrm{vol}(S \cap \mathfrak{t}_\mathbb{R}^+) = |W_\mathcal{E}| \quad \text{and} \quad \mathrm{vol}(S) / \mathrm{vol}(S \cap \mathfrak{t}_\mathbb{R}^{+,t}) = |W(\tilde{Z_G}(t)^\circ, T)|.$$

With part (a) it follows that

$$|W_\mathcal{E}^t| \; \mathrm{vol}(S \cap \mathfrak{t}_\mathbb{R}^+) = |W_\mathcal{E}| \; \mathrm{vol}(S \cap \mathfrak{t}_\mathbb{R}^+) / |W(\tilde{Z_G}(t)^\circ, T)| = \mathrm{vol}(S \cap \mathfrak{t}_\mathbb{R}^{+,t}). \tag{2.7}$$

Since $\mathfrak{t}_\mathbb{R}^+$ is a Weyl chamber for $W_\mathcal{E}$, the translates $w \mathfrak{t}_\mathbb{R}^+$ intersect $\mathfrak{t}_\mathbb{R}^+$ only in a set of measure zero. Hence the left hand side of (2.7) is the volume of $S \cap \bigcup_{w \in W_\mathcal{E}^t} w^{-1} \mathfrak{t}_\mathbb{R}^+$. As $\bigcup_{w \in W_\mathcal{E}^t} w^{-1} \mathfrak{t}_\mathbb{R}^+ \subset \mathfrak{t}_\mathbb{R}^{+,t}$ and both are cones defined by linear equations coming from roots, the equality (2.7) shows that they coincide.

The same reasoning applies to $\mathfrak{t}_\mathbb{R}^*$ and the dual root systems.

(c) The definition of $W_\mathcal{E}^t$ entails $W_\mathcal{E}^t \mathfrak{t}_\mathbb{R}^{-,t} \subset \mathfrak{t}_\mathbb{R}^-$. Conversely, suppose that $x \in \mathfrak{t}_\mathbb{R}$ and that $W_\mathcal{E}^t x \subset \mathfrak{t}_\mathbb{R}^-$. For every $w \in W_\mathcal{E}^t$ and every $\lambda \in \mathfrak{t}_\mathbb{R}^{*,+}$:

$$\langle x , w^{-1} \lambda \rangle = \langle w x , \lambda \rangle \leq 0.$$

In view of part (b) for $\mathfrak{t}_\mathbb{R}^{*,+}$, this means that $x \in \mathfrak{t}_\mathbb{R}^{-,t}$. □

Theorem 2.5. *Let $t \in T_{\mathrm{un}}$.*

(a) *There is a canonical equivalence between the following categories:*
- *finite dimensional* $\mathcal{H}(\tilde{Z_G}(t), M, q\mathcal{E}, \vec{\mathbf{z}})$*-modules with* $\mathcal{O}(T \times (\mathbb{C}^\times)^d)$*-weights in* $tT_{\mathrm{rs}} \times \mathbb{R}^d_{>0}$*;*
- *finite dimensional* $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$*-modules with* $\mathcal{O}(T \times (\mathbb{C}^\times)^d)$*-weights in* $W_{q\mathcal{E}} tT_{\mathrm{rs}} \times \mathbb{R}^d_{>0}$.

It is given by localization of the centre and induction, and we denote it (suggestively) by $\mathrm{ind}_{\mathcal{H}(\tilde{Z_G}(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})}$.

(b) *The above equivalences are compatible with parabolic induction, in the following sense. Let* $Q \subset G$ *be an algebraic subgroup such that* $Q \cap G^\circ$ *is a Levi subgroup of* G° *and* $Q \supset M$*. Then*

$$\mathrm{ind}_{\mathcal{H}(\tilde{Z_G}(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})} \circ \mathrm{ind}_{\mathcal{H}(\tilde{Z_Q}(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(\tilde{Z_G}(t),M,q\mathcal{E},\vec{\mathbf{z}})} = \mathrm{ind}_{\mathcal{H}(Q,M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})} \circ \mathrm{ind}_{\mathcal{H}(\tilde{Z_Q}(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(Q,M,q\mathcal{E},\vec{\mathbf{z}})}.$$

(c) *The set of* $\mathcal{O}(T \times (\mathbb{C}^\times)^d)$*-weights of* $\mathrm{ind}_{\mathcal{H}(\tilde{Z_G}(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})}(V)$ *is*

$$\big\{ (wt', \vec{z}) : w \in \mathfrak{R}_{q\mathcal{E}} W_{\mathcal{E}}^{\circ,t}, (t',\vec{z}) \text{ is a } \mathcal{O}(T \times (\mathbb{C}^\times)^d)\text{-weight of } V \big\}.$$

(d) *Parts (a)–(c) hold more generally for any twisted affine Hecke algebra* $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \vec{\mathbf{z}}) \rtimes \mathbb{C}[\Gamma, \natural]$ *as in Proposition 2.2. Then the algebra associated to* $t \in \mathrm{Hom}(X, S^1)$ *is* $\mathcal{H}(R_t, X, R_t^\vee, Y, \lambda, \lambda^*, \vec{\mathbf{z}}) \rtimes \mathbb{C}[\Gamma_t, \natural]$*, where* $R_t = \{\alpha \in R : s_\alpha(t) = t\}$ *and* $(W(R)\Gamma)_t = W(R_t) \rtimes \Gamma_t$.

Proof. (a) The case $d = 1, \natural = 1$ was proven in [Sol3, Theorem 2.1.2], building upon [Lus3, Theorem 8.6] when $\mathfrak{R}_{q\mathcal{E}} = 1$.

Let $\mathfrak{R}_{q\mathcal{E}}^+ \to \mathfrak{R}_{q\mathcal{E}}$ be a central extension as in (2.2). Extend it trivially to a central extension $\mathfrak{R}_{q\mathcal{E}}^+ W_{q\mathcal{E}}^\circ \to W_{q\mathcal{E}}$ and let $\mathfrak{R}_{q\mathcal{E},t}^+$ be the inverse image of $\mathfrak{R}_{q\mathcal{E},t} \subset W_{q\mathcal{E},t}$ in $\mathfrak{R}_{q\mathcal{E}}^+ W_{q\mathcal{E}}^\circ$. Then

$$\begin{aligned} \mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}}) &= \mathcal{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes p_\natural \mathbb{C}[\mathfrak{R}_{q\mathcal{E}}^+], \\ \mathcal{H}(\tilde{Z_G}(t), M, q\mathcal{E}, \vec{\mathbf{z}}) &= \mathcal{H}(\tilde{Z_{G^\circ}}(t), M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes p_\natural \mathbb{C}[\mathfrak{R}_{q\mathcal{E},t}^+]. \end{aligned} \tag{2.8}$$

As $p_\natural \in \mathbb{C}[\ker(\mathfrak{R}_{q\mathcal{E}}^+ \to \mathfrak{R}_{q\mathcal{E}})]$ is a central idempotent, we may just as well establish the analogous result for the algebras

$$\mathcal{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes \mathfrak{R}_{q\mathcal{E}}^+ \quad \text{and} \quad \mathcal{H}(\tilde{Z_{G^\circ}}(t), M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes \mathfrak{R}_{q\mathcal{E},t}^+. \tag{2.9}$$

Since we are dealing with finite dimensional representations only, we can decompose them according to the (generalized) weights for the action of the centre. Fix $(x, \vec{z}) \in T_{\mathrm{rs}} \times \mathbb{R}^d_{>0}$. Denote the category of finite dimensional A-modules with weights in U by $\mathrm{Mod}_{f,U}(A)$. We compare the categories

$$\begin{aligned} &\mathrm{Mod}_{f, W_{q\mathcal{E},t} tx \times \{\vec{z}\}} \big(\mathcal{H}(\tilde{Z_{G^\circ}}(t), M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes \mathfrak{R}_{q\mathcal{E},t}^+ \big), \\ &\mathrm{Mod}_{f, W_{q\mathcal{E},t} tx \times \{\vec{z}\}} \big(\mathcal{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes \mathfrak{R}_{q\mathcal{E}}^+ \big). \end{aligned} \tag{2.10}$$

The most appropriate technique to handle the general case is analytic localization, as in [Opd1, §4] (but there with fixed parameters $z_1, \ldots, z_d$). For an open subset $U \subset T \times (\mathbb{C}^\times)^d$, let $C^{\mathrm{an}}(U)$ and $C^{\mathrm{me}}(U)$ denote the algebras of complex analytic, respectively meromorphic, functions on U. We assume that U is $W_{q\mathcal{E}}$-stable. The restriction map $\mathcal{O}(T \times (\mathbb{C}^\times)^d) \to C^{\mathrm{an}}(U)$ is injective because U is Zariski-dense, and we can form the algebras

$$\mathcal{H}^{an/me}(U) := C^{an/me}(U)^{W_{q\mathcal{E}}} \underset{\mathcal{O}(T\times(\mathbb{C}^\times)^d)^{W_{q\mathcal{E}}}}{\otimes} \mathcal{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes \mathfrak{R}^+_{q\mathcal{E}}. \tag{2.11}$$

As observed in [Opd1, Proposition 4.3], the finite dimensional modules of $\mathcal{H}^{\mathrm{an}}(U)$ can be identified with the finite dimensional modules of $\mathcal{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes \mathfrak{R}^+_{q\mathcal{E}}$ with $\mathcal{O}(T \times (\mathbb{C}^\times)^d)$-weights in U.

In [Sol3, Conditions 2.1] it is described how one can find an open neighborhood $U_0 \subset T \times (\mathbb{C}^\times)^d$ of $(t', \vec{z})$, which is so small that localization to U_0 is more or less equivalent to localization at $(t', \vec{z})$. We take $U = W_{q\mathcal{E}} U_0$ and $\tilde{U} = W_{q\mathcal{E},t} U_0$. By Lusztig's first reduction theorem [Lus3], in the version [Sol3, Theorem 2.1.2], there is an embedding of

$$\mathcal{H}^{\mathrm{an}}_t(\tilde{U}) := C^{\mathrm{an}}(\tilde{U})^{W_{q\mathcal{E},t}} \underset{\mathcal{O}(T\times(\mathbb{C}^\times)^d)^{W_{q\mathcal{E},t}}}{\otimes} \mathcal{H}(\tilde{\mathrm{Z}}_{G^\circ}(t), M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes \mathfrak{R}^+_{q\mathcal{E},t}$$

in $\mathcal{H}^{\mathrm{an}}(U)$, which moreover is a Morita equivalence. The canonicity of this construction is hardly documented in the literature, so we address that here. Let $\mathbb{C}(T)$ be the quotient field of $\mathcal{O}(T)$. By [Lus3, §5], the identity on $\mathcal{O}(T \times (\mathbb{C}^\times)^d)$ extends to an algebra isomorphism

$$\big(\mathbb{C}(T) \rtimes W^\circ_{q\mathcal{E}}\big) \otimes \mathbb{C}[\vec{\mathbf{z}}, \vec{\mathbf{z}}^{-1}] \xrightarrow{\sim} \mathcal{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \otimes_{\mathcal{O}(T)^{W^\circ_{q\mathcal{E}}}} \mathbb{C}(T)^{W^\circ_{q\mathcal{E}}}. \tag{2.12}$$

The image of $w \in W^\circ_{q\mathcal{E}}$ under this isomorphism is denoted τ_w. For a simple root $\alpha \in R(G_j T, T)$, [Lus3, §5] gives the explicit formula

$$\tau_{s_\alpha} + 1 = (\mathbf{z}_j^{\lambda(\alpha)} N_{s_\alpha} + 1) \frac{\theta_\alpha - 1}{\theta_\alpha \mathbf{z}_j^{\lambda(\alpha)+\lambda^*(\alpha)} - 1} \, \frac{\theta_\alpha + 1}{\theta_\alpha \mathbf{z}_j^{\lambda(\alpha)-\lambda^*(\alpha)} + 1}. \tag{2.13}$$

where $\lambda^*(\alpha)$ is interpreted as $\lambda(\alpha)$ if $\alpha^\vee \notin 2X_*(T)$. As observed in [Lus3, Proposition 3.9], the multiplication relation (v) from Proposition 2.2 can be rewritten as

$$f(\tau_{s_\alpha} + 1) - (\tau_{s_\alpha} + 1) s_\alpha(f) = f - s_\alpha(f) \qquad f \in \mathcal{O}(T).$$

Thus $\tau_{s_\alpha} + 1$ arises directly from the multiplication rules in $\mathcal{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{z}})$, and in that sense it is canonically associated to α. Since $w \mapsto \tau_w$ is multiplicative and $W^\circ_{q\mathcal{E}}$ is generated by simple reflections, this entails that all the τ_w are canonical.

The isomorphism (2.12) can be extended with $\rtimes \mathfrak{R}^+_{q\mathcal{E}}$ on both sides, and then we put $\tau_{w\gamma} = \tau_w N_\gamma$ for $w \in W^\circ_{q\mathcal{E}}$ and $\gamma \in \mathfrak{R}^+_{q\mathcal{E}}$. All this can also be done for $\mathcal{H}(\tilde{\mathrm{Z}}_G(t)^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes \mathcal{R}^+_{q\mathcal{E},t}$, which involves analogous elements $\tau_{w,t}$ for $w \in W_{q\mathcal{E},t}$. Then (2.12) shows that

$$\mathcal{H}^{\mathrm{me}}_t(U) \to \mathcal{H}^{\mathrm{me}}(U) : f\tau_{w,t} \mapsto f\tau_w \qquad f \in C^{\mathrm{me}}(U), w \in W_{q\mathcal{E},t} \tag{2.14}$$

is an injective algebra homomorphism. It is canonical because the elements $\tau_{w,t}$ and τ_w are so. It is shown in [Sol3, Theorem 2.1.2] that (2.14) restricts to the aforementioned embedding $\mathcal{H}^{\mathrm{an}}_t(U) \to \mathcal{H}^{\mathrm{an}}(U)$, which is therefore canonical.

It follows that the composed functor

$$\begin{aligned} \mathrm{ind}_{\mathcal{H}(\tilde{\mathrm{Z}}_{G^\circ}(t), M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes \mathfrak{R}^+_{q\mathcal{E},t}}^{\mathcal{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes \mathfrak{R}^+_{q\mathcal{E}}} : \mathrm{Mod}_{f,\tilde{U}}\big(\mathcal{H}(\tilde{\mathrm{Z}}_{G^\circ}(t), M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes \mathfrak{R}^+_{q\mathcal{E},t}\big) \to \\ \mathrm{Mod}_f\big(\mathcal{H}^{\mathrm{an}}_t(\tilde{U})\big) \to \mathrm{Mod}_f\big(\mathcal{H}^{\mathrm{an}}(U)\big) \to \mathrm{Mod}_{f,U}\big(\mathcal{H}(G^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes \mathfrak{R}^+_{q\mathcal{E}}\big) \end{aligned}$$

is a canonical equivalence of categories. We specialize this at $W_{q\mathcal{E},t} t t' \times \{\vec{z}\} \subset U$ and we restrict to modules on which $p_\natural$ acts as the identity. Via (2.10) and (2.8) this gives the required equivalence of categories $\mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})}$.
(b) We just showed that the above functor is really induction between localizations of the indicated algebras. Similar remarks apply to the functor $\mathrm{ind}_{\mathcal{H}(Q,M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})}$. Thus the acclaimed compatibility with parabolic induction is just an instance of the transitivity of induction.
(c) Lemma 2.4.a and the constructions in [Sol3, §2.1] entail that

$$\mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})}(V) \cong \mathbb{C}[\mathfrak{R}_{q\mathcal{E}} W_{\mathcal{E}}^t, \natural_{q\mathcal{E}}] \underset{\mathbb{C}[\mathfrak{R}_{q\mathcal{E},t}, \natural_{q\mathcal{E}}]}{\otimes} V \tag{2.15}$$

as $\mathcal{O}(T \times (\mathbb{C}^\times)^d)$-modules. Notice that the group $\mathfrak{R}_{q\mathcal{E},t}$ acts from the right on $\mathfrak{R}_{q\mathcal{E}} W_{\mathcal{E}}^t$, because it stabilizes $R(\tilde{Z}_G(t)^\circ, T) \cap R(P, T)$. Since

$$\mathcal{H}(\tilde{Z}_G(t), M, q\mathcal{E}, \vec{\mathbf{z}}) \cong \mathcal{H}(\tilde{Z}_G(t)^\circ M, M, q\mathcal{E}, \vec{\mathbf{z}}) \rtimes \mathbb{C}[\mathfrak{R}_{q\mathcal{E},t}, \natural_{q\mathcal{E}}],$$

the $\mathcal{O}(T \times (\mathbb{C}^\times)^d)$-weights of V come in full $\mathfrak{R}_{q\mathcal{E},t}$-orbits. It was observed in the proof of [Opd1, Proposition 4.20] that the $\mathcal{O}(T \times \mathbb{C}^\times)^d)$-weights of $\mathbb{C}w \otimes V$ ($w \in W_{\mathcal{E}}^{\circ,t}$) are precisely $(wt', \vec{z})$ with $(t', \vec{z})$ a $\mathcal{O}(T \times (\mathbb{C}^\times)^d)$-weight of V. Multiplication by N_γ ($\gamma \in \mathfrak{R}_{q\mathcal{E}}$) just changes a weight $(t', \vec{z})$ to $(\gamma t', \vec{z})$. These observations and (2.15) prove that the $\mathcal{O}(T \times (\mathbb{C}^\times)^d)$-weights of $\mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})}(V)$ are as stated.
(d) In the arguments for parts (a)–(c), the specific shapes of $\lambda, \lambda^*, \natural$ and $\mathfrak{R}_{q\mathcal{E}}$ from G do not play a role. Further, all the arguments from [Sol3, §2.1] work just as well when $\mathfrak{R}_{q\mathcal{E}}$ does not fix $1 \in T$ but $\mathfrak{R}_{q\mathcal{E}} 1 \subset T$ consists of elements in the kernel of all $\alpha \in R(G^\circ, T)$. Therefore the entire proof goes through in the generality of Proposition 2.2. □

As a consequence of Theorem 2.5, the study of irreducible $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$-modules on which each $\mathbf{z}_j$ acts by a positive number can be reduced to the study of irreducible $\mathcal{H}(\tilde{Z}_G(t), M, q\mathcal{E}, \vec{\mathbf{z}})$-modules with a central character in $tT_{\mathrm{rs}}/W_{q\mathcal{E},t} \times \mathbb{R}_{>0}^d$. The advantage is that the entire group $W_{q\mathcal{E},t} = W(\tilde{Z}_G(t)^\circ, T) \rtimes \mathfrak{R}_{q\mathcal{E},t}$ fixes t, which implies that we can continue the analysis in the same way as for central characters in $T_{\mathrm{rs}}/W_{q\mathcal{E},t} \times \mathbb{R}_{>0}^d$.

In our reduction process we would like to preserve the analytic properties from [AMS2, §3.5]. Just as in [AMS2, (79)], we can define $\mathcal{O}(T)$-weights for modules of affine Hecke algebras or extended versions such as $\mathcal{H}(\tilde{Z}_G(t), M, q\mathcal{E}, \vec{\mathbf{z}})$. We denote the set of $\mathcal{O}(T)$-weights of a module V for such an algebra by $\mathrm{Wt}(V)$. We can apply the polar decomposition (2.6) to it, which gives a set $|\mathrm{Wt}(V)| \subset T_{\mathrm{rs}}$.

Let us recall the definitions of temperedness and discrete series from [Opd1, §2.7].

Definition 2.6. Let V be a finite dimensional $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$-module. We say that V is tempered (respectively anti-tempered) if $|\mathrm{Wt}(V)| \subset \exp(\mathfrak{t}_{\mathbb{R}}^-)$, respectively $\subset \exp(-\mathfrak{t}_{\mathbb{R}}^-)$.

Let $\mathfrak{t}_{\mathbb{R}}^{--}$ be the interior of $\mathfrak{t}_{\mathbb{R}}^-$ in $\mathfrak{t}_{\mathbb{R}}$. We call V discrete series (resp. anti-discrete series) if $|\mathrm{Wt}(V)| \subset \exp(\mathfrak{t}_{\mathbb{R}}^{--})$, respectively $\subset \exp(-\mathfrak{t}_{\mathbb{R}}^{--})$. The module V is essentially discrete series if its restriction to $\mathcal{H}(G/\mathrm{Z}(G^\circ)^\circ, M/\mathrm{Z}(G^\circ)^\circ, q\mathcal{E}, \vec{\mathbf{z}})$ is discrete series, or equivalently if $|\mathrm{Wt}(V)| \subset \exp(\mathrm{Z}(\mathfrak{g}) \oplus \mathfrak{t}_{\mathbb{R}}^{--})$.

The next result fills a gap in [Sol3, Theorem 2.3.1], where it was used between the lines. Similar results, for G°_{der} only and with somewhat different notions of temperedness and discrete series, were proven in [Lus7, Lemmas 3.4 and 3.5].

Proposition 2.7. *The equivalence from Theorem 2.5.a, and its inverse, preserve:*
(*a*) *(anti-)temperedness,*
(*b*) *the discrete series.*
(*c*) *The $\mathcal{H}(\tilde{Z_G}(t), M, q\mathcal{E}, \vec{\mathbf{z}})$-module $\mathrm{ind}_{\mathcal{H}(\tilde{Z_G}(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})}(V)$ is essentially discrete series if and only if V is essentially discrete series and $R(\tilde{Z_G}(t)^\circ, T)$ has full rank in $R(G^\circ, T)$.*

Remark 2.8. The extra condition for essentially discrete series representations is necessary, for the centre of $\tilde{Z_G}(t)^\circ$ can be of higher dimension than that of G°.

Proof. Let V be a finite dimensional $\mathcal{H}(\tilde{Z_G}(t), M, q\mathcal{E}, \vec{\mathbf{z}})$-module with $\mathcal{O}(T \times (\mathbb{C}^\times)^d)$-weights in $tT_{\mathrm{rs}} \times \mathbb{R}^d_{>0}$.
(a) The $\mathcal{O}(T)$-weights of $\mathrm{ind}_{\mathcal{H}(\tilde{Z_G}(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})}(V)$ were given in Theorem 2.5.c. As $\log = \exp^{-1} : T_{\mathrm{rs}} \to \mathfrak{t}_\mathbb{R}$ is $W_{q\mathcal{E}}$-equivariant, it entails that

$$\log \big|\mathrm{Wt}\big(\mathrm{ind}_{\mathcal{H}(\tilde{Z_G}(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})} V\big)\big| = \mathfrak{R}_{q\mathcal{E}} W^t_\mathcal{E} \log |\mathrm{Wt}(V)|.$$

Recall from Lemma 2.4.c that

$$\mathfrak{t}_\mathbb{R}^{-,t} = \{x \in \mathfrak{t}_\mathbb{R} : W^t_\mathcal{E} x \subset \mathfrak{t}_\mathbb{R}^-\} = \{x \in \mathfrak{t}_\mathbb{R} : \mathfrak{R}_{q\mathcal{E}} W^t_\mathcal{E} x \subset \mathfrak{t}_\mathbb{R}^-\}.$$

Comparing these with the definition of (anti-)temperedness for G and for $\tilde{Z_G}(t)$, we see that V is (anti-)tempered if and only if $\mathrm{ind}_{\mathcal{H}(\tilde{Z_G}(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})}(V)$ is so.
(b) We have to assume that $Z(G^\circ)$ is finite, for otherwise $\exp(\mathfrak{t}_\mathbb{R}^{--})$ is empty and there are no discrete series representations on any side of the equivalences.

Suppose that V is discrete series. Then $\tilde{Z_G}(t)^\circ$ is semisimple, so $R(\tilde{Z_G}(t)^\circ, T)$ is of full rank in $R(G^\circ, T)$. This implies that $\mathfrak{t}_\mathbb{R}^{--,t}$ is an open subset of $\mathfrak{t}_\mathbb{R}^{--}$. The same argument as for part (a) shows that $\mathrm{ind}_{\mathcal{H}(\tilde{Z_G}(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})}(V)$ is discrete series.

Conversely, suppose that $\mathrm{ind}_{\mathcal{H}(\tilde{Z_G}(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})}(V)$ is discrete series. It is tempered, so V is tempered and $|\mathrm{Wt}(V)| \subset \exp(\mathfrak{t}_\mathbb{R}^{-,t})$. Assume that $\tilde{Z_G}(t)^\circ$ is not semisimple. Then

$$\mathfrak{t}_Z := \mathrm{Lie}\big(Z(\tilde{Z_G}(t)^\circ)\big) = \bigcap\nolimits_{\alpha \in R(\tilde{Z_G}(t)^\circ, T)} \ker \alpha$$

has positive dimension. In particular $\mathfrak{t}_Z^*$ contains nonzero elements $\lambda \in \mathfrak{t}_\mathbb{R}^{*,+}$, for example the sum of the fundamental weights for simple roots not in $\mathbb{R}R(\tilde{Z_G}(t)^\circ, T)$. Let $t' \in T$ be any weight of V. Then $\log |t'| \in \mathfrak{t}_\mathbb{R}^{-,t} \subset \mathrm{Lie}(\tilde{Z_G}(t)^\circ_{\mathrm{der}})$. Hence $\langle \log |t'|, \lambda \rangle = 0$, which means that $\log |t'| \in \mathfrak{t}_\mathbb{R}^- \setminus \mathfrak{t}_\mathbb{R}^{--}$. But t' is also a weight of $\mathrm{ind}_{\mathcal{H}(\tilde{Z_G}(t),M,q\mathcal{E})}^{\mathcal{H}(G,M,q\mathcal{E})}(V)$, and that is a discrete series representation, so $\log |t'| \in \mathfrak{t}_\mathbb{R}^{--}$. This contractiction shows that $\tilde{Z_G}(t)^\circ$ is semisimple.

Suppose now that $\log |t'|$ does not lie in the interior of $\mathfrak{t}_\mathbb{R}^{-,t}$. Then it is orthogonal to a nonzero element λ' in the boundary of $\mathfrak{t}_\mathbb{R}^{*,+,t}$. By Lemma 2.4.b we can choose a $w \in W^t_\mathcal{E}$ such that $w\lambda' \in \mathfrak{t}_\mathbb{R}^{*,+}$. Theorem 2.5.c wt' is a weight of $\mathrm{ind}_{\mathcal{H}(\tilde{Z_G}(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})}(V)$, and it satisfies

$$\langle \log |wt'|, w\lambda' \rangle = \langle \log |t'|, \lambda' \rangle = 0.$$

This shows that $\log |wt'| \notin \mathfrak{t}_{\mathbb{R}}^{--}$, which contradicts that $\mathrm{ind}_{\mathcal{H}(\tilde{Z_G}(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})}(V)$ is discrete series. Therefore $\log |t'|$ belongs to $\mathfrak{t}_{\mathbb{R}}^{--,t}$. As t' was an arbitrary weight of V, this proves that V is discrete series.
(c) Suppose that $\mathrm{ind}_{\mathcal{H}(\tilde{Z_G}(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})}(V)$ is essentially discrete series. Its restriction to $\mathcal{H}(G/\mathrm{Z}(G^\circ)^\circ, M/\mathrm{Z}(G^\circ)^\circ, q\mathcal{E}, \vec{\mathbf{z}})$ is discrete series, so by what we have just proven V is discrete series as a module for $\mathcal{H}(\widetilde{\mathrm{Z}_{G/\mathrm{Z}(G^\circ)^\circ}}(t), M/\mathrm{Z}(G^\circ)^\circ, q\mathcal{E}, \vec{\mathbf{z}})$, and $\widetilde{\mathrm{Z}_{G/\mathrm{Z}(G^\circ)^\circ}}(t)^\circ$ is semisimple. Then $R(\tilde{Z_G}(t)^\circ, T)$ has full rank in $R(G^\circ, T)$ and the restriction of V to the smaller algebra $\mathcal{H}(\tilde{Z_G}(t)^\circ/\mathrm{Z}(G^\circ)^\circ, M/\mathrm{Z}(G^\circ)^\circ, q\mathcal{E}, \vec{\mathbf{z}})$ is also discrete series, so V is essentially discrete series.

Conversely, suppose that V is essentially discrete series and that $R(\tilde{Z_G}(t)^\circ, T)$ has full rank in $R(G^\circ, T)$. The second assumption implies that $\mathrm{Z}(G^\circ)^\circ$ is also the connected centre of $\tilde{Z_G}(t)^\circ$. The same argument as in the tempered and the discrete series case shows that

$$\big|\mathrm{Wt}\big(\mathrm{ind}_{\mathcal{H}(\tilde{Z_G}(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})} V\big)\big| \subset \exp(\mathfrak{t}_{\mathbb{R}}^{--} \oplus \mathrm{Z}(\mathfrak{g})).$$

This means that $\mathrm{ind}_{\mathcal{H}(\tilde{Z_G}(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})}(V)$ is essentially discrete series. $\square$

Suppose that $t' \in W_{q\mathcal{E}} t$. Then we can apply Theorem 2.5.a also with t' instead of t, and that should give essentially the same equivalence of categories. We check this in a slightly more general setting, which covers all $t' \in T \cap \mathrm{Ad}(G)t$. (Recall that for $g, h \in G$ we write $\mathrm{Ad}(g)(h) = ghg^{-1}$.) By [Lus4, §8.13.b] and Proposition 1.4.a

$$T \cap \mathrm{Ad}(G)t \text{ equals } T \cap \mathrm{Ad}(\mathrm{N}_G(T))t \supset W_{q\mathcal{E}} t. \tag{2.16}$$

Let $g \in \mathrm{N}_G(M) = \mathrm{N}_G(T)$, with image $\bar{g}$ in $\mathrm{N}_G(M)/M$. Conjugation with g yields an algebra isomorphism

$$\begin{aligned} &\mathrm{Ad}(g) \colon \mathcal{H}(\tilde{Z_G}(t), M, q\mathcal{E}, \vec{\mathbf{z}}) \to \mathcal{H}(\tilde{Z_G}(gtg^{-1}), M, \mathrm{Ad}(g^{-1})^* q\mathcal{E}, \vec{\mathbf{z}}), \\ &\mathrm{Ad}(g)(N_w) = N_{\bar{g}w\bar{g}^{-1}}, \quad \mathrm{Ad}(g)\theta_x = \theta_{x \circ \mathrm{Ad}(g^{-1})} = \theta_{\bar{g}x}, \quad \mathrm{Ad}(g)\mathbf{z}_j = \mathbf{z}_j, \end{aligned} \tag{2.17}$$

where $w \in W_{q\mathcal{E}}$ and $x \in X^*(T)$. Notice that this depends only on g through its class in $\mathrm{N}_G(M)/M$.

Lemma 2.9. *Let $t \in T_{\mathrm{un}}$ and $g \in \mathrm{N}_G(M)$. Then*

$$\mathrm{ind}_{\mathcal{H}(\tilde{Z_G}(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})} = \mathrm{Ad}(g)^* \circ \mathrm{ind}_{\mathcal{H}(\tilde{Z_G}(gtg^{-1}),M,\mathrm{Ad}(g^{-1})^* q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,\mathrm{Ad}(g^{-1})^* q\mathcal{E},\vec{\mathbf{z}})} \circ \mathrm{Ad}(g^{-1})^*$$

as functors between the appropriate categories of modules of these algebras (as specified in Theorem 2.5).

Remark 2.10. This result was used, but not proven, in [Lus5, §4.9 and §5.20] and [Sol3, Theorem 2.3.1].

Proof. Our argument for Theorem 2.5.a, with (2.2), shows how several relevant results can be extended from $\mathcal{H}(G^\circ M, M, q\mathcal{E}, \vec{\mathbf{z}})$ to $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$. This justifies the below use of some results from [Lus3], which were formulated only for $\mathcal{H}(G^\circ M, M, q\mathcal{E})$.

Let (π, V) be a finite dimensional $\mathcal{H}(G, M, q\mathcal{E})$-module with $\mathcal{O}(T \times \mathbb{C}^\times)$-weights in $W_{q\mathcal{E}} t T_{\mathrm{rs}} \times \mathbb{R}_{>0}$. In [Lus3, §8] V is decomposed canonically as $\bigoplus_{t' \in W_{q\mathcal{E}} t} V_{t' T_{\mathrm{rs}}}$, where

$V_{t'T_{\rm rs}}$ is the sum of all generalized $\mathcal{O}(T)$-weight spaces with weights in $t'T_{\rm rs}$. Then $V_{t'T_{\rm rs}}$ is a module for $\mathcal{H}(\tilde{Z}_G(t'), M, q\mathcal{E})$ and

$$V = \mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(t'),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})} (V_{t'T_{\rm rs}}). \tag{2.18}$$

Assume that $g \in \mathrm{N}_G(M, q\mathcal{E})$, so $\bar{g} \in W_{q\mathcal{E}}$. Then $V_{tT_{\rm rs}}$ and $V_{gtg^{-1}T_{\rm rs}}$ are related via multiplication with an element $\tau_{\bar{g}}$, which lives in a suitable localization of $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$ [Lus3, §5]. We can rewrite the right hand side of (2.18) as

$$\mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})} \big(\tau_{\bar{g}} V_{g^{-1}tgT_{\rm rs}}\big) = \tau_{\bar{g}} \big(\mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(gtg^{-1}),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})} (V_{g^{-1}tgT_{\rm rs}})\big). \tag{2.19}$$

From [Lus3, §8.8] and [Sol1, Lemma 4.2] we see that the effect of conjugation by $\tau_{\bar{g}}$ on $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$ and $\mathcal{H}(\tilde{Z}_G(t), M, q\mathcal{E}, \vec{\mathbf{z}})$ boils down to the algebra isomorphism (2.17). The right hand side of (2.19) becomes

$$\mathrm{Ad}(g)^* \circ \mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(gtg^{-1}),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}}} \circ \mathrm{Ad}(g^{-1})^* \big(V_{tT_{\rm rs}}\big),$$

which proves the lemma for such g.

Now we consider a general $g \in \mathrm{N}_G(M)$. We will analyse

$$\mathrm{Ad}(g)^* \circ \mathrm{ind}_{\mathcal{H}(\tilde{Z}_G(gtg^{-1}),M,\mathrm{Ad}(g^{-1})^* q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,\mathrm{Ad}(g^{-1})^* q\mathcal{E},\vec{\mathbf{z}})} \circ \mathrm{Ad}(g^{-1})^* \big(V_{tT_{\rm rs}}\big). \tag{2.20}$$

From the above we see that the underlying vector space is

$$\bigoplus_{w \in gW_{q\mathcal{E}}g^{-1}/gW_{q\mathcal{E},t}g^{-1}} \tau_w \big(\mathrm{Ad}(g^{-1})^* V_{tT_{\rm rs}}\big) = \bigoplus_{w \in W_{q\mathcal{E}}/W_{q\mathcal{E},t}} \mathrm{Ad}(g^{-1})^* \tau_w V_{tT_{\rm rs}} = \mathrm{Ad}(g^{-1})^* V.$$

The action of $\mathcal{H}(\tilde{Z}_G(gtg^{-1}), M, \mathrm{Ad}(g^{-1})^* q\mathcal{E}, \vec{\mathbf{z}}) = \mathrm{Ad}(g)\mathcal{H}(\tilde{Z}_G(t), M, q\mathcal{E}, \vec{\mathbf{z}})$ works out to

$$(\mathrm{Ad}(g)h) \cdot (\mathrm{Ad}(g^{-1})^* v) = \mathrm{Ad}(g^{-1})^* (h \cdot v).$$

Thus (2.20) can be identified with V. □

2.2. Parametrization of irreducible representations.

Next we want to reduce from $\mathcal{H}(\tilde{Z}_G(t), M, q\mathcal{E}, \vec{\mathbf{z}})$-modules to modules over $\mathbb{H}(G_t, M, q\mathcal{E}, \vec{\mathbf{r}})$. The exponential map for $T \times \mathbb{C}^\times$ gives a $W_{q\mathcal{E},t}$-equivariant map

$$\exp_t \colon \mathfrak{t} \oplus \mathbb{C}^d \to T \times (\mathbb{C}^\times)^d, \qquad \exp_t(x, r_1, \ldots, r_d) = (\exp(x)t, \exp r_1, \ldots, \exp r_d).$$

Notice that the restriction $\exp_t \colon \mathfrak{t}_\mathbb{R} \oplus \mathbb{R}^d \to tT_{\rm rs} \times \mathbb{R}^d_{>0}$ is a diffeomorphism.

Theorem 2.11. *Let $t \in T_{\rm un}$.*

(*a*) *There is a canonical equivalence between the following categories:*
- *finite dimensional $\mathbb{H}(G_t, M, q\mathcal{E}, \vec{\mathbf{r}})$-modules with $\mathcal{O}(\mathfrak{t} \oplus \mathbb{C}^d)$-weights in $\mathfrak{t}_\mathbb{R} \oplus \mathbb{R}^d$;*
- *finite dimensional $\mathcal{H}(\tilde{Z}_G(t), M, q\mathcal{E}, \vec{\mathbf{z}})$-modules with $\mathcal{O}(T \times (\mathbb{C}^\times)^d)$-weights in $tT_{\rm rs} \times \mathbb{R}^d_{>0}$.*

It is given by localization with respect to central ideals in combination with the map $\exp_t$. We denote this equivalence by $(\exp_t)_*$.

(*b*) *The functor $(\exp_t)_*$ is compatible with parabolic induction, in the following sense. Let $Q \subset G$ be an algebraic subgroup such that $Q \cap G^\circ$ is a Levi subgroup of G° and $Q \supset M$. Then*

$$\mathrm{ind}_{\mathcal{H}(\tilde{Z}_Q(t),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(\tilde{Z}_G(t),M,q\mathcal{E},\vec{\mathbf{z}})} \circ (\exp_t^Q)_* = (\exp_t)_* \circ \mathrm{ind}_{\mathbb{H}(Q_t,M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathbb{H}(G_t,M,q\mathcal{E},\vec{\mathbf{z}})}.$$

(c) *The functor* $(\exp_t)_*$ *preserves the underlying vector space of a representation, and it transforms a* $S(\mathfrak{t}^* \oplus \mathbb{C}^d)$*-weight* $(x, \vec{r})$ *into a* $\mathcal{O}(T \times (\mathbb{C}^\times)^d)$*-weight* $\exp_t(x, \vec{r})$.

(d) *The functors* $(\exp_t)_*$ *and* $(\exp_t)_*^{-1}$ *preserve (anti-)temperedness and (essentially) discrete series.*

(e) *Parts (a)–(d) also apply to a twisted affine Hecke algebra* $\mathcal{H}(R, X, R^\vee, Y, \lambda, \lambda^*, \vec{\mathbf{z}}) \rtimes \mathbb{C}[\Gamma, \natural]$ *as in Proposition 2.2, and a* $t \in \mathrm{Hom}(X, S^1)$ *fixed by* $W(R) \rtimes \Gamma$*. Then the associated twisted graded Hecke algebra is* $\mathbb{H}(Y \otimes_{\mathbb{Z}} \mathbb{C}, W(R), k, \vec{\mathbf{r}}) \rtimes \mathbb{C}[\Gamma, \natural]$*, where* $k(\alpha) = \lambda(\alpha) + \alpha(t)\lambda^*(\alpha)$*, with the convention that* $\lambda^*(\alpha) = \lambda(\alpha)$ *if* $2\alpha \notin R$.

Proof. (a) The case $d = 1, \natural = 1$ was proven in [Sol3, Corollary 2.1.5], building on [Lus3, Theorem 9.3] when $\mathfrak{R}_{q\mathcal{E}} = 1$.

We use the similar techniques and notations as in the proof of Theorem 2.5.a. By the same argument as over there, it suffices to compare the categories

$$\begin{aligned} &\mathrm{Mod}_{f, W_{q\mathcal{E},t} tt' \times \{\vec{z}\}}\big(\mathcal{H}(\tilde{Z_G}(t)^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{z}}) \rtimes \mathfrak{R}^+_{q\mathcal{E},t}\big), \\ &\mathrm{Mod}_{f, W_{q\mathcal{E},t} \log(t') \times \{\log(\vec{z})\}}\big(\mathbb{H}(Z_G(t)^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}}) \rtimes \mathfrak{R}^+_{q\mathcal{E},t}\big). \end{aligned} \tag{2.21}$$

Recall from (2.3) that the parameter functions for these algebras are related by

$$\begin{array}{llll} c_t(\alpha) & = & 2\lambda(\alpha) & 2\alpha \notin R(\tilde{Z_G}(t)^\circ, T), \\ c_t(\alpha) & = & \lambda(\alpha) + \lambda^*(\alpha) & 2\alpha \in R(\tilde{Z_G}(t)^\circ, T), \alpha(t) = 1, \\ c_t(2\alpha)/2 & = & \lambda(\alpha) - \lambda^*(\alpha) & 2\alpha \in R(\tilde{Z_G}(t)^\circ, T), \alpha(t) = -1. \end{array} \tag{2.22}$$

Let us define $k \colon R(\tilde{Z_G}(t)^\circ, T)_{\mathrm{red}} \to \mathbb{R}$ by

$$\begin{array}{llll} k(\alpha) & = & 2\lambda(\alpha) & 2\alpha \notin R(\tilde{Z_G}(t)^\circ, T), \\ k(\alpha) & = & \lambda(\alpha) + \alpha(t)\lambda^*(\alpha) & 2\alpha \in R(\tilde{Z_G}(t)^\circ, T). \end{array} \tag{2.23}$$

The only difference between $\mathbb{H}(\mathfrak{t}, W(\tilde{Z_G}(t)^\circ, T), k\vec{\mathbf{r}})$ and $\mathbb{H}(Z_G(t)^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}})$ arises from roots $\alpha \in R(\tilde{Z_G}(t)^\circ, T) \setminus R(Z_G(t)^\circ, T)$ with $\alpha(t) = -1$. The corresponding braid relations are

$$\begin{array}{llll} N_{s_\alpha}\xi - {}^{s_\alpha}\xi N_{s_\alpha} & = & (\lambda(\alpha) - \lambda^*(\alpha))\mathbf{r}_j(\xi - {}^{s_\alpha}\xi)/\alpha & \text{in } \mathbb{H}(\mathfrak{t}, W(\tilde{Z_G}(t)^\circ, T), k\vec{\mathbf{r}}), \\ N_{s_{2\alpha}}\xi - {}^{s_{2\alpha}}\xi N_{s_{2\alpha}} & = & c_t(2\alpha)\mathbf{r}_j(\xi - {}^{s_{2\alpha}}\xi)/(2\alpha) & \text{in } \mathbb{H}(Z_G(t)^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}}). \end{array}$$

Since $s_\alpha = s_{2\alpha}$ and $c_t(2\alpha) = 2(\lambda(\alpha) - \lambda^*(\alpha))$, these two braid relations are equivalent, and we may identify

$$\mathbb{H}(\mathfrak{t}, W(\tilde{Z_G}(t)^\circ, T), k\vec{\mathbf{r}}) \rtimes \mathfrak{R}^+_{q\mathcal{E},t} = \mathbb{H}(Z_G(t)^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}}) \rtimes \mathfrak{R}^+_{q\mathcal{E},t}. \tag{2.24}$$

Let $V \subset \mathfrak{t} \times \mathbb{C}^d$ be a $W_{q\mathcal{E},t}$-stable open subset. Recall $\mathcal{H}_t^{\mathrm{an}}(U)$ from (2.11). Similarly we can form the algebra

$$\mathbb{H}_t^{\mathrm{an}}(V) := C^{\mathrm{an}}(V)^{W_{q\mathcal{E},t}} \underset{\mathcal{O}(\mathfrak{t} \oplus \mathbb{C}^d)^{W_{q\mathcal{E},t}}}{\otimes} \mathbb{H}(\mathfrak{t}, W(\tilde{Z_G}(t)^\circ, T), k\vec{\mathbf{r}}) \rtimes \mathfrak{R}^+_{q\mathcal{E},t}.$$

The argument for [Opd1, Proposition 4.3] shows that its finite dimensional modules are precisely the finite dimensional $\mathbb{H}(\mathfrak{t}, W(\tilde{Z_G}(t)^\circ, T), k\vec{\mathbf{r}}) \rtimes \mathfrak{R}^+_{q\mathcal{E},t}$-modules with $\mathcal{O}(\mathfrak{t} \oplus \mathbb{C}^d)$-weights in V. If $\exp_t$ is injective on V, it induces an algebra isomorphism

$$\exp_t^* \colon C^{\mathrm{an}}(\exp_t(V))^{W_{q\mathcal{E},t}} \to C^{\mathrm{an}}(V)^{W_{q\mathcal{E},t}}. \tag{2.25}$$

We suppose in addition that V is contained in a sufficiently small open neighborhood of $\mathfrak{t}_{\mathbb{R}} \oplus \mathbb{R}^d$. In view of the relations between the parameters (2.22) and (2.23), we can apply [Sol3, Theorem 2.1.4.b]. It shows that (2.25) extends to an isomorphism of $C^{an}(V)^{W_{q\mathcal{E},t}}$-algebras

$$\Phi_t \colon \mathcal{H}_t^{an}(\exp_t(V)) \to \mathbb{H}_t^{an}(V),$$

which is the identity on $\mathbb{C}[\mathfrak{R}^+_{q\mathcal{E},t}]$. To explain the canonicity of Φ_t, we use notations from the proof of Theorem 2.5 for algebras involving meromorphic functions. By construction Φ_t is the restriction of an isomorphism

$$\begin{array}{ccccc} \mathcal{H}_t^{me}(\exp_t(V)) & \to & \mathbb{H}_t^{me}(V) & \cong & C^{me}(V) \rtimes \mathbb{C}[W_{q\mathcal{E},t}, \natural_{q\mathcal{E}}] \\ f\tau_w & \mapsto & (f \circ \exp_t)\tilde{\tau}_w & & \end{array}, \tag{2.26}$$

where $f \in C^{me}(\exp_t(V))$ and $\tilde{\tau}_w \in \mathbb{H}_t^{me}(V)$ is an analogue of

$$\tau_w \in \mathcal{H}_t^{me}(\exp_t(V)) \cong C^{me}(\exp_t(V)) \rtimes \mathbb{C}[W_{q\mathcal{E},t}, \natural_{q\mathcal{E}}].$$

For a simple root $\alpha \in R(G_j T, T)$, it is known from [Lus3, §5] that

$$\tilde{\tau}_{s_\alpha} + 1 = (N_{s_\alpha} + 1)\frac{\alpha}{\alpha + k(\alpha)\mathbf{r}_j}. \tag{2.27}$$

This element arises canonically from the definition of $\mathbb{H}(\tilde{Z}_G(t)^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{r}})$, because the multiplication relation (iv) in Proposition 1.1 can be rewritten as

$$\xi(N_{s_\alpha} + 1)\frac{\alpha}{\alpha + k(\alpha)\mathbf{r}_j} - (N_{s_\alpha} + 1)\frac{\alpha}{\alpha + k(\alpha)\mathbf{r}_j}{}^{s_\alpha}\xi = \xi - {}^{s_\alpha}\xi.$$

As $w \mapsto \tilde{\tau}_w$ is multiplicative on $W(\tilde{Z}_G(t)^\circ, T)$ and $\tilde{\tau}_{w\gamma} = \tilde{\tau}_w N_\gamma$ for $w \in W(\tilde{Z}_G(t)^\circ, T)$ and $\gamma \in \mathfrak{R}_{q\mathcal{E},t}$, it follows that all $\tilde{\tau}_w$ with $w \in W_{q\mathcal{E},t}$ are canonical. This is shows that (2.26) and Φ_t are canonical.

Choosing for V a small neighborhood of $W_{q\mathcal{E},t}\log(t') \times \{\log(\vec{z})\}$ in $\mathfrak{t} \oplus \mathbb{C}^d$, Φ_t induces an equivalence between the categories of modules with weights in, respectively, $W_{q\mathcal{E},t}tt' \times \{\vec{z}\}$ and $W_{q\mathcal{E},t}\log(t') \times \{\log(\vec{z})\}$. In view of [Opd1, Proposition 4.3] and (2.24), this provides the equivalence between the categories (2.21).

Since Φ_t fixes $p_\natural \in \mathbb{C}[\mathfrak{R}^+_{q\mathcal{E},t}]$, we can restrict that equivalence to modules on which $p_\natural$ acts as the identity.

(b) For G° this is shown in [BaMo, Theorem 6.2] and [Sol2, Proposition 6.4]. Extending G° to a disconnected group boils down to extending the involved algebras by $\mathbb{C}[\mathfrak{R}_{q\mathcal{E},t}, \natural_{q\mathcal{E}}]$ or $\mathbb{C}[\mathfrak{R}^Q_{q\mathcal{E},t}, \natural_{q\mathcal{E}}]$. As we noted in proof of part (a), the algebra homomorphism Φ_t used to define $(\exp_t)_*$ is the identity on $\mathbb{C}[\mathfrak{R}_{q\mathcal{E},t}, \natural_{q\mathcal{E}}] \subset \mathbb{C}[\mathfrak{R}^+_{q\mathcal{E},t}]$. Hence this extension works the same on both sides of the equivalence, and the argument given in [Sol2, §6] generalizes to the current setting.

(c) By construction [Sol3, §2.1] $(\exp_t)_*\pi = \pi \circ \exp_t^*$ as $\mathcal{O}(T \times (\mathbb{C}^\times)^d)$-representations. (For $f \in \mathcal{O}(T \times (\mathbb{C}^\times)^d)$ the action of $f \circ \exp_t$ on the vector space underlying π is defined via a suitable localization.) This immediately implies that $(\exp_t)_*$ has the effect of $\exp_t$ on weights.

(d) This result generalizes the observations made in [Slo, (2.11)]. Let V be a finite dimensional $\mathcal{H}(\tilde{Z}_G(t), M, q\mathcal{E})$-module with $\mathcal{O}(T \times (\mathbb{C}^\times)^d)$-weights in $tT_{rs} \times \mathbb{R}_{>0}$. By part (b)

$$\mathrm{Wt}((\exp_t)_*^{-1} V) = \exp_t^{-1}(\mathrm{Wt}(V)) \subset \mathfrak{t}_{\mathbb{R}}.$$

By assumption $t \in T_{un}$, so we get

$$|\mathrm{Wt}(V)| = \exp\big(\Re\big(\mathrm{Wt}((\exp_t)_*^{-1} V)\big)\big).$$

Comparing [AMS2, Definition 3.24] and Definition 2.6, we see that $(\exp_t)_*$ and $(\exp_t)_*^{-1}$ preserve (anti-)temperedness and the discrete series. With [AMS2, Definition 3.27] we see that "essentially discrete series" is also respected.
(e) From $\mathcal{H}(G_t^\circ, M^\circ, \mathcal{E}, \vec{\mathbf{z}})$ to $\mathcal{H}(R, X, R^\vee, Y, \lambda, \lambda^*, \vec{\mathbf{z}})$ is only a notational change. The particular shape of $W_{q\mathcal{E},t}$ and of $\natural_{q\mathcal{E}}$ does not play a role in the proof, so when we replace them by more general Γ and $\natural$, the above arguments remain valid. ☐

Theorems 2.5 and 2.11 together provide an equivalence between $\mathbb{H}(G_t, M, q\mathcal{E}, \vec{\mathbf{r}})$-modules with central character in $\mathfrak{t}_\mathbb{R}/W_{q\mathcal{E},t} \times \mathbb{R}^d$ and $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$-modules with central character in $W_{q\mathcal{E}} t T_{\mathrm{rs}}/W_{q\mathcal{E}} \times \mathbb{R}^d_{>0}$, where $t \in T_{\mathrm{un}}$.

Recall from [AMS2, Corollary 3.23] and Theorem 1.6 that we can parametrize $\mathrm{Irr}_{\vec{r}}(\mathbb{H}(G_t, M, q\mathcal{E}, \vec{\mathbf{r}}))$ with $\mathrm{N}_{G_t}(M)/M$-orbits of triples $(\sigma_0, \mathcal{C}, \mathcal{F})$, where $\sigma_0 \in \mathfrak{t}$, $\mathcal{C}$ is a nilpotent $\mathrm{Z}_{G_t}(\sigma_0)$-orbit in $\mathrm{Z}_\mathfrak{g}(\sigma_0)$ and $\mathcal{F}$ is an irreducible $\mathrm{Z}_{G_t}(\sigma_0)$-equivariant local system on $\mathcal{C}$ such that $\Psi_{\mathrm{Z}_{G_t}(\sigma_0)}(\mathcal{C}, \mathcal{F}) = (M, \mathcal{C}_v^M, q\mathcal{E})$, up to $\mathrm{Z}_{G_t}(\sigma_0)$-conjugacy.

To find all irreducible representations with $S(\mathfrak{t}^*)^{W_{q\mathcal{E}}}$-character in $\mathfrak{t}_\mathbb{R}$ (those are all we need for the relation with affine Hecke algebras) it suffices to consider such triples $(\sigma_0, \mathcal{C}, \mathcal{F})$ with $\sigma_0 \in \mathfrak{t}_\mathbb{R}$. To phrase things more directly in terms of the group G, we allow t to vary in T_{un} and we replace σ_0 by $t' = t \exp(\sigma_0) \in t T_{\mathrm{rs}}$. In other words, we consider triples $(t', \mathcal{C}, \mathcal{F})$ such that:

- $t' \in T$ with unitary part $t = t'|t'|^{-1}$;
- $\mathcal{C}$ is a nilpotent $\mathrm{Z}_G(t')$-orbit in $\mathrm{Z}_\mathfrak{g}(t') = \mathrm{Lie}(G_{t'})$.
- $\mathcal{F}$ is an irreducible $\mathrm{Z}_G(t')$-equivariant local system on $\mathcal{C}$ with $q\Psi_{\mathrm{Z}_G(t')}(\mathcal{C}, \mathcal{F}) = (M, \mathcal{C}_v^M, q\mathcal{E})$, up to $\mathrm{Z}_G(t')$-conjugacy.

To such a triple we can associate the standard $\mathbb{H}(G_t, M, q\mathcal{E}, \vec{\mathbf{r}})$-modules

$$E_{y, \log|t'| + \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right), \vec{r}, \rho} \quad \text{and} \quad \mathrm{IM}^* E_{y, -\log|t'| + \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right), \vec{r}, \rho}, \tag{2.28}$$

where $y \in \mathcal{C}$ and ρ is the representation of $\pi_0(\mathrm{Z}_G(t', y))$ on $\mathcal{F}_y$. Furthermore $\gamma \colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Z}_G(t')^\circ$ is an algebraic homomorphism with

$$\mathrm{d}\gamma\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = y \quad \text{and} \quad \mathrm{d}\gamma\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathfrak{t} + \sigma_v, \tag{2.29}$$

where σ_v is as in (1.20) and $\mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right)$ is given by (1.10). The modules (2.28) have distinguished irreducible quotients

$$M_{y, \log|t'| + \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right), \vec{r}, \rho} \quad \text{and} \quad \mathrm{IM}^* M_{y, -\log|t'| + \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right), \vec{r}, \rho}.$$

By [AMS2, Corollary 3.23] all these representations depend only on the $\mathrm{N}_{G_t}(M)/M$-orbit of $(t', \mathcal{C}, \mathcal{F})$, not on the additional choices.

For $\vec{z} \in \mathbb{R}^d_{>0}$ we consider the irreducible $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$-module

$$\mathrm{ind}_{\mathcal{H}(\tilde{\mathrm{Z}_G}(t), M, q\mathcal{E}, \vec{\mathbf{z}})}^{\mathcal{H}(G, M, \mathcal{E}, \vec{\mathbf{z}})} (\exp_t)_* \, \mathrm{IM}^* \, M_{y, \mathrm{d}\vec{\gamma}\left(\begin{smallmatrix} \log \vec{z} & 0 \\ 0 & -\log \vec{z} \end{smallmatrix}\right) - \log|t'|, \log \vec{z}, \rho}. \tag{2.30}$$

Lemma 2.12. *Fix $\vec{z} \in \mathbb{R}^d_{>0}$. The representations (2.30) provide a bijection between $\mathrm{Irr}_{\vec{z}}(\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}}))$ and $\mathrm{N}_G(M)/M$-orbits of triples $(t', \mathcal{C}, \mathcal{F})$ with arbitrary $t' \in T$ and $(\mathcal{C}, \mathcal{F})$ as above.*

Proof. For irreducible $\mathcal{H}(\tilde{\mathrm{Z}_G}(t), M, q\mathcal{E}, \vec{\mathbf{z}})$-representations with central character in $W_{q\mathcal{E},t} t T_{\mathrm{rs}} \times \mathbb{R}_{>0}$ this follows from [AMS2, Corollary 3.23] and Theorems 2.11 and 2.5. We note that at this point we still have to consider $\mathrm{N}_{G_t}(M)/M$-conjugacy classes of parameters $(t', \mathcal{C}, \mathcal{F})$.

With Theorem 2.5.a we extend this to the whole of $\mathrm{Irr}_{\vec{z}}(\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}}))$. In view of (2.16), this involves the choice of a unitary element t in a $\mathrm{N}_G(M)$-orbit in T. But by Lemma 2.9 the parametrization does not depend on that choice. Hence the representation (2.30) depends, up to isomorphism, only on the $\mathrm{N}_G(M)/M$-orbit of $(t',\mathcal{C},\mathcal{F})$. $\square$

To simplify the parameters, we would like to get rid of the restriction $t' \in T$ – we would rather allow any semisimple element of G°. It is also convenient to replace $\mathcal{C}$ by a single unipotent element (contained in $\exp \mathcal{C}$) in G°, and $\mathcal{F}$ by the associated representation of the correct component group.

As new parameters we take triples (s,u,ρ) such that:

- $s \in G^\circ$ is semisimple;
- $u \in \mathrm{Z}_G(s)^\circ$ is unipotent;
- $\rho \in \mathrm{Irr}\big(\pi_0(\mathrm{Z}_G(s,u))\big)$ with $q\Psi_{\mathrm{Z}_G(s)}(u,\rho) = (M,\mathcal{C}_v^M,q\mathcal{E})$ up to G-conjugacy.

Assume that $s \in T$ and choose an algebraic homomorphism $\gamma_u : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Z}_G(s)^\circ$ with

$$\gamma_u \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = u \quad \text{and} \quad \mathrm{d}\gamma_u \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{t} + \sigma_v. \tag{2.31}$$

Using the decomposition (1.4) of $\mathfrak{g}$ we write, like in (1.10),

$$\vec{\gamma_u} \begin{pmatrix} \vec{z} & 0 \\ 0 & \vec{z}^{-1} \end{pmatrix} = \exp \left(\mathrm{d}\vec{\gamma_u} \begin{pmatrix} \log \vec{z} & 0 \\ 0 & -\log \vec{z} \end{pmatrix} \right) \in M. \tag{2.32}$$

For $\vec{z} \in \mathbb{R}^d_{>0}$ we define the standard $\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})$-module

$$\bar{E}_{s,u,\rho,\vec{z}} = \mathrm{ind}_{\mathcal{H}(\tilde{Z_G}(s|s|^{-1}),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})} (\exp_{s|s|^{-1}})_* \, \mathrm{IM}^* \, E_{\log u, \mathrm{d}\vec{\gamma_u} \begin{pmatrix} \log \vec{z} & 0 \\ 0 & -\log \vec{z} \end{pmatrix} - \log |s|, \log \vec{z}, \rho}.$$

and its irreducible quotient

$$\bar{M}_{s,u,\rho,\vec{z}} = \mathrm{ind}_{\mathcal{H}(\tilde{Z_G}(s|s|^{-1}),M,q\mathcal{E},\vec{\mathbf{z}})}^{\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})} (\exp_{s|s|^{-1}})_* \, \mathrm{IM}^* \, M_{\log u, \mathrm{d}\vec{\gamma_u} \begin{pmatrix} \log \vec{z} & 0 \\ 0 & -\log \vec{z} \end{pmatrix} - \log |s|, \log \vec{z}, \rho}.$$

Starting from $(s,u,\rho,\vec{z})$, this expression means that first we take logarithms to obtain parameters for representations of the twisted graded Hecke algebra $\mathbb{H}(G_{s\,|s|^{-1}},M,q\mathcal{E},\vec{\mathbf{r}})$. The associated module of that algebra, from Theorem 1.6, is composed with the Iwahori–Matsumoto involution IM and subsequently transformed into a module of the twisted affine Hecke algebra $\mathcal{H}\big(\tilde{Z_G}(s\,|s|^{-1}),M,q\mathcal{E},\vec{\mathbf{z}}\big)$ by the functor $(\exp_{s\,|s|^{-1}})_*$ from Theorem 2.11. Finally, this module is induced to $\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})$ in the sense of Theorem 2.5.

Even when $s \notin T$, the condition on ρ and [AMS2, Propositions 3.5.a and 3.7] guarantee the existence of a $g_0 \in G^\circ$ such that $g_0 s g_0^{-1} \in T$. In this case we put

$$\bar{E}_{s,u,\rho,\vec{z}} := \bar{E}_{g_0 s g_0^{-1}, g_0 u g_0^{-1}, g_0 \cdot \rho, \vec{z}} \quad \text{and} \quad \bar{M}_{s,u,\rho,\vec{z}} := \bar{M}_{g_0 s g_0^{-1}, g_0 u g_0^{-1}, g_0 \cdot \rho, \vec{z}}. \tag{2.33}$$

We extend the polar decomposition (2.6) to this setting by

$$|s| := g_0^{-1} \, |g_0 s g_0^{-1}| \, g_0.$$

With the Jordan decomposition in G° it is possible to combine s and u in a single element $g = su \in G^\circ$. Then s equals the semisimple part g_S, u becomes the unipotent part g_U and $\rho \in \mathrm{Irr}\big(\pi_0(\mathrm{Z}_G(g))\big)$.

Now we come to our main result about affine Hecke algebras. In the case that G is connected, it is almost the same parametrization as in [Lus5, §5.20] and [Lus7, Theorems 10.4]. The only difference is that we twist by the Iwahori–Matsumoto involution. This is necessary to improve the unsatisfactory notions of ζ-tempered and ζ-square integrable in [Lus7, Theorem 10.5].

Theorem 2.13. *Let $\vec{z} \in \mathbb{R}^d_{>0}$.*

(*a*) *The maps*

$$(g, \rho) \;\mapsto\; (s = g_S, u = g_U, \rho) \;\mapsto \bar{M}_{s,u,\rho,\vec{z}}$$

provide canonical bijections between the following sets:

- *G-conjugacy classes of pairs (g, ρ) with $g \in G^\circ$ and $\rho \in \mathrm{Irr}\big(\pi_0(\mathrm{Z}_G(g))\big)$ such that $q\Psi_{\mathrm{Z}_G(g_S)}(g_U, \rho) = (M, \mathcal{C}_v^M, q\mathcal{E})$ up to G-conjugacy;*
- *G-conjugacy classes of triples (s, u, ρ) as above;*
- $\mathrm{Irr}_{\vec{z}}(\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}}))$.

(*b*) *Suppose that $s \in T$. The representations $\bar{E}_{s,u,\rho,\vec{z}}$ and $\bar{M}_{s,u,\rho,\vec{z}}$ admit the $\mathcal{O}(T)^{W_{q\mathcal{E}}}$-character $W_{q\mathcal{E}} s\, \vec{\chi}_{u,v}(\vec{z})^{\pm 1}$, with $\chi_{u,v}$ as in (1.25) and the arrow defined as in (1.10).*

(*c*) *Suppose that $\vec{z} \in \mathbb{R}^d_{\geq 1}$. The following are equivalent:*

- *s is contained in a compact subgroup of G°;*
- *$|s| = 1$;*
- *$\bar{M}_{s,u,\rho,z}$ is tempered;*
- *$\bar{E}_{s,u,\rho,z}$ is tempered.*

(*d*) *When $\vec{z} \in \mathbb{R}^d_{>1}$, $\bar{M}_{s,u,\rho,\vec{z}}$ is essentially discrete series if and only if u is distinguished in G°. In this case $|s| \in \mathrm{Z}(G^\circ)$.*

There are no essentially discrete series representations on which at least one $\mathbf{z}_j$ acts as 1.

(*e*) *Let $\zeta \in \mathrm{Z}(G) \cap G^\circ$. Then*

$$\bar{M}_{\zeta s,u,\rho,\vec{z}} = \zeta \otimes \bar{M}_{s,u,\rho,\vec{z}} \quad \text{and} \quad \bar{E}_{\zeta s,u,\rho,\vec{z}} = \zeta \otimes \bar{E}_{s,u,\rho,\vec{z}},$$

where $\zeta\otimes$ is as defined after Lemma 2.3.

(*f*) *Suppose that $\vec{z} \in \mathbb{R}^d_{>1}$ and $|s| \in \mathrm{Z}(G^\circ)$. Then $\bar{E}_{s,u,\rho,\vec{z}} = \bar{M}_{s,u,\rho,\vec{z}}$.*

Proof. (a) The uniqueness in the Jordan decomposition entails that the first map is a canonical bijection.

We already noted in (2.33) that, for every eligible triple (s, u, ρ), s lies in $\mathrm{Ad}(G^\circ)T$. Therefore we may restrict to triples with $s \in T$. Consider the map

$$(s, u, \rho) \mapsto \big(s, \mathcal{C}^{\mathrm{Z}_G(s)}_{\log u}, \mathcal{F}\big),$$

where $\mathcal{F}$ is determined by $\mathcal{F}_{\log u} = \rho$. As in the proof of [AMS2, Corollary 3.23], this gives a canonical bijection between G-conjugacy classes of triples (s, u, ρ) and the parameters used in Lemma 2.12. Furthermore (2.31) just reflects (2.29), so Lemma 2.12 yields the desired canonical bijection with $\mathrm{Irr}_{\vec{z}}(\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}}))$.
(b) By Proposition 1.8.f the $\mathbb{H}(\mathrm{Z}_G(s|s|^{-1}), M, q\mathcal{E}, \vec{\mathbf{r}})$-representation

$$\mathrm{IM}^* E_{\log u, \mathrm{d}\vec{\gamma_u} \begin{pmatrix} \log \vec{z} & 0 \\ 0 & -\log \vec{z} \end{pmatrix} - \log |s|, \log \vec{z}, \rho} \tag{2.34}$$

admits the central character $W_{q\mathcal{E}, s|s|^{-1}}\big(\log |s| \pm \mathrm{d}\vec{\chi}_{u,v}(\log \vec{z}), \log \vec{z}\big)$.

By Theorems 2.11.c and 2.5.c the central character of $\bar{E}_{s,u,\rho,\vec{z}}$ becomes

$$\big(W_{q\mathcal{E}} s\, \vec{\chi}_{u,v}(\vec{z}), \vec{z}\big) = W_{q\mathcal{E}}\big(s\, \vec{\chi}_{u,v}(\vec{z})^{-1}, \vec{z}\big).$$

The same holds for the quotient $\bar{M}_{s,u,\rho,\vec{z}}$.
(c) Suppose that $s \in T$. By [AMS2, (84)] the representation (2.35) and its quotient

$$\mathrm{IM}^* \, M_{\log u, \mathrm{d}\vec{\gamma_u}\begin{pmatrix} \log \vec{z} & 0 \\ 0 & -\log \vec{z} \end{pmatrix} - \log |s|, \log \vec{z}, \rho} \tag{2.35}$$

are tempered if and only if $\log |s| \in i\mathfrak{t}_\mathbb{R}$. By definition $\log |s| \in \mathfrak{t}_\mathbb{R}$, so this condition is equivalent to $\log |s| = 0$. This is turn is equivalent to $|s| = 1$ and to $s \in T_{\mathrm{un}}$. By Theorem 2.11.d and Proposition 2.7.b this is also equivalent to temperedness of $\bar{E}_{s,u,\rho,\vec{z}}$ or $\bar{M}_{s,u,\rho,\vec{z}}$.

The proof of part (a) shows that also for general s, temperedness is equivalent to $|s| = 1$. This happens if and only if s lies in the unitary part of a torus conjugate to T, which in turn is equivalent to s lying in a compact subgroup of G°.
(d) As in part (c), it suffices to consider the case $s \in T$.

Suppose that $\bar{M}_{s,u,\rho,\vec{z}}$ is essentially discrete series. By Proposition 2.7.c and Theorem 2.11.d the representation (2.35) has the same property. Moreover we saw in the proof of Proposition 2.7.c that $\widetilde{Z_{G^\circ_{\mathrm{der}}}}(s|s|^{-1})^\circ$ is semisimple. Up to doubling some roots (with respect to T), $Z_{G^\circ_{\mathrm{der}}}(s|s|^{-1})^\circ$ has the same root system, so that group is semisimple as well.

By assumption $\log \vec{z} \in \mathbb{R}^d_{>0}$. Now [AMS2, (85)] says that $\log u$ is distinguished in $\mathrm{Lie}\big(Z_G(s|s|^{-1})^\circ\big)$. In view of the aforementioned semisimplicity, this is the same as distinguished in $\mathfrak{g}$. So u is distinguished in G°.

Conversely, suppose that u is distinguished in G°, or equivalently that $\log u$ is distinguished in $\mathfrak{g}$. As u commutes with s, it also commutes with $|s|$ and with $s|s|^{-1}$. This implies that $R(Z_G(s|s|^{-1})^\circ, T)$ and $R(\tilde{Z_G}(s|s|^{-1})^\circ, T)$ have full rank in $R(G^\circ, T)$. By [AMS2, (85)], Theorem 2.11.d and Proposition 2.7.c $\bar{M}_{s,u,\rho,\vec{z}}$ is essentially discrete series.

Suppose that either of the above two conditions holds. Then $|s| \in T_{\mathrm{rs}}$ commutes with the distinguished unipotent element $u \in G^\circ$. This implies that the semisimple subalgebra $\mathbb{C} \log |s| \subset \mathfrak{g}$ is contained in $Z(\mathfrak{g})$. Hence $|s| \in Z(G^\circ)$. Moreover [AMS2, Theorem 3.26.b] and Lemma 1.3 imply that $\bar{E}_{s,u,\rho,\vec{z}} = \bar{M}_{s,u,\rho,\vec{z}}$.

Finally, suppose that $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$ has an essentially discrete series representation on which Z_j acts as 1. Its dimension is finite, so it has an irreducible subquotient, say $\bar{M}_{s,u,\rho,\vec{z}}$. Then $\mathrm{IM}^* M_{\log u, -\log |s|, \log \vec{z}, \rho}$ restricts to an essentially discrete series representation of $\mathbb{H}(Z_G(s|s|^{-1})^\circ, M^\circ, \mathcal{E})$, which is annihilated by $\mathbf{r}_j$. By (1.8) and (1.9) it contains a $\mathbb{H}(G_j, M_j, \mathcal{E}_j)$-representation with the same properties. But [AMS2, Theorem 3.26.c] says that this is impossible.
(e) By Proposition 1.8.d

$$(\exp_{\zeta s|\zeta s|^{-1}})_* \, \mathrm{IM}^* \, M_{\log u, \mathrm{d}\vec{\gamma_u}\begin{pmatrix} \log \vec{z} & 0 \\ 0 & -\log \vec{z} \end{pmatrix} - \log |\zeta s|, \log \vec{z}, \rho} =$$
$$(\exp_{\zeta|\zeta|^{-1} s|s|^{-1}})_* \, \log |\zeta| \otimes \mathrm{IM}^* \, M_{\log u, \mathrm{d}\vec{\gamma_u}\begin{pmatrix} \log \vec{z} & 0 \\ 0 & -\log \vec{z} \end{pmatrix} - \log |\zeta s|, \log \vec{z}, \rho}.$$

From Theorem 2.11.a and the definitions of $\zeta \otimes, \log |\zeta| \otimes$ we see that this equals

$$\zeta \otimes (\exp_{s|s|^{-1}})_* \, \mathrm{IM}^* \, M_{\log u, \mathrm{d}\vec{\gamma_u}\begin{pmatrix} \log \vec{z} & 0 \\ 0 & -\log \vec{z} \end{pmatrix} - \log |s|, \log \vec{z}, \rho}.$$

Since ζ is central in G, $\mathcal{H}(\tilde{Z_G}(s|s|^{-1}), M, q\mathcal{E}, \vec{\mathbf{z}})$ does not change upon replacing s by ζs, and $\zeta \otimes$ is preserved by $\mathrm{ind}_{\mathcal{H}(\tilde{Z_G}(s|s|^{-1}), M, q\mathcal{E}, \vec{\mathbf{z}})}^{\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})}$. This proves the claim for $\bar{M}_{s,u,\rho,\vec{z}}$,

while the argument for $\bar{E}_{s,u,\rho,\vec{z}}$ is analogous.
(f) We use Theorems 2.5.a and 2.11.a to translate the statement to modules over $\mathbb{H}(G, M, q\mathcal{E}, \vec{\mathbf{r}})$, with $\vec{\mathbf{r}}$ acting as $\log(\vec{z}) \in \mathbb{R}^d_{>0}$. Then we apply Proposition 1.8.e. □

Let us discuss the relation between the parametrization from Theorem 2.13.a and parabolic induction. Suppose that $Q \subset G$ is an algebraic subgroup such that $Q \cap G^\circ$ is a Levi subgroup of G° and $M \subset Q$. Let (s, u, ρ) be as above, with $s, u \in Q^\circ$. Also take $\rho^Q \in \mathrm{Irr}\big(\pi_0(\mathrm{Z}_Q(s, u))\big)$ with $q\Psi_{\mathrm{Z}_Q(s)}(u, \rho^Q) = (M, \mathcal{C}_v^M, q\mathcal{E})$ up to Q-conjugation.

Recall ϵ_j from 17. We extend it to the current setting by defining

$$\epsilon_{u,j}(s, \vec{z}) = \epsilon_{\log u, j}\Big(\mathrm{d}\vec{\gamma}_u \begin{pmatrix} \log \vec{z} & 0 \\ 0 & -\log \vec{z} \end{pmatrix} - \log |s|, \log \vec{z}\Big).$$

Corollary 2.14. *Assume that $\epsilon_{u,j}(s, \vec{z}) \neq 0$ for each $j = 1, \ldots, d$.*

(a) *There is a natural isomorphism of $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$-modules*

$$\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}}) \underset{\mathcal{H}(Q,M,q\mathcal{E},\vec{\mathbf{z}})}{\otimes} \bar{E}^Q_{s,u,\rho^Q,\vec{z}} \cong \bigoplus\nolimits_\rho \mathrm{Hom}_{\pi_0(\mathrm{Z}_Q(s,u))}(\rho^Q, \rho) \otimes \bar{E}_{s,u,\rho,\vec{z}},$$

where the sum runs over all $\rho \in \mathrm{Irr}\big(\pi_0(\mathrm{Z}_G(s, u))\big)$ with $q\Psi_{\mathrm{Z}_G(s)}(u, \rho) = (M, \mathcal{C}_v^M, q\mathcal{E})$ up to G-conjugation. For $\vec{z} = \vec{1}$ this isomorphism contains

$$\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}}) \underset{\mathcal{H}(Q,M,q\mathcal{E},\vec{\mathbf{z}})}{\otimes} \bar{M}^Q_{s,u,\rho^Q,\vec{z}} \cong \bigoplus\nolimits_\rho \mathrm{Hom}_{\pi_0(\mathrm{Z}_Q(s,u))}(\rho^Q, \rho) \otimes \bar{M}_{s,u,\rho,\vec{z}}.$$

(b) *The multiplicity of $\bar{M}_{s,u,\rho,\vec{z}}$ in $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}}) \underset{\mathcal{H}(Q,M,q\mathcal{E},\vec{\mathbf{z}})}{\otimes} \bar{E}^Q_{s,u,\rho^Q,\vec{z}}$ is $[\rho^Q : \rho]_{\pi_0(\mathrm{Z}_Q(s,u))}$. It already appears that many times as a quotient, via $\bar{E}^Q_{s,u,\rho^Q,\vec{z}} \to \bar{M}^Q_{s,u,\rho^Q,\vec{z}}$. More precisely, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{H}(Q,M,q\mathcal{E},\vec{\mathbf{z}})}(\bar{M}^Q_{s,u,\rho^Q,\vec{z}}, \bar{M}_{s,u,\rho,\vec{z}}) \cong \mathrm{Hom}_{\pi_0(\mathrm{Z}_Q(s,u))}(\rho^Q, \rho)^*.$$

Proof. Recall that the analogous statement for twisted graded Hecke algebras is Proposition 1.7. To that we can apply the Iwahori–Matsumoto involution, supported by [AMS2, (83)]. Next, part (b) of Theorem 2.11 allows us to apply part (a) while retaining the desired properties. The same goes for Theorem 2.5. Then we have transferred Proposition 1.7 to the representations $\bar{E}_{s,u,\rho,\vec{z}}$ and $\bar{M}_{s,u,\rho,\vec{z}}$. □

Notice that the parameters in Theorem 2.13.a do not depend on $\vec{z}$. This enables us to relate $\mathrm{Irr}_{\vec{z}}(\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}}))$ to an extended quotient of T by $W_{q\mathcal{E}}$, as in [ABPS5, §2.3] and [AMS2, (87)]. The 2-cocycle $\natural_{q\mathcal{E}}$ of $W_{q\mathcal{E}}$ gives rise to a twisted version of the extended quotient $T//W_{q\mathcal{E}}$, see [ABPS5, §2.1].

Theorem 2.15. *Let $\vec{z} \in \mathbb{R}^d_{>0}$. There exists a canonical bijection*

$$\mu_{G,M,q\mathcal{E}} : (T//W_{q\mathcal{E}})_{\natural_{q\mathcal{E}}} \to \mathrm{Irr}_{\vec{z}}(\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}}))$$

such that:

- *$\mu_{G,M,q\mathcal{E}}(T_{\mathrm{un}}//W_{q\mathcal{E}})_{\natural_{q\mathcal{E}}} = \mathrm{Irr}_{\vec{z},\mathrm{temp}}(\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}}))$ when $\vec{z} \in \mathbb{R}^d_{\geq 1}$;*
- *the central character of $\mu_{G,M,q\mathcal{E}}(t, \pi_t)$ is $\big(W_{q\mathcal{E}} t\, \vec{\chi}(\vec{z}), \vec{z}\big)$, for some algebraic cocharacter χ of $\mathrm{Z}_G(t)^\circ$.*

Remark 2.16. Together with [Sol3, Theorem 5.4.2] this proves a substantial part of the ABPS conjectures [ABPS1, §15] for the twisted affine Hecke algebra $\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}})$. For $\vec{z} \in (0, 1]^d$, $\mu_{G,M,q\mathcal{E}}(T_{\mathrm{un}}//W_{q\mathcal{E}})_{\natural_{q\mathcal{E}}}$ is the anti-tempered part of $\mathrm{Irr}_{\vec{z}}(\mathcal{H}(G, M, q\mathcal{E}, \vec{\mathbf{z}}))$, compare with [AMS2, Theorem 3.29].

Proof. From Proposition 2.2 we see that

$$\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})/(\mathrm{Z}_1-1,\ldots,\mathrm{Z}_d-1)\cong\mathcal{O}(T)\rtimes\mathbb{C}[W_{q\mathcal{E}},\natural_{q\mathcal{E}}].$$

By [ABPS5, Lemma 2.3] there exists a canonical bijection

$$\begin{array}{ccc}(T/\!/W_{q\mathcal{E}})_{\natural_{q\mathcal{E}}} & \to & \mathrm{Irr}(\mathcal{O}(T)\rtimes\mathbb{C}[W_{q\mathcal{E}},\natural_{q\mathcal{E}}]) \\ (t,\pi_t) & \mapsto & \mathbb{C}_t\rtimes\pi_t=\mathrm{ind}_{\mathcal{O}(T)\rtimes\mathbb{C}[W_{q\mathcal{E},t},\natural_{q\mathcal{E}}]}^{\mathcal{O}(T)\rtimes\mathbb{C}[W_{q\mathcal{E}},\natural_{q\mathcal{E}}]}(\mathbb{C}_t\otimes V_{\pi_t})\end{array}.$$

We consider $\mathbb{C}_t\rtimes\pi_t$ as an irreducible $\mathcal{H}(G,M,q\mathcal{E},\vec{\mathbf{z}})$-representation with central character $(W_{q\mathcal{E}}t,1)$. By Theorem 2.15 there exist u and ρ, unique up to $\mathrm{Z}_G(t)$-conjugation, such that $\mathbb{C}_t\rtimes\pi_t\cong\bar{M}_{t,u,\rho,1}$. Now we define

$$\mu_{G,M,q\mathcal{E}}(t,\pi_t)=\bar{M}_{t,u,\rho,\vec{z}}.$$

This is canonical because Theorem 2.13.a is. The properties involving temperedness and the central character follow from parts (c) and (b) of Theorem 2.13. □

2.3. Comparison with the Kazhdan–Lusztig parametrization.

Irreducible representations of affine Hecke algebras were also classified in [KaLu, Ree], in terms of equivariant K-theory. This concerns the cases with only one complex parameter $q=\mathbf{z}^2$, which is not a root of unity. In terms of Proposition 2.2 this means that $\lambda=\lambda^*=1$. In view of (2.3) and [Lus2, Proposition 2.8], this happens if and only if $T=M^\circ$ is a maximal torus of G° and $v=1$. For the upcoming comparison we assume that $M=\mathrm{Z}_G(T)$ equals T. Then $\pi_0(\mathrm{Z}_M(v))=1$, $q\mathcal{E}$ is the trivial representation and

$$\mathfrak{R}_{q\mathcal{E}}=\mathrm{N}_G(T,B)/T\cong G/G^\circ,$$

where B is a Borel subgroup of G° containing T (called P before). The Kazhdan–Lusztig parametrization was extended to algebras of the form

$$\mathcal{H}(G,T,q\mathcal{E}=\mathrm{triv})=\mathcal{H}(\mathcal{R}(G^\circ,T),\lambda=1,\lambda^*=1,\mathbf{z})\rtimes\mathfrak{R}_{q\mathcal{E}}$$

in [ABPS4, §9]. The parameters are triples (t_q,u,ρ), where

- $t_q\in T$ is semisimple;
- $u\in G^\circ$ is unipotent and $t_q u t_q^{-1}=u^q$;
- $\mathcal{B}_{G^\circ}^{t_q,u}$ is the variety of Borel subgroups of G° containing t_q and u;
- $\rho\in\mathrm{Irr}\big(\pi_0(\mathrm{Z}_G(t_q,u))\big)$ such that every irreducible component of $\rho|_{\pi_0(\mathrm{Z}_{G^\circ}(t_q,u))}$ appears in $H_*(\mathcal{B}_{G^\circ}^{t_q,u},\mathbb{C})$.

Two triples of this kind are considered equivalent if they are G-conjugate. The representation $\bar{M}(t_q,u,\rho)$ attached to these data is the unique irreducible quotient of the standard module

$$\bar{E}_{t_q,u,\rho}:=\mathrm{Hom}_{\pi_0(\mathrm{Z}_G(t_q,u))}\big(\rho,H_*(\mathcal{B}_{G^\circ}^{t_q,u}\times\mathfrak{R}_{q\mathcal{E}},\mathbb{C})\big).\tag{2.36}$$

The classification of $\mathcal{H}(G^\circ,T,\mathcal{E}=\mathrm{triv})$ with $q=\mathbf{z}=1$ goes back to Kato [Kat, Theorem 4.1], see also [ABPS4, §8]. With [ABPS4, Remark 9.2] and the subsequent argument (which underlies the above for $q\neq1$) it can be extended to $\mathcal{H}(G,T,q\mathcal{E}=\mathrm{triv})$. The parameters are the same as above (only with $q=1$), and the irreducible module is

$$\bar{M}(t_1,u,\rho)=\mathrm{Hom}_{\pi_0(\mathrm{Z}_G(t_1,u))}\big(\rho,H_{d(u)}(\mathcal{B}_{G^\circ}^{t_1,u}\times\mathfrak{R}_{q\mathcal{E}},\mathbb{C})\big),\tag{2.37}$$

where $d(u)$ refers to the dimension of $\mathcal{B}_{G^\circ}^{t_1,u}$ as a real variety. Clearly $\bar{M}(t_1,u,\rho)$ is again a quotient of $\bar{E}_{t_1,u,\rho}$, but for $q=1$ (2.36) has other irreducible quotients as well, in lower homological degree.

Lemma 2.17. *The above set of parameters (t_q,u,ρ) is naturally in bijection with the sets of parameters used in Theorem 2.13.a.*

Proof. By [ABPS4, Lemma 7.1], we obtain the same G-conjugacy classes of parameters if we replace the above t_q by a semisimple element $s \in \mathrm{Z}_{G^\circ}(u)$. In Theorem 2.13 we also have parameters (s,u,ρ), but with a different condition on ρ, namely that

$$q\Psi_{\mathrm{Z}_G(s)}(u,\rho) = (T, v=1, q\epsilon = \mathrm{triv}).$$

By definition this is equivalent to

$$\Psi_{\mathrm{Z}_G(s)^\circ}(u,\rho_s) = (T, v=1, \epsilon = \mathrm{triv}), \tag{2.38}$$

for any irreducible constituent ρ_s of $\rho|_{\pi_0(\mathrm{Z}_{\mathrm{Z}_G(s)^\circ}(u))}$. Write $r = \log z \in \mathbb{R}$ and $y = \log(u) \in \mathrm{Lie}(\mathrm{Z}_G(s))$. According to [AMS2, Proposition 3.7] for the group $\mathrm{Z}_G(s)^\circ$, (2.38) is equivalent to ρ_s appearing in

$$E^\circ_{y,0,r} = \mathbb{C}_{0,r} \underset{H_*^{M(y)^\circ}(\{y\})}{\otimes} H_*^{M(y)^\circ}(\mathcal{P}^\circ_y,\mathbb{C}) = H_*(\mathcal{P}^\circ_y,\mathbb{C}).$$

To make this more explicit, we assume (as we may) that $s \in T$. Then $\mathrm{Z}_B(s) = \mathrm{Z}_G(s)^\circ \cap B$ is a Borel subgroup of $\mathrm{Z}_G(s)^\circ$ and

$$\begin{aligned} \mathcal{P}^\circ_y = \{ g\mathrm{Z}_B(s) \in \mathrm{Z}_G(s)^\circ/\mathrm{Z}_B(s) : \mathrm{Ad}(g^{-1})y \in \mathrm{Lie}(\mathrm{Z}_B(s))\} = \\ \{ g\mathrm{Z}_B(s) \in \mathrm{Z}_G(s)^\circ/\mathrm{Z}_B(s) : u \in g\mathrm{Z}_B(s)g^{-1}\} = \mathcal{B}^u_{\mathrm{Z}_G(s)^\circ}. \end{aligned} \tag{2.39}$$

Hence (2.38) is equivalent to ρ_s appearing in $H_*(\mathcal{B}^u_{\mathrm{Z}_G(s)^\circ},\mathbb{C})$. Let ρ° be a $\pi_0(\mathrm{Z}_{G^\circ}(s,u))$-constituent of ρ containing ρ_s. By [ABPS4, Proposition 6.2] there are isomorphisms of $\mathrm{Z}_{G^\circ}(s,u)$-varieties

$$\mathcal{B}^{t_q,u}_{G^\circ} \cong \mathcal{B}^{s,u}_{G^\circ} \cong \mathcal{B}^u_{\mathrm{Z}_G(s)^\circ} \times \mathrm{Z}_{G^\circ}(s,u)/\mathrm{Z}_{\mathrm{Z}_G(s)^\circ}(u). \tag{2.40}$$

With this and Frobenius reciprocity we see that the condition on ρ_s is also equivalent to ρ° appearing in $H_*(\mathcal{B}^{s,u}_{G^\circ},\mathbb{C})$. We conclude that the parameters (s,u,ρ) in Theorem 2.13 are equivalent to those in [ABPS4, §9], the only change being $s \leftrightarrow t_q$. □

Proposition 2.18. *The parametrization of $\mathrm{Irr}_z(\mathcal{H}(G,T,q\mathcal{E} = \mathrm{triv}))$ obtained in Theorem 2.13.a agrees with the above parametrization by the representations $\bar{M}(t_q,u,\rho)$, when we set $q = z^2 \in \mathbb{R}_{>0}$ and take Lemma 2.17 into account. Moreover the standard modules $\bar{E}_{s,u,\rho,z}$ and $\bar{E}_{t_q,u,\rho}$ are isomorphic.*

In other words, our classification of irreducible representations of affine Hecke algebras agrees with that of Kazhdan–Lusztig and the extended versions thereof.

Remark 2.19. Our parametrization differs from the one used by Lusztig in [Lus5, §5.20] and [Lus7, Theorem 10.4], namely by the Iwahori–Matsumoto involution. Thus Proposition 2.18 shows that the classification of unipotent representations of adjoint simple groups in [Lus5, Lus7] does not agree with the earlier classification of Iwahori–spherical representations in [KaLu].

Proof. Let (s, u, ρ) be a triple as above, and choose an algebra homomorphism $\gamma_u \colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Z}_G(s)^\circ$ with $\gamma_u \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = u$. Then we can take $t_q = s\gamma_u \left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right)$, where $z^2 = q$. Recall that $\bar{M}(t_q, u, \rho)$ is a quotient of $\bar{E}_{t_q,u,\rho}$ from (2.36). Write $\rho = \rho^\circ \rtimes \tau^*$, where

$$\tau^* \in \mathrm{Irr}(\mathfrak{R}_{q\mathcal{E},u,s,\rho^\circ}) \quad \text{with} \quad \mathfrak{R}_{q\mathcal{E},u,s,\rho^\circ} = \pi_0(\mathrm{Z}_G(s,u))_{\rho^\circ} / \pi_0(\mathrm{Z}_{G^\circ}(s,u)).$$

From [ABPS4, (72)] we see that $\bar{E}_{t_q,u,\rho}$ equals

$$\mathrm{Hom}_{\pi_0(\mathrm{Z}_{G^\circ}(s,u))}\big(\rho^\circ, H_*(\mathcal{B}^{s,u}_{G^\circ}, \mathbb{C})\big) \rtimes \tau. \tag{2.41}$$

To the part without $\rtimes\tau$ we can apply [EvMi], which compares the two parametrizations. In [EvMi] both the Iwahori–Matsumoto involution and a related "shift" are mentioned. This involution is necessary to get temperedness for the same parameters in both classifications. Unfortunately, it is not entirely clear what Evens and Mirkovich mean by a "shift", for signs can be inserted at various places. In any case their argument is based on temperedness and a comparison of weights [EvMi, Theorem 5.5], and it will work once we arrange the modules such that these two aspects match. With this in mind, [EvMi, Theorem 6.10] says that the $\mathbb{H}(\mathrm{Z}_{G^\circ}(s|s|^{-1}), T, \mathrm{triv})$-module obtained from $\mathrm{Hom}_{\pi_0(\mathrm{Z}_{G^\circ}(s,u))}\big(\rho^\circ, H_*(\mathcal{B}^{s,u}_{G^\circ}, \mathbb{C})\big)$ via Theorems 2.5 and 2.11 is $\mathrm{IM}^* E_{y, \mathrm{d}\gamma_u \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right) - \log|s|, r, \rho^\circ}$. The extension with the group $\mathfrak{R}_{q\mathcal{E}}$ is handled in the same way for all algebras under consideration here, namely with Clifford theory. It follows that applying Theorems 2.5 and 2.11 to (2.41) yields

$$\Big(\mathrm{IM}^* E_{y, \mathrm{d}\gamma_u \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right) - \log|s|, r, \rho^\circ}\Big) \rtimes \tau. \tag{2.42}$$

Moreover IM is the identity on $\mathbb{C}[\mathfrak{R}_{q\mathcal{E}}]$, so the large brackets are actually superfluous here. Notice that the subgroup of Γ appearing in $\mathrm{Z}_G(s|s|^{-1})$ is $\Gamma_{\mathrm{Ad}(G^\circ)s|s|^{-1}}$, the stabilizer of the $\mathrm{Ad}(G^\circ)$-orbit of $s|s|^{-1}$. The action of $\mathfrak{R}_{q\mathcal{E},u,s,\rho^\circ}$ underlying $\rtimes\tau$ in (2.41) comes from the action of $\pi_0(\mathrm{Z}_G(s,u))$ on $H_*(\mathcal{B}^u_{\mathrm{Z}_G(s|s|^{-1})^\circ} \times \Gamma_{\mathrm{Ad}(G^\circ)s|s|^{-1}}, \mathbb{C})$. By (2.39) for the group $\mathrm{Z}_G(s|s|^{-1})$:

$$\mathcal{B}^u_{\mathrm{Z}_G(s|s|^{-1})^\circ} \times \Gamma_{\mathrm{Ad}(G^\circ)s|s|^{-1}} = \mathcal{P}_y.$$

Via this equality the $\pi_0(\mathrm{Z}_G(s,u))$-action on $H_*(\mathcal{B}^u_{\mathrm{Z}_G(s|s|^{-1})^\circ} \times \Gamma_{\mathrm{Ad}(G^\circ)s|s|^{-1}}, \mathbb{C})$ agrees with the action on

$$H_*(\mathcal{P}_y, \mathbb{C}) \cong \mathbb{C}_{|s|,r} \underset{H_*^{M(y)^\circ}(\{y\})}{\otimes} H_*^{M(y)^\circ}(\mathcal{P}_y, \mathbb{C})$$

from [AMS2, Theorem 3.2.d]. Hence

$$\begin{aligned} \big(\mathrm{IM}^* E_{y, \mathrm{d}\gamma_u \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right) - \log|s|, r, \rho^\circ}\big) \rtimes \tau &= \mathrm{IM}^*\big(E_{y, \mathrm{d}\gamma_u \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right) - \log|s|, r, \rho^\circ} \rtimes \tau\big) \\ &= \mathrm{IM}^*\big(E_{y, \mathrm{d}\gamma_u \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right) - \log|s|, r, \rho^\circ \rtimes \tau^*}\big) = \mathrm{IM}^* E_{y, \mathrm{d}\gamma_u \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right) - \log|s|, r, \rho}. \end{aligned}$$

We see that the standard modules $\bar{E}_{t_q,u,\rho}$ and $\bar{E}_{s,u,\rho,z}$ give the same module upon applying Theorems 2.5 and 2.11. Hence they are isomorphic.

From here on we have to assume that $q = z^2 \in \mathbb{R}_{>0}$ is not a root of unity. We recognize the unique irreducible quotient of the right hand side as (2.35), a part of the definition of $\bar{M}_{s,u,\rho,z}$. Using Theorems 2.11 and 2.5 again, but now in the opposite direction, we see that both $\bar{M}_{s,u,\rho,z}$ and $\bar{M}(t_q, u, \rho)$ are the unique irreducible quotient of

$$\mathrm{ind}_{\mathcal{H}(\tilde{Z_G}(s|s|^{-1}),M,q\mathcal{E})}^{\mathcal{H}(G,M,\mathcal{E})} (\exp_{s|s|^{-1}})_* \, \mathrm{IM}^* \, E_{\log u, \mathrm{d}\gamma_u \begin{pmatrix} \log z & 0 \\ 0 & -\log z \end{pmatrix} - \log|s|, \log z, \rho}.$$

Thus the two parametrizations agree when $q = z^2 \neq 1$.

For $q = z = 1$ a different argument is needed. We note that (2.41) still applies, which enables us to write

$$\bar{M}(t_1 = s, u, \rho) = \mathrm{Hom}_{\pi_0(Z_{G^\circ}(s,u))}\big(\rho^\circ, H_{d(u)}(\mathcal{B}^{s,u}_{G^\circ}, \mathbb{C})\big) \rtimes \tau.$$

From the definition of the $X^*(T)$-action in [Kat, §3] we see that $H_*(\mathcal{B}^{s,u}_{G^\circ}, \mathbb{C})$ is completely reducible as a $X^*(T)$-module. With [ABPS4, Theorem 8.2] we deduce that the weight space for $s \in T$ is, as $(W_{q\mathcal{E}})_s$-representation, equal to

$$\mathrm{Hom}_{\pi_0(Z_G(s,u))}\big(\rho, H_{d(u)}(\mathcal{B}^u_{Z_G(s)^\circ} \times \Gamma_{\mathrm{Ad}(G^\circ)s}, \mathbb{C})\big) = \\ \mathrm{Hom}_{\pi_0(Z_{Z_G(s)^\circ}(u))}\big(\rho^\circ, H_{d(u)}(\mathcal{B}^u_{Z_G(s)^\circ}, \mathbb{C})\big) \rtimes \tau.$$

From [AMS2, (34)] we can also determine the $X^*(T)$-weight space for s in $\bar{M}_{s,u,\rho,1}$. First we look at the $S(\mathfrak{t}^*)$-weight $-\log|s|$ in $M^\circ_{y,-\log|s|,0,\rho^\circ}$, that gives $M^{Q^\circ}_{y,-\log|s|,0,\rho^\circ}$. As in [AMS2, Section 3.2], we denote the underlying $W(Z_G(s)^\circ, T)$-representation by M_{y,ρ°. Next we replace $Z_G(s)^\circ$ by $Z_G(s)$ and ρ° by $\rho = \rho^\circ \rtimes \tau^*$, obtaining the $(W_{q\mathcal{E}})_s$-representation

$$M^{Q^\circ}_{y,-\log|s|,0,\rho^\circ} \rtimes \tau = M_{y,\rho^\circ} \rtimes \tau. \tag{2.43}$$

Applying the Iwahori–Matsumoto involution and Theorem 2.11, we get

$$(\exp_{s|s|^{-1}})_* \mathrm{IM}^*(M^{Q^\circ}_{y,-\log|s|,0,\rho^\circ} \rtimes \tau). \tag{2.44}$$

The previous $S(\mathfrak{t}^*)$-weight space (2.43) for $-\log|s|$ has now been transformed into the $X^*(T)$-weight space for s in the representation $\bar{M}_{s,u,\rho,1}$ with respect to the group $Z_G(s)$. To land inside $\bar{M}_{s,u,\rho,1}$ with respect to G, we must still apply Theorem 2.5. But that does not change the $X^*(T)$-weight space for s, so we can stick to (2.44).

For $r = 0, z = 1$ the map $(\exp_{s|s|^{-1}})_*$ becomes the identity on $\mathbb{C}[W_\mathcal{E}]$, see [Sol3, (2.5) and (1.25)]. It remains to compare the $\mathbb{C}[W_\mathcal{E}]$-modules

$$\mathrm{IM}^*(M_{y,\rho^\circ} \rtimes \tau) \quad \text{and} \quad \mathrm{Hom}_{\pi_0(Z_{Z_G(s)^\circ}(u))}\big(\rho^\circ, H_{d(u)}(\mathcal{B}^u_{Z_G(s)^\circ}, \mathbb{C})\big) \rtimes \tau. \tag{2.45}$$

By definition [AMS2, Section 3.2] M_{y,ρ° is the $W(Z_G(s)^\circ, T)$-representation associated to (y, ρ°) by the generalized Springer correspondence from [Lus1]. It differs from the classical Springer correspondence by the sign representation, so

$$M_{y,\rho^\circ} = \mathrm{sign} \, \otimes \mathrm{Hom}_{\pi_0(Z_{Z_G(s)^\circ}(u))}\big(\rho^\circ, H_{d(u)}(\mathcal{B}^u_{Z_G(s)^\circ}, \mathbb{C})\big).$$

On both sides of (2.45) the actions underlying $\rtimes\tau$ come from the action of $Z_G(s,u)$ on $H_*(\mathcal{B}^u_{Z_G(s)^\circ} \times \Gamma_{\mathrm{Ad}(G^\circ)s}, \mathbb{C}) \cong H_*(\mathcal{P}_u, \mathbb{C})$. Moreover $\mathrm{IM}(w) = \mathrm{sign}(w)w$ for $w \in W(Z_G(s)^\circ, T)$ and IM is the identity on the group $\mathfrak{R}$ for $Z_G(s)$. We conclude that the two representations in (2.45) are equal.

This proves that $\bar{M}(t_1 = s, u, \rho)$ and $\bar{M}_{s,u,\rho,1}$ have the same $X(T)$-weight space for the weight s. Since both representations are irreducible, that implies that they are isomorphic. $\square$

3. Langlands parameters

Let F be a non-archimedean local field and let $\mathcal{G}$ be a connected reductive group defined over F. In this section we construct a bijection between enhanced Langlands parameters for $\mathcal{G}(F)$ and a certain collection of irreducible representations of twisted Hecke algebras.

We have to collect several notions about L-parameters, for which we follow [AMS1]. For the background we refer to that paper, here we do little more than recalling the necessary notations. Let $\mathcal{G}^\vee$ be the complex dual group of $\mathcal{G}$. It is endowed with an action of the Weil group $\mathbf{W}_F$, which preserves a pinning of $\mathcal{G}^\vee$. The Langlands dual group is ${}^L\mathcal{G} = \mathcal{G}^\vee \rtimes \mathbf{W}_F$.

Definition 3.1. A Langlands parameter for ${}^L\mathcal{G}$ is a continuous group homomorphism

$$\phi\colon \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathcal{G}^\vee \rtimes \mathbf{W}_F$$

such that:

- $\phi(w) \in \mathcal{G}^\vee w$ for all $w \in \mathbf{W}_F$;
- $\phi(\mathbf{W}_F)$ consists of semisimple elements;
- $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is algebraic.

We call a L-parameter:

- bounded, if $\phi(\mathrm{Frob}_F) = (c, \mathrm{Frob}_F)$ with c in a compact subgroup of $\mathcal{G}^\vee$;
- discrete, if $\mathrm{Z}_{\mathcal{G}^\vee}(\phi)^\circ = \mathrm{Z}(\mathcal{G}^\vee)^{\mathbf{W}_F,\circ}$.

With [Bor, §3] it is easily seen that this definition of discreteness is equivalent to the usual definition with proper Levi subgroups.

Let $\mathcal{G}^\vee_{\mathrm{sc}}$ be the simply connected cover of the derived group $\mathcal{G}^\vee_{\mathrm{der}}$. Let $\mathrm{Z}_{\mathcal{G}^\vee_{\mathrm{ad}}}(\phi)$ be the image of $\mathrm{Z}_{\mathcal{G}^\vee}(\phi)$ in the adjoint group $\mathcal{G}^\vee_{\mathrm{ad}}$. We define

$$\mathrm{Z}^1_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi) = \text{ inverse image of } \mathrm{Z}_{\mathcal{G}^\vee_{\mathrm{ad}}}(\phi) \text{ under } \mathcal{G}^\vee_{\mathrm{sc}} \to \mathcal{G}^\vee_{\mathrm{ad}}. \tag{3.1}$$

Notice that the conjugation action of $\mathcal{G}^\vee_{\mathrm{sc}} \rtimes \mathbf{W}_F$ on $\mathcal{G}^\vee_{\mathrm{sc}}$ descends to an action of $\mathcal{G}^\vee \rtimes \mathbf{W}_F$ on $\mathcal{G}^\vee_{\mathrm{sc}}$.

Definition 3.2. To ϕ we associate the finite group $\mathcal{S}_\phi := \pi_0(\mathrm{Z}^1_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi))$. An enhancement of ϕ is an irreducible representation of $\mathcal{S}_\phi$.

The group $\mathcal{G}^\vee$ acts on the collection of enhanced L-parameters for ${}^L\mathcal{G}$ by

$$g \cdot (\phi, \rho) = (g\phi g^{-1}, g \cdot \rho), \quad \text{where } g \cdot \rho(a) = \rho(g^{-1}ag) \text{ for } a \in \mathcal{S}_\phi.$$

Let $\Phi_{\mathrm{e}}({}^L\mathcal{G})$ be the collection of $\mathcal{G}^\vee$-orbits of enhanced L-parameters.

Let us consider $\mathcal{G}(F)$ as an inner twist of a quasi-split group. Via the Kottwitz isomorphism it is parametrized by a character of $\mathrm{Z}(\mathcal{G}^\vee_{\mathrm{sc}})^{\mathbf{W}_F}$, say $\zeta_{\mathcal{G}}$. We say that $(\phi, \rho) \in \Phi_{\mathrm{e}}({}^L\mathcal{G})$ is relevant for $\mathcal{G}(F)$ if $\mathrm{Z}(\mathcal{G}^\vee_{\mathrm{sc}})^{\mathbf{W}_F}$ acts on ρ as $\zeta_{\mathcal{G}}$. The subset of $\Phi_{\mathrm{e}}({}^L\mathcal{G})$ which is relevant for $\mathcal{G}(F)$ is denoted $\Phi_{\mathrm{e}}(\mathcal{G}(F))$.

As is well-known, $(\phi, \rho) \in \Phi_{\mathrm{e}}({}^L\mathcal{G})$ is already determined by $\phi|_{\mathbf{W}_F}$ (the restriction to the first factor of $\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C})$), the unipotent element $u_\phi := \phi\big(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\big)$ and the enhancement ρ. Sometimes we will also consider $\mathcal{G}^\vee$-conjugacy classes of such triples $(\phi|_{\mathbf{W}_F}, u_\phi, \rho)$ as enhanced L-parameters. An enhanced L-parameter $(\phi|_{\mathbf{W}_F}, v, q\epsilon)$ will often be abbreviated to $(\phi_v, q\epsilon)$. We will study enhanced Langlands parameters via their cuspidal support, as introduced in [AMS1].

Definition 3.3. For $(\phi,\rho) \in \Phi_{\mathrm{e}}({}^L\mathcal{G})$ we write $G_\phi = \mathrm{Z}^1_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi|_{\mathbf{W}_F})$, a complex reductive group. We say that (ϕ,ρ) is cuspidal if ϕ is discrete and $(u_\phi = \phi\big(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\big), \rho)$ is a cuspidal pair for G_ϕ in the sense of [AMS1, §3]. (This means that $\rho = \mathcal{F}_{u_\phi}$, for a G_ϕ-equivariant cuspidal local system $\mathcal{F}$ on $\mathcal{C}^{G_\phi}_{u_\phi}$.) We denote the collection of cuspidal L-parameters for ${}^L\mathcal{G}$ by $\Phi_{\mathrm{cusp}}({}^L\mathcal{G})$, and the subset which is relevant for $\mathcal{G}(F)$ by $\Phi_{\mathrm{cusp}}(\mathcal{G}(F))$.

We denote the cuspidal quasi-support of (u_ϕ,ρ), in the sense of [AMS1, §5], by $[M, v, q\epsilon]_{G_\phi}$. In particular $v \in M \subset G_\phi \subset \mathcal{G}^\vee_{\mathrm{sc}}$.

Proposition 3.4. [AMS1, Proposition 7.3]
Let $(\phi,\rho) \in \Phi_{\mathrm{e}}(\mathcal{G}(F))$. Upon replacing (ϕ,ρ) by $\mathcal{G}^\vee$-conjugate and replacing $(M, v, q\epsilon)$ by a G_ϕ-conjugate, there exists a Levi subgroup $\mathcal{L}(F) \subset \mathcal{G}(F)$ such that $(\phi|_{\mathbf{W}_F}, v, q\epsilon)$ is a cuspidal L-parameter for $\mathcal{L}(F)$. Moreover

$$\mathcal{L}^\vee \rtimes \mathbf{W}_F = \mathrm{Z}_{\mathcal{G}^\vee \rtimes \mathbf{W}_F}(\mathrm{Z}(M)^\circ),$$

and this group is uniquely determined by (ϕ,ρ) up to $\mathcal{G}^\vee$-conjugation.

Inside ${}^L\mathcal{G}$, we can conjugate $\mathcal{L}^\vee \rtimes \mathbf{W}_F$ with elements of $\mathcal{G}^\vee$. A subgroup of the form $g(\mathcal{L}^\vee \rtimes \mathbf{W}_F)g^{-1}$ projects naturally onto $\mathbf{W}_F$, but unlike ${}^L\mathcal{L}$ it does not necessarily contain $\mathbf{W}_F$. If $(\phi',\rho') \in \Phi_{\mathrm{e}}({}^L\mathcal{L})$, then $g \cdot (\phi',\rho')$ is an enhanced L-parameter for $g(\mathcal{L}^\vee \rtimes \mathbf{W}_F)g^{-1}$.

Suppose that (ϕ,ρ) is as in Proposition 3.4. We define its modified cuspidal support as

$${}^L\Psi(\phi,\rho) = (\mathcal{L}^\vee \rtimes \mathbf{W}_F, \phi|_{\mathbf{W}_F}, v, q\epsilon)/\mathcal{G}^\vee\text{-conjugacy}.$$

The right hand side consists of a Langlands dual group and a cuspidal enhanced L-parameter for that (up to $\mathcal{G}^\vee$-conjugacy). Every enhanced L-parameter for ${}^L\mathcal{G}$ is conjugate to one as above, so ${}^L\Psi$ can be considered as a well-defined map from $\Phi_{\mathrm{e}}({}^L\mathcal{G})$ to $\mathcal{G}^\vee$-conjugacy of pairs consisting of a $\mathbf{W}_F$-stable Levi subgroup of $\mathcal{G}^\vee$ and a cuspidal L-parameter for the associated L-group. Notice that ${}^L\Psi$ preserves boundedness of enhanced L-parameters.

We also need Bernstein components of enhanced L-parameters. Recall from [Hai, §3.3.1] that the group of unramified characters of $\mathcal{L}(F)$ is naturally isomorphic to $\big((\mathrm{Z}(\mathcal{L}^\vee)^{\mathbf{I}_F})_{\mathbf{W}_F}\big)^\circ$. We consider this as an object on the Galois side of the local Langlands correspondence and with Lemma A.1 we write

$$X_{\mathrm{nr}}({}^L\mathcal{L}) = \big((Z(\mathcal{L}^\vee)^{\mathbf{I}_F})_{\mathbf{W}_F}\big)^\circ = (Z(\mathcal{L}^\vee \rtimes \mathbf{I}_F)_{\mathbf{W}_F})^\circ. \tag{3.2}$$

Given $(\phi',\rho') \in \Phi_{\mathrm{e}}(\mathcal{L}(F))$ and $z \in Z(\mathcal{L}^\vee \rtimes \mathbf{I}_F)_{\mathbf{W}_F}$, we define $(z\phi',\rho') \in \Phi_{\mathrm{e}}(\mathcal{L}(F))$ by

$$z\phi' = \phi' \text{ on } \mathbf{I}_F \times \mathrm{SL}_2(\mathbb{C}) \text{ and } (z\phi')(\mathrm{Frob}_F) = \tilde{z}\phi'(\mathrm{Frob}_F),$$

where $\tilde{z} \in Z(\mathcal{L}^\vee \rtimes \mathbf{I}_F)$ represents z.

Definition 3.5. An inertial equivalence class for $\Phi_{\mathrm{e}}(\mathcal{G}(F))$ is the $\mathcal{G}^\vee$-conjugacy class $\mathfrak{s}^\vee$ of a pair $(\mathcal{L}^\vee \rtimes \mathbf{W}_F, \mathfrak{s}^\vee_{\mathcal{L}})$, where $\mathcal{L}(F)$ is a Levi subgroup of $\mathcal{G}(F)$ and $\mathfrak{s}^\vee_{\mathcal{L}}$ is a $X_{\mathrm{nr}}({}^L\mathcal{L})$-orbit in $\Phi_{\mathrm{cusp}}(\mathcal{L}(F))$.

The Bernstein component of $\Phi_{\mathrm{e}}(\mathcal{G}(F))$ associated to $\mathfrak{s}^\vee$ is

$$\Phi_{\mathrm{e}}(\mathcal{G}(F))^{\mathfrak{s}^\vee} := {}^L\Psi^{-1}(\mathcal{L}^\vee \rtimes \mathbf{W}_F, \mathfrak{s}^\vee_{\mathcal{L}}). \tag{3.3}$$

We denote the set of inertial equivalence classes for $\Phi_{\mathrm{e}}(\mathcal{G}(F))$ by $\mathfrak{B}^\vee(\mathcal{G}(F))$.

In this way, we obtain a partition of the set $\Phi_e(\mathcal{G}(F))$ analogous to the partition of $\mathrm{Irr}(\mathcal{G}(F))$ induced by its Bernstein decomposition:

$$\Phi_e(\mathcal{G}(F)) = \bigsqcup\nolimits_{\mathfrak{s}^\vee \in \mathfrak{B}^\vee(\mathcal{G}(F))} \Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee}, \tag{3.4}$$

We note that $\Phi_e(\mathcal{L}(F))^{\mathfrak{s}^\vee_{\mathcal{L}}}$ is a torsor for the quotient of the complex torus $X_{\mathrm{nr}}({}^L\mathcal{L})$ by a finite subgroup. In particular $\Phi_e(\mathcal{L}(F))^{\mathfrak{s}^\vee_{\mathcal{L}}}$ is isomorphic to a torus as complex algebraic variety, albeit not in a canonical way.

With an inertial equivalence class $\mathfrak{s}^\vee$ for $\Phi_e(\mathcal{G}(F))$ we associate the finite group

$$W_{\mathfrak{s}^\vee} := \text{ stabilizer of } \mathfrak{s}^\vee_{\mathcal{L}} \text{ in } \mathrm{N}_{\mathcal{G}^\vee}(\mathcal{L}^\vee \rtimes \mathbf{W}_F)/\mathcal{L}^\vee. \tag{3.5}$$

Let $W_{\mathfrak{s}^\vee,\phi}$ be the isotropy group in $W_{\mathfrak{s}^\vee}$ of $(\phi|_{\mathbf{W}_F}, v, q\epsilon) \in \mathfrak{s}^\vee_{\mathcal{L}}$. With the generalized Springer correspondence [AMS1, Theorem 5.5] we can attach to any element of ${}^L\Psi^{-1}(\mathcal{L}^\vee \rtimes \mathbf{W}_F, \phi_v, q\epsilon)$ an irreducible projective representation of $W_{\mathfrak{s}^\vee,\phi}$. More precisely, consider the cuspidal quasi-support

$$q\mathfrak{t} = [G_\phi \cap \mathcal{L}^\vee_c, v, q\epsilon]_{G_\phi},$$

where $\mathcal{L}^\vee_c \subset \mathcal{G}^\vee_{\mathrm{sc}}$ is the preimage of $\mathcal{L}^\vee$ under $\mathcal{G}^\vee_{\mathrm{sc}} \to \mathcal{G}^\vee$. In this setting we write the group $W_{q\mathcal{E}}$ from (1.1) as $W_{q\mathfrak{t}}$. By [AMS1, Lemma 8.2] $W_{q\mathfrak{t}}$ is canonically isomorphic to $W_{\mathfrak{s}^\vee,\phi_v,q\epsilon}$. According to [AMS1, Proposition 9.1] there exist a 2-cocycle $\kappa_{q\mathfrak{t}}$ of $W_{q\mathfrak{t}}$ and a bijection (canonical up to the choice of $\kappa_{q\mathfrak{t}}$ in its cohomology class)

$${}^L\Sigma_{q\mathfrak{t}} \colon {}^L\Psi^{-1}(\mathcal{L}^\vee \rtimes \mathbf{W}_F, \phi_v, q\epsilon) \to \mathrm{Irr}(\mathbb{C}[W_{q\mathfrak{t}}, \kappa_{q\mathfrak{t}}]).$$

It is given by applying the generalized Springer correspondence for $(G_\phi, q\mathfrak{t})$ to (u_ϕ, ρ).

Theorem 3.6. [AMS1, Theorem 9.3]
There exists a bijection

$$\begin{array}{ccc} \Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee} & \longleftrightarrow & \big(\Phi_e(\mathcal{L}(F))^{\mathfrak{s}^\vee_{\mathcal{L}}} // W_{\mathfrak{s}^\vee}\big)_\kappa, \\ (\phi, \rho) & \mapsto & \big({}^L\Psi(\phi,\rho), {}^L\Sigma_{q\mathfrak{t}}(\phi,\rho)\big). \end{array}$$

It is almost canonical, in the sense that it depends only on the choices of 2-cocycles $\kappa_{q\mathfrak{t}}$ as above.

3.1. Graded Hecke algebras.

In Theorem 2.15 we saw that the irreducible representations of a (twisted) affine Hecke algebra can be parametrized with a (twisted) extended quotient of a torus by a finite group. Motivated by the analogy with Theorem 3.6, we want to associate to any Bernstein component $\Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee}$ a twisted affine Hecke algebra, whose irreducible representations are naturally parametrized by $\Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee}$. As this turns out to be complicated, we first do something similar with twisted graded Hecke algebras. From a Bernstein component we will construct a family of algebras, such that a suitable subset of their irreducible representations is canonically in bijection with $\Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee}$. Of course this will be based on the cuspidal quasi-support $[M, v, q\epsilon]_{G_\phi}$ for the group

$$G_\phi := \mathrm{Z}^1_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi|_{\mathbf{W}_F}). \tag{3.6}$$

As before, we abbreviate $T = \mathrm{Z}(M)^\circ$. Let $\mathcal{L}^\vee_c$ be the preimage of $\mathcal{L}^\vee$ in $\mathcal{G}^\vee_{\mathrm{sc}}$. We record that by [AMS1, (99)]

$$M = G_\phi \cap \mathcal{L}^\vee_c. \tag{3.7}$$

Let $q\mathcal{E}$ be the M-equivariant cuspidal local system on $\mathcal{C}^M_{\log v}$ with $q\mathcal{E}_{\log v} = q\epsilon$ as representations of $\pi_0(\mathrm{Z}_M(v)) = \pi_0(\mathrm{Z}_M(\log v))$. We compute the group $W_{q\mathcal{E}}$ from (1.1) for G_ϕ. Let $M_{AD} = Z_{\mathcal{G}^\vee_{\mathrm{ad}}}(\phi|_{\mathbf{W}_F}) \cap \mathcal{L}^\vee / \mathrm{Z}(\mathcal{G}^\vee)$ be the image of M in $\mathcal{G}^\vee_{\mathrm{ad}}$, then the definition of $\mathrm{Z}^1_{\mathcal{G}_{\mathrm{sc}}}(\phi|_{\mathbf{W}_F})$ in (3.1) shows that

$$N_{G_\phi}(M, q\mathcal{E})/M \cong \big(\mathrm{Z}_{\mathcal{G}^\vee_{\mathrm{ad}}}(\phi|_{\mathbf{W}_F}) \cap N_{\mathcal{G}^\vee_{\mathrm{ad}}}(M_{AD}, q\mathcal{E})\big)/M_{AD}. \tag{3.8}$$

Every element of $\mathcal{G}^\vee$ which normalizes M normalizes $\mathrm{Z}(M)^\circ$, and then Proposition 3.4 shows that it also normalizes $\mathcal{L}^\vee \rtimes \mathbf{W}_F$. Hence (3.8) is isomorphic to

$$\begin{aligned} &\big(\mathrm{Z}_{\mathcal{G}^\vee}(\phi|_{\mathbf{W}_F}) \cap N_{\mathcal{G}^\vee}(\mathcal{L}^\vee \rtimes \mathbf{W}_F, q\mathcal{E})\big)/\mathrm{Z}_{\mathcal{L}^\vee}(\phi|_{\mathbf{W}_F}) \;\cong \\ &\big(\mathrm{Z}_{\mathcal{G}^\vee}(\phi|_{\mathbf{W}_F}) \cap N_{\mathcal{G}^\vee}(\mathcal{L}^\vee \rtimes \mathbf{W}_F, q\mathcal{E})\big)\mathcal{L}^\vee/\mathcal{L}^\vee \;=\; W_{\mathfrak{s}^\vee,\phi}, \end{aligned} \tag{3.9}$$

where the equality sign follows from $\mathcal{L}^\vee \cdot (\phi|_{\mathbf{W}_F}, \mathcal{C}^M_{\log v}, q\mathcal{E}) = \mathcal{L}^\vee \cdot (\phi, q\epsilon) \in \Phi_{\mathrm{e}}({}^L\mathcal{L})$.

One problem for the construction of twisted graded Hecke algebras is that $\mathrm{Z}(\mathcal{G}^\vee)^\circ$ was left out of $\mathcal{G}^\vee_{\mathrm{sc}}$, so we can never see it when working in G_ϕ. We resolve this in a crude way, replacing G_ϕ by $G_\phi \times X_{\mathrm{nr}}({}^L\mathcal{G})$. Although that is not a subgroup of $\mathcal{G}^\vee$ or $\mathcal{G}^\vee_{\mathrm{sc}}$, the next result implies that its Lie algebra and the real split part of its centre have the desired shape.

Lemma 3.7. *We use the notations from Proposition 3.4. The natural map*

$$T \times X_{\mathrm{nr}}({}^L\mathcal{G}) \to X_{\mathrm{nr}}({}^L\mathcal{L})$$

is a finite covering of complex tori.

Proof. In Proposition 3.4 we saw that

$$\mathcal{L}^\vee \rtimes \mathbf{W}_F = \mathrm{Z}_{\mathcal{G}^\vee \rtimes \mathbf{W}_F}(T). \tag{3.10}$$

Hence the image of M° under the covering $\mathcal{G}^\vee_{\mathrm{sc}} \to \mathcal{G}^\vee_{\mathrm{der}}$ is contained in $\mathcal{L}^\vee$. It also shows that $\mathbf{W}_F$ fixes T pointwise, so

$$T = (\mathrm{Z}(M)^{\mathbf{I}_F})^\circ_{\mathbf{W}_F}.$$

As $\mathcal{L}^\vee$ is a Levi subgroup of $\mathcal{G}^\vee$, it contains $Z(\mathcal{G}^\vee)^\circ$. Hence there exists a natural map

$$T \times X_{\mathrm{nr}}(\mathcal{G}) = \big(\mathrm{Z}(M)^{\mathbf{I}_F} \times \mathrm{Z}(\mathcal{G}^\vee)^{\mathbf{I}_F}\big)^\circ_{\mathbf{W}_F} \to (Z(\mathcal{L}^\vee)^{\mathbf{I}_F})^\circ_{\mathbf{W}_F} = X_{\mathrm{nr}}({}^L\mathcal{L}). \tag{3.11}$$

The intersection of $\mathrm{Z}(\mathcal{G}^\vee)^\circ$ and $\mathcal{G}^\vee_{\mathrm{der}}$ is finite and T lands in $\mathcal{G}^\vee_{\mathrm{der}} \cap \mathcal{L}^\vee$, so the kernel of (3.11) is finite.

Recall from Proposition 3.4 that $\phi(\mathbf{W}_F) \subset \mathcal{L}^\vee \rtimes \mathbf{W}_F$. Hence

$$\mathrm{Z}(\mathcal{L}^\vee \rtimes \mathbf{W}_F) \subset \mathrm{Z}_{\mathcal{G}^\vee}(\phi(\mathbf{W}_F)) \quad \text{and} \quad \mathrm{Z}(\mathcal{L}^\vee_c \rtimes \mathbf{W}_F)^\circ \subset \mathrm{Z}_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi(\mathbf{W}_F))^\circ.$$

Since M° is a Levi subgroup of $\mathrm{Z}_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi(\mathbf{W}_F))^\circ$ and by (3.10), T equals $\mathrm{Z}(\mathcal{L}^\vee_c \rtimes \mathbf{W}_F)^\circ$. In particular

$$\begin{aligned} \dim T = \dim \mathrm{Z}(\mathcal{L}^\vee_c \rtimes \mathbf{W}_F)^\circ &= \dim \mathrm{Z}(\mathcal{L}^\vee_c \rtimes \mathbf{I}_F)^\circ_{\mathbf{W}_F} \\ &= \dim \mathrm{Z}(\mathcal{L}^\vee \rtimes \mathbf{I}_F)^\circ_{\mathbf{W}_F} - \dim \mathrm{Z}(\mathcal{G}^\vee \rtimes \mathbf{I}_F)^\circ_{\mathbf{W}_F}, \end{aligned}$$

showing that both sides of (3.11) have the same dimension. As the map is an algebraic homomorphism between complex tori and has finite kernel, it is surjective. □

Recall that $\mathfrak{s}_{\mathcal{L}}^\vee$ came from the cuspidal quasi-support $(M, v, q\epsilon)$. For $(\phi_b|_{\mathbf{W}_F}, v, q\epsilon) \in \Phi_{\mathrm{e}}(\mathcal{L}(F))^{\mathfrak{s}_{\mathcal{L}}^\vee}$ we can consider the group

$$\mathrm{Z}^1_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi_b|_{\mathbf{W}_F}) \times X_{\mathrm{nr}}({}^L\mathcal{G}) = G_{\phi_b} \times X_{\mathrm{nr}}({}^L\mathcal{G}),$$

which contains $M \times X_{\mathrm{nr}}({}^L\mathcal{G})$ as a quasi-Levi subgroup. We choose an almost direct factorization for $G_\phi \times X_{\mathrm{nr}}({}^L\mathcal{G})$ as in (1.2) and we put

$$\begin{aligned}\mathbb{H}(\phi_b, v, q\epsilon, \vec{\mathbf{r}}) &:= \mathbb{H}\big(G_{\phi_b} \times X_{\mathrm{nr}}({}^L\mathcal{G}), M \times X_{\mathrm{nr}}({}^L\mathcal{G}), q\mathcal{E}, \vec{\mathbf{r}}\big) \\ &= \mathbb{H}\big(\mathrm{Lie}(X_{\mathrm{nr}}({}^L\mathcal{L})), W_{\mathfrak{s}^\vee,\phi_b}, c\vec{\mathbf{r}}, \natural_{q\mathcal{E}}\big),\end{aligned} \tag{3.12}$$

where $W_{\mathfrak{s}^\vee,\phi_b}$ comes from (3.8)–(3.9). From Lemma 3.7 we see that

$$\begin{aligned}\mathbb{H}(\phi_b, v, q\epsilon, \vec{\mathbf{r}}) &= \mathbb{H}(\mathrm{Z}^1_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi_b|_{\mathbf{W}_F}), M, q\mathcal{E}, \vec{\mathbf{r}}) \otimes S\big(\mathrm{Lie}(X_{\mathrm{nr}}({}^L\mathcal{G}))^*\big) \\ &= \mathbb{H}(G_{\phi_b}, M, q\mathcal{E}, \vec{\mathbf{r}}) \otimes S\big(\mathrm{Lie}(\mathrm{Z}(\mathcal{G}^\vee \rtimes \mathbf{I}_F)^\circ_{\mathbf{W}_F})^*\big).\end{aligned}$$

We say that a representation of $\mathbb{H}(\phi_b, v, q\epsilon, \vec{\mathbf{r}})$ is essentially discrete series if its restriction to $\mathbb{H}(G_{\phi_b}, M, q\mathcal{E}, \vec{\mathbf{r}})$ is so, in the sense of [AMS2, Definition 3.27]. That means that the real parts of its weights (as $\mathbb{H}(G_{\phi_b}, M, q\mathcal{E}, \vec{\mathbf{r}})$-representation) must lie in $\mathrm{Lie}(\mathrm{Z}(G_{\phi_b})^\circ) \oplus \mathfrak{t}_{\mathbb{R}}^{--}$.

Let $X_{\mathrm{nr}}({}^L\mathcal{L}) = X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{un}} \times X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}}$ be the polar decomposition of the complex torus $X_{\mathrm{nr}}({}^L\mathcal{L})$. Let $(\phi_b|_{\mathbf{W}_F}, v, q\epsilon) \in \Phi_{\mathrm{e}}(\mathcal{L}(F))^{\mathfrak{s}_{\mathcal{L}}^\vee}$ with ϕ_b bounded. Suppose that $(\phi, \rho) \in \Phi_{\mathrm{e}}(\mathcal{G}(F))^{\mathfrak{s}^\vee}$ with:

(3.13)
- $\phi|_{\mathbf{I}_F} = \phi_b|_{\mathbf{I}_F}$;
- $\phi(\mathrm{Frob}_F)\phi_b(\mathrm{Frob}_F)^{-1} \in X_{\mathrm{nr}}({}^L\mathcal{L}^\vee)_{\mathrm{rs}}$;
- $\mathrm{d}\phi|_{\mathrm{SL}_2(\mathbb{C})} \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{Lie}(M)$.

For such (ϕ, ρ) and $\vec{r} \in \mathbb{C}^d$ we define

$$E(\phi, \rho, \vec{r}) = \mathrm{IM}^* E_{\log(u_\phi), \log(\phi(\mathrm{Frob}_F)^{-1}\phi_b(\mathrm{Frob}_F)) + \mathrm{d}\vec{\phi}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right), \vec{r}, \rho} \in \mathrm{Mod}(\mathbb{H}(\phi_b, v, q\epsilon, \vec{\mathbf{r}})),$$
$$M(\phi, \rho, \vec{r}) = \mathrm{IM}^* M_{\log(u_\phi), \log(\phi(\mathrm{Frob}_F)^{-1}\phi_b(\mathrm{Frob}_F)) + \mathrm{d}\vec{\phi}\left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right), \vec{r}, \rho} \in \mathrm{Irr}(\mathbb{H}(\phi_b, v, q\epsilon, \vec{\mathbf{r}})).$$

If in addition $\mathrm{d}\phi \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{Lie}(T) + \sigma_v$, as can always be arranged by Proposition 1.4.c, then we define an algebraic cocharacter $\chi_{\phi,v} = \chi_{u_\phi,v}$ of T by

$$\chi_{\phi,v}(z) = \phi\big(1, \left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right)\big) \gamma_v \left(\begin{smallmatrix} z^{-1} & 0 \\ 0 & z \end{smallmatrix}\right). \tag{3.14}$$

We note that $\chi_{\phi,v}$ stems from [AMS1, Lemma 7.6] and that

$$\mathrm{d}\vec{\chi}_{\phi,v}(\vec{r}) = \mathrm{d}\vec{\phi} \left(\begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix}\right) - \vec{r}\sigma_v.$$

Theorem 3.8. *Fix $\vec{r} \in \mathbb{C}^d$ and $(\phi_b|_{\mathbf{W}_F}, v, q\epsilon) \in \Phi_{\mathrm{e}}(\mathcal{L}(F))^{\mathfrak{s}_{\mathcal{L}}^\vee}$ with ϕ_b bounded.*

(a) *The map $(\phi, \rho) \mapsto M(\phi, \rho, \vec{r})$ defines a canonical bijection between*
- ${}^L\Psi^{-1}(\mathcal{L}^\vee \rtimes \mathbf{W}_F, X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}}\phi_b|_{\mathbf{W}_F}, v, q\epsilon)$;
- *the irreducible representations of $\mathbb{H}(\phi_b, v, q\epsilon, \vec{\mathbf{r}})$ with central character in $\mathrm{Lie}(X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}})/W_{\mathfrak{s}^\vee,\phi_b} \times \{\vec{r}\}$.*

(b) *Assume that $\Re(\vec{r}) \in \mathbb{R}^d_{\geq 0}$. The following are equivalent:*
- *ϕ is bounded;*
- ${}^L\Psi(\phi, \rho) = (\mathcal{L}^\vee \rtimes \mathbf{W}_F, \phi_b|_{\mathbf{W}_F}, v, q\epsilon)$;
- *$E(\phi, \rho, \vec{r})$ is tempered;*
- *$M(\phi, \rho, \vec{r})$ is tempered.*

(c) *Suppose that* $\Re(\vec{r}) \in \mathbb{R}^d_{>0}$. *Then* ϕ *is discrete if and only if* $M(\phi, \rho, \vec{r})$ *is essentially discrete series and the rank of* $R(G^\circ_{\phi_b}, T)$ *equals* $\dim_{\mathbb{C}}(T)$.

In this case $\phi(\mathrm{Frob}_F)\phi_b(\mathrm{Frob}_F)^{-1}$ *comes from an element of* $Z(G^\circ_{\phi_b}) \times X_{\mathrm{nr}}({}^L\mathcal{G})$ *via Lemma 3.7.*

(d) *Let* $\zeta \in X_{\mathrm{nr}}({}^L\mathcal{G})_{\mathrm{rs}}$. *Then*

$$M(\zeta\phi, \rho, \vec{r}) = \log(\zeta) \otimes M(\phi, \rho, \vec{r}) \quad \text{and} \quad E(\zeta\phi, \rho, \vec{r}) = \log(\zeta) \otimes E(\phi, \rho, \vec{r}).$$

(e) *Suppose that* $\Re(\vec{r}) \in \mathbb{R}^d_{>0}$ *and that* $\phi(\mathrm{Frob}_F)\phi_b(\mathrm{Frob}_F)^{-1}$ *comes from* $Z(G^\circ_{\phi_b}) \times X_{\mathrm{nr}}({}^L\mathcal{G})$ *via Lemma 3.7. Then* $M(\phi, \rho, \vec{r}) = E(\phi, \rho, \vec{r})$.

(f) *If* $\mathrm{d}\phi \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{Lie}(T) + \sigma_v$, *then* $E(\phi, \rho, \vec{r})$ *and* $M(\phi, \rho, \vec{r})$ *admit the central character* $W_{\mathfrak{s}^\vee, \phi_b}(\log(\phi(\mathrm{Frob}_F)\phi_b(\mathrm{Frob}_F)^{-1}) \pm \mathrm{d}\vec{\chi}_{\phi,v}(\vec{r}), \vec{r})$.

Proof. (a) By Theorem 3.6 every element of ${}^L\Psi^{-1}(\mathcal{L}^\vee \rtimes \mathbf{W}_F, X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}}\phi_b|_{\mathbf{W}_F}, v, q\epsilon)$ has a representative (ϕ, ρ) with $\phi|_{\mathbf{W}_F}$ in $X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}}\phi_b|_{\mathbf{W}_F}$. Then $\phi|_{\mathbf{I}_F}$ is fixed, so $\phi|_{\mathbf{W}_F}$ can be described by the single element $\phi(\mathrm{Frob}_F)\phi_b(\mathrm{Frob}_F)^{-1} \in X_{\mathrm{nr}}({}^L\mathcal{L}^\vee)_{\mathrm{rs}}$. Since $X_{\mathrm{nr}}({}^L\mathcal{L}^\vee)_{\mathrm{rs}}$ is the real split part of a complex torus, there is a unique logarithm

$$\sigma_0 = \log\left(\phi(\mathrm{Frob}_F)\phi_b(\mathrm{Frob}_F)^{-1}\right) \in \mathrm{Lie}(X_{\mathrm{nr}}({}^L\mathcal{L}^\vee)_{\mathrm{rs}}). \tag{3.15}$$

Clearly (ϕ_b, v) is the unique bounded L-parameter in $X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}}(\phi_b, v)$. Hence every element of $\mathcal{G}^\vee_{\mathrm{ad}}$ that centralizes ϕ also centralizes ϕ_b, which implies

$$G_\phi = \mathrm{Z}^1_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi|_{\mathbf{W}_F}) \subset \mathrm{Z}^1_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi_b|_{\mathbf{W}_F}) = G_{\phi_b}.$$

In particular $\phi(\mathrm{SL}_2(\mathbb{C})) \subset G_{\phi_b}$ and

$$\pi_0(\mathrm{Z}_{G_\phi}(u_\phi)) = \pi_0\big(\mathrm{Z}_{G_{\phi_b}}(\sigma_0, \log(u_\phi))\big).$$

By assumption $q\Psi_{G_\phi}(u_\phi, \rho) = (v, q\epsilon)$, and by [AMS2, Proposition 3.7] this cuspidal quasi-support is relevant for

$$\mathbb{H}(\phi_b, v, q\epsilon, \vec{\mathbf{r}}) = \mathbb{H}\big(G_{\phi_b} \times X_{\mathrm{nr}}({}^L\mathcal{G}), M \times X_{\mathrm{nr}}({}^L\mathcal{G}), q\mathcal{E}, \vec{\mathbf{r}}\big).$$

By Proposition 1.4.c, (ϕ, ρ) is conjugate to an enhanced L-parameter with all the above properties, which in addition satisfies

$$\mathrm{d}\phi|_{\mathrm{SL}_2(\mathbb{C})} \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{Lie}(M).$$

Consequently $(\log(u_\phi), \sigma_0, \vec{r}, \rho)$ is a parameter of the kind considered in Section 1, and $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ can play the role of γ from (1.10). By reversing the above procedure every parameter $(y, \sigma', \vec{r}, \rho')$ for $\mathbb{H}(\phi_b, v, q\epsilon, \vec{\mathbf{r}})$ gives rise to an element of

$${}^L\Psi^{-1}(\mathcal{L}^\vee \rtimes \mathbf{W}_F, X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}}\phi_b|_{\mathbf{W}_F}, v, q\epsilon).$$

The equivalence relations on these two sets of parameters agree, for both come from conjugation by G_{ϕ_b}.

Now it follows from Theorem 1.6, Proposition 1.4 and Proposition 1.8.f that

$${}^L\Psi^{-1}(\mathcal{L}^\vee \rtimes \mathbf{W}_F, X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}}\phi_b|_{\mathbf{W}_F}, v, q\epsilon)$$

parametrizes the part of $\mathrm{Irr}_r(\mathbb{H}(\phi_b, v, q\epsilon))$ with central character in

$$\mathrm{Lie}(X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}})/W_{\mathfrak{s}^\vee, \phi_b, v, q\epsilon} \times \{\vec{r}\}.$$

As in [AMS2, Theorem 3.29] and Proposition 1.8, we compose this parametrization with the Iwahori–Matsumoto involution from (1.28). Then the representation associated to (ϕ, ρ) becomes $\pi(\phi, \rho, r)$.

(b) By [AMS2, Theorem 3.25] and [AMS2, (84)] the third and the fourth statements are both equivalent to

$$\phi(\mathrm{Frob}_F)\phi_b(\mathrm{Frob}_F)^{-1} \in \mathrm{Lie}(X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{un}}).$$

But by construction this lies in $\mathrm{Lie}(X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}})$, so the statement becomes $\phi(\mathrm{Frob}_F) = \phi_b(\mathrm{Frob}_F)$. As (ϕ_b, v) is the only bounded L-parameter in $X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}}(\phi_b, v)$, this holds if and only if ϕ is bounded. Since the map ${}^L\Psi$ preserves $\phi|_{\mathbf{W}_F}$, the statement $\phi(\mathrm{Frob}_F) = \phi_b(\mathrm{Frob}_F)$ is also equivalent to

$${}^L\Psi(\phi, \rho) = (\mathcal{L}^\vee \rtimes \mathbf{W}_F, \phi_b|_{\mathbf{W}_F}, v, q\epsilon).$$

Knowing these equivalences, the equality $M(\phi, \rho, \vec{r}) = E(\phi, \rho, \vec{r})$ is given in Proposition 1.8.b.

(c) Suppose that ϕ is discrete. Then

$$G_\phi^\circ = \mathrm{Z}_{\mathcal{G}_{\mathrm{sc}}^\vee}(\phi(\mathbf{W}_F))^\circ = \mathrm{Z}_{\mathcal{G}_{\mathrm{sc}}^\vee}(\phi_b(\mathbf{W}_F), \sigma)^\circ$$

is a reductive group in which $\phi(\mathrm{SL}_2(\mathbb{C}))$ has finite centralizer. This implies that G_ϕ° is semisimple and that u_ϕ is distinguished in it. The first of these two properties implies that G_ϕ° is a full rank subgroup of G_{ϕ_b}, and that $G_{\phi_b}^\circ$ is also semisimple. In other words, $R(G_{\phi_b}^\circ, T)$ has rank equal to the dimension of T. Then u_ϕ is distinguished in $G_{\phi_b}^\circ$ as well, and [AMS2, (85)] says that $M(\phi, \rho, \vec{r})$ is essentially discrete series.

Conversely, suppose that $M(\phi, \rho, \vec{r})$ is essentially discrete series and that the rank of $R(G_{\phi_b}^\circ, T)$ equals $\dim_{\mathbb{C}}(T)$. Then $G_{\phi_b}^\circ$ is semisimple and by [AMS2, (85)] $u_\phi \in G_\phi^\circ$ is distinguished in $G_{\phi_b}^\circ$. Hence $\mathrm{Z}_{G_\phi}(u_\phi)^\circ$ is contained in the unipotent group $\mathrm{Z}_{G_{\phi_b}}(u_\phi)^\circ$, and itself unipotent. It is known (see for example [Ree, §4.3]) that

$$\mathrm{Z}_{\mathcal{G}_{\mathrm{sc}}^\vee}(\phi)^\circ = \mathrm{Z}_{G_\phi}\big(\phi(\mathrm{SL}_2(\mathbb{C}))\big)^\circ$$

is the maximal reductive quotient of $\mathrm{Z}_{G_\phi}(u_\phi)^\circ$. Hence $\mathrm{Z}_{\mathcal{G}_{\mathrm{sc}}^\vee}(\phi)^\circ$ is trivial, which means that ϕ is discrete.

In this case Proposition 1.7.c says that $\sigma_0 \in Z\big(\mathrm{Lie}(G_{\phi_b} \times X_{\mathrm{nr}}({}^L\mathcal{G}))\big)$. Via the exponential map, that translates to the statement about $\phi(\mathrm{Frob}_F)\phi_b(\mathrm{Frob}_F)^{-1}$.

(d) This is a direct consequence of Proposition 1.8.d (and, for $E(\phi, \rho, \vec{r})$, also the proof thereof).

(e) Via (3.15), the condition becomes $\sigma_0 \in Z\big(\mathrm{Lie}(G_{\phi_b} \times X_{\mathrm{nr}}({}^L\mathcal{G}))\big)$. Apply Proposition 1.8.e.

(f) This follows from Proposition 1.4.e with $\gamma = \phi|_{SL_2(\mathbb{C})}$. □

We conclude this paragraph with some remarks about parabolic induction. Suppose that $\mathcal{Q}(F) \subset \mathcal{G}(F)$ is a Levi subgroup such that ϕ has image in ${}^L\mathcal{Q}$. Let $\mathcal{Q}_c^\vee$ be the inverse image of $\mathcal{Q}^\vee$ in $\mathcal{G}_{\mathrm{sc}}^\vee$, by [Bor, §3] it equals $\mathrm{Z}_{\mathcal{G}_{\mathrm{sc}}^\vee}(\mathrm{Z}(\mathcal{Q}_c^\vee \rtimes \mathbf{W}_F)^\circ)$. Therefore

$$\begin{aligned} \mathrm{Z}^1_{\mathcal{Q}_c^\vee}(\phi_b|_{\mathbf{W}_F}) &= \mathrm{Z}^1_{\mathcal{G}_{\mathrm{sc}}^\vee}(\phi_b|_{\mathbf{W}_F}) \cap \mathrm{Z}_{\mathcal{G}_{\mathrm{sc}}^\vee}(\mathrm{Z}(\mathcal{Q}_c^\vee \rtimes \mathbf{W}_F)^\circ) \\ &= G_{\phi_b} \cap \mathrm{Z}_{\mathcal{G}_{\mathrm{sc}}^\vee}(\mathrm{Z}(\mathcal{Q}_c^\vee \rtimes \mathbf{W}_F)^\circ). \end{aligned} \tag{3.16}$$

This in turn shows that

$$G_{\phi_b}^\circ \cap \mathrm{Z}^1_{\mathcal{Q}_c^\vee}(\phi_b|_{\mathbf{W}_F}) = \mathrm{Z}_{\mathcal{Q}_c^\vee}(\phi_b(\mathbf{W}_F))^\circ$$

is a Levi subgroup of $G_{\phi_b}^\circ$. Furthermore $\mathrm{Z}^1_{\mathcal{Q}_c^\vee}(\phi_b|_{\mathbf{W}_F})$ contains M, for the cuspidal quasi-support of (ϕ, ρ) with respect to ${}^L\mathcal{G}$ is the same as the cuspidal quasi-support of (ϕ, ρ^Q) with respect to ${}^L\mathcal{Q}$, for a suitable $\rho^Q \in \mathrm{Irr}(\mathcal{S}_\phi^{\mathcal{Q}})$ [AMS1, Proposition 5.6.a].

Let ζ be the character of $\mathrm{Z}(\mathcal{G}^\vee_{\mathrm{sc}})$ determined by ρ, an extension of the character $\zeta_{\mathcal{G}} \in \mathrm{Irr}(\mathrm{Z}(\mathcal{G}^\vee_{\mathrm{sc}})^{\mathbf{W}_F})$ which was used to define $\mathcal{G}(F)$-relevance. Let $\zeta^{\mathcal{Q}} \in \mathrm{Irr}(\mathrm{Z}(\mathcal{Q}^\vee_{\mathrm{sc}}))$ be derived from ζ as in [AMS1, Lemma 7.4]. Let $p_\zeta \in \mathbb{C}[\mathcal{S}_\phi]$ and $p_{\zeta^{\mathcal{Q}}} \in \mathbb{C}[\mathcal{S}_\phi^{\mathcal{Q}}]$ be the central idempotents associated to these characters.

Let $\mathcal{S}_{\phi,\mathcal{Q}}$ be the component group of the centralizer of u_ϕ in $\mathrm{Z}^1_{\mathcal{Q}_c^\vee}(\phi|_{\mathbf{W}_F})$, or equivalently the component group of the centralizer of $(\phi(\mathrm{Frob}), u_\phi)$ in (3.16). By [AMS1, Lemma 7.4.c] there exist a canonical isomorphism and a canonical injection

$$p_{\zeta^{\mathcal{Q}}}\mathbb{C}[\mathcal{S}_\phi^{\mathcal{Q}}] \cong p_\zeta \mathbb{C}[\mathcal{S}_{\phi,\mathcal{Q}}] \to p_\zeta \mathbb{C}[\mathcal{S}_\phi].$$

This enables to restrict representations of $\mathcal{S}_\phi$ to $\mathcal{S}_\phi^{\mathcal{Q}}$, and it shows that enhancements for $\phi \in \Phi(\mathcal{Q}(F))$ can just as well be constructed via (3.16) and $\mathcal{S}_{\phi,Q}$.

That is, $G_{\phi_b} \times X_{\mathrm{nr}}({}^L\mathcal{G})$ and $\mathrm{Z}^1_{\mathcal{Q}_c^\vee}(\phi_b|_{\mathbf{W}_F}) \times X_{\mathrm{nr}}({}^L\mathcal{G})$ fulfill the conditions of [AMS2, Proposition 3.22] and Corollary 2.14. It follows that the families of representations

$$(\phi,\rho,\vec{r}) \mapsto E_{\log(u_\phi),\log(\phi(\mathrm{Frob}_F)^{-1}\phi_b(\mathrm{Frob}_F))+\mathrm{d}\vec{\phi}\left(\begin{smallmatrix}\vec{r} & 0\\ 0 & -\vec{r}\end{smallmatrix}\right),\vec{r},\rho} \in \mathrm{Mod}(\mathbb{H}(\phi_b, v, q\epsilon, \vec{\mathbf{r}})),$$
$$(\phi,\rho,\vec{r}) \mapsto M_{\log(u_\phi),\log(\phi(\mathrm{Frob}_F)^{-1}\phi_b(\mathrm{Frob}_F))+\mathrm{d}\vec{\phi}\left(\begin{smallmatrix}\vec{r} & 0\\ 0 & -\vec{r}\end{smallmatrix}\right),\vec{r},\rho} \in \mathrm{Irr}(\mathbb{H}(\phi_b, v, q\epsilon, \vec{\mathbf{r}}))$$

are compatible with parabolic induction in the same sense as [AMS2, Proposition 3.22] and Corollary 2.14. In view of [AMS2, (83)] this does not change upon applying the Iwahori–Matsumoto involution, so it also goes for the representations $E(\phi,\rho,\vec{r})$ and $M(\phi,\rho,\vec{r})$ considered in Theorem 3.8.

3.2. **Root systems.**

We fix an inertial equivalence class $\mathfrak{s}^\vee$ for $\Phi_{\mathrm{e}}(\mathcal{G}(F))$, represented by a cuspidal L-parameter $(\phi|_{\mathbf{W}_F}, v, q\epsilon)$ for $\mathcal{L}(F)$. We use the notations from Proposition 3.4 and (3.7), in particular $T = \mathrm{Z}(M)^\circ = \mathrm{Z}(\mathcal{L}_c^\vee)^{\mathbf{W}_F,\circ}$. We define

$$J := \mathrm{Z}^1_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi|_{\mathbf{I}_F}), \tag{3.17}$$

a variation on G_ϕ from (3.6). The groups T, J and $M = G_\phi \cap \mathcal{L}_c^\vee$ depend only on $(\mathcal{L}^\vee, \mathfrak{s}^\vee_{\mathcal{L}})$. We note that J is reductive, possibly disconnected and

$$G_\phi \subset J \quad \text{and} \quad G_\phi^\circ = \mathrm{Z}_J(\phi(\mathrm{Frob}_F))^\circ. \tag{3.18}$$

In this paragraph, we use the convention that a root system is a finite and integral root system.

Proposition 3.9. *Define $R(J^\circ, T)$ as the set of $\alpha \in X^*(T) \setminus \{0\}$ which appear in the adjoint action of T on Lie(J°).*

(*a*) *$R(J^\circ,T)$ is a root system.*
(*b*) *There exists a $(\phi_1|_{\mathbf{W}_F}, v, q\epsilon)$ such that $R(G^\circ_{\phi_1}, T)_{\mathrm{red}} = R(J^\circ,T)_{\mathrm{red}}$.*
(*c*) *If $t \in T$ commutes with $\mathcal{G}^\vee$, then it lies in the kernel of every $\alpha \in R(J^\circ, T)$.*

Remark 3.10. This result does not imply that $R(G^\circ_{\phi_1}, T)$ equals $R(J^\circ, T)$. For example, suppose that $\mathcal{G} = \mathrm{U}_{2n+1}$ is an unramified unitary group and $\phi_1(\mathbf{I}_F) = 1$. Let $\mathcal{L}^\vee$ be the diagonal torus in $\mathcal{G}^\vee = \mathrm{GL}_{2n+1}(\mathbb{C})$. For $\mathfrak{s}^\vee_{\mathcal{L}}$ we can take the set of enhanced L-parameters corresponding to the unramified characters of $\mathcal{L}(F)$. Then

$$J^\circ = \mathrm{SL}_{2n+1}(\mathbb{C}),\ G^\circ_{\phi_1} = \mathrm{SO}_{2n+1}(\mathbb{C}) \text{ and } T = \mathcal{L}^\vee \cap \mathrm{SO}_{2n+1}(\mathbb{C}).$$

In this case $R(G^\circ_{\phi_1}, T)$ has type B_n, whereas $R(J^\circ, T)$ has type BC_n.

Proof. (a) From [Lus2, Proposition 2.2] we know that every $R(G^\circ_\phi, T)$ is a root system. However, this result does not apply to our current J°, as $(M, v, q\epsilon)$ need not be a cuspidal quasi-support for a group with neutral component J°.

We will check that $R(J^\circ, T)$ satisfies the axioms of a root system. We fix a $\mathrm{N}_J(T)$-invariant inner product on $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ (which exists because $\mathrm{N}_J(T)/\mathrm{Z}_J(T)$ is finite). For $\alpha \in R(J^\circ, T)$ we define $\alpha^\vee \in X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ as the unique element which is orthogonal to

$$\{x \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{R} : \langle x, \alpha \rangle = 0\}$$

and satisfies $\langle \alpha^\vee, \alpha \rangle = 2$. Every single $\alpha \in R(J^\circ, T)$ appears in $R(G^\circ_\phi, T)$ for a suitable choice of ϕ (see the construction of ϕ_t below), which entails that $\alpha^\vee \in X_*(T)$.

For arbitrary $\alpha, \beta \in R(J^\circ, T)$ we have to show that

1. $\langle \alpha^\vee, \beta \rangle \in \mathbb{Z}$;
2. $s_\alpha(\beta) \in R(J^\circ, T)$, where $s_\alpha : X^*(T) \to X^*(T)$ is the reflection associated to α and $\alpha^\vee$.

Assume first that α and β are linearly independent in $X^*(T)$. The element $\phi(\mathrm{Frob}_F) \in \mathcal{L}^\vee \rtimes \mathbf{W}_F$ centralizes T and normalizes J°, so it stabilizes each of the root subspaces $\mathfrak{g}_\alpha \subset \mathrm{Lie}(J^\circ)$. Let λ_α (respectively λ_β) be an eigenvalue of $\mathrm{Ad}(\phi(\mathrm{Frob}_F))|_{\mathfrak{g}_\alpha}$ (respectively $\mathrm{Ad}(\phi(\mathrm{Frob}_F))|_{\mathfrak{g}_\beta}$). Since α and β are linearly independent, we can find a $t \in T$ with $\alpha(t^{-1}) = \lambda_\alpha$ and $\beta(t^{-1}) = \lambda_\beta$. Define $(\phi_t|_{\mathbf{W}_F}, v, q\epsilon) \in \mathfrak{s}^\vee_{\mathcal{L}}$ by

$$\phi_t|_{\mathbf{I}_F} = \phi|_{\mathbf{I}_F} \quad \text{and} \quad \phi_t(\mathrm{Frob}_F) = \phi(\mathrm{Frob}_F)(\text{image of } t \text{ in } \mathcal{G}^\vee_{\mathrm{der}}). \tag{3.19}$$

Clearly $\alpha, \beta \in R(G^\circ_{\phi_t}, T)$. Since this is a root system, (i) and (ii) hold for α and β inside $R(G^\circ_{\phi_t}, T)$. Then they are also valid in the larger set $R(J^\circ, T)$.

Next we consider linearly dependent α, β. Then $s_\alpha(\beta) = -\beta$, so (ii) is automatically fulfilled.

Suppose that there exists a $\gamma \in R(J^\circ, T) \setminus \mathbb{Q}\alpha$ which is not orthogonal to α. As before, we can find ϕ_2, ϕ_3 such that $\alpha, \gamma \in R(G^\circ_{\phi_2}, T)$ and $\beta, \gamma \in R(G^\circ_{\phi_3}, T)$. Hence both $\{\alpha, \gamma\}$ and $\{\beta, \gamma\}$ generate rank two irreducible root systems in $X^*(T)$, and these root systems have the same $\mathbb{Q}$-span. From the classification of rank two root systems we see that $\mathbb{Q}\alpha \cap R(J^\circ, T)$ is either $\{\pm\tilde{\alpha}\}$ or $\{\pm\tilde{\alpha}, \pm 2\tilde{\alpha}\}$ for a suitable $\tilde{\alpha}$. In particular (i) holds, because

$$\langle \alpha^\vee, \beta \rangle \in \pm\{1, 2, 4\} \subset \mathbb{Z}.$$

Finally we suppose that $\mathbb{Q}\alpha \cap R(J^\circ, T)$ is orthogonal to $R(J^\circ, T) \setminus \mathbb{Q}\alpha$. As above, we may pick ϕ such that $\alpha \in R(G^\circ_\phi, T)$. By assumption $\beta = c\alpha$ for some $c \in \mathbb{Q}^\times$. Pick ϕ_t so that $\beta \in R(G^\circ_{\phi_t}, T)$. As

$$\langle \beta^\vee, \beta \rangle = 2 = \langle \alpha^\vee, \alpha \rangle,$$

we have $c\alpha = \beta \in X^*(T)$ and $c^{-1}\alpha^\vee = \beta^\vee \in X_*(T)$. It follows that $c \in \pm\{1/2, 1, 2\}$ and $\langle \alpha^\vee, \beta \rangle \in \pm\{1, 2, 4\}$.

(b) Let Δ be a basis of the reduced root system $R(J^\circ, T)_{\mathrm{red}}$ – which is well-defined by part (a). Let $\lambda_\alpha \in \mathbb{C}$ ($\alpha \in \Delta$) be an eigenvalue of $\mathrm{Ad}(\phi(\mathrm{Frob}_F))$ on $\mathfrak{g}_\alpha$. Since Δ is linearly independent, we can find $t_1 \in T$ with $\alpha(t_1^{-1}) = \lambda_\alpha$ for all $\alpha \in \Delta$. We put $\phi_1 := \phi_{t_1}$, where ϕ_{t_1} is defined by (3.19). Then Δ is contained in the root system $R(G^\circ_{\phi_1}, T)$. The Weyl group of (J°, T) is generated by the reflections s_α with $\alpha \in \Delta$, so it equals the Weyl group of $(G^\circ_{\phi_1}, T)$. In particular it stabilizes $R(G^\circ_{\phi_1}, T)$. Every element of $R(J^\circ, T)_{\mathrm{red}}$ is in the Weyl group orbit of some $\alpha \in \Delta$, so $R(G^\circ_{\phi_1}, T)$

contains $R(J^\circ, T)_{\mathrm{red}}$.
(c) Such a t commutes with $\mathcal{G}^\vee_{\mathrm{sc}}$ and with J, so its image under the adjoint representation of J° is trivial. $\square$

By [Lus1, Theorem 9.2] and Proposition 3.9,

$$\mathrm{N}_{G^\circ_{\phi_1}}(T)/\mathrm{Z}_{G^\circ_{\phi_1}}(T) = W(R(G^\circ_{\phi_1}, T)) = W(R(J^\circ, T)). \tag{3.20}$$

The group $\mathrm{N}_J(T)$ acts naturally on $R(J^\circ, T)$. Let $\mathrm{N}_{J'}(T)$ be the preimage of $W(R(J^\circ, T))$ in $\mathrm{N}_{J^\circ}(T)$, by (3.20) it surjects onto $W(R(J^\circ, T))$. We write

$$W^\circ_{\mathfrak{s}^\vee} := W(R(J^\circ, T)) = \mathrm{N}_{J'}(T)/\mathrm{Z}_{J^\circ}(T). \tag{3.21}$$

Since $\mathcal{L}^\vee_c = \mathrm{Z}_{\mathcal{G}^\vee_{\mathrm{sc}}}(T)$ and $J^\circ = \mathrm{Z}_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi|_{\mathbf{I}_F})^\circ$

$$W^\circ_{\mathfrak{s}^\vee} = \mathrm{N}_{J^\circ}(T)/\mathrm{Z}_{\mathcal{L}^\vee_c}(\phi(\mathbf{I}_F))^\circ = \mathrm{N}_{J^\circ}(\mathcal{L}^\vee_c)/(J^\circ \cap \mathcal{L}^\vee_c).$$

Any element of $G^\circ_{\phi_1}$ which normalizes $T = T^{\mathbf{W}_F}$ will also normalize $\mathcal{L}^\vee \rtimes \mathbf{W}_F = \mathrm{Z}_{\mathcal{G}^\vee \rtimes \mathbf{W}_F}(T)$ and $M = \mathrm{Z}_{G_{\phi_1}}(T) = \mathrm{Z}_{G_\phi}(T)$, while by [AMS2, Lemma 2.1] it stabilizes $\mathcal{C}^M_v$ and $q\mathcal{E}$. The group

$$W_{\mathfrak{s}^\vee} \subset \mathrm{N}_{\mathcal{G}^\vee}(\mathcal{L}^\vee \rtimes \mathbf{W}_F)/\mathcal{L}^\vee = \mathrm{N}_{\mathcal{G}^\vee}(T)/\mathcal{L}^\vee \tag{3.22}$$

from (3.5) stabilizes the $\mathcal{L}^\vee$-conjugacy classes of $X_{\mathrm{nr}}(\mathcal{L}^\vee)(\phi|_{\mathbf{W}_F}, v, q\epsilon)$ and of

$$M = G_{\phi_1} \cap \mathcal{L}^\vee_c = G_\phi \cap \mathcal{L}^\vee_c. \tag{3.23}$$

Further, the group $Z_{\mathcal{G}^\vee}(\phi|_{\mathbf{I}_F})$ automatically normalizes $J = \mathrm{Z}^1_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi|_{\mathbf{I}_F})$. Hence we can express $W_{\mathfrak{s}^\vee}$ as

$$W_{\mathfrak{s}^\vee} \cong \big(\mathrm{N}_{\mathcal{G}^\vee}(T) \cap \mathrm{Z}_{\mathcal{G}^\vee}(\phi|_{\mathbf{I}_F})\big)/\mathrm{Z}_{\mathcal{L}^\vee}(\phi|_{\mathbf{I}_F}). \tag{3.24}$$

As $\mathcal{L}^\vee = \mathrm{Z}_{\mathcal{G}^\vee}(T)$ and $J/\mathrm{Z}(\mathcal{G}^1_{\mathrm{sc}}) = \mathrm{Z}_{\mathcal{G}_{\mathrm{ad}}}(\phi|_{\mathbf{I}_F})$, we deduce from (3.24) that there is a canonical isomorphism

$$\mathrm{N}_J(T)/\mathrm{Z}_J(T) \to W_{\mathfrak{s}^\vee}. \tag{3.25}$$

In particular $W_{\mathfrak{s}^\vee}$ acts on $R(J^\circ, T)$ and naturally contains $W^\circ_{\mathfrak{s}^\vee}$. We choose a ϕ_1 as in Proposition 3.9, which will play the role of a basepoint on $\mathfrak{s}^\vee_{\mathcal{L}}$. Then $W^\circ_{\mathfrak{s}^\vee} = W(R(G^\circ_{\phi_1}, T))$ fixes $(\phi_1|_{\mathbf{W}_F}, v, q\epsilon) \in \mathfrak{s}^\vee_{\mathcal{L}}$, but $W_{\mathfrak{s}^\vee}$ need not fix $\phi_1|_{\mathbf{W}_F}$.

Clearly the set

$$X_{\mathrm{nr}}({}^L\mathcal{L})_{\phi_1} := \{z \in X_{\mathrm{nr}}({}^L\mathcal{L}) : z\phi_1 \equiv_{\mathcal{L}^\vee} \phi_1\} \tag{3.26}$$

only depends on $\mathfrak{s}^\vee_{\mathcal{L}}$, not on ϕ_1. Moreover it is finite, for it consists of elements coming from the finite group $\mathcal{L}^\vee_{\mathrm{der}} \cap \mathrm{Z}(\mathcal{L}^\vee)$. Writing

$$T_{\mathfrak{s}^\vee} = X_{\mathrm{nr}}({}^L\mathcal{L})/X_{\mathrm{nr}}({}^L\mathcal{L})_{\phi_1}, \tag{3.27}$$

we obtain a bijection

$$T_{\mathfrak{s}^\vee} \to \mathfrak{s}^\vee_{\mathcal{L}} \colon z \mapsto [z\phi_1|_{\mathbf{W}_F}, v, q\epsilon]. \tag{3.28}$$

Via this bijection we can retract the action of $W_{\mathfrak{s}^\vee}$ on $\mathfrak{s}^\vee_{\mathcal{L}}$ to $T_{\mathfrak{s}^\vee}$. Then $W^\circ_{\mathfrak{s}^\vee}$ fixes $1 \in T_{\mathfrak{s}^\vee}$. If $\phi_0 = t_0\phi_1$ is another basepoint, like ϕ_1, then also $W(R(G^\circ_{\phi_0}, T)) \cong W^\circ_{\mathfrak{s}^\vee}$, so $t_0 \in (T)^{W^\circ_{\mathfrak{s}^\vee}}$. Consequently the action of $W^\circ_{\mathfrak{s}^\vee}$ on $T_{\mathfrak{s}^\vee}$ is independent of the choice of ϕ_1. On the other hand, the action of $W_{\mathfrak{s}^\vee}$ on $T_{\mathfrak{s}^\vee}$ may very well depend on the choice of the basepoint ϕ_1.

Analogous to (3.26), we consider the finite group

$$
\begin{aligned}
T_{\phi_1} &:= \{t \in T : t\phi_1 \equiv_{\mathcal{L}^\vee} \phi_1\} \\
&= T \cap \{l\phi_1(\mathrm{Frob}_F) l^{-1} \phi_1(\mathrm{Frob}_F)^{-1} : l \in \mathrm{Z}_{\mathcal{L}_c^\vee}(\phi_1(\mathbf{I}_F), v, q\epsilon)\}.
\end{aligned}
$$

From Lemma 3.7 we get a natural, finite covering of tori

$$
T/T_{\phi_1} \times X_{\mathrm{nr}}({}^L\mathcal{G}) \to T_{\mathfrak{s}^\vee}, \tag{3.29}
$$

which is injective on T/T_{ϕ_1}. In general the elements of $R(J^\circ, T)$ do not descend to characters of $X_{\mathrm{nr}}({}^L\mathcal{L})$, and even if they do, they need not descend further to characters of $T_{\mathfrak{s}^\vee}$. The former problem arises in the setting of Remark 3.10, and the latter problem already occurs for the Levi subgroup $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$ of $\mathrm{GL}_4(F)$.

To set things up properly, extend $\phi_1|_{\mathbf{W}_F}$ to

$$
(\phi_1|_{\mathbf{W}_F}, u_{\phi_1}, \rho) \in \Phi_{\mathrm{e}}({}^L\mathcal{G}) \quad \text{with} \quad q\Psi_{G_{\phi_1}}(u_{\phi_1}, \rho) = [M, v, q\epsilon]_{G_{\phi_1}}.
$$

We consider $\phi_1(\mathrm{Frob}_F)$ as a semisimple automorphism of J. By [Ste, Theorem 7.5] $\phi_1(\mathrm{Frob}_F)$ stabilizes a Borel subgroup B_J of J°, and a maximal torus T_J thereof. Then $B_J^{\phi_1(\mathrm{Frob}_F),\circ}$ is a Borel subgroup of $G^\circ_{\phi_1} = J^{\phi_1(\mathrm{Frob}_F),\circ}$, containing $T_J^{\phi_1(\mathrm{Frob}_F),\circ}$ as maximal torus. By conjugation in $G^\circ_{\phi_1}$ we may assume that $u_{\phi_1} \in B_J^{\phi_1(\mathrm{Frob}_F),\circ}$, and then

$$
M \supset T_J^{\phi_1(\mathrm{Frob}_F)} \supset T = \mathrm{Z}(M)^\circ.
$$

We recall that T is the centre of the Levi subgroup M° of G°_ϕ. Hence the restriction map

$$
R\big(G^\circ_{\phi_1}, T_J^{\phi_1(\mathrm{Frob}_F),\circ}\big) \cup \{0\} \to R(G^\circ_{\phi_1}, T) \cup \{0\} \tag{3.30}
$$

has the property that the full preimage of any $\alpha \in R(G^\circ_{\phi_1}, T)$ is contained in a single irreducible component of $R\big(G^\circ_{\phi_1}, T_J^{\phi_1(\mathrm{Frob}_F),\circ}\big)$. As $\phi_1(\mathrm{Frob}_F)$ stabilizes (B_J, T_J), the restriction map

$$
R(J^\circ, T_J) \to R\big(G^\circ_{\phi_1}, T_J^{\phi_1(\mathrm{Frob}_F),\circ}\big) \tag{3.31}
$$

induces a bijection between the $\phi_1(\mathrm{Frob}_F)$-orbits of irreducible components of $R(J^\circ, T_J)$ and the set of irreducible components of $R\big(G^\circ_{\phi_1}, T_J^{\phi_1(\mathrm{Frob}_F),\circ}\big)$. From (3.30) and (3.31) we see that the preimage in $R(J^\circ, T_J)$ of any $\alpha \in R(G^\circ_{\phi_1}, T)$ is contained in single $\phi_1(\mathrm{Frob}_F)$-orbit of irreducible components of $R(J^\circ, T_J)$. That paves the way for the following definition.

Definition 3.11. For each $\alpha \in R(J^\circ, T)_{\mathrm{red}}$, we define $m_\alpha \in \mathbb{Z}_{>0}$ by the following requirements:

- Suppose that the preimage of α in $R(J^\circ, T_J)$ lies in a single irreducible component of that root system. Then m_α is the smallest positive integer such that $T_{\phi_1} \subset \ker(m_\alpha \alpha)$.
- Suppose that the preimage of α in $R(J^\circ, T_J)$ meets $k > 1$ irreducible components of that root system, permuted transitively by the action of $\phi_1(\mathrm{Frob}_F)$. Then $m_\alpha = m_\alpha(\mathrm{Frob}_F)$ equals k times the number $m_\alpha(\mathrm{Frob}_F^k)$ computed (as in the first bullet) with respect to the action of $\phi_1(\mathrm{Frob}_F^k)$. Equivalently, m_α is k times the analogous number obtained by replacing $\mathbf{W}_F$ with the Weil group of the degree k unramified extension of F.

These conditions guarantee that $m_\alpha \alpha$ descends to a character of T/T_{ϕ_1}. Moreover m_α is the minimal such natural number, unless maybe when $\alpha \in 2X^*(T)$ and k even. Extend $m_\alpha \alpha$ to a character of $T/T_{\phi_1} \times X_{\mathrm{nr}}({}^L \mathcal{G})$, trivial on the second factor. In view of Proposition 3.9.c, $m_\alpha \alpha$ is trivial on the kernel of (3.29), and hence descends naturally to a character of $T_{\mathfrak{s}^\vee}$. We define

$$R_{\mathfrak{s}^\vee} = \{ m_\alpha \alpha : \alpha \in R(J^\circ, T)_{\mathrm{red}} \} \subset X^*(T_{\mathfrak{s}^\vee}). \tag{3.32}$$

Recall that in the proof of Proposition 3.9 we fixed an inner product on $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$, invariant under $\mathrm{N}_J(T)/\mathrm{Z}_J(T) \cong W_{\mathfrak{s}^\vee}$.

Lemma 3.12. (*a*) *$R_{\mathfrak{s}^\vee}$ is a reduced root system, and it is stable under the action of $W_{\mathfrak{s}^\vee}$ on $X^*(T_{\mathfrak{s}^\vee})$.*
(*b*) *For α and β in the same irreducible component of $R(J^\circ, T)_{\mathrm{red}}$, $m_\alpha = m_\beta$ or $m_\alpha = \|\alpha\|^{-2} \|\beta\|^2 m_\beta$.*
(*c*) *For each $\alpha \in R(J^\circ, T)_{\mathrm{red}}$, $\alpha^\vee / m_\alpha \in X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ defines a cocharacter of $T_{\mathfrak{s}^\vee}$.*

Proof. (a) The realization of $W_{\mathfrak{s}^\vee}$ in (3.24) makes that it normalizes T_{ϕ_1}. From $T_J \supset T$ and (3.25) we see that $W_{\mathfrak{s}^\vee}$ can be represented by elements of $\mathrm{N}_J(T)$ that normalize T_J. Then $W_{\mathfrak{s}^\vee}$ stabilizes all the data that go into the definition of m_α, so the map $\alpha \mapsto m_\alpha$ is constant on $W_{\mathfrak{s}^\vee}$-orbits. Consequently $R_{\mathfrak{s}^\vee}$ is $W_{\mathfrak{s}^\vee}$-stable. In particular $R_{\mathfrak{s}^\vee}$ is stable under all the reflections $s_{m_\beta \beta} = s_\beta$ with $m_\beta \beta \in R_{\mathfrak{s}^\vee}$. Hence $R_{\mathfrak{s}^\vee}$ is a (possibly non-integral) root system with Weyl group

$$W(R_{\mathfrak{s}^\vee}) = W(R(J^\circ, T)) = W_{\mathfrak{s}^\vee}^\circ. \tag{3.33}$$

By construction $R_{\mathfrak{s}^\vee}$ is reduced, it remains to see that it is integral. The lattice $\mathbb{Z}R_{\mathfrak{s}^\vee}$ is stable under $W(R_{\mathfrak{s}^\vee})$, so we can form the semidirect product $W_a = W(R_{\mathfrak{s}^\vee}) \ltimes \mathbb{Z}R_{\mathfrak{s}^\vee}$. We let any $x \in \mathbb{Z}R_{\mathfrak{s}^\vee}$ act on $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ by the translation t_x. Then W_a is generated by the affine reflections $s_\alpha t_{n\alpha}$ with $\alpha \in R(J^\circ, T)_{\mathrm{red}}$ and $n \in m_\alpha \mathbb{Z}$. If we consider W_a as a group of affine transformations of $\mathbb{Z}R_{\mathfrak{s}^\vee} \otimes_{\mathbb{Z}} \mathbb{R}$, we are in the setting of [Bou, Proposition VI.2.5.8]. It says that W_a is the affine Weyl group of a reduced integral root system, namely $R_{\mathfrak{s}^\vee}^\vee$. Hence $R_{\mathfrak{s}^\vee}$ is integral as well.
(b) By definition, in $R(J^\circ, T)_{\mathrm{red}}$:

$$s_\alpha(m_\beta \beta) = m_\beta \beta - m_\beta \langle \alpha^\vee , \beta \rangle \alpha. \tag{3.34}$$

On the other hand, in the integral root system $R_{\mathfrak{s}^\vee}$:

$$s_{m_\alpha \alpha}(m_\beta \beta) = m_\beta \beta - \langle (m_\alpha \alpha)^\vee , m_\beta \beta \rangle m_\alpha \alpha \tag{3.35}$$

Comparing (3.34) and (3.35), we see that $m_\beta \langle \alpha^\vee , \beta \rangle \in m_\alpha \mathbb{Z}$.

By the $W_{\mathfrak{s}^\vee}$-invariance of $\alpha \mapsto m_\alpha$, it suffices to consider simple roots α, β in one irreducible component of $R(J^\circ, T)_{\mathrm{red}}$. If they have the same length, then they are $W_{\mathfrak{s}^\vee}$-associate and $m_\alpha = m_\beta$. That leaves the case where their lengths differ, say α is longer. Replacing α and β by $W(R(J^\circ, T))$-associate simple roots, we can achieve that they are not orthogonal and $\langle \alpha^\vee , \beta \rangle = -1$. Then (3.34) and (3.35) entail that $m_\beta \geq m_\alpha$. More precisely, in that case

$$\langle \beta^\vee , \alpha \rangle = - \|\alpha\|^2 \|\beta\|^{-2} \in \{-1, -2, -3\}$$

and $m_\alpha \in m_\beta \langle \beta^\vee , \alpha \rangle^{-1} \mathbb{Z}$, so $m_\alpha = m_\beta$ or $m_\alpha = \|\alpha\|^{-2} \|\beta\|^2 m_\beta$.
(c) In view of the covering map (3.29), it suffices to show that $\alpha^\vee / m_\alpha$ defines a

cocharacter of T/T_{ϕ_1}. From the finite covering $T \to T/T_{\phi_1}$ we obtain inclusions

$$\begin{array}{lllll} X_*(T) & \subset & X_*(T/T_{\phi_1}) & \subset & X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}, \\ X^*(T/T_{\phi_1}) & \subset & X^*(T) & \subset & X^*(T/T_{\phi_1}) \otimes_{\mathbb{Z}} \mathbb{Q}. \end{array}$$

The reflection s_α acts on $X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ by

$$x \mapsto s_\alpha(x) = x - \langle x, \alpha \rangle \alpha^\vee = x - \langle x, m_\alpha \alpha \rangle \alpha^\vee / m_\alpha. \tag{3.36}$$

Since $s_\alpha \in W_{\mathfrak{s}^\vee}$ normalizes T_{ϕ_1}, the action (3.36) stabilizes $X_*(T/T_{\phi_1})$.

Suppose that m_α is the smallest positive integer such that $m_\alpha \alpha$ descends to a character of T/T_{ϕ_1}. Then $m_\alpha \alpha$ is indivisible in $X_*(T/T_{\phi_1})$, so there exists an $x \in X^*(T/T_{\phi_1})$ such that $\langle x, m_\alpha \alpha \rangle = 1$. For that x, (3.36) shows that

$$\alpha^\vee / m_\alpha = x - s_\alpha(x) \in X_*(T/T_{\phi_1}).$$

When this characterization of m_α does not hold, then $\alpha/2 \in X^*(T)$ and $m_\alpha \alpha/2$ is the smallest multiple of α that descends to a character of T/T_{ϕ_1} (as remarked after Definition 3.11). As $\alpha \in 2X^*(T)$, the irreducible component R_1 of $R(J^\circ, T)_{\mathrm{red}}$ that contains α has type C_n, and it is contained in a direct summand $\mathbb{Z}^n$ of $X^*(T)$. Then $X^*(T/T_{\phi_1})$ contains $m_\alpha \alpha'/2$ for every long root $\alpha' \in R_1$, so it contains $m_\alpha \beta$ for every short root $\beta \in R_1$. From this and part (b) we deduce that $m_\beta = m_\alpha$ (provided that C_n has rank > 1 so that it has short roots). It follows that $\mathbb{Q}R_1 \cap X^*(T/T_{\phi_1}) = m_\alpha \mathbb{Z}^n$. In particular

$$X^*(T/T_{\phi_1}) = (\alpha^\vee)^\perp \oplus m_\alpha \alpha/2.$$

Since the pairing between $X_*(T/T_{\phi_1})$ and $X^*(T/T_{\phi_1})$ is perfect, there exists $y \in X_*(T/T_{\phi_1}) \cap (\alpha^\vee)^{\perp\perp}$ with $\langle y, m_\alpha \alpha/2 \rangle = 1$. This y is $\alpha^\vee / m_\alpha$. $\square$

Lemma 3.12 implies that

$$\mathcal{R}_{\mathfrak{s}^\vee} := \big(R_{\mathfrak{s}^\vee}, X^*(T_{\mathfrak{s}^\vee}), R_{\mathfrak{s}^\vee}^\vee, X_*(T_{\mathfrak{s}^\vee}) \big) \tag{3.37}$$

is a root datum with an action of $W_{\mathfrak{s}^\vee} \supset W_{\mathfrak{s}^\vee}^\circ = W(R_{\mathfrak{s}^\vee})$.

Lemma 3.13. *Let $\alpha \in R(J^\circ, T)_{\mathrm{red}}$ and $t \in T$.*

(*a*) *If $(m_\alpha \alpha)(t) = 1$, then $\alpha \in R(G^\circ_{t\phi_1}, T)$.*

(*b*) *Suppose that $R(G^\circ_{t\phi_1}, T)$ contains α or 2α. Then $(m_\alpha \alpha)(t) = 1$ or $(m_\alpha \alpha)(t) = -1$ and $(m_\alpha \alpha)^\vee \in 2X_*(T_{\mathfrak{s}^\vee})$.*

Proof. (a) Suppose that $m_\alpha \alpha$ is the smallest multiple of α that descends to a character of T/T_{ϕ_1}. From $\alpha(t)^{m_\alpha} = 1$ we see that $\alpha(t) \in \alpha(T_{\phi_1})$. In particular there exists a $t' \in tT_{\phi_1}$ with $\alpha(t') = 1$. By the definition of T_{ϕ_1}, $t\phi_1$ and $t'\phi_1$ are $\mathcal{L}^\vee$-conjugate. Hence $\alpha \in R(G^\circ_{t'\phi_1}, T) = R(G^\circ_{t\phi_1}, T)$.

When this characterization of m_α does not hold, we need a more involved argument. Following Definition 3.11, we write $m_\alpha = k m_\alpha(\mathrm{Frob}_F^k)$. This means that k irreducible components of $R(J^\circ, T_J)$ are relevant for α, and they are permuted transitively by $\phi_1(\mathrm{Frob}_F)$. Now α is a root for $(G^\circ_{t\phi_1}, T)$ if and only if α is a root for $\mathrm{Z}_J((t\phi_1(\mathrm{Frob}_F))^k)^\circ$, the version of $G^\circ_{t\phi_1}$ with Frob_F^k instead of Frob_F. More precisely, the root subspaces for α in these two groups are naturally in bijection. Since T centralizes $\phi_1(\mathrm{Frob}_F) \in \mathcal{L}^\vee \rtimes \mathbf{W}_F$,

$$(t\phi_1(\mathrm{Frob}_F))^k = t^k \phi_1(\mathrm{Frob}_F^k).$$

The root subspace for α and $\mathrm{Z}_J((t\phi_1(\mathrm{Frob}_F))^k)^\circ$ depends only t^k (regarding ϕ_1 as fixed). By assumption

$$(m_\alpha(\mathrm{Frob}_F^k)\alpha)(t^k) = (k^{-1}m_\alpha\alpha)(t^k) = (m_\alpha\alpha)(t) = 1.$$

This brings us back to a situation analogous to the first part of the proof, and we conclude as over there.
(b) The reflection s_α stabilizes $t\phi_1 \in \mathfrak{s}_L^\vee \cong T_{\mathfrak{s}^\vee}$. Thus s_α fixes t considered as element of $T_{\mathfrak{s}^\vee}$. As $\mathcal{R}_{\mathfrak{s}^\vee}$ is a root datum, we have reduced to the well-known setting of Weyl groups acting on complex tori associated to root data. In that setting we conclude with [Lus3, Lemma 3.15]. □

We endow $\mathcal{R}_{\mathfrak{s}^\vee}$ with the set of simple roots determined by the Borel subgroup $B_J \subset J^\circ$. We look for parameter functions λ and λ^* on $\mathcal{R}_{\mathfrak{s}^\vee}$ which are compatible with specialization to the graded Hecke algebras from Paragraph 3.1. Recall from (2.3) that $\lambda^*(\alpha)$ is defined to be $\lambda(\alpha)$ unless α is a short root in a type B root subsystem of $R_{\mathfrak{s}^\vee}$.

Proposition 3.14. (*a*) *There exist unique $W_{\mathfrak{s}^\vee}$-invariant parameter functions*

$$\lambda\colon R_{\mathfrak{s}^\vee} \to \mathbb{Q}_{>0}, \qquad \lambda^*\colon \{m_\alpha\alpha \in R_{\mathfrak{s}^\vee} : (m_\alpha\alpha)^\vee \in 2X_*(T_{\mathfrak{s}^\vee})\} \to \mathbb{Q}$$

such that, for every $(\phi_b|_{\mathbf{W}_F}, v, q\epsilon) \in \mathfrak{s}_{\mathcal{L}}^\vee$ with ϕ_b bounded, the reduction of $\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}})$ via Theorems 2.5 and 2.11 gives the subalgebra $\mathbb{H}\big(\mathrm{Lie}(X_{\mathrm{nr}}(^L\mathcal{L})), (W_{\mathfrak{s}^\vee}^\circ)_{\phi_b}, c\vec{\mathbf{r}}\big)$ of $\mathbb{H}(\phi_b, v, q\epsilon, \vec{\mathbf{r}})$ from (3.12).

(*b*) *The basepoint ϕ_1 of $\mathfrak{s}_{\mathcal{L}}^\vee$ can be chosen so that λ has image in $\mathbb{Z}_{>0}$, λ^* has image in $\mathbb{Z}_{\geq 0}$ and $\lambda \geq \lambda^*$ on the domain of λ^*.*

(*c*) *Let $t'\phi_1$ be another basepoint as in part (b), and let $m_\alpha\alpha \in R_{\mathfrak{s}^\vee}$. Then $(m_\alpha\alpha)(t')$ equals 1 or $(m_\alpha\alpha)(t') = -1, s_\alpha(t') = t'$ and $\lambda^*(m_\alpha\alpha) = 0$.*

Proof. (a) The aforementioned reduction produces twisted graded Hecke algebras with vector space $\mathrm{Lie}(T_{\mathfrak{s}^\vee}) = \mathrm{Lie}(X_{\mathrm{nr}}(^L\mathcal{L}))$, finite group $(W_{\mathfrak{s}^\vee}^\circ)_{\phi_b}$ and trivial 2-cocycle, as required. However, the roots for the resulting graded Hecke algebra are $m_\alpha\alpha$, whereas in (3.12) the root system is contained in $R(J^\circ, T)$. We reconcile this by imposing $c(m_\alpha\alpha) = m_\alpha c(\alpha)$, which is allowed because it preserves the braid relations in a graded Hecke algebra (Proposition 1.1).

For $\phi_b = \phi_1$, (2.23) imposes the conditions

$$\lambda(m_\alpha\alpha) + \lambda^*(m_\alpha\alpha) = m_\alpha c(\alpha) \qquad \alpha \in R(J^\circ, T)_{\mathrm{red}}, \tag{3.38}$$

where $c(\alpha) \in \mathbb{Z}_{>0}$ is computed as in Proposition 2.1, with respect to $G_{\phi_1}^\circ$. Given $\phi_1\big|_{\mathbf{I}_F}, v$ and $q\epsilon$, the value of $c(\alpha)$ depends only on the root subspaces for α and 2α in $G_{t\phi_1}$. The proof of Lemma 3.13.a shows that these root subspaces depend (up the isomorphism) only on $(m_\alpha\alpha)(t)$.

In view of Lemma 3.13.b, we need to consider at most two values of $c(\alpha)$ for $\phi_b \in \mathfrak{s}_{\mathcal{L}}^\vee$: one for ϕ_1 and maybe another one, say $c^*(\alpha)$, for a $t\phi_1$ with $(m_\alpha\alpha)(t) = -1$. When $R(G_{t\phi_1}^\circ, T)$ contains 2α but not α, we must rescale $c^*(\alpha) = c(2\alpha)/2$ so that it really refers to α like $c(\alpha)$.

When $(m_\alpha\alpha)^\vee \notin 2X_*(T_{\mathfrak{s}^\vee})$, Lemma 3.13 says that $R(G_{t\phi_1}^\circ, T)$ contains α or 2α if and only if $(m_\alpha\alpha)(t) = 1$. By convention $\lambda^*(m_\alpha\alpha) = \lambda(m_\alpha\alpha)$, and the only way to solve (3.38) is setting

$$\lambda(m_\alpha\alpha) = c(\alpha)m_\alpha/2 \in \mathbb{Q}_{>0}. \tag{3.39}$$

Next consider an $\alpha \in R(J^\circ, T)_{\text{red}}$ with $(m_\alpha \alpha)^\vee \in 2X_*(T_{\mathfrak{s}^\vee})$. Then $s_{m_\alpha \alpha}$ fixes $t\phi_1$ if $(m_\alpha \alpha)(t) = -1$, so we have to consider $c^*(\alpha) \in \mathbb{Z}_{\geq 0}$. If α and 2α do not belong to $R(G^\circ_{t\phi_1}, T)$ for one such t, then the above argument shows that they do not lie in $R(G^\circ_{t\phi_1}, T)$ for any such t. In that case, in the twisted graded Hecke algebra $\mathbb{H}(t\phi_1, v, q\epsilon, \vec{\mathbf{r}})$ the element N_{s_α} satisfies a braid relation with trivial parameter $c^*(\alpha) := 0$.

For any $t \in T$ with $(m_\alpha \alpha)(t) = -1$, (2.23) imposes the new condition

$$\lambda(m_\alpha \alpha) - \lambda^*(m_\alpha \alpha) = m_\alpha c^*(\alpha) \in \mathbb{Z}_{>0}. \tag{3.40}$$

Clearly (3.38) and (3.40) admit the unique solution

$$\lambda(m_\alpha \alpha) = (c(\alpha) + c^*(\alpha)) m_\alpha / 2, \quad \lambda^*(m_\alpha \alpha) = (c(\alpha) - c^*(\alpha)) m_\alpha / 2. \tag{3.41}$$

We address the $W_{\mathfrak{s}^\vee}$-invariance. Represent $\gamma \in W_{\mathfrak{s}^\vee}$ in $\mathrm{N}_J(T)$ as in (3.25). Then it acts on the entire setting by conjugation, so $\lambda \circ \gamma$ and $\lambda^* \circ \gamma$ are parameter functions which also fulfill the requirements with respect to reduction to graded Hecke algebras. With the uniqueness of the solutions to the above equations, we find that $\lambda \circ \gamma = \lambda$ and $\lambda^* \circ \gamma = \lambda^*$.

(b) By (3.39) and (3.40) we have

$$\lambda(m_\alpha \alpha) > 0 \quad \text{and} \quad \lambda(m_\alpha \alpha) \geq \lambda^*(m_\alpha \alpha) \quad \text{for all } m_\alpha \alpha \in R_{\mathfrak{s}^\vee}.$$

If $c^*(\alpha) > c(\alpha)$, then we exchange them. This can be achieved with the method from the proof of Proposition 3.9.b: take a new basepoint $\phi_{t'}$ such that $(m_\alpha \alpha)(t') = -1$ while t' lies in the kernel of every other simple roots of $R(J^\circ, T)$. This assures that λ^* takes values in $\mathbb{Q}_{\geq 0}$.

Case 1: $(m_\alpha \alpha)^\vee \notin 2X_*(T_{\mathfrak{s}^\vee})$

When $2\alpha \notin R(J^\circ, T)$, Proposition 2.1.a ensures that $c(\alpha)$ is even. When $2\alpha \in R(J^\circ, T)$ and still $(m_\alpha \alpha)^\vee \notin 2X_*(T_{\mathfrak{s}^\vee})$, Lemma 3.12 shows that the relevant irreducible components of $R(J^\circ, T)$ and $R_{\mathfrak{s}^\vee}$ have type BC_n and C_n, respectively. In particular $m_\alpha = 2m_\beta$ for any other simple root in the same component of $R(J^\circ, T)$, and m_α is even. Hence (3.39) is always an integer.

Case 2: $(m_\alpha \alpha)^\vee \in 2X_*(T_{\mathfrak{s}^\vee}), 2\alpha \notin R(G^\circ_{\phi_1}, T)$

In view of (3.41), we need to show that

$$(c(\alpha) \pm c^*(\alpha)) m_\alpha \qquad \text{is even.} \tag{3.42}$$

By Proposition 2.1.a, $c(\alpha)$ is even. Select $t \in T$ with $(m_\alpha \alpha)(t) = -1$.

- If α lies in $R(G^\circ_{t\phi_1}, T)$ but 2α does not, then $c^*(\alpha)$ is also even.
- If $\alpha, 2\alpha \notin R(G^\circ_{t\phi_1}, T)$ then we argued in the proof of part (a) that $c^*(\alpha) = 0$.
- Suppose 2α lies in $R(G^\circ_{t\phi_1}, T)$. If m_α would be odd, we could arrange that $\alpha(t) = -1$. Then $(2\alpha)(t) = 1$, so 2α would lie in both $R(G^\circ_{t\phi_1}, T)$ and $R(G^\circ_{\phi_1}, T)$. That contradicts our assumptions, so m_α is even. By Proposition 2.1 either $c^*(\alpha) = c_t(2\alpha)/2$ or $c^*(\alpha) = c(\alpha)$ and it is always an integer.

In all these three instances (3.42) holds.

Case 3: $(m_\alpha \alpha)^\vee \in 2X_*(T_{\mathfrak{s}^\vee}), 2\alpha \in R(G^\circ_{\phi_1}, T)$

Again we need to verify (3.42), and we pick a $t \in T$ with $(m_\alpha \alpha)(t) = -1$. By Proposition 2.1.b, $c(\alpha)$ is odd.

- If $\alpha, 2\alpha \in R(G^\circ_{t\phi_1}, T)$, then $c^*(\alpha)$ is also odd.

- Suppose m_α is even and not $\alpha, 2\alpha \in R(G^\circ_{t\phi_1}, T)$. If $\alpha, 2\alpha \notin R(G^\circ_{t\phi_1}, T)$, then we argued in the proof of part (a) that $c^*(\alpha) = 0$. Otherwise, by Proposition 2.1 either $c^*(\alpha) = c(\alpha)$ or $c^*(\alpha) = c_t(2\alpha)/2$, and this is always an integer.
- Suppose m_α is odd and not $\alpha, 2\alpha \in R(G^\circ_{t\phi_1}, T)$. Here we can arrange that $\alpha(t) = -1$, so that $(2\alpha)(t) = 1$. Then the root subspace $\mathfrak{g}_{2\alpha}$ is the same for $G^\circ_{\phi_1}$ as for $G^\circ_{t\phi_1}$, so $2\alpha \in R(G^\circ_{t\phi_1}, T) \not\ni \alpha$ and $c^*(\alpha)$ can be computed from $G^\circ_{\phi_1}$ alone. By Proposition 2.1.b $c^*(\alpha) = c_t(2\alpha)/2 = 1$, which is odd.

In these three instances, (3.42) is indeed valid.
(c) By Proposition 3.9.b $s_\alpha(t') = t'$, so $(m_\alpha \alpha)(t') \in \{\pm 1\}$. Suppose that $(m_\alpha \alpha)(t') = -1$ and $\lambda^*(m_\alpha \alpha) \neq 0$. Then $c(\alpha) > c^*(\alpha)$ with respect to ϕ_1, but $c(\alpha) < c^*(\alpha)$ with respect to $t'\phi_1$. That would give $\lambda^*(m_\alpha \alpha) < 0$ with respect to $t'\phi_1$, which disqualifies $t'\phi_1$ as basepoint in the sense of part (b). So if $(m_\alpha \alpha)(t') = -1$, then we must have $\lambda^*(m_\alpha \alpha) = 0$. □

3.3. Affine Hecke algebras.

Recall that $W_{\mathfrak{s}^\vee}$ acts naturally on the root system $R(J^\circ, T)$. Let $R^+(J^\circ, T)$ be the positive system defined by the $\phi(\mathrm{Frob}_F)$-stable Borel subgroup B_J of J°. Any two such B_J are J°-conjugate, so the choice is inessential. This also determines the positive (co-)roots for the root datum $\mathcal{R}_{\mathfrak{s}^\vee}$ from (3.27), (3.32) and (3.37).

Since $W^\circ_{\mathfrak{s}^\vee}$ acts simply transitively on the collection of positive systems for $R(J^\circ, T)$, we obtain a semi-direct factorization

$$W_{\mathfrak{s}^\vee} = W^\circ_{\mathfrak{s}^\vee} \rtimes \mathfrak{R}_{\mathfrak{s}^\vee},$$
$$\mathfrak{R}_{\mathfrak{s}^\vee} = \{w \in W_{\mathfrak{s}^\vee} : w R^+(J^\circ, T) = R^+(J^\circ, T)\}.$$

To $\mathfrak{s}^\vee$ we can associate the affine Hecke algebra $\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}})$, where ϕ_1 is as in Proposition 3.14 and λ and λ^* satisfy (3.38) and (3.40). However, this algebra takes only the subgroup $W^\circ_{\mathfrak{s}^\vee}$ of $W_{\mathfrak{s}^\vee}$ into account. To see $W_{\mathfrak{s}^\vee, \phi_1}$, we can enlarge it to

$$\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}}) \rtimes \mathbb{C}[\mathfrak{R}_{\mathfrak{s}^\vee, \phi_1, v, q\epsilon}, \natural_{\mathfrak{s}^\vee, \phi_1, v, q\epsilon}] \ = \tag{3.43}$$

$$\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}}) \rtimes \mathrm{End}^+_{\mathcal{P}_{G_{\phi_1}} \mathrm{Lie}(G_{\phi_1})_{\mathrm{RS}}} \big(q\pi_* (\widetilde{q\mathcal{E}}) \big). \tag{3.44}$$

But $W_{\mathfrak{s}^\vee}$ can also contain elements that do not fix ϕ_1. In fact, in some cases $W_{\mathfrak{s}^\vee}$ even acts freely on $T_{\mathfrak{s}^\vee}$, see [ABPS3, Example 5.3].

Proposition 3.15. *Assume that the almost direct factorization* (1.2) *of* J° *induces a decomposition of* $R(J^\circ, T)$ *which is* $W_{\mathfrak{s}^\vee}$*-stable.*
(*a*) *The group* $\mathfrak{R}_{\mathfrak{s}^\vee}$ *acts canonically on* $\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}})$*, by algebra automorphisms.*
(*b*) *This can be realized in a twisted affine Hecke algebra*

$$\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}}) \rtimes \mathbb{C}[\mathfrak{R}_{\mathfrak{s}^\vee}, \natural_{\mathfrak{s}^\vee}] = \mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}}) \rtimes \mathrm{End}^+_{\mathcal{P}_J \mathrm{Lie}(J)_{\mathrm{RS}}} \big(q\pi_* (\widetilde{q\mathcal{E}}) \big)$$

in which (3.43) *is canonically embedded.*

Proof. (a) The action of $\mathfrak{R}_{\mathfrak{s}^\vee}$ on $T_{\mathfrak{s}^\vee}$ comes from (3.28). This determines an action on $\mathcal{O}(T_{\mathfrak{s}^\vee}) \cong \mathbb{C}[X^*(T_{\mathfrak{s}^\vee})]$. Any $\gamma \in \mathfrak{R}_{\mathfrak{s}^\vee}$ maps θ_x to an invertible element of $\mathbb{C}[X^*(T_{\mathfrak{s}^\vee})]$. That is,

$$\gamma \cdot \theta_x = \theta_{\gamma x} \lambda_{\gamma, x} \quad \text{with} \quad \lambda_{\gamma, x} \in \mathbb{C}^\times.$$

The linear part $x \mapsto \gamma x$ is an automorphism of $X^*(T_{\mathfrak{s}^\vee})$, and the translation part of $\gamma \colon T_{\mathfrak{s}^\vee} \to T_{\mathfrak{s}^\vee}$ is given by $\lambda_{\gamma^{-1}, x} = x(\gamma(1))$. Since $W^\circ_{\mathfrak{s}^\vee}$ is normal in $W_{\mathfrak{s}^\vee}$,

$$(W^\circ_{\mathfrak{s}^\vee})_{\gamma(1)} = \big(\gamma W^\circ_{\mathfrak{s}^\vee} \gamma^{-1} \big)_1 = (W^\circ_{\mathfrak{s}^\vee})_1 = W^\circ_{\mathfrak{s}^\vee}.$$

More precisely, writing

$$R_{\mathfrak{s}^\vee} = \{m_\alpha \alpha : \alpha \in R(G^\circ_{\phi_1}, T)_{\mathrm{red}}\} =: mR(G^\circ_{\phi_1}, T)_{\mathrm{red}}$$

as in (3.32) and Proposition 3.9.b, we have

$$\begin{aligned} \{\alpha \in R_{\mathfrak{s}^\vee} : \alpha(\gamma(1)) = 1\} &\cong mR(G^\circ_{\gamma(1)\phi_1}, T)_{\mathrm{red}} = mR(\gamma G^\circ_{\phi_1} \gamma^{-1}, T)_{\mathrm{red}} \\ &= \gamma\big(mR(G^\circ_{\phi_1}, T)_{\mathrm{red}}\big) \cong \gamma(R_{\mathfrak{s}^\vee}) = R_{\mathfrak{s}^\vee}. \end{aligned}$$

Hence the elements $\lambda_{\gamma^{-1}} = \gamma(1) \in T$ fulfill the conditions in Proposition 2.2.

According to [AMS1, Lemma 9.2] there exists a canonical algebra isomorphism

$$\psi_{\gamma,\phi_1} \colon \mathbb{C}[W_{\mathfrak{s}^\vee,\phi_1}, \kappa_{\phi_1,v,q\epsilon}] \to \mathbb{C}[W_{\mathfrak{s}^\vee,\gamma(\phi_1)}, \kappa_{\gamma(\phi_1),v,q\epsilon}].$$

Let us recall its construction. There is a G_{ϕ_1}-equivariant local system $q\pi_*(\widetilde{q\mathcal{E}})$ on $(G_{\phi_1})_{\mathrm{RS}}$, an analogue of K and K^*. It satisfies

$$\mathbb{C}[W_{\mathfrak{s}^\vee,\phi_1}, \kappa_{\phi_1,v,q\epsilon}] \cong \mathrm{End}_{\mathcal{D}(G_{\phi_1})_{\mathrm{RS}}}(q\pi_*(\widetilde{q\mathcal{E}})). \tag{3.45}$$

Choosing a lift $n_\gamma \in \mathrm{N}_{G_{\phi_1}}(M)$ of γ and following the proof of [AMS1, Lemma 5.4], we find an isomorphism

$$qb_\gamma \colon q\pi_*(\widetilde{q\mathcal{E}}) \to q\pi_*\big(\widetilde{\mathrm{Ad}(n_\gamma)^* q\mathcal{E}}\big). \tag{3.46}$$

Then $\psi_{\gamma,\phi_1,v,q\epsilon}$ is conjugation with qb_γ.

In this context [AMS1, Lemma 5.4] says that there are canonical elements $qb_w \in \mathrm{End}_{\mathcal{D}(G_{\phi_1})_{\mathrm{RS}}}(q\pi_*(\widetilde{q\mathcal{E}}))$ $(w \in W^\circ_{\mathfrak{s}^\vee})$ which via (3.45) become a basis of $\mathbb{C}[W^\circ_{\mathfrak{s}^\vee}]$. Since $W^\circ_{\mathfrak{s}^\vee}$ is normal in $W_{\mathfrak{s}^\vee}$, $\psi_{\gamma,\phi_1,v,q\epsilon}$ stabilizes the set $\{qb_w \colon W^\circ_{\mathfrak{s}^\vee}\}$. Moreover $\gamma \in \mathfrak{R}_{\mathfrak{s}^\vee}$, so $\psi_{\gamma,\phi_1,v,q\epsilon}$ permutes the set of simple reflections in $W^\circ_{\mathfrak{s}^\vee}$.

By Proposition 3.14 the parameter functions λ and λ^* are $W_{\mathfrak{s}^\vee}$-invariant, so by Proposition 2.2 $\mathfrak{R}_{\mathfrak{s}^\vee}$ acts canonically on $\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}})$ by algebra automorphisms.
(b) The same construction as in the proof of Proposition 2.2 yields an algebra

$$\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}}) \rtimes \mathbb{C}[\mathfrak{R}_{\mathfrak{s}^\vee}, \natural_{\mathfrak{s}^\vee}], \tag{3.47}$$

in which the action of $\mathfrak{R}_{\mathfrak{s}^\vee}$ on $\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}})$ has become an inner automorphism. This works for any 2-cocycle $\natural_{\mathfrak{s}^\vee}$. It only remains to pick it in a good way, such that $\natural_{\mathfrak{s}^\vee}|_{(W_{\mathfrak{s}^\vee,\phi,v,q\epsilon})^2}$ equals $\kappa_{\mathfrak{s}^\vee,\phi,v,q\epsilon}$ (which is $\natural_{q\mathcal{E}}$ for G_ϕ). For this we, again, use the maps qb_γ from (3.46). The cuspidal local system $\mathrm{Ad}(n_\gamma)^* q\mathcal{E}$ does not depend on the choice of n_γ, because $q\mathcal{E}$ is M-equivariant. Furthermore qb_γ is unique up to scalars, so

$$qb_\gamma \cdot qb_{\gamma'} = \lambda_{\gamma,\gamma'}\, qb_{\gamma\gamma'} \text{ for a unique } \lambda_{\gamma,\gamma'} \in \mathbb{C}^\times.$$

We define $\natural_{\mathfrak{s}^\vee}$ by $\natural_{\mathfrak{s}^\vee}(\gamma, \gamma') = \lambda_{\gamma,\gamma'}$. This is a slight generalization of the construction in Section 1 and in [AMS1, Lemma 5.4]. As over there,

$$\begin{aligned} \mathrm{End}_{\mathcal{P}_J \mathrm{Lie}(J)_{\mathrm{RS}}}\big(q\pi_*(\widetilde{q\mathcal{E}})\big) &\cong \mathbb{C}[W_{\mathfrak{s}^\vee}, \natural_{\mathfrak{s}^\vee}], \\ \mathrm{End}^+_{\mathcal{P}_J \mathrm{Lie}(J)_{\mathrm{RS}}}\big(q\pi_*(\widetilde{q\mathcal{E}})\big) &\cong \mathbb{C}[\mathfrak{R}_{\mathfrak{s}^\vee}, \natural_{\mathfrak{s}^\vee}]. \end{aligned}$$

As the J-equivariant sheaf $q\pi_*(\widetilde{q\mathcal{E}})$ on $\mathrm{Lie}(J)_{\mathrm{RS}}$ contains the G_ϕ-equivariant sheaf $q\pi_*(\widetilde{q\mathcal{E}})$ on $\mathrm{Lie}(G_\phi)_{\mathrm{RS}}$,

$$\natural_{\mathfrak{s}^\vee} \colon (W_{\mathfrak{s}^\vee})^2 \to (W_{\mathfrak{s}^\vee}/W^\circ_{\mathfrak{s}^\vee})^2 = \mathfrak{R}^2_{\mathfrak{s}^\vee} \to \mathbb{C}^\times \tag{3.48}$$

extends $\kappa_{\mathfrak{s}^\vee,\phi,v,q\epsilon} \colon (W_{\mathfrak{s}^\vee,\phi})^2 \to \mathbb{C}^\times$, for every $(\phi|_{\mathbf{W}_F}, v, q\epsilon) \in \mathfrak{s}^\vee_{\mathcal{L}}$. For $\phi = \phi_1$ this means that

$$\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}}) \rtimes \mathbb{C}[W_{\mathfrak{s}^\vee,\phi_1}, \kappa_{\mathfrak{s}^\vee,\phi_1,v,q\epsilon}].$$

is canonically embedded in (3.47). □

The algebra from Proposition 3.15.b is attached to $\mathfrak{s}^\vee$ and the basepoint ϕ_1 of $\mathfrak{s}^\vee_{\mathcal{L}}$, chosen as in Proposition 3.14.b. To remove the dependence on the basepoint, we reinterpret $\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}})$. Recall that $W_{\mathfrak{s}^\vee}$ acts naturally on $\mathfrak{s}^\vee_{\mathcal{L}}$ (which is diffeomorphic to $T_{\mathfrak{s}^\vee}$). We can replace $\mathbb{C}[X^*(T_{\mathfrak{s}^\vee})]$ by $\mathcal{O}(\mathfrak{s}^\vee_{\mathcal{L}})$, then (i)–(iv) in Proposition 2.2 still work equally well.

Every $\alpha \in R_{\mathfrak{s}^\vee}$ is by definition a character of $T_{\mathfrak{s}^\vee}$, so via the choice of ϕ_1 it becomes a function on $\mathfrak{s}^\vee_{\mathcal{L}}$. By Proposition 3.14.c, an alternative basepoint $t'\phi_1$ gives the same function α on $\mathfrak{s}^\vee_{\mathcal{L}}$, or leads to the new coordinate $\alpha(t')\alpha = -\alpha$ on $\mathfrak{s}^\vee_{\mathcal{L}}$. In the latter case, the property $\lambda^*(\alpha) = 0$ from Proposition 3.14.c ensures that the multiplication rule (v) from Proposition 2.2 is the same with respect to the coordinates α and $-\alpha$. This entails that Proposition 2.2 defines an algebra structure on

$$\mathcal{O}(\mathfrak{s}^\vee_{\mathcal{L}}) \otimes \mathbb{C}[\vec{\mathbf{z}}, \vec{\mathbf{z}}^{-1}] \otimes \mathbb{C}[W^\circ_{\mathfrak{s}^\vee}],$$

independent of ϕ_1. We call this algebra $\mathcal{H}(\mathfrak{s}^\vee_{\mathcal{L}}, W^\circ_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}})$. It is isomorphic to $\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}})$, but only via the choice of a basepoint of $\mathfrak{s}^\vee_{\mathcal{L}}$. In Proposition 3.15.a we showed that $\mathfrak{R}_{\mathfrak{s}^\vee}$ acts naturally on $\mathcal{H}(\mathfrak{s}^\vee_{\mathcal{L}}, W^\circ_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}})$. Applying Proposition 3.15.b, we obtain an algebra

$$\mathcal{H}(\mathfrak{s}^\vee_{\mathcal{L}}, W^\circ_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}}) \rtimes \mathrm{End}^+_{\mathcal{P}_J \mathrm{Lie}(J)_{\mathrm{RS}}}\big(q\pi_*(\widetilde{q\mathcal{E}})\big), \text{ where } J = \mathrm{Z}^1_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi|_{\mathbf{I}_F}). \tag{3.49}$$

Now we suppose that the almost direct factorization of J° induces a $W_{\mathfrak{s}^\vee}$-stable decomposition of $R(J^\circ, T)$ (and, equivalently, of $R_{\mathfrak{s}^\vee}$). We focus on two algebras obtained in this way:

- $\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z})$, the algebra (3.49) when $J_1 = J^\circ_{\mathrm{der}}$, with only one variable $\mathbf{z}$;
- $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$, the algebra (3.49) when (1.2) induces the finest possible $W_{\mathfrak{s}^\vee}$-stable decomposition of $R(J^\circ, T)$.

Lemma 3.16. *The algebras $\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z})$ and $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$ depend only on $\mathfrak{s}^\vee$, up to canonical isomorphisms.*

Proof. The above construction shows that $\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z})$ and $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$ are uniquely determined by $(\mathfrak{s}^\vee_{\mathcal{L}}, B_J)$, where B_J serves only to determine the positive roots in $R(J^\circ, T)$. Up to $\mathcal{G}^\vee$-conjugation, this pair is completely determined by $\mathfrak{s}^\vee$.

The $\mathcal{G}^\vee_{\mathrm{sc}}$-normalizer of $\mathfrak{s}^\vee_{\mathcal{L}}$ is contained in J, and the pointwise stabilizer of $\mathfrak{s}^\vee_{\mathcal{L}}$ in J is just M, see (3.18) and (3.23). Given $\mathfrak{s}^\vee_{\mathcal{L}}$ and M, [AMS2, Lemma 2.1.a] shows that all possible choices for B_J are conjugate by unique elements of $\mathrm{N}_{J^\circ}(M^\circ)/M^\circ$. Thus all possible $({\mathfrak{s}'}^\vee_{\mathcal{L}}, B'_J)$ underlying $\mathfrak{s}^\vee$ are conjugate to $(\mathfrak{s}^\vee_{\mathcal{L}}, B_J)$ in a canonical way. Any element of $\mathcal{G}^\vee_{\mathrm{sc}}$ which realizes such a conjugation provides a canonical isomorphism between $\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z})$ (respectively $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$) and its version based on $({\mathfrak{s}'}^\vee_{\mathcal{L}}, B'_J)$. □

Example 3.17. Suppose that (ϕ, ρ) is itself cuspidal, so $\mathcal{L}^\vee = \mathcal{G}^\vee$ and $q\epsilon = \rho$. Then $J^\circ = M^\circ$, v is distinguished in that group, $T = 1$ and $R(J^\circ, T)$ is empty. Furthermore $W_{\mathfrak{s}^\vee} = 1$ because $\mathrm{N}_{\mathcal{G}^\vee}(\mathcal{L}^\vee \rtimes \mathbf{W}_F)/\mathcal{L}^\vee = 1$. Consequently

$$\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z}) = \mathcal{O}(T_{\mathfrak{s}^\vee}) \otimes \mathbb{C}[\mathbf{z}, \mathbf{z}^{-1}] \quad \text{and} \quad \mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}}) = \mathcal{O}(T_{\mathfrak{s}^\vee}) \otimes \mathbb{C}[\mathbf{z}_1, \mathbf{z}_1^{-1}, \ldots, \mathbf{z}_d, \mathbf{z}_d^{-1}],$$

where d is the number of simple factors of J°_{der}.

For (ϕ,ρ) as in (3.13), let $\bar M(\phi,\rho,\vec z)$ be the irreducible $\mathcal H(\mathfrak s^\vee,\vec{\mathbf z})$-module obtained from $M(\phi,\rho,\log\vec z) \in \mathrm{Irr}(\mathbb H(\phi_b,v,q\epsilon,\vec{\mathbf r}))$ via Theorems 2.5 and 2.11. Up to $\mathcal G^\vee$-conjugation, every element of $\Phi_{\mathrm e}(\mathcal G(F))^{\mathfrak s^\vee}$ is of the form described in (3.13), so this definition extends naturally to all possible (ϕ,ρ). Similarly we define $\bar E(\phi,\rho,\vec z)$ as the "standard" $\mathcal H(\mathfrak s^\vee,\vec{\mathbf z})$-module obtained from $E(\phi,\rho,\log\vec z)\in \mathrm{Mod}(\mathbb H(\phi_b,v,q\epsilon,\vec{\mathbf r}))$ via Theorems 2.5 and 2.11.

We formulate the next result only for $\mathcal H(\mathfrak s^\vee,\vec{\mathbf z})$, but there is also a version for $\mathcal H(\mathfrak s^\vee,\mathbf z)$. In view of (2.4), the latter can be obtained by assuming that all z_j are equal.

Theorem 3.18. (*a*) *For every $\vec z\in\mathbb R^d_{>0}$ there exists a canonical bijection*

$$\Phi_{\mathrm e}(\mathcal G(F))^{\mathfrak s^\vee} \to \mathrm{Irr}_{\vec z}(\mathcal H(\mathfrak s^\vee,\vec{\mathbf z})) : \quad (\phi,\rho)\mapsto \bar M(\phi,\rho,\vec z).$$

(*b*) *Both $\bar M(\phi,\rho,\vec z)$ and $\bar E(\phi,\rho,\vec z)$ admit the central character $W_{\mathfrak s^\vee}(\tilde\phi|_{\mathbf W_F},v,q\epsilon)\in \Phi_{\mathrm e}(\mathcal L(F))^{\mathfrak s^\vee_{\mathcal L}}/W_{\mathfrak s^\vee}$, where $\tilde\phi|_{\mathbf I_F}=\phi|_{\mathbf I_F}$ and $\tilde\phi(\mathrm{Frob}_F)=\phi(\mathrm{Frob}_F)\vec\chi_{\phi,v}(\vec z)$ with $\chi_{\phi,v}$ as in (3.14). We may also take $\chi_{\phi,v}^{-1}$ instead of $\chi_{\phi,v}$.*

(*c*) *Suppose that $\vec z\in\mathbb R^d_{\geq 1}$. Equivalent are:*
- *ϕ is bounded;*
- *$\bar E(\phi,\rho,\vec z)$ is tempered;*
- *$\bar M(\phi,\rho,\vec z)$ is tempered.*

(*d*) *Suppose that $\vec z\in\mathbb R^d_{>1}$. Then ϕ is discrete if and only if $\bar M(\phi,\rho,\vec z)$ is essentially discrete series and the rank of $R_{\mathfrak s^\vee}$ equals $\dim_{\mathbb C}(T_{\mathfrak s^\vee}/X_{\mathrm{nr}}({}^L\mathcal G))$.*
In this case $\phi(\mathrm{Frob}_F)\phi_b(\mathrm{Frob}_F)^{-1}$ comes from an element of $\mathrm Z(J^\circ)\times X_{\mathrm{nr}}({}^L\mathcal G)$ via Lemma 3.7 and (3.28).

(*e*) *Suppose that $\zeta\in \mathrm Z(\mathcal G^\vee\rtimes\mathbf I_F)_{\mathbf W_F}$ stabilizes $\Phi_{\mathrm e}(\mathcal G(F))^{\mathfrak s^\vee}$. Via (3.28) ζ determines a unique element $t_\zeta\in T_{\mathfrak s^\vee}$. (For instance $\zeta\in X_{\mathrm{nr}}({}^L\mathcal G)$, in which case $t_\zeta=\zeta X_{\mathrm{nr}}({}^L\mathcal L)_{\mathfrak s^\vee}$.) Then*

$$\bar M(\zeta\phi,\rho,\vec z)=t_\zeta\otimes\bar M(\phi,\rho,\vec z) \quad \textit{and} \quad \bar E(\zeta\phi,\rho,\vec z)=t_\zeta\otimes\bar E(\phi,\rho,\vec z).$$

(*f*) *Suppose that $\vec z\in\mathbb R^d_{>1}$ and that $\phi(\mathrm{Frob}_F)\phi_b(\mathrm{Frob}_F)^{-1}$ comes from an element of $\mathrm Z(J^\circ)\times X_{\mathrm{nr}}({}^L\mathcal G)$ via Lemma 3.7 and (3.28). Then $\bar E(\phi,\rho,\vec z)=\bar M(\phi,\rho,\vec z)$.*

Proof. (a) Let us fix the bounded part ϕ_b and consider only ϕ in $X_{\mathrm{nr}}({}^L\mathcal L)_{\mathrm{rs}}\phi_b$. We need to construct a bijection between such (ϕ,ρ) and the set of irreducible $\mathcal H(\mathfrak s^\vee,\vec{\mathbf z})$-modules on which $\vec{\mathbf z}$ acts as $\vec z$ and with $\mathcal O(\mathfrak s^\vee_{\mathcal L})$-weights in

$$W_{\mathfrak s^\vee}(X_{\mathrm{nr}}({}^L\mathcal L)_{\mathrm{rs}}\phi_b,v,q\epsilon)\subset\mathfrak s^\vee_{\mathcal L}.$$

We want to apply Theorem 2.5.a here, but we need to check that it can be done in a canonical way. Let $\mathcal H(\mathfrak s^\vee,\phi_b,\vec{\mathbf z})$ be the twisted affine Hecke algebra with the same $\mathcal O(\mathfrak s^\vee_{\mathcal L})$, parameters λ,λ^* and 2-cocycle $\natural_{\mathfrak s^\vee}$ as $\mathcal H(\mathfrak s^\vee,\vec{\mathbf z})$, but with root system

$$R_{\mathfrak s^\vee,\phi_b}=\{\alpha\in R_{\mathfrak s^\vee}: s_\alpha(\phi_b)=\phi_b\}$$

and finite group

$$W_{\mathfrak s^\vee,\phi_b}=W(R_{\mathfrak s^\vee,\phi_b})\rtimes\mathfrak R_{\mathfrak s^\vee,\phi_b},$$

where $\mathfrak R_{\mathfrak s^\vee,\phi_b}$ is the stabilizer of $R_{\mathfrak s^\vee,\phi_b}\cap R^+_{\mathfrak s^\vee}$ in $W_{\mathfrak s^\vee,\phi_b}$. For an open $W_{\mathfrak s^\vee}$-stable subset $U\subset\mathfrak s^\vee_{\mathcal L}$ we can extend our algebras with analytic or meromorphic functions on U, like (2.11) that yields the algebras

$$\begin{aligned} \mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})^{an/me}(U) &= \mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}}) \otimes_{\mathcal{O}(\mathfrak{s}_{\mathcal{L}}^\vee)^{W_{\mathfrak{s}^\vee}}} C^{an/me}(U), \\ \mathcal{H}(\mathfrak{s}^\vee, \phi_b, \vec{\mathbf{z}})^{an/me}(U) &= \mathcal{H}(\mathfrak{s}^\vee, \phi_b, \vec{\mathbf{z}}) \otimes_{\mathcal{O}(\mathfrak{s}_{\mathcal{L}}^\vee)^{W_{\mathfrak{s}^\vee}}} C^{an/me}(U). \end{aligned}$$

If we rewrite this in terms of $\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}}) \rtimes \mathbb{C}[\mathfrak{R}_{\mathfrak{s}^\vee}, \natural_{\mathfrak{s}^\vee}]$ using a basepoint ϕ_1, then (2.12) shows that there is an isomorphism of $C^{\mathrm{me}}(U)^{W_{\mathfrak{s}^\vee}}$-algebras

$$(3.50)\qquad \begin{array}{ccc} (C^{\mathrm{me}}(U) \rtimes \mathbb{C}[W_{\mathfrak{s}^\vee}, \natural_{q\mathcal{E}}]) \otimes \mathbb{C}[\vec{\mathbf{z}}, \vec{\mathbf{z}}^{-1}] & \to & \mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})^{\mathrm{me}}(U) \\ fw & \mapsto & f\tau_w \qquad f \in C^{\mathrm{me}}(U), w \in W_{\mathfrak{s}^\vee} \end{array} .$$

Here the τ_w for $w \in W_{\mathfrak{s}^\vee}$ are determined by the $\tau_\gamma = N_\gamma$ for $\gamma \in \mathfrak{R}_{\mathfrak{s}^\vee}$ and the τ_{s_α} for simple roots α. From Proposition 3.14.c and the explicit formula (2.13) we see that τ_{s_α} does not change if we choose a different basepoint, say $t'\phi_1$. Namely, if $\alpha(t') = 1$, then ϕ_1 and $t'\phi_1$ give the same coordinate θ_α. Otherwise $\alpha(t') = -1$ and $t'\phi_1$ gives the coordinate $\alpha(t')\theta_\alpha = -\theta_\alpha$, but then $\lambda^*(\alpha) = 0$ and (2.13) is again the same with respect to ϕ_1 and with respect to $t'\phi_1$. Hence (3.50) is canonical.

Similarly, (3.50) holds for $\mathcal{H}(\mathfrak{s}^\vee, \phi_b, \vec{\mathbf{z}})$, then with canonical elements τ_{w,ϕ_b} for $w \in W_{\mathfrak{s}^\vee,\phi_b}$. Like in (2.14), this leads to a canonical embedding

$$(3.51)\qquad \begin{array}{ccc} \mathcal{H}(\mathfrak{s}^\vee, \phi_b, \vec{\mathbf{z}})^{\mathrm{me}}(U) & \to & \mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})^{\mathrm{me}}(U) \\ f\tau_{w,\phi_b} & \mapsto & f\tau_w \end{array} ,$$

which for suitable U restricts to $\mathcal{H}(\mathfrak{s}^\vee, \phi_b, \vec{\mathbf{z}})^{\mathrm{an}}(U) \to \mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})^{\mathrm{an}}(U)$. This entails that the entire proof of Theorem 2.5 can be applied to $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$ and $\mathcal{H}(\mathfrak{s}^\vee, \phi_b, \vec{\mathbf{z}})$, independent of a basepoint of $\mathfrak{s}_{\mathcal{L}}^\vee$. In particular Theorem 2.5.(a,d) yields a canonical equivalence between

(i) the category of finite length $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$-modules with weights in $W_{\mathfrak{s}^\vee}(X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}}\phi_b, v, q\epsilon) \times \{\vec{z}\}$,

(ii) the category of finite length $\mathcal{H}(\mathfrak{s}^\vee, \phi_b, \vec{\mathbf{z}})$-modules with weights in $(X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}}\phi_b, v, q\epsilon) \times \{\vec{z}\}$.

To (ii) we will apply Theorem 2.11, with the $W_{\mathfrak{s}^\vee,\phi_b}$-fixed point $(\phi_b, v, q\epsilon) \in \mathfrak{s}_{\mathcal{L}}^\vee$ in the role of t. The resulting twisted graded Hecke algebra clearly has vector space $\mathrm{Lie}(X_{\mathrm{nr}}({}^L\mathcal{L}))$ and finite group $W_{\mathfrak{s}^\vee,\phi_b}$, while by (3.48) the 2-cocycle $\natural_{\mathfrak{s}^\vee}$ restricts to $\kappa_{\mathfrak{s}^\vee,\phi_b,v,q\epsilon} = \natural_{q\mathcal{E}}$ on $W^2_{\mathfrak{s}^\vee,\phi_b}$.

We have to be careful because we need to carry Theorem 2.11 out in a canonical way, independent of the choice of a basepoint of $\mathfrak{s}_{\mathcal{L}}^\vee$. First we look at the parameter function $k\colon R_{\mathfrak{s}^\vee,\phi_b,\mathrm{red}} \to \mathbb{R}$ from (2.23), with respect to basepoints $\phi_1, t'\phi_1$ as in Proposition 3.14.b. If $\alpha(t') = 1$, then clearly $k(\alpha)$ is the same for ϕ_1 and $t'\phi_1$. Otherwise $\alpha(t') = -1$, then Proposition 3.14.c says that $\lambda^*(\alpha) = 0$, so again $k(\alpha)$ is the same for ϕ_1 and for $t'\phi_1$. Now (2.23) and (3.38)–(3.41) show that with $t = \phi_b(\mathrm{Frob}_F)\phi_1(\mathrm{Frob}_F)^{-1}$

$$k(\alpha) = \begin{cases} m_\alpha c(\alpha/m_\alpha) & \text{if } \alpha(t) = 1, \\ m_\alpha c^*(\alpha/m_\alpha) & \text{if } \alpha(t) = -1. \end{cases}$$

In view of the conventions for $c^*(\alpha)$ in the proof of Proposition 3.14, $k(\alpha)$ always equals $m_\alpha c(\alpha/m_\alpha)$ for $\phi_b = t\phi_1$. It follows that in this application of Theorem 2.11 we have to use the twisted graded Hecke algebra

$$\mathbb{H}\big(\mathrm{Lie}(X_{\mathrm{nr}}({}^L\mathcal{L})), W_{\mathfrak{s}^\vee,\phi_b}, c\vec{\mathbf{r}}, \natural_{q\mathcal{E}}\big) = \mathbb{H}(\phi_b, v, q\epsilon, \vec{\mathbf{r}}).$$

For a small open neighborhood V of $\mathrm{Lie}(X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}}) \oplus \mathbb{R}^d$ in $\mathrm{Lie}(X_{\mathrm{nr}}({}^L\mathcal{L})) \oplus \mathbb{C}^d$, the map

$$\exp_{\phi_b} \colon (x, r_1, \ldots, r_d) \mapsto (\exp(x)\phi_b, \exp r_1, \ldots, \exp r_d)$$

provides a diffeomorphism between V and $\exp_{\phi_b}(V) \subset \mathfrak{s}^\vee_{\mathcal{L}} \times (\mathbb{C}^\times)^d$. Upon choosing a basepoint ϕ_1 for $\mathfrak{s}^\vee_{\mathcal{L}}$, (2.26) shows that there is an algebra isomorphism

$$\begin{array}{ccc} \mathcal{H}(\mathfrak{s}^\vee, \phi_b, \vec{\mathbf{z}})^{\mathrm{me}}(\exp_{\phi_b} V) & \xrightarrow{\sim} & \mathbb{H}(\phi_b, v, q\epsilon, \vec{\mathbf{r}})^{\mathrm{me}}(V) \cong C^{\mathrm{me}}(V) \rtimes \mathbb{C}[W_{\mathfrak{s}^\vee,\phi_b}, \natural_{q\mathcal{E}}] \\ f\tau_{w,\phi_b} & \mapsto & (f \circ \exp_{\phi_b})\tilde{\tau}_w, \end{array} \tag{3.52}$$

where $f \in C^{\mathrm{me}}(\exp_{\phi_b} V)$ and $w \in W_{\mathfrak{s}^\vee,\phi_b}$. Between (3.37) and (3.50), we saw that τ_{w,ϕ_b} does not depend on the choice of a basepoint of $\mathfrak{s}^\vee_{\mathcal{L}}$, and we already argued after (2.27) that the elements $\tilde{\tau}_w$ are canonical. Therefore the map (3.52) is canonical. This means that we can carry out the entire proof of Theorem 2.11 for $\mathcal{H}(\mathfrak{s}^\vee, \phi_b, \vec{\mathbf{z}})$ with the basepoint ϕ_1, and then the outcome does not depend on the choice of ϕ_1. Thus Theorem 2.11 yields a canonical equivalence between (ii) above and

(iii) the category of finite length $\mathbb{H}(\phi_b, v, q\epsilon, \vec{\mathbf{r}})$-modules with weights in $\mathrm{Lie}(X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}}) \times \{\log \vec{z}\}$.

Of course, the equivalences between the categories (i), (ii) and (iii) (from Theorems 2.5 and 2.11) can be restricted to irreducible modules in each of these categories. By Theorem 3.8 the set of isomorphism classes of irreducible objects in (iii) is canonically in bijection with

$$ {}^L\Psi^{-1}(\mathcal{L}^\vee \rtimes \mathbf{W}_F, X_{\mathrm{nr}}({}^L\mathcal{L})_{\mathrm{rs}}\phi_b|_{\mathbf{W}_F}, v, q\epsilon). \tag{3.53}$$

The resulting bijection between (3.53) and the subset of $\mathrm{Irr}(\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}}))$ with the appropriate central character could depend on the choice of an element in the $W_{\mathfrak{s}^\vee}$-orbit of ϕ_b. Fortunately, the proof of Lemma 2.9 applies also in this setting, and it entails that the bijection does not depend on such choices. Now we combine all these bijections, for the various ϕ_b. This gives a canonical bijection between $\Phi_{\mathrm{e}}(\mathcal{G}(F))^{\mathfrak{s}^\vee}$ and $\mathrm{Irr}_{\vec{z}}(\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}}))$.

(b) By Theorem 3.8.f $E(\phi, \rho, \log \vec{z})$ admits the central character

$$W_{\mathfrak{s}^\vee,\phi_b,v,q\epsilon}\big(\sigma_0 \pm d\vec{\chi}_{\phi,v}(\log \vec{z}), \log \vec{z}\big),$$

where σ_0 is given by (3.15). Applying Theorems 2.11 and 2.5 produces the representation $\bar{E}(\phi, \rho, \vec{z})$, with the central character that sends Frob_F to $\phi(\mathrm{Frob}_F)\vec{\chi}_{\phi,v}(\vec{z})^{\pm 1}$. That is just $W_{\mathfrak{s}^\vee}(\tilde{\phi}|_{\mathbf{W}_F}, v, q\epsilon)$. The same holds for the quotient $\bar{M}(\phi, \rho, \vec{z})$ of $\bar{E}(\phi, \rho, \vec{z})$.

(c) This follows from Theorem 3.8.b, Theorem 2.11.d and Proposition 2.7.a.

(d) Notice that by the very definition of $R_{\mathfrak{s}^\vee}$, it has the same rank as $R(J^\circ, T)$.

Suppose that ϕ is discrete. By Theorem 3.8.c $\bar{M}(\phi, \rho, \log \vec{z})$ is essentially discrete series as a module for $\mathbb{H}(\phi_b, v, q\epsilon, \log \vec{z})$, and the rank of $R(G^\circ_{\phi_b}, T)$ equals $\dim_{\mathbb{C}}(T)$. Now Theorem 2.11.d and Proposition 2.7.c say that $\bar{M}(\phi, \rho, \vec{z})$ is essentially discrete series. The root system $R(J^\circ, T)$ contains $R(G^\circ_{\phi_b}, T)$, so its rank is at least $\dim_{\mathbb{C}}(T)$ – and hence precisely that, for it obviously cannot be strictly larger. By Lemma 3.7 T is a finite cover of $T_{\mathfrak{s}^\vee}/X_{\mathrm{nr}}({}^L\mathcal{G})$, so both these tori have the same dimension.

Conversely, suppose that $\bar{M}(\phi, \rho, \vec{z})$ is essentially discrete series and that the rank of $R(J^\circ, T)$ equals $\dim_{\mathbb{C}}(T_{\mathfrak{s}^\vee}/X_{\mathrm{nr}}({}^L\mathcal{G}))$. By Proposition 2.7.c the root system $R(G^\circ_{\phi_b}, T)$ has the same rank, which we already saw equals $\dim_{\mathbb{C}}(T)$. In combination with Theorem 2.11 we also obtain that the $\mathbb{H}(\phi_b, v, q\epsilon, \log \vec{z})$-module $\bar{M}(\phi, \rho, \log \vec{z})$ is essentially discrete series. Now Theorem 3.8.c tells us that ϕ is discrete.

By Theorem 2.13.d $\phi(\mathrm{Frob}_F)\phi_b(\mathrm{Frob}_F)^{-1}$ lies in $\mathrm{Z}(G^\circ)$ for a complex reductive group G° with maximal torus $T_{\mathfrak{s}^\vee}$ and Weyl group $W^\circ_{\mathfrak{s}^\vee}$. That is the Weyl group of (J°, T), so via Lemma 3.7 $\phi(\mathrm{Frob}_F)\phi_b(\mathrm{Frob}_F)^{-1}$ must come from an element of $G^\circ_{\phi_1} \times X_{\mathrm{nr}}({}^L\mathcal{G})$ which is centralized by J°.
(e) As ${}^L\Psi(\zeta\phi,\rho) = \zeta {}^L\Psi(\zeta,\rho) \in \mathfrak{s}^\vee_{\mathcal{L}}$, ζ determines a unique element of $T_{\mathfrak{s}^\vee}$. It is invariant under $\mathcal{G}^\vee$ and $\mathcal{G}^\vee_{\mathrm{sc}}$, because ζ comes from $\mathrm{Z}(\mathcal{G}^\vee)$. Now the claim follows from Theorem 3.8.d in the same way as Theorem 2.13.e was derived from Proposition 1.8.d.
(f) Reasoning as in the last lines of the proof of part (d), we see that
$\phi(\mathrm{Frob}_F)\phi_b(\mathrm{Frob}_F)^{-1} \in \mathrm{Z}(G^\circ)$. Apply Theorem 2.13.f. □

Comparing Theorem 3.18.b with [AMS1, Definition 7.7] we see that, when $\vec{z} = q_F^{\pm 1/2}$, the central character of $\bar{M}(\phi,\rho,q_F^{\pm 1/2})$ equals the cuspidal support of (ϕ,ρ). Part (e) says that Theorem 3.18 is equivariant with respect to twists by $X_{\mathrm{nr}}({}^L\mathcal{G})$, that is, equivariant with respect to twisting by unramified characters of $\mathcal{G}(F)$.

The bijection obtained in part (a) is compatible with parabolic induction in the same sense as Corollary 2.14. For reference, we formulate this precisely. We use the notations as in (3.16) and after that. Recall from pages 36 and 17 that

$$\epsilon_{u_\phi,j}\big(\phi(\mathrm{Frob}_F)\phi_b(\mathrm{Frob}_F)^{-1},\vec{z}\big) = \epsilon_{\log(u_\phi),j}\Big(\mathrm{d}\vec{\phi}\left(\begin{smallmatrix}\vec{r} & 0\\ 0 & -\vec{r}\end{smallmatrix}\right) + \log(\phi(\mathrm{Frob}_F)^{-1}\phi_b(\mathrm{Frob}_F)),\vec{r}\Big)$$

is a function which detects parameters for which parabolic induction could behave undesirably.

Lemma 3.19. *Let $Q = \mathcal{Q}(F)$ be a Levi subgroup of $\mathcal{G}(F)$ and assume that $\epsilon_{u_\phi,j}\big(\phi(\mathrm{Frob}_F)\phi_b(\mathrm{Frob}_F)^{-1},\vec{z}\big) \neq 0$ for each $j = 1,\ldots,d$.*
(*a*) *There is a natural isomorphism of $\mathcal{H}(\mathfrak{s}^\vee,\vec{\mathbf{z}})$-modules*

$$\mathcal{H}(\mathfrak{s}^\vee,\vec{\mathbf{z}}) \underset{\mathcal{H}(\mathfrak{s}^\vee_Q,\vec{\mathbf{z}})}{\otimes} \bar{E}^Q(\phi,\rho^Q,\vec{z}) \cong \bigoplus\nolimits_\rho \mathrm{Hom}_{\mathcal{S}^Q_\phi}(\rho^Q,\rho) \otimes \bar{E}(\phi,\rho,\vec{z}),$$

where the sum runs over all $\rho \in \mathrm{Irr}(\mathcal{S}_\phi)$ with ${}^L\Psi^Q(\phi,\rho^Q) = {}^L\Psi(\phi,\rho)$.
(*b*) *The multiplicity of $\bar{M}(\phi,\rho,\vec{z})$ in $\mathcal{H}(\mathfrak{s}^\vee,\vec{\mathbf{z}}) \underset{\mathcal{H}(\mathfrak{s}^\vee_Q,\vec{\mathbf{z}})}{\otimes} \bar{E}^Q(\phi,\rho^Q,\vec{z})$ is $[\rho^Q : \rho]_{\mathcal{S}^Q_\phi}$. It already appears that many times as a quotient of $\mathcal{H}(\mathfrak{s}^\vee,\vec{\mathbf{z}}) \underset{\mathcal{H}(\mathfrak{s}^\vee_Q,\vec{\mathbf{z}})}{\otimes} \bar{M}^Q(\phi,\rho^Q,\vec{z})$.*

Proof. As observed after (3.16), the bijection in Theorem 3.8.a is compatible with parabolic induction in the sense of Corollary 2.14. The bijection in Theorem 3.18.a is obtained from Theorem 3.8 by means of the reduction Theorems 2.5 and 2.11. Since these reduction theorems respect parabolic induction, Corollary 2.14 remains valid in the setting of Theorem 3.8, and it gives the desired results. □

4. The relation with the stable Bernstein center

Let $\Phi({}^L\mathcal{G})$ be the collection of $\mathcal{G}^\vee$-orbits of L-parameters for ${}^L\mathcal{G}$. Recently, inspired by [Vog], Haines has considered the stable Bernstein center in [Hai]. We will explore below the relation of the latter with the Bernstein components $\Phi_e({}^L\mathcal{G})^{\mathfrak{s}^\vee}$.

The notion of stable Bernstein center which we employ here naturally lives on the Galois side. In principle it should be related to stable distributions on $\mathcal{G}(F)$ [Hai, §5.5], but that connection is currently highly conjectural. Because of that, we will consider it for all inner twists of a given reductive connected p-adic group

$\mathcal{G}(F)$ simultaneously. Let $\mathcal{G}^*(F)$ be a quasi-split F-group which is an inner twist of $\mathcal{G}(F)$. The equivalence classes of inner twists of $\mathcal{G}^*$ are parametrized by the Galois cohomology group $H^1(F, \mathcal{G}^*_{\mathrm{ad}})$. For every $\alpha \in H^1(F, \mathcal{G}^*_{\mathrm{ad}})$, we will denote by $\mathcal{G}_\alpha(F)$ an inner twist of $\mathcal{G}^*(F)$ which is parametrized by α. By construction

$$\Phi_{\mathrm{e}}({}^L\mathcal{G}) = \bigsqcup_{\alpha \in H^1(F, \mathcal{G}^*_{\mathrm{ad}})} \Phi_{\mathrm{e}}(\mathcal{G}_\alpha(F)).$$

Definition 4.1. The infinitesimal character of an L-parameter $\phi \in \Phi({}^L\mathcal{G})$ (or an enhanced L-parameter $(\phi, \rho) \in \Phi_{\mathrm{e}}({}^L\mathcal{G})$) is the $\mathcal{G}^\vee$-conjugacy class of the admissible morphism $\lambda_\phi \colon \mathbf{W}_F \to \mathcal{G}^\vee \rtimes \mathbf{W}_F$ (trivial on $\mathrm{SL}_2(\mathbb{C})$) defined by

$$\lambda_\phi(w) := \phi\left(w, \begin{pmatrix} \|w\|^{1/2} & 0 \\ 0 & \|w\|^{-1/2} \end{pmatrix}\right) \qquad w \in \mathbf{W}_F.$$

With this notion we can reinterpret Theorem 3.18.b as: the infinitesimal character of (ϕ, ρ) equals the infinitesimal character of the central character of $\bar{M}(\phi, \rho, q_F^{\pm 1/2})$.

Remark 4.2. As noticed in [Hai, §5], if ϕ is relevant for $\mathcal{G}(F)$, it may happen that λ_ϕ is not relevant for $\mathcal{G}(F)$ anymore. This is why λ_ϕ is called an admissible morphism, *i.e.* an L-parameter without the relevance condition. In contrast, for every $\phi \in \Phi({}^L\mathcal{G})$, we have $\lambda_\phi \in \Phi({}^L\mathcal{G})$, for λ_ϕ is relevant for $\mathcal{G}^*(F)$.

Definition 4.3. An inertial infinitesimal datum $\mathfrak{i}$ for $\Phi({}^L\mathcal{G})$ is a pair $({}^L\mathcal{M}, \mathfrak{i}_{{}^L\mathcal{M}})$, where ${}^L\mathcal{M}$ is a Levi L-subgroup of ${}^L\mathcal{G}$, *i.e.* ${}^L\mathcal{M} = \mathcal{M}^\vee \rtimes \mathbf{W}_F$ with $\mathcal{M}^\vee$ a $\mathbf{W}_F$-stable Levi subgroup of $\mathcal{G}^\vee$ and $\mathfrak{i}_{{}^L\mathcal{M}}$ is the $\mathcal{M}^\vee$-conjugacy class of the $X_{\mathrm{nr}}({}^L\mathcal{M})$-orbit of a discrete admissible morphism $\lambda \colon \mathbf{W}_F \to \mathcal{M}^\vee \rtimes \mathbf{W}_F$ (trivial on $\mathrm{SL}_2(\mathbb{C})$). Another such object is regarded as equivalent if the two are conjugate by an element of $\mathcal{G}^\vee$. The equivalence class is denoted

$$\mathfrak{i} = (\mathcal{M}^\vee \rtimes \mathbf{W}_F, \mathfrak{i}_{{}^L\mathcal{M}})_{\mathcal{G}^\vee} = [\mathcal{M}^\vee \rtimes \mathbf{W}_F, \lambda]_{\mathcal{G}^\vee}.$$

We will write $\mathfrak{B}^\vee_{\mathrm{st}}({}^L\mathcal{G})$ for the set of inertial infinitesimal equivalence classes.

For every inertial infinitesimal datum $\mathfrak{i} = (\mathcal{M}^\vee \rtimes \mathbf{W}_F, \mathfrak{i}_{{}^L\mathcal{M}})_{\mathcal{G}^\vee}$, $\mathfrak{i}_{{}^L\mathcal{M}}$ has the structure of an affine variety over $\mathbb{C}$ (see [Hai, § 5.3]). The stable Bernstein center for ${}^L\mathcal{G}$ is the ring of regular functions on the disjoint union $\bigsqcup_{\mathfrak{i} = ({}^L\mathcal{M}, \mathfrak{i}_{{}^L\mathcal{M}}) \in \mathfrak{B}^\vee_{\mathrm{st}}({}^L\mathcal{G})} \mathfrak{i}_{{}^L\mathcal{M}}$.

We will attach to each inertial equivalence class for $\Phi_{\mathrm{e}}(\mathcal{G}(F))$ an inertial infinitesimal datum, as follows:

Definition 4.4. For every cuspidal inertial equivalence class $\mathfrak{s}^\vee = (\mathcal{L} \rtimes \mathbf{W}_F, X_{\mathrm{nr}}({}^L\mathcal{L}) \cdot (\phi, \rho)) \in \mathfrak{B}^\vee(\mathcal{G}(F))$, we set

$$\inf(\mathfrak{s}^\vee) := (\mathcal{M}^\vee \rtimes \mathbf{W}_F, (X_{\mathrm{nr}}({}^L\mathcal{M}) \cdot \lambda_\phi)_{\mathcal{M}})_{\mathcal{G}^\vee},$$

where $\mathcal{M}^\vee \rtimes \mathbf{W}_F$ is a Levi L-subgroup of ${}^L\mathcal{G}$ which minimally contains $\lambda_\phi(\mathbf{W}_F)$.

We remark that if ϕ has nontrivial restriction to $\mathrm{SL}_2(\mathbb{C})$, then we may have $\mathcal{M}^\vee \rtimes \mathbf{W}_F \subsetneq \mathcal{L}^\vee \rtimes \mathbf{W}_F$ and $X_{\mathrm{nr}}({}^L\mathcal{L}) \subsetneq X_{\mathrm{nr}}({}^L\mathcal{M})$.

For every $\mathfrak{i} = [\mathcal{M}^\vee \rtimes \mathbf{W}_F, \lambda]_{\mathcal{G}^\vee} \in \mathfrak{B}^\vee_{\mathrm{st}}({}^L\mathcal{G})$ we set:

$$\Phi_{\mathrm{e}}({}^L\mathcal{G})^{\mathfrak{i}} := \left\{ (\phi, \rho) \in \Phi_{\mathrm{e}}({}^L\mathcal{G}) \ : \ \begin{array}{l} \lambda_\phi \text{ is minimally contained in } \mathcal{M}^\vee \rtimes \mathbf{W}_F \\ \text{and } \lambda_\phi \in (X_{\mathrm{nr}}({}^L\mathcal{M}) \cdot \lambda)_{\mathcal{M}^\vee} \end{array} \right\}.$$

In this way, we obtain a partition of the set $\Phi_{\mathrm{e}}({}^L\mathcal{G})$ (a "stable Bernstein decomposition"):

$$\Phi_{\mathrm{e}}({}^L\mathcal{G}) = \bigsqcup_{\mathfrak{i} \in \mathfrak{B}^\vee_{\mathrm{st}}({}^L\mathcal{G})} \Phi_{\mathrm{e}}({}^L\mathcal{G})^{\mathfrak{i}}. \qquad (4.1)$$

It is worth to observe that, in contrast with Section 3, the above definitions involve only the Langlands parameter $\phi \in \Phi({}^L\mathcal{G})$ and not the enhancement of ϕ. In particular (ϕ, ρ) and (ϕ, ρ') are always contained in the same $\Phi_{\mathrm{e}}({}^L\mathcal{G})^{\mathfrak{i}}$. Consequently the decomposition (4.1) is coarser than the Bernstein decomposition of $\Phi_{\mathrm{e}}({}^L\mathcal{G})$ from (3.4). However, under the local Langlands conjecture, it is a union of L-packets. Indeed, let $\mathfrak{i} = [\mathcal{M}^\vee \rtimes \mathbf{W}_F, \lambda]_{\mathcal{G}^\vee} \in \mathfrak{B}^\vee_{\mathrm{st}}({}^L\mathcal{G})$. From the definition of $\mathfrak{i}$ we see that

$$\Phi_{\mathrm{e}}({}^L\mathcal{G})^{\mathfrak{i}} = \bigsqcup_{\alpha \in H^1(F,\mathcal{G}^*_{\mathrm{ad}})} \bigsqcup_{(\lambda\chi)_{\mathcal{M}^\vee} \in \mathfrak{i}_{L_\mathcal{M}}} \bigsqcup_{\phi \in \Phi({}^L\mathcal{G}), (\lambda_\phi)_{\mathcal{G}^\vee} = (\lambda\chi)_{\mathcal{G}^\vee}} \Pi_\phi(\mathcal{G}_\alpha(F)).$$

Define

$$\mathfrak{B}^\vee({}^L\mathcal{G}) := \bigsqcup\nolimits_{\alpha \in H^1(F,\mathcal{G}^*_{\mathrm{ad}})} \mathfrak{B}^\vee(\mathcal{G}_\alpha(F)).$$

Theorem 4.5. *For* $\mathfrak{i} \in \mathfrak{B}^\vee_{\mathrm{st}}({}^L\mathcal{G})$, *we write* $\mathfrak{B}^\vee({}^L\mathcal{G})_{\mathfrak{i}} := \{\mathfrak{s}^\vee \in \mathfrak{B}^\vee({}^L\mathcal{G}) : \inf(\mathfrak{s}^\vee) = \mathfrak{i}\}$. *Then*

$$\Phi_{\mathrm{e}}({}^L\mathcal{G})^{\mathfrak{i}} = \bigsqcup\nolimits_{\mathfrak{s}^\vee \in \mathfrak{B}^\vee({}^L\mathcal{G})_{\mathfrak{i}}} \Phi_{\mathrm{e}}({}^L\mathcal{G})^{\mathfrak{s}^\vee}.$$

Proof. Use that for any enhanced Langlands parameter $(\phi, \rho) \in \Phi_{\mathrm{e}}({}^L\mathcal{G})$, the infinitesimal character λ_ϕ of ϕ coincides with the infinitesimal character λ_φ of its cuspidal support $(\varphi, q\epsilon)$ [AMS1, (108)]. □

This theorem implies that (4.1) is a partition of $\Phi_{\mathrm{e}}({}^L\mathcal{G})$ in subsets which are at the same time unions of Bernstein components and unions of L-packets.

Combining Theorems 4.5 and 3.18, we obtain:

Corollary 4.6. *For every* $\mathfrak{i} \in \mathfrak{B}^\vee_{\mathrm{st}}({}^L\mathcal{G})$ *and every* $\vec{z} \in \mathbb{R}^d_{>0}$, *there is a canonical bijection*

$$\Phi_{\mathrm{e}}({}^L\mathcal{G})^{\mathfrak{i}} \longleftrightarrow \bigsqcup\nolimits_{\mathfrak{s}^\vee \in \mathfrak{B}^\vee({}^L\mathcal{G})_{\mathfrak{i}}} \mathrm{Irr}_{\vec{z}}(\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})).$$

Remark 4.7. It is natural to expect that a certain compatibility should exist between the algebras $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$ when $\mathfrak{s}^\vee$ runs over the set $\mathfrak{B}^\vee({}^L\mathcal{G})_{\mathfrak{i}}$, for a fixed $\mathfrak{i} = [\mathcal{M}^\vee \rtimes \mathbf{W}_F, \lambda]_{\mathcal{G}^\vee}$. A naive guess would be that there exist "spectral transfer morphisms" (as introduced for affine Hecke algebras by Opdam [Opd2]) between the algebras $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$ for $\mathfrak{s}^\vee \in \mathfrak{B}^\vee({}^L\mathcal{G})_{\mathfrak{i}}$, the role of the lowest algebra being played by an algebra $\mathcal{H}(\mathfrak{s}^\vee_1, \vec{\mathbf{z}})$, with $\mathfrak{s}^\vee_1 = [\mathcal{M}^\vee \rtimes \mathbf{W}_F, \lambda, 1]_{\mathcal{G}^\vee}$.

5. Examples

In this section we will work out some affine Hecke algebras attached to Bernstein components of Langlands parameters. In the examples that we consider the local Langlands correspondence is known, and it matches Bernstein components on the Galois side with Bernstein components on the p-adic side. We will compare the Hecke algebras associated to Bernstein components that correspond under the LLC.

All our examples are inner forms of split groups, so $X_{\mathrm{nr}}({}^L\mathcal{L}) = \mathrm{Z}(\mathcal{L}^\vee)^\circ$ and we may replace ${}^L\mathcal{G}$ by $\mathcal{G}^\vee$.

5.1. Inner twists of $\mathrm{GL}_n(F)$.

Recall that F is a local non-archimedean field, and let q_F be the cardinality of its residue field. Let D be a division algebra with centre F and $\dim_F(D) = d^2$. Take $m \in N$ and consider $\mathcal{G}(F) = \mathrm{GL}_m(D)$. It is an inner form of $\mathrm{GL}_n(F)$ with $n = md$. In fact $\mathcal{G}(F)$ becomes an inner twist if we regard D, the Hasse invariant

$h(D) \in \{z \in \mathbb{C}^\times : z^d = 1\}$ or the associated character χ_D of $\mathrm{Z}(\mathrm{SL}_n(\mathbb{C}))$ as part of the data. Up to conjugacy every Levi subgroup of $\mathcal{G}(F)$ is of the form

$$\mathcal{L}(F) = \prod\nolimits_j \mathrm{GL}_{m_j}(D) \quad \text{with } \sum\nolimits_j m_j = m.$$

Let $(\phi = \bigoplus_j \phi_j, \rho = \otimes_j \rho_j) \in \Phi_{\mathrm{cusp}}(\mathcal{L}(F))$. In [AMS1, Example 6.11] we worked out the shape of cuspidal Langlands parameters (ϕ_j, ρ_j) for $\mathrm{GL}_{m_j}(D)$. Namely

- $\phi_j = \phi_j|_{\mathbf{W}_F} \otimes S_{d_j}$ where S_{d_j} is the irreducible d_j-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$ and $\phi_j|_{\mathbf{W}_F}$ is an irreducible representation of dimension $m_j d/d_j$. (This says that ϕ_j is discrete.)
- $\mathcal{S}_{\phi_j} = \mathrm{Z}(\mathrm{SL}_{m_j d}(\mathbb{C}))$ and ρ_j is the character associated to $\mathrm{GL}_{m_j}(D)$, that is, $\rho_j(\exp(2\pi i k/(m_j d)) I_{m_j d}) = h(D)^k$. (So (ϕ_j, ρ_j) is relevant for $\mathrm{GL}_{m_j}(D)$.)
- lcm $(d, m_j d/d_j) = m_j d$, or equivalently $\gcd(d, m_j d/d_j) = d/d_j$. (This guarantees cuspidality.)

It is known that two irreducible representation ϕ_j and ϕ_k of $\mathbf{W}_F$ are isomorphic up to an unramified character twist if and only if their restrictions to $\mathbf{I}_F$ are isomorphic. Hence we can adjust the indexing so that $\phi|_{\mathbf{I}_F} = \bigoplus_i \phi_i^{\oplus e_i}|_{\mathbf{I}_F}$. Because the restriction of each ϕ_i to $\mathbf{I}_F$ decomposes as sum of irreducible representations of $\mathbf{I}_F$ with multiplicity one, we find that $R(J^\circ, T) \cong \prod_i A_{e_i - 1}$. To determine the Hecke algebra of the associated Bernstein component $\mathfrak{s}^\vee$ of $\Phi_{\mathrm{e}}(\mathcal{G}(F))$, we make a simplifying assumption: if $m_i = m_j$ and ϕ_i differs from ϕ_j by an unramified twist, then $\phi_i = \phi_j$.

We adjust the indexing so that

$$\mathcal{L}(F) = \prod\nolimits_i \mathrm{GL}_{m_i}(D)^{e_i}, \quad \phi = \bigoplus\nolimits_i \phi_i^{\oplus e_i}, \quad \rho = \bigotimes\nolimits_i \rho_i^{\otimes e_i},$$

where ϕ_i and ϕ_j are not inertially equivalent if $i \neq j$. Let $\mathfrak{s}_i^\vee$ be the Bernstein component of $\Phi_{\mathrm{e}}(\mathrm{GL}_{m_i e_i}(D))$ determined by $(\phi_i^{\oplus e_i}, \rho_i^{\otimes e_i})$. Choose an isomorphism $M_{d e_i m_i}(\mathbb{C}) \cong M_{m_i d/d_i}(\mathbb{C}) \otimes M_{d_i e_i}(\mathbb{C})$ and let 1_m be the multiplicative unit of the matrix algebra $M_m(\mathbb{C})$. Then

$$\begin{aligned}
&G_\phi = \mathrm{Z}_{\mathrm{SL}_n(\mathbb{C})}(\phi(\mathbf{W}_F)) \cong \mathrm{SL}_n(\mathbb{C}) \cap \prod\nolimits_i (1_{m_i d/d_i} \otimes \mathrm{GL}_{d_i e_i}(\mathbb{C})) = \mathrm{SL}_n(\mathbb{C}) \cap \prod\nolimits_i G_{\phi,i},\\
&M \cong \mathrm{SL}_n(\mathbb{C}) \cap \prod\nolimits_i (1_{m_i d/d_i} \otimes \mathrm{GL}_{d_i}(\mathbb{C})^{e_i}),\\
&T \cong \mathrm{SL}_n(\mathbb{C}) \cap \prod\nolimits_i (1_{m_i d/d_i} \otimes \mathrm{Z}(\mathrm{GL}_{d_i}(\mathbb{C}))^{e_i}), \quad R(G_\phi, T) \cong \prod\nolimits_i A_{e_i - 1},\\
&T_i = \big\{\phi_i \otimes \chi_i \in \Phi(\mathrm{GL}_{m_i}(D)) : \chi_i \in X_{\mathrm{nr}}({}^L\mathrm{GL}_{m_i}(D))\big\} / \mu_{t_{\phi_i}}(\mathbb{C}),\\
&T_{\mathfrak{s}^\vee} = \prod\nolimits_i T_{\mathfrak{s}_i^\vee} = \prod\nolimits_i T_i^{e_i}, \quad W_{\mathfrak{s}^\vee} = W_{\mathfrak{s}^\vee,\phi} \cong \prod\nolimits_i S_{e_i}.
\end{aligned}$$

Here μ_k denotes the functor of taking k-th roots of unity and t_{ϕ_i} denotes the number of unramified twists $z_i \in X_{\mathrm{nr}}({}^L\mathrm{GL}_{m_i}(D))$ such that $z_i \phi_i \cong \phi_i$ in $\Phi_{\mathrm{cusp}}(\mathrm{GL}_{m_i}(D))$. The cyclic group $\mu_{t_{\phi_i}}(\mathbb{C})$ is naturally embedded in the onedimensional complex torus $X_{\mathrm{nr}}({}^L\mathrm{GL}_{m_i}(D))$. Furthermore we can decompose $u_\phi = \prod_i u_{\phi,i}$, where $u_{\phi,i}$ belongs to the unique distinguished unipotent class of $1_{m_i d/d_i} \otimes \mathrm{GL}_{d_i}(\mathbb{C})^{e_i}$. By [Lus2, 2.13] this implies $c(\alpha) = 2d_i$ for all $\alpha \in R(G_{\phi,i}T, T)$. Then $\lambda(\alpha) = t_{\phi_i} d_i$ on $R(G_{\phi,i}T, T)$, whereas λ^* does not occur. We conclude that

$$\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}}) = \mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \vec{\mathbf{z}}) \cong \bigotimes\nolimits_i \mathcal{H}(\mathrm{GL}_{e_i d_i}(\mathbb{C}), \mathrm{GL}_{d_i}(\mathbb{C})^{e_i}, v_i, \rho_i^{\otimes e_i}, \mathbf{z}_i), \tag{5.1}$$

a tensor product of affine Hecke algebras of type GL_{e_i} with parameters $\mathbf{z}_i^{t_{\phi_i} d_i}$. The most appropriate specialization of (5.1) is at $\mathbf{z}_i = q_F^{1/2}$. Indeed this recovers the exact parameters found by Sécherre in [Sec1, Théorème 4.6], see (5.3).

Now we consider Hecke algebras on the p-adic side. By the local Langlands correspondence for $\mathrm{GL}_{m_i}(D)$ (see [HiSa, §11] and [ABPS2, §2]), (ϕ_i, ρ_i) is associated to a unique essentially square-integrable representation $\sigma_i \in \mathrm{Irr}(\mathrm{GL}_{m_i}(D))$. Moreover the condition $\mathrm{lcm}(d, m_i d/d_i) = m_i d$ guarantees that σ_i is supercuspidal, by [DKV, Théorème B.2.b]. (This is a formal consequence of the Jacquet–Langlands correspondence, so in view of [Bad] it also holds in positive characteristic.) Hence

$$(\phi_i^{\oplus e_i}, \rho_i^{\otimes e_i}) \in \Phi_{\mathrm{cusp}}(\mathrm{GL}_{m_i}(D)^{e_i}) \quad \text{corresponds to} \quad \sigma_i^{\otimes e_i} \in \mathrm{Irr}_{\mathrm{cusp}}(\mathrm{GL}_{m_i}(D)^{e_i}).$$

Let $\mathfrak{s}_i$ denote the inertial equivalence class for $\mathrm{GL}_{m_i e_i}(D)$ determined by $(\mathrm{GL}_{m_i}(D)^{e_i}, \sigma_i^{\otimes e_i})$. In [SeSt1, Théorème 5.23] a $\mathfrak{s}_i$-type (J_i, τ_i) was constructed. The Hecke algebra for (J_i, τ_i) was analysed in [Sec1, Théorème 4.6], Sécherre found an isomorphism

$$\mathcal{H}(\mathrm{GL}_{m_i e_i}(D), J_i, \tau_i) \cong \mathcal{H}(\mathrm{GL}_{e_i}, q_F^{f_i}), \tag{5.2}$$

where the right hand side denotes an affine Hecke algebra of type GL_{e_i} with parameter $q_F^{f_i}$ (for a suitable $f_i \in \mathbb{N}$ depending only on σ_i or ϕ_i, see below). From the explicit description in [Sec1, §4] one sees readily that the isomorphism (5.2) respects the natural Hilbert algebra structures on both sides.

Remark 5.1. Let t_{σ_i} denote the torsion number of σ_i, *i.e.*, the number of unramified characters χ_i of $\mathrm{GL}_{m_i}(D)$ such that $\chi_i \otimes \sigma_i \cong \sigma_i$. It equals t_{ϕ_i}.

If $D = F$, then $f_i = t_{\sigma_i}$. In general, $f_i = s_{\sigma_i} t_{\sigma_i}$, where s_{σ_i} is the reducibility number of σ_i, as defined in [SeSt2, Introduction] (see also [Sec2, Theorem 4.6]). The number s_{σ_i} coincides with the invariant introduced in [DKV, Théorème B.2.b] (as it follows for instance from [BHLS, Eqn. (1.1) and Definition 2.2]), itself equal to the integer d_i. Hence f_i admits the following description in terms of Langlands parameters:

$$f_i = s_{\sigma_i} t_{\sigma_i} = d_i t_{\phi_i}. \tag{5.3}$$

Write $\mathcal{M}(F) = \prod_i \mathrm{GL}_{m_i}(D)^{e_i}$, $\sigma = \bigotimes_i \sigma_i^{\otimes e_i}$ and let $\mathfrak{s}$ be the inertial equivalence class of $(\mathcal{M}(F), \sigma)$ for $\mathrm{GL}_m(D)$. In [SeSt2, Theorem C] a $\mathfrak{s}$-type (J, τ) was constructed, as a cover of the product of the types (J_i, τ_i) for $\mathfrak{s}_i$. Moreover it was shown that

$$\mathcal{H}(\mathrm{GL}_m(D), J, \tau) \cong \bigotimes\nolimits_i \mathcal{H}(\mathrm{GL}_{e_i}, q_F^{f_i}). \tag{5.4}$$

Since (5.2) was an isomorphism of Hilbert algebras, so is (5.4). Notice that the right hand side is also the specialization of $\mathcal{H}(\mathfrak{s}^\vee, \vec{z})$ at $\mathbf{z}_i = q_F^{1/2}$. Thus there are equivalences of categories

$$\mathrm{Rep}(\mathrm{GL}_m(D))^{\mathfrak{s}} \cong \mathrm{Mod}\Big(\bigotimes\nolimits_i \mathcal{H}(\mathrm{GL}_{e_i}, q_F^{f_i}) \Big) \cong \mathrm{Mod}\Big(\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}}) / \big(\{\mathbf{z}_i - q_F^{1/2}\}_i \big) \Big). \tag{5.5}$$

It was shown in [BaCi, §5.4] that, since these equivalences come from isomorphisms of Hilbert algebras, they preserve temperedness of representations. Then [ABPS4, Lemma 16.5] proves that (5.5) maps essentially square-integrable representations to essentially discrete series representations and conversely.

The torus underlying $\bigotimes_i \mathcal{H}(\mathrm{GL}_{e_i}, q_F^{f_i})$ is $T_{\mathfrak{s}} = [\mathcal{M}(F), \sigma]_{\mathcal{M}(F)}$, which by the LLC for $\mathrm{GL}_{m_i}(D)$ is naturally isomorphic to the torus $T_{\mathfrak{s}^\vee}$ underlying $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})$. Then [ABPS3, Theorem 4.1] shows that, with the interpretation as in Lemma 3.16 (which highlights the tori in these affine Hecke algebras), the equivalences (5.5) become canonical. This means in essence that we use the local Langlands correspondence for supercuspidal representations as input. With Theorem 3.18 we obtain canonical bijections

$$\mathrm{Irr}(\mathrm{GL}_m(D))^{\mathfrak{s}} \longleftrightarrow \mathrm{Irr}\big(\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})/\big(\{\mathbf{z}_i - q_F^{1/2}\}_i\big)\big) \longleftrightarrow \Phi_{\mathrm{e}}(\mathrm{GL}_m(D))^{\mathfrak{s}^\vee}. \tag{5.6}$$

Proposition 5.2. *The union of the bijections (5.6) over all Bernstein components for $\mathrm{GL}_m(D)$ equals the local Langlands correspondence for $\mathrm{GL}_m(D)$.*

Proof. In [ABPS2, §2] the LLC for $\mathrm{GL}_m(D)$ was constructed by starting with irreducible essentially square-integrable representations of Levi subgroups, then applying parabolic induction and finally taking Langlands quotients. In the context of types and covers thereof, [BuKu1, Corollary 8.4] shows that the maps (5.5) commute with parabolic induction. They also commute with taking Langlands quotients, because for these groups every Langlands quotient is the unique irreducible quotient of a suitable representation.

Thus we have reduced the claim to the case of irreducible essentially square-integrable representations. From [DKV, §B.2] we see that $\mathrm{Rep}(\mathrm{GL}_m(D))^{\mathfrak{s}}$ only contains such representations if $m_1 e_1 = m$. We may just as well consider the group $\mathrm{GL}_{m_i e_i}(D)$, which we prefer because then we can stick to the above notation. All its irreducible essentially square-integrable representations are generalized Steinberg representations built from $T_{\mathfrak{s}_i}$. By construction the bijection (5.6) for $\mathrm{GL}_{m_i}(D)^{e_i}$ sends $T_{\mathfrak{s}_i}$ to $T_{\mathfrak{s}_i^\vee}$.

Let $\chi_i \in X_{\mathrm{nr}}(\mathrm{GL}_{m_i}(D))$, with Langlands parameter $t_i \in X_{\mathrm{nr}}({}^L\mathrm{GL}_{m_i}(D))$. The generalized Steinberg representation $\mathrm{St}(\sigma')$ based on $\sigma' = (\chi_i \sigma_i)^{\otimes e_i}$ is the irreducible essentially square-integrable subrepresentation of the parabolic induction of

$$\nu_i^{(1-e_i)/2} \chi_i \sigma_i \otimes \cdots \otimes \nu_i^{(e_i-1)/2} \chi_i \sigma_i \tag{5.7}$$

to $\prod_i \mathrm{GL}_{m_i e_i}(D)$, where ν_i denotes the absolute value of reduced norm map for $\mathrm{GL}_{m_i}(D)$. There is a unique such subrepresentation by [DKV, Théorème B.2.b]. By definition [ABPS2, (12)] $\mathrm{St}(\sigma')$ has Langlands parameter $t_i \phi_i \otimes S_{e_i}$.

Now we plug $\mathrm{St}(\sigma')$ in (5.6) and we use the property discussed under (5.5). Thus we end up with an essentially discrete series representation of $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}})/\big(\{\mathbf{z}_i - q_F^{1/2}\}_i\big)$. By Theorem 3.18 it corresponds to a discrete element of $\Phi_{\mathrm{e}}(\mathrm{GL}_{m_i e_i}(D))^{\mathfrak{s}_i^\vee}$. Its enhancement ρ_i is uniquely determined by the requirement that it is relevant for $\mathrm{GL}_{m_i e_i}(D)$, so we can ignore that and focus on the L-parameter. The image of $\mathbf{W}_F$ under this L-parameter is contained in $\mathrm{GL}_{m_i}(D)^{e_i,\vee} = \mathrm{GL}_{m_i d}(\mathbb{C})^{e_i}$, so it can only be discrete if it is of the form $\psi_i \otimes \pi_{e_i, \mathrm{SL}_2(\mathbb{C})}$ for some irreducible $m_i d$-dimensional representation of $\mathbf{W}_F$. Since the cuspidal support of the enhanced L-parameter lies in $T_{\mathfrak{s}_i^\vee}$, ψ_i must be an unramified twist of ϕ_i. From (5.7) and the expression for the central character of $M(\psi_i \otimes \pi_{e_i, \mathrm{SL}_2(\mathbb{C})}, \rho_i, z_i)$ given in Theorem 3.18.b we deduce that $\psi_i = t_i \phi_i$. Thus (5.6) agrees with the local Langlands correspondence for essentially square-integrable representations. □

5.2. **Inner twists of $\mathrm{SL}_n(F)$.**

This paragraph is largely based on [ABPS2, ABPS3]. We keep the notations from the previous paragraph. For any subgroup of $\mathrm{GL}_m(D)$, we indicate the subgroup of elements of reduced norm 1 by a $\sharp$. Thus

$$\mathcal{G}^\sharp(F) = \mathrm{GL}_m(D)^\sharp = \{g \in \mathrm{GL}_m(D) : \mathrm{Nrd}(g) = 1\} = \mathrm{SL}_m(D).$$

The inner twists of $\mathrm{GL}_n(F)$ are in bijection with the inner twists of $\mathrm{SL}_n(F)$, via

$$\mathrm{GL}_m(D) \leftrightarrow \mathrm{GL}_m(D)^\sharp = \mathrm{SL}_m(D).$$

The L-parameters for $\mathrm{GL}_m(D)^\sharp$ are the same as for $\mathrm{GL}_m(D)$, only their image is considered in $\mathrm{PGL}_n(\mathbb{C})$. In particular every discrete L-parameter

$$\phi^\sharp \colon \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C})$$

lifts to an irreducible n-dimensional representation of $\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C})$. The local Langlands correspondence for these groups was worked out in [HiSa, ABPS2]. It provides a bijection between the Bernstein components on both sides of the LLC, which will use implicitly as $\mathfrak{s}^\sharp \leftrightarrow \mathfrak{s}^{\sharp\vee}$.

Let $\phi = \otimes_i \phi_i^{\otimes e_i}$ be as before, and let $\phi^\sharp \in \Phi(\mathcal{L}^\sharp(F))$ be the obtained by composition with the projection $\mathrm{GL}_n(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C})$. Every Bernstein component contains L-parameters of this form. There is a central extension

$$1 \to \mathrm{Z}_{\phi^\sharp} \to \mathcal{S}_{\phi^\sharp} \to \mathcal{R}_{\phi^\sharp} \to 1$$

where $\mathfrak{R}_{\phi^\sharp} = \pi_0(\mathrm{Z}_{\mathrm{PGL}_n(\mathbb{C})}(\mathrm{im}\phi^\sharp))$ and

$$\mathcal{Z}_{\phi^\sharp} = \mathrm{Z}(\mathrm{SL}_n(\mathbb{C}))/\mathrm{Z}(\mathrm{SL}_n(\mathbb{C})) \cap \mathrm{Z}_{\mathrm{SL}_n(\mathbb{C})}(\phi^\sharp)^\circ.$$

Let $\rho^\sharp$ be an enhancement of $\phi^\sharp$. The restriction $\rho = \rho^\sharp|_{\mathcal{Z}_{\phi^\sharp}}$ is an enhancement of ϕ, so as before we may assume that it has the form $\rho = \otimes_i \rho_i^{\otimes e_i}$. Cuspidality of $(\phi^\sharp, \rho^\sharp)$ depends only (ϕ, ρ), it holds whenever ρ_i is associated to the inner twist $\mathrm{GL}_{m_i}(D)$ of $\mathrm{GL}_n(F)$ via the Kottwitz isomorphism. We assume that this is the case, and that $(\phi^\sharp, \rho^\sharp) \in \Phi_{\mathrm{cusp}}(\mathcal{L}^\sharp(F))$. We note that $\mathcal{G}^\vee_{\mathrm{sc}}$ is the same for $\mathrm{GL}_m(D)$ and $\mathrm{SL}_m(D)$, and that ϕ and $\phi^\sharp$ have the same connected centralizer. Consequently

$$\begin{aligned} &G^\circ_{\phi^\sharp} = G^\circ_\phi, \quad G_{\phi^\sharp}/G^\circ_{\phi^\sharp} \cong \mathfrak{R}_{\phi^\sharp}, \quad M^\circ_{\phi^\sharp} = M^\circ_\phi, \\ &R(G^\circ_{\phi^\sharp}, T) = \prod\nolimits_i A_{e_i - 1}, \quad \lambda(\alpha) = t_{\phi_i} d_i \; \forall \alpha \in R(G_{\phi,i}T, T) \subset R(G^\circ_{\phi^\sharp}, T). \end{aligned}$$

Let $\mathfrak{s}^{\sharp\vee}$ be the inertial equivalence class for $\Phi_{\mathrm{e}}(\mathrm{GL}_m(D)^\sharp)$ determined by $(\phi^\sharp, \rho^\sharp)$. (In spite of the notation $\mathfrak{s}^\vee$ does not determine it uniquely.) Then

$$T_{\mathfrak{s}^{\sharp\vee}} = \big(\prod\nolimits_i T_{\phi_i}^{e_i}\big)/\mathrm{Z}(\mathrm{GL}_n(\mathbb{C})), \quad W^\circ_{\mathfrak{s}^{\sharp\vee}} \cong \prod\nolimits_i S_{e_i}.$$

The cuspidal local system $q\mathcal{E}$ associated to $(\phi^\sharp, \rho^\sharp)$ satisfies

$$\mathfrak{R}_{q\mathcal{E}} \cong W_{\mathfrak{s}^{\sharp\vee}}/W^\circ_{\mathfrak{s}^{\sharp\vee}} = \mathfrak{R}_{\mathfrak{s}^{\sharp\vee}} \cong \mathfrak{R}_{\phi^\sharp}.$$

The algebra

$$\mathcal{H}(\mathcal{R}_{\mathfrak{s}^{\sharp\vee}}, \lambda, \vec{\mathbf{z}}) = \mathcal{H}(G^\circ_{\phi^\sharp}, M^\circ_{\phi^\sharp}, v, \rho, \vec{\mathbf{z}}) \tag{5.8}$$

is a subalgebra of $\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \vec{\mathbf{z}})$, corresponding to the projection $T_{\mathfrak{s}^\vee} \to T_{\mathfrak{s}^{\sharp\vee}}$. It is contained in

$$\mathcal{H}(\mathfrak{s}^{\vee\sharp}, \vec{\mathbf{z}}) = \mathcal{H}(\mathcal{R}_{\mathfrak{s}^{\sharp\vee}}, \lambda, \vec{\mathbf{z}}) \rtimes \mathbb{C}[\mathfrak{R}_{\phi^\sharp}, \natural_{\phi^\sharp}].$$

Here the twisted group algebra and the 2-cocycle $\natural_{\phi^\sharp} = \natural_{\mathfrak{s}^{\sharp\vee}}$ are given by

$$\mathbb{C}[\mathfrak{R}_{\phi^\sharp}, \natural_{\phi^\sharp}] = p_\rho \mathbb{C}[\mathcal{S}_{\phi^\sharp}],$$

while the action of $\mathfrak{R}_{\phi^\sharp}$ on (5.8) comes from its natural action on $\mathcal{R}_{\mathfrak{s}^{\sharp\vee}}$.

For better comparison with the p-adic side we also determine the graded Hecke algebras attached to $\mathfrak{s}^{\sharp\vee}$. Let $(\phi_b^\sharp, \rho^\sharp) \in \Phi_{\mathrm{cusp}}(\mathcal{L}^\sharp(F))$ be an unramified twist of $(\phi^\sharp, \rho^\sharp)$ which is bounded. Let $W_{\phi_b^\sharp}$ be the stabilizer of $\phi_b^\sharp$ in $W_{\mathfrak{s}^{\sharp\vee}}$. Then $W^\circ_{\phi_b^\sharp} = W(G^\circ_{\phi_b^\sharp}, T)$ is the subgroup of $W_{\phi_b^\sharp} \cap W^\circ_{\mathfrak{s}^{\sharp\vee}}$ generated by the reflections it contains. The parabolic subgroup of $G^\circ_{\phi_b^\sharp}$ generated by $M^\circ_{\phi_b^\sharp}$ and upper triangular matrices determines a group $\mathfrak{R}_{\phi_b^{\sharp\vee}}$ such that

$$W_{\phi_b^\sharp} = W^\circ_{\phi_b^\sharp} \rtimes \mathfrak{R}_{\phi_b^\sharp}.$$

The 2-cocycle $\natural_{\phi_b^\sharp}$ on $W_{\phi_b^\sharp}$ is the restriction of $\natural_{\mathfrak{s}^{\sharp\vee}} : W^2_{\mathfrak{s}^{\sharp\vee}} \to \mathbb{C}^\times$. The root system $R_{\phi_b^\sharp}$ is again a product of systems of type A, namely $\prod_j A_{\epsilon_j - 1}$ if $\phi_b^\sharp = \otimes_j \phi_j^{\epsilon_j}$. Then

$$W^\circ_{\phi_b^\sharp} \cong \prod\nolimits_j S_{\epsilon_j} \quad \text{and} \quad \mathfrak{t}_{\mathfrak{s}^{\sharp\vee}} = \mathrm{Lie}(T_{\mathfrak{s}^{\sharp\vee}}) = \big(\sum\nolimits_i \mathrm{Lie}(T^{e_i}_{\phi_i})\big) / \mathrm{Z}(\mathfrak{gl}_n(\mathbb{C})).$$

It follows that

$$\mathbb{H}(\phi_b, v, q\epsilon, \vec{\mathbf{r}}) \cong \mathbb{H}(\mathfrak{t}_{\mathfrak{s}^{\sharp\vee}}, W_{\phi_b^\sharp}, \vec{\mathbf{r}}, \natural_{\phi_b^\sharp}) \cong \mathbb{H}(\mathfrak{t}_{\mathfrak{s}^{\sharp\vee}}, W_{\phi_b^\sharp}, \vec{\mathbf{r}}) \rtimes \mathbb{C}[\mathfrak{R}_{\phi_b^\sharp}, \natural_{\phi_b^\sharp}]. \tag{5.9}$$

The Hecke algebras for Bernstein components of $\mathrm{SL}_m(D)$ were computed in [ABPS3]. They are substantially more complicated than their counterparts for $\mathrm{GL}_m(D)$, and in particular do not match entirely with the above affine Hecke algebras for Langlands parameters. To describe them, we need some notations. Let $\mathcal{P}$ be a parabolic subgroup of $\mathrm{GL}_m(D)$, with Levi factor $\mathcal{M}$. Consider the inertial equivalence classes $\mathfrak{s}_\mathcal{M} = [\mathcal{M}, \sigma]_\mathcal{M}$ and $\mathfrak{s} = [\mathcal{M}, \sigma]_{\mathrm{GL}_m(D)}$. Recall from (5.4) that $\mathcal{H}(\mathrm{GL}_m(D))^\mathfrak{s}$ is Morita equivalent with

$$\mathcal{H}(\mathcal{R}_\mathfrak{s}, \lambda, q_\mathfrak{s}) = \bigotimes\nolimits_i \mathcal{H}(\mathrm{GL}_{e_i}, q_F^{f_i}).$$

We need the groups

$$\begin{aligned}
&X^\mathcal{M}(\mathfrak{s}) = \big\{\gamma \in \mathrm{Irr}\big(\mathcal{M}/\mathcal{M}^\sharp \mathrm{Z}(\mathrm{GL}_m(D))\big) : \gamma \otimes \sigma \in \mathfrak{s}_\mathcal{M}\big\},\\
&X^{\mathrm{GL}_m(D)}(\mathfrak{s}) = \big\{\gamma \in \mathrm{Irr}\big(\mathrm{GL}_m(D)/\mathrm{GL}_m(D)^\sharp \mathrm{Z}(\mathrm{GL}_m(D))\big) : \gamma \otimes I_\mathcal{P}^{\mathrm{GL}_m(D)}(\sigma) \in \mathfrak{s}\big\},\\
&W_\mathfrak{s}^\sharp = \big\{w \in \mathrm{N}_{\mathrm{GL}_m(D)}(\mathcal{M})/(\mathcal{M}) : \exists \gamma \in \mathrm{Irr}\big(\mathcal{M}/\mathcal{M}^\sharp \mathrm{Z}(\mathrm{GL}_m(D))\big) : w(\gamma \otimes \sigma) \in \mathfrak{s}_\mathcal{M}\big\}.
\end{aligned}$$

By [ABPS3, Lemma 2.3] $W_\mathfrak{s}^\sharp = W_\mathfrak{s} \rtimes \mathfrak{R}_\mathfrak{s}^\sharp$ for a suitable subgroup $\mathfrak{R}_\mathfrak{s}^\sharp$, and

$$X^{\mathrm{GL}_m(D)}(\mathfrak{s}) / X^\mathcal{M}(\mathfrak{s}) \cong \mathfrak{R}_\mathfrak{s}^\sharp$$

by [ABPS3, Lemma 2.4]. The group $X^{\mathrm{GL}_m(D)}(\mathfrak{s})$ acts naturally on $T_\mathfrak{s} \rtimes W_\mathfrak{s}$.

Let $\sigma^\sharp$ be an irreducible constituent of $\sigma|_{\mathcal{M}^\sharp}$. Every inertial equivalence class for $\mathrm{SL}_m(D) = \mathrm{GL}_m(D)^\sharp$ is of the form $\mathfrak{s}^\sharp = [\mathcal{M}^\sharp, \sigma^\sharp]_{\mathrm{GL}_m(D)^\sharp}$. By [ABPS3, Theorem 1] there exists a finite dimensional projective representation V_μ of $X^{\mathrm{GL}_m}\mathfrak{s})$ such that $\mathcal{H}(\mathrm{GL}_m(D)^\sharp)^{\mathfrak{s}^\sharp}$ is Morita equivalent with one direct summand of

$$\big(\mathcal{H}(\mathcal{R}_\mathfrak{s}, \lambda, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu)\big)^{X^\mathcal{M}(\mathfrak{s}) X_{\mathrm{nr}}(\mathcal{M}/\mathcal{M}^\sharp)} \rtimes \mathfrak{R}_\mathfrak{s}^\sharp. \tag{5.10}$$

The other direct summands correspond to different constituents of $\sigma|_{\mathcal{M}^\sharp}$. In (5.10) the group

$$X_{\mathrm{nr}}(\mathcal{M}/\mathcal{M}^\sharp) = \{\chi \in X_{\mathrm{nr}}(\mathcal{M}) : \mathcal{M}^\sharp \subset \ker \chi\}$$

acts only via translations of $T_\mathfrak{s}$. We denote the quotient torus $T_\mathfrak{s}/X_{\mathrm{nr}}(\mathcal{M}/\mathcal{M}^\sharp)$ by $T_\mathfrak{s}^\sharp$ and its Lie algebra by $\mathfrak{t}_\mathfrak{s}^\sharp$.

From now on we will be more sketchy. The below can be made precise, but for that one would have to delve into some of the technicalities of [ABPS3], which are not so relevant for this paper. Although it is not so easy to write down all direct summands of (5.10) explicitly, we can say that they look like

$$\big(\mathcal{H}(X^*(T_\mathfrak{s}^\sharp), R_\mathfrak{s}, X_*(T_\mathfrak{s}^\sharp), R_\mathfrak{s}^\vee, \lambda, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_{\mu^\sharp})\big)^{X^\mathcal{M}(\mathfrak{s},\sigma^\sharp)} \rtimes \mathfrak{R}_{\mathfrak{s}^\sharp} \tag{5.11}$$

for suitable $X^\mathcal{M}(\mathfrak{s},\sigma^\sharp) \subset X^\mathcal{M}(\mathfrak{s})$ and $V_{\mu^\sharp} \subset V_\mu$. (From the below argument for graded Hecke algebras one sees approximately how (5.11) arises from (5.10).) This algebra need not be Morita equivalent to a twisted affine Hecke algebra as studied in this paper. The problem comes from the simultaneous action of $X^\mathcal{M}(\mathfrak{s},\sigma^\sharp)$ on $T_\mathfrak{s}^\sharp$ and $V_{\mu^\sharp}$: if that is complicated, it prevents (5.11) from being Morita equivalent to a similar algebra without $\mathrm{End}_\mathbb{C}(V_{\mu^\sharp})$. If we consider (5.11) as a kind of algebra bundle over $T_\mathfrak{s}^\sharp$, then these remarks mean that $V_{\mu^\sharp}$ could introduce some extra twists in this bundle, which take the algebra outside the scope of this paper. Examples can be constructed by combining the ideas in [ABPS3, Examples 5.2 and 5.5].

That being said, the other data involved in (5.11) are as desired. It was checked in [ABPS5, Lemma 5.5] that:

(i) The underlying torus $T_{\mathfrak{s}^\sharp} = T_\mathfrak{s}^\sharp / X^\mathcal{M}(\mathfrak{s},\sigma^\sharp)$ is naturally isomorphic to $T_{\mathfrak{s}^{\sharp\vee}} = \Phi_{\mathrm{e}}(\mathcal{M}^\sharp)^{[\mathcal{M}^\sharp,\sigma^\sharp]_{\mathcal{M}^\sharp}}$.
(ii) $W_\mathfrak{s} \rtimes \mathfrak{R}_{\mathfrak{s}^\sharp} = W_{\mathfrak{s}^\sharp}$ is isomorphic to $W_{\mathfrak{s}^{\sharp\vee}} = W_{\mathfrak{s}^\vee} \rtimes \mathfrak{R}_{\mathfrak{s}^{\sharp\vee}}$.
(iii) The space of irreducible representations of (5.11) is isomorphic to a twisted extended quotient

$$(T_{\mathfrak{s}^\sharp} /\!/ W_{\mathfrak{s}^\sharp})_{\kappa_{\sigma^\sharp}} \cong (T_{\mathfrak{s}^{\sharp\vee}} /\!/ W_{\mathfrak{s}^{\sharp\vee}})_{\kappa_{\sigma^\sharp}},$$

and the 2-cocycle $\kappa_{\sigma^\sharp}$ of $W_{\mathfrak{s}^\sharp}$ is equivalent to the 2-cocycle $\natural_{\mathfrak{s}^{\sharp\vee}}$ of $W_{\mathfrak{s}^{\sharp\vee}}$.

Let us also discuss the graded Hecke algebras which can be derived from (5.10) and (5.11). The algebra $\mathcal{O}(T_\mathfrak{s}^\sharp)^{X^\mathcal{M}(\mathfrak{s}) W_\mathfrak{s}}$ is naturally contained in the centre of (5.10). This entails that we can localize at suitable subsets of $T_\mathfrak{s}^\sharp / W_\mathfrak{s}^\sharp X^\mathcal{M}(\mathfrak{s})$. Fix $t \in (T_\mathfrak{s}^\sharp)_{\mathrm{un}}$. By localization at a small neighborhood of U of $W_\mathfrak{s}^\sharp X^\mathcal{M}(\mathfrak{s}) t (T_\mathfrak{s}^\sharp)_{\mathrm{rs}}$, we can effectively replace $X^\mathcal{M}(\mathfrak{s})$ by the stabilizer of $X^\mathcal{M}(\mathfrak{s})_t$, and $\mathfrak{R}_\mathfrak{s}^\sharp$ by the stabilizer $\mathfrak{R}_\mathfrak{s}^\sharp(t)$ of $W_\mathfrak{s} X^\mathcal{M}(\mathfrak{s}) t$. Then (5.10) is transformed into the algebra

$$C_{\mathrm{an}}(U)^{X^\mathcal{M}(\mathfrak{s}) W_\mathfrak{s}^\sharp} \underset{\mathcal{O}(T_\mathfrak{s}^\sharp)^{X^\mathcal{M}(\mathfrak{s}) W_\mathfrak{s}^\sharp}}{\otimes} \big(\mathcal{H}(\mathcal{R}_\mathfrak{s}^\sharp, \lambda, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu)\big)^{X^\mathcal{M}(\mathfrak{s})_t} \rtimes \mathfrak{R}_{\mathfrak{s}^\sharp}(t) \tag{5.12}$$

where $\mathcal{R}_\mathfrak{s}^\sharp = (X^*(T_\mathfrak{s}^\sharp), R_\mathfrak{s}, X_*(T_\mathfrak{s}^\sharp), R_\mathfrak{s}^\vee)$. But $X^\mathcal{M}(\mathfrak{s})$ acts by translations on $T_\mathfrak{s}^\sharp$, so $X^\mathcal{M}(\mathfrak{s})_t$ consists of all the elements that fix $T_\mathfrak{s}^\sharp$ entirely. From the description of the actions on (5.10) in [ABPS3, Lemma 4.11] we see that $X^\mathcal{M}(\mathfrak{s})_t$ acts only on $\mathrm{End}_\mathbb{C}(V_\mu)$. Then

$$\mathrm{End}_\mathbb{C}(V_\mu)^{X^\mathcal{M}(\mathfrak{s})_t} = \mathrm{End}_{X^\mathcal{M}(\mathfrak{s})_t}(V_\mu) \cong \bigoplus\nolimits_{\mu^\sharp} \mathrm{End}_\mathbb{C}(V_{\mu^\sharp}) \tag{5.13}$$

is a finite dimensional semisimple algebra. The direct summands of (5.10) and of (5.12) are in bijection with the $\mathfrak{R}^\sharp_\mathfrak{s}(t)$-orbits on the set of direct summands of (5.13). That holds for any $t \in (T^\sharp_\mathfrak{s})_{\mathrm{un}}$, in particular for s ome t with $\mathfrak{R}^\sharp_\mathfrak{s}(t) = 1$, so in fact the direct summands $\mathrm{End}_\mathbb{C}(V_{\mu^\sharp})$ of (5.13) parametrize the direct summands of (5.10) and of (5.12). Thus (5.12) is a direct sum of algebras

$$C_{\mathrm{an}}(U)^{X^{\mathcal{M}}(\mathfrak{s}) W^\sharp_\mathfrak{s}} \underset{\mathcal{O}(T^\sharp_\mathfrak{s})^{X^{\mathcal{M}}(\mathfrak{s}) W^\sharp_\mathfrak{s}}}{\otimes} \big(\mathcal{H}(\mathcal{R}^\sharp_\mathfrak{s}, \lambda, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_{\mu^\sharp}) \big) \rtimes \mathfrak{R}_{\mathfrak{s}^\sharp}(t). \tag{5.14}$$

Here $(\mu^\sharp, V_{\mu^\sharp})$ is a projective representation of $\mathfrak{R}_{\mathfrak{s}^\sharp}(t)$. In such situations there is a Morita equivalent algebra embedding

$$\begin{array}{ccc} \mathbb{C}[\mathfrak{R}_{\mathfrak{s}^\sharp}(t), \natural] & \to & \mathrm{End}_\mathbb{C}(V_{\mu^\sharp}) \rtimes \mathfrak{R}_{\mathfrak{s}^\sharp}(t) \\ r & \mapsto & \mu^\sharp(r)^{-1} r, \end{array}$$

for a suitable 2-cocycle $\natural$. Via this method (5.14) is Morita equivalent with

$$C_{\mathrm{an}}(U)^{X^{\mathcal{M}}(\mathfrak{s}) W^\sharp_\mathfrak{s}} \underset{\mathcal{O}(T^\sharp_\mathfrak{s})^{X^{\mathcal{M}}(\mathfrak{s}) W^\sharp_\mathfrak{s}}}{\otimes} \mathcal{H}(\mathcal{R}^\sharp_\mathfrak{s}, \lambda, q_\mathfrak{s}) \rtimes \mathbb{C}[\mathfrak{R}_{\mathfrak{s}^\sharp}(t), \natural]. \tag{5.15}$$

From the property (iii) of the algebra (5.11) we see that $\natural$ has to be the restriction of $\natural_{\mathfrak{s}^{\sharp\vee}}$ to $\mathfrak{R}_{\mathfrak{s}^\sharp}(t)^2$. By Theorems 2.5.a and 2.11.a the algebra (5.15) is Morita equivalent with

$$C_{\mathrm{an}}(U)^{X^{\mathcal{M}}(\mathfrak{s}) W^\sharp_\mathfrak{s}} \underset{\mathcal{O}(\mathfrak{t}^\sharp_\mathfrak{s})^{X^{\mathcal{M}}(\mathfrak{s}) W^\sharp_\mathfrak{s}}}{\otimes} \mathbb{H}(\mathfrak{t}^\sharp_\mathfrak{s}, W(R_\mathfrak{s})_t, q_\mathfrak{s}) \rtimes \mathbb{C}[\mathfrak{R}_{\mathfrak{s}^\sharp}(t), \natural_{\mathfrak{s}^{\sharp\vee}}]. \tag{5.16}$$

Hence the equivalence between $\mathrm{Rep}(\mathrm{SL}_m(D))^{\mathfrak{s}^\sharp} \cong \mathrm{Mod}\big(\mathcal{H}(\mathrm{GL}_m(D)^\sharp)^{\mathfrak{s}^\sharp}\big)$ and the module category of (5.11) restricts to an equivalence between

$$\begin{array}{l} \mathrm{Mod}_{f, W^\sharp_\mathfrak{s} X^{\mathcal{M}}(\mathfrak{s}) t (T^\sharp_\mathfrak{s})_{\mathrm{rs}}} \big(\mathcal{H}\big(\mathrm{GL}_m(D)^\sharp\big)^{\mathfrak{s}^\sharp} \big) \quad \text{and} \\ \mathrm{Mod}_{f, (\mathfrak{t}^\sharp_\mathfrak{s})_{\mathrm{rs}}} \big(\mathbb{H}(\mathfrak{t}^\sharp_\mathfrak{s}, W(R_\mathfrak{s})_t, q_\mathfrak{s}) \rtimes \mathbb{C}[\mathfrak{R}_{\mathfrak{s}^\sharp}(t), \natural_{\mathfrak{s}^{\sharp\vee}}] \big). \end{array}$$

Every finite length representation in $\mathrm{Rep}(\mathrm{SL}_m(D))^{\mathfrak{s}^\sharp}$ decomposes canonically as a direct sum of generalized weight spaces for $\mathcal{O}(T^\sharp_\mathfrak{s})^{X^{\mathcal{M}}(\mathfrak{s}) W^\sharp_\mathfrak{s}}$, so by varying t in $(T^\sharp_\mathfrak{s})_{\mathrm{un}}$ we can describe all such representations in terms of these equivalences of categories. In this sense

$$\mathbb{H}(\mathfrak{t}^\sharp_\mathfrak{s}, W(R_\mathfrak{s})_t, q_\mathfrak{s}) \rtimes \mathbb{C}[\mathfrak{R}_{\mathfrak{s}^\sharp}(t), \natural_{\mathfrak{s}^{\sharp\vee}}] \tag{5.17}$$

is the graded Hecke algebra attached to $(\mathfrak{s}^\sharp, t)$. Suppose that t corresponds to $(\phi^\sharp_b, \rho^\sharp) \in \Phi_{\mathrm{cusp}}(\mathcal{L}^\sharp(F))$, where $\mathcal{M} = \mathcal{L}(F)$. Then we can compare (5.17) with (5.9). Using the earlier comparison results (i), (ii) and (iii), we see that (5.17) is the specialization of (5.9) at $\vec{\mathbf{r}} = \log(q_\mathfrak{s})$.

We conclude that, for a Bernstein component $\mathfrak{s}^\sharp$ of $\mathrm{SL}_m(D)$, corresponding to a Bernstein component $\mathfrak{s}^{\sharp\vee}$ of enhanced L-parameters:

- The twisted graded Hecke algebras attached to $\mathfrak{s}^\sharp$ and to $\mathfrak{s}^{\sharp\vee}$ are isomorphic.
- The twisted affine Hecke algebras attached to $\mathfrak{s}^\sharp$ and to $\mathfrak{s}^{\sharp\vee}$ need not be isomorphic, but they are sufficiently close, so that their categories of finite length modules are equivalent.

5.3. **Pure inner twists of classical groups.**

Take $n \in \mathbb{N}$ and let $\mathcal{G}_n^*$ be a F-split connected classical group of rank n. That is, $\mathcal{G}_n^*$ is one the following groups:

(i) Sp_{2n}, the symplectic group in $2n$ variables defined over F,
(ii) SO_{2n+1}, the split special orthogonal group in $2n+1$ variables defined over F,
(iii) SO_{2n}, the split special orthogonal group in $2n$ variables defined over F,

Let V^* be a finite dimensional F-vector space equipped with a non-degenerate symplectic or orthogonal form such that $\mathcal{G}_n^*(F)$ equals $\mathrm{Sp}(V^*)$ or $\mathrm{SO}(V^*)$. The pure inner twists $\mathcal{G}_n$ of $\mathcal{G}_n^*$ correspond bijectively to forms V of the space V^* with its bilinear form $\langle \, , \, \rangle$ [KMRT, §29D–E]. If $\mathcal{G}_n^*(F) = \mathrm{Sp}(V^*)$, then the pointed set $H_1(F, \mathcal{G}_n^*)$ has only one element and there are no nontrivial pure inner twists of $\mathcal{G}_n^*$. If $\mathcal{G}_n^*(F) = \mathrm{SO}(V^*)$, then elements of $H_1(F, \mathcal{G}_n^*)$ correspond bijectively to the isomorphism classes of orthogonal spaces V over F with $\dim(V) = \dim(V^*)$ and $\mathrm{disc}(V) = \mathrm{disc}(V^*)$. The corresponding pure inner twist of $\mathcal{G}_n^*(F)$ is the special orthogonal group $\mathrm{SO}(V)$.

Let $\mathcal{G}_n(F)$ be a pure inner twist of $\mathcal{G}_n^*(F)$ (we allow $\mathcal{G}_n(F) = \mathcal{G}_n^*(F)$). It is known (see for instance [ChGo]), that up to conjugacy every Levi subgroup of $\mathcal{G}_n(F)$ is of the form

$$\mathcal{L}(F) = \mathcal{G}_{n^-}(F) \times \prod\nolimits_j \mathrm{GL}_{m_j}(F), \tag{5.18}$$

where $\sum_j m_j + n^- = n$ and $\mathcal{G}_{n^-}(F)$ is an inner twist of the split connected classical group $\mathcal{G}_{n^-}^*$ defined over F, of rank n^-, which has the same type as $\mathcal{G}_n^*(F)$. There is a natural embedding $\mathrm{Std}_{^L\mathcal{G}}$ of $^L\mathcal{G}$ into $\mathrm{GL}_{N^\vee}(\mathbb{C}) \rtimes \mathbf{W}_F$, where $N^\vee = 2n+1$ if $\mathcal{G}_n^* = \mathrm{Sp}_{2n}$, and $N^\vee = 2n$ otherwise.

Let $(\phi, \rho) \in \Phi_{\mathrm{cusp}}(\mathcal{L}(F))$. The factorization (5.18) leads to

$$\mathrm{Std}_{^L\mathcal{G}} \circ \phi = \varphi \oplus \bigoplus\nolimits_j (\phi_j \oplus \phi_j^\vee). \tag{5.19}$$

Because we consider only pure inner twists in this section, it would be superfluous to replace $\mathcal{G}^\vee$ by $\mathcal{G}^\vee_{\mathrm{sc}}$. We refrain from doing so in this section, and we use the objects, which before where defined in terms of $\mathcal{G}^\vee_{\mathrm{sc}}$, now with the same definition involving just $\mathcal{G}^\vee$. For instance, instead of the group $\mathcal{S}_\phi$ defined in Definition 3.2, we will take the component group $\pi_0(\mathrm{Z}_{\mathcal{L}^\vee}(\phi))$ and we use a variation on $\Phi_{\mathrm{e}}(^L\mathcal{G})$ with that component group. The restriction of an enhancement ρ to the center of $\mathcal{L}^\vee$ still determines the relevance. For instance, if the restriction to $\mathrm{Z}(\mathcal{L}^\vee)$ is trivial, then it corresponds to the split form, otherwise it corresponds to a non-split form. Hence, we can decompose $\rho = \varrho \otimes \bigotimes_j \rho_j$, where $(\varphi, \varrho) \in \Phi_{\mathrm{cusp}}(\mathcal{G}_{n^-}(F))$ and $(\phi_j, \rho_j) \in \Phi_{\mathrm{cusp}}(\mathrm{GL}_{m_j}(F))$ for each j.

Let I_ϕ^+ (resp. I_ϕ^-) be the set of (classes of) self-dual irreducible representations of $\mathbf{W}_F$ which occur in $\mathrm{Std}_{^L\mathcal{G}} \circ \phi$ and which factor through a group of the type of $\mathcal{G}^\vee$ (resp. of opposite type of $\mathcal{G}^\vee$). Let I_ϕ^0 be a set of (classes of) non self-dual irreducible representations of $\mathbf{W}_F$ which occur in $\mathrm{Std}_{^L\mathcal{G}} \circ \phi$, such that if $\tau \in I_\phi^0$ then $\tau^\vee \notin I_\phi^0$, and maximal for this property. We denote the irreducible a-dimensional representation of $\mathrm{SL}_n(\mathbb{C})$ by S_a.

On the one hand (ϕ_j, ρ_j) satisfy the conditions stated in Paragraph 5.1, i.e. ϕ_j is an irreducible representation of $\mathbf{W}_F$ and ρ_j is the trivial representation of

$\pi_0(\mathrm{Z}_{\mathrm{GL}_{m_j}(\mathbb{C})}(\phi_j))$. On the other hand, by [Mou, Proposition 3.6] we have

$$\mathrm{Std}_{{}^L\mathcal{G}_{n^-}} \circ \varphi = \bigoplus_{\tau \in I_\varphi^+} \bigoplus_{a \text{ odd}, a=1}^{a_\tau} (\tau \otimes S_a) \oplus \bigoplus_{\tau \in I_\varphi^-} \bigoplus_{a \text{ even}, a=2}^{a_\tau} (\tau \otimes S_a), \tag{5.20}$$

where $a_\tau \in \mathbb{Z}_{\geq 0}$. As introduced by Mœglin, let $\mathrm{Jord}(\varphi)$ be the set of pairs (τ, a) with $\tau \in \mathrm{Irr}(\mathbf{W}_F)$, $a \in \mathbb{Z}_{>0}$ such that $\tau \boxtimes S_a$ is an irreducible subrepresentation of $\mathrm{Std}_{{}^L\mathcal{G}_{n^-}} \circ \varphi$.

The group $\mathcal{S}_\phi$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^p$ for some integer p. It is generated by elements of order two, by $z_{\tau,a} z_{\tau',a'}$ where $(\tau, a), (\tau', a') \in \mathrm{Jord}(\phi)$ without hypothesis on the parity of a and by $z_{\tau,a}$ when a is even. The character ρ satisfies $\rho(z_{\tau,2i-1} z_{\tau,2i+1}) = -1$ for all $\tau \in I_\varphi^+$ and $i \in [\![1, \frac{a_\tau - 1}{2}]\!]$ and $\rho(z_{\tau,2i}) = (-1)^i$ for all $\tau \in I_\varphi^-$ and $i \in [\![1, \frac{a_\tau}{2}]\!]$.

If τ is an irreducible representation of $\mathbf{W}_F$ and of dimension m such that $\tau|_{\mathbf{I}_F} \cong \tau^\vee|_{\mathbf{I}_F}$, then $\tau \cong \tau^\vee z$ with $z \in X_{\mathrm{nr}}({}^L\mathrm{GL}_m(F))$. Replacing τ by $\tau z^{1/2}$ (where $z^{1/2}$ is any square root of z), we can assume that $\tau \cong \tau^\vee$. In the following, for all j we assume that, if $\phi_j^\vee$ is inertially equivalent to ϕ_j, then $\phi_j^\vee \cong \phi_j$. Note that a self-dual irreducible representation of $\mathbf{W}_F$ is necessarily of symplectic-type or of orthogonal-type.

We choose a basepoint ϕ (inside its inertial equivalence class) as follows:

- if $m_i = m_j$ and ϕ_i differs from ϕ_j by an unramified twist, then $\phi_i = \phi_j$;
- if $\phi_i^\vee$ is an unramified twist of ϕ_i, then we can assume that $\phi_i^\vee \cong \phi_i$;
- if $\phi_i^\vee \cong \phi_j$, then $i = j$.

For an irreducible representation τ of $\mathbf{W}_F$, we will denote by e_τ the number of times that τ appears in a GL factor of $\mathcal{L}^\vee$ and by ℓ_τ the multiplicity of τ in $\varphi|_{\mathbf{W}_F}$, so that

$$\begin{aligned} \phi &= \bigoplus_{\tau \in I_\phi^+ \sqcup I_\phi^-} 2e_\tau \tau \oplus \bigoplus_{\tau \in I_\phi^0} e_\tau (\tau \oplus \tau^\vee) \oplus \varphi, \\ \phi|_{\mathbf{W}_F} &= \bigoplus_{\tau \in I_\phi^+ \sqcup I_\phi^-} (2e_\tau + \ell_\tau)\tau \oplus \bigoplus_{\tau \in I_\phi^0} e_\tau (\tau \oplus \tau^\vee). \end{aligned}$$

The following groups are associated to this ϕ:

$$G_\phi \cong \prod_{\tau \in I_\phi^-} \mathrm{Sp}_{2e_\tau + \ell_\tau}(\mathbb{C}) \times \prod_{\tau \in I_\phi^+, \dim \tau \text{ even}} \mathrm{O}_{2e_\tau + \ell_\tau}(\mathbb{C}) \times S\Big(\prod_{\tau \in I_\phi^+, \dim \tau \text{ odd}} \mathrm{O}_{2e_\tau + \ell_\tau}(\mathbb{C})\Big) \times \prod_{\tau \in I_\phi^0} \mathrm{GL}_{e_\tau}(\mathbb{C}),$$

$$\begin{aligned} M \cong & \prod_{\tau \in I_\phi^-} \big((\mathbb{C}^\times)^{e_\tau} \times \mathrm{Sp}_{\ell_\tau}(\mathbb{C})\big) \times \prod_{\tau \in I_\phi^+} (\mathbb{C}^\times)^{e_\tau} \times \\ & \prod_{\tau \in I_\phi^+, \dim \tau \text{ even}} \mathrm{O}_{\ell_\tau}(\mathbb{C}) \times S\Big(\prod_{\tau \in I_\phi^+, \dim \tau \text{ odd}} \mathrm{O}_{\ell_\tau}(\mathbb{C})\Big) \times \prod_{\tau \in I_\phi^0} (\mathbb{C}^\times)^{e_\tau}. \end{aligned}$$

Here $S(H)$, for a matrix group H, means the elements of determinant 1 in H. The above expression for G_ϕ° naturally factors as $\prod_{\tau \in I_\phi^- \sqcup I_\phi^+ \sqcup I_\phi^0} G_\tau^\circ$, and similarly for M°. This is an almost direct factorization of G_ϕ° in the sense of (1.2). With that we can write

$$T \cong \prod_{\tau \in I_\phi^- \sqcup I_\phi^+ \sqcup I_\phi^0} (\mathbb{C}^\times)^{e_\tau}, \quad R(G_\phi^\circ, T) \cong \prod_{\tau \in I_\phi^- \sqcup I_\phi^+ \sqcup I_\phi^0} R(G_\tau^\circ T, T). \tag{5.21}$$

Let us record the root systems $R_\tau = R(G^\circ_\tau T, T)$:

	condition	R_τ	$R_{\tau,\mathrm{red}}$
$\tau \in I^-_\phi$	$e_\tau = 0$	$\varnothing$	$\varnothing$
	$e_\tau \neq 0, \ell_\tau = 0$	C_{e_τ}	C_{e_τ}
	$e_\tau \neq 0, \ell_\tau \neq 0$	BC_{e_τ}	B_{e_τ}
$\tau \in I^+_\phi$	$e_\tau = 0$	$\varnothing$	$\varnothing$
	$e_\tau \neq 0, \ell_\tau = 0$	D_{e_τ}	D_{e_τ}
	$e_\tau \neq 0, \ell_\tau \neq 0$	B_{e_τ}	B_{e_τ}
$\tau \in I^0_\phi$	$e_\tau \leqslant 1$	$\varnothing$	$\varnothing$
	$e_\tau \geqslant 2$	$A_{e_\tau -1}$	$A_{e_\tau -1}$

To justify the above choice of a basepoint ϕ, we need to check that G°_ϕ detects as many roots as possible. Let us consider the restriction $\phi|_{\mathbf{I}_F}$:

$$\begin{aligned}\mathrm{Std}_{{}^L\mathcal{G}} \circ \phi|_{\mathbf{I}_F} &= \varphi|_{\mathbf{I}_F} \oplus \bigoplus\nolimits_j (\phi_j|_{\mathbf{I}_F} \oplus \phi_j^\vee|_{\mathbf{I}_F}) \\ &= \bigoplus_{\tau \in I^+_\phi \sqcup I^-_\phi} (2e_\tau + \ell_\tau)\tau|_{\mathbf{I}_F} \oplus \bigoplus_{\tau \in I^0_\phi} e_\tau(\tau|_{\mathbf{I}_F} \oplus \tau^\vee|_{\mathbf{I}_F}).\end{aligned}$$

We have assumed that for $\tau \in I^0_\phi$, $\tau|_{\mathbf{I}_F} \not\cong \tau^\vee|_{\mathbf{I}_F}$ and we know that an irreducible representation τ of $\mathbf{W}_F$ decomposes upon restriction to $\mathbf{I}_F$ as

$$\tau|_{\mathbf{I}_F} = \theta \oplus \theta^{\mathrm{Frob}_F} \oplus \ldots \oplus \theta^{\mathrm{Frob}_F^{t_\tau -1}}, \tag{5.22}$$

for some irreducible representation θ of $\mathbf{I}_F$. Here for all $w \in \mathbf{I}_F$, $\theta^{\mathrm{Frob}_F^k}(w) = \theta(\mathrm{Frob}_F^{-k} w \mathrm{Frob}_F^k)$. If we assume $\tau|_{\mathbf{I}_F} \cong \tau^\vee|_{\mathbf{I}_F}$, then $\theta^\vee \cong \theta^{\mathrm{Frob}_F^i}$ for some integer i between 0 and $t_\tau - 1$. Then we have $\theta \cong \theta^{\mathrm{Frob}_F^i\,\vee} \cong \theta^{\mathrm{Frob}_F^{2i}}$. This implies that $i = 0$ or t_τ is even and $i = t_\tau/2$. In the first case, $\theta^\vee \cong \theta$ and in the second case $\theta^\vee \cong \theta^{\mathrm{Frob}_F^{t_\tau/2}}$. We denote by I^{++}_ϕ (resp. I^{--}_ϕ) the subset of I^+_ϕ (resp. I^-_ϕ) corresponding to the first case, and define I^{+-}_ϕ as the remaining subset of $I^+_\phi \cup I^-_\phi$. For any τ, let τ' be a twist of τ by an unramified character of $\mathbf{W}_F$, such that τ' is self-dual but not isomorphic to τ. For $\tau \in I^{+-}_\phi$ the type of τ' is opposite to that of τ, which motivates the superscipt $+-$. The three sets $I^{++}_\phi, I^{--}_\phi, I^{+-}_\phi$ are considered modulo the relation $\tau \sim \tau'$. We find that

$$\begin{aligned} J^\circ = \mathrm{Z}_{\mathcal{G}^\vee}(\phi|_{\mathbf{I}_F})^\circ \;\cong\; & \prod\nolimits_{\tau \in I^{--}_\phi} \mathrm{Sp}_{2e_\tau + \ell_\tau + \ell_{\tau'}}(\mathbb{C})^{t_\tau} \times \prod\nolimits_{\tau \in I^{++}_\phi} \mathrm{SO}_{2e_\tau + \ell_\tau + \ell_{\tau'}}(\mathbb{C})^{t_\tau} \times \\ & \prod\nolimits_{\tau \in I^{+-}_\phi} \mathrm{GL}_{2e_\tau + \ell_\tau + \ell_{\tau'}}(\mathbb{C})^{t_\tau/2} \times \prod\nolimits_{\tau \in I^0_\phi} \mathrm{GL}_{e_\tau}(\mathbb{C})^{t_\tau}. \end{aligned} \tag{5.23}$$

For all $\tau \in I^{++}_\phi$, we have an embedding of $(\mathbb{C}^\times)^{e_\tau}$ into $(\mathbb{C}^\times)^{e_\tau} \times \mathrm{SO}_{\ell_\tau + \ell_{\tau'}}(\mathbb{C})^{t_\tau}$ and the latter is embedded diagonally as Levi subgroup in $\mathrm{SO}_{2e_\tau + \ell_\tau + \ell_{\tau'}}(\mathbb{C})^{t_\tau}$. We have the same kind of embedding for $\tau \in I^{--}_\phi$ and $\tau \in I^0_\phi$. For $\tau \in I^{+-}_\phi$, the embedding of $(\mathbb{C}^\times)^{e_\tau}$ in $\mathrm{GL}_{2e_\tau + \ell_\tau + \ell_{\tau'}}(\mathbb{C})$ is given by

$$(z_1, \ldots, z_{e_\tau}) \mapsto \mathrm{diag}(z_1, \ldots, z_{e_\tau}, 1, \ldots, 1, z_{e_\tau}^{-1}, \ldots, z_1^{-1}),$$

with $\ell_\tau + \ell_{\tau'}$ times 1 in the middle.

From (5.23) we see that $R(J^\circ, T)$ is a union of irreducible components $R(J^\circ, T)_\tau$. Comparing these data with the earlier description from (5.21) and the subsequent table, we deduce that $R(J^\circ, T)_{\tau,\mathrm{red}} = R(G_\phi^\circ, T)_{\tau,\mathrm{red}}$ for all τ. Hence $R(J^\circ, T)_{\mathrm{red}} = R(G_\phi^\circ, T)_{\mathrm{red}}$, as required for a good basepoint ϕ. In particular $W_{\mathfrak{s}^\vee}^\circ \cong W(G_\phi^\circ, T)$.

We note that $\mathrm{Z}(\mathcal{L}^\vee)^\circ = T$, see (5.21). Since $\phi_j \colon \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_{m_j}(\mathbb{C})$ is cuspidal, it is irreducible and trivial on $\mathrm{SL}_2(\mathbb{C})$. Thus we can write

$$\mathcal{L}^\vee = \mathcal{G}_{n^-}^\vee(\mathbb{C}) \times \prod\nolimits_j \mathrm{GL}_{m_j}(\mathbb{C}) = \mathcal{G}_{n^-}^\vee(\mathbb{C}) \times \prod\nolimits_{\tau \in I_\phi^- \sqcup I_\phi^+ \sqcup I_\phi^0} \mathrm{GL}_{\dim(\tau)}(\mathbb{C})^{e_\tau}.$$

It follows from (5.22) that $\dim(\tau) = t_\tau \dim(\theta)$ with $\theta \in \mathrm{Irr}(\mathbf{I}_F)$, and that

$$X_{\mathrm{nr}}({}^L\mathcal{L})_\phi \cong \prod\nolimits_{\tau \in I_\phi^- \sqcup I_\phi^+ \sqcup I_\phi^0} \mu_{t_\tau}(\mathbb{C})^{e_\tau}.$$

Here μ_k denotes the functor of taking the k-th roots of unity in ring. In particular t_τ equals $|X_{\mathrm{nr}}({}^L\mathrm{GL}_m)_\tau|$, the number of unramified characters χ such that $\chi\tau \equiv \tau$. In the following table, which stems largely from [Mou, §4.1], we describe the root systems and the Weyl groups. We may omit the cases $e_\tau = 0$, because there all the root systems and Weyl groups are trivial.

$\tau \in$	$M_\tau, M_{\tau'}$	condition	$R(J^\circ, T)_\tau$	$W_{M_\tau^\circ}^{G_\tau^\circ}$	$W_{M_\tau}^{G_\tau}$
I_ϕ^{--}	$(\mathbb{C}^\times)^{e_\tau} \times \mathrm{Sp}_\ell(\mathbb{C})$	$\ell_\tau = 0$	C_{e_τ}	$S_{e_\tau} \rtimes (\mathbb{Z}/2\mathbb{Z})^{e_\tau}$	$S_{e_\tau} \rtimes (\mathbb{Z}/2\mathbb{Z})^{e_\tau}$
		$\ell_\tau \neq 0$	BC_{e_τ}	$S_{e_\tau} \rtimes (\mathbb{Z}/2\mathbb{Z})^{e_\tau}$	$S_{e_\tau} \rtimes (\mathbb{Z}/2\mathbb{Z})^{e_\tau}$
I_ϕ^{+-}	$(\mathbb{C}^\times)^{e_\tau} \times \mathrm{Sp}_\ell(\mathbb{C})$,	$\ell_\tau = 0$	C_{e_τ}	$S_{e_\tau} \rtimes (\mathbb{Z}/2\mathbb{Z})^{e_\tau}$	$S_{e_\tau} \rtimes (\mathbb{Z}/2\mathbb{Z})^{e_\tau}$
	$(\mathbb{C}^\times)^{e_\tau} \times \mathrm{O}_\ell(\mathbb{C})$	$\ell_\tau \neq 0$	BC_{e_τ}	$S_{e_\tau} \rtimes (\mathbb{Z}/2\mathbb{Z})^{e_\tau}$	$S_{e_\tau} \rtimes (\mathbb{Z}/2\mathbb{Z})^{e_\tau}$
I_ϕ^{++}	$(\mathbb{C}^\times)^{e_\tau} \times \mathrm{O}_\ell(\mathbb{C})$	$\ell_\tau = 0$	D_{e_τ}	$S_{e_\tau} \rtimes (\mathbb{Z}/2\mathbb{Z})^{e_\tau - 1}$	$S_{e_\tau} \rtimes (\mathbb{Z}/2\mathbb{Z})^{e_\tau}$
		$\ell_\tau \neq 0$	B_{e_τ}	$S_{e_\tau} \rtimes (\mathbb{Z}/2\mathbb{Z})^{e_\tau}$	$S_{e_\tau} \rtimes (\mathbb{Z}/2\mathbb{Z})^{e_\tau}$
I_ϕ^0	$(\mathbb{C}^\times)^{e_\tau}$	$e_\tau \leqslant 1$	$\varnothing$	$\{1\}$	$\{1\}$
		$e_\tau \geqslant 2$	$A_{e_\tau - 1}$	S_{e_τ}	S_{e_τ}

For all $\tau \in I_\phi^{++}$ such that $\ell_\tau = 0 \neq e_\tau$, take

$$r_\tau = \mathrm{diag}\big(1, \ldots, 1, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), 1, \ldots, 1\big) \in O_{2e_\tau}(\mathbb{C}) \setminus \mathrm{SO}2e_\tau(\mathbb{C}).$$

It normalizes M, T, ϕ and generates $W_{M_\tau}^{G_\tau} / W_{M_\tau^\circ}^{G_\tau^\circ}$. The finite group $\mathfrak{R}_{\mathfrak{s}^\vee}$ is generated by such elements r_τ. More precisely, let

$$\begin{aligned} C &:= \{\tau \in I_\phi^+ \mid \ell_\tau = 0\}, \\ C_{\mathrm{even}} &:= \{\tau \in C \mid \dim \tau \text{ even}\}, \\ C_{\mathrm{odd}} &:= \{\tau \in C \mid \dim \tau \text{ odd}\}. \end{aligned}$$

It was shown in [Mou, §4.1] that:

- if $\mathcal{G} = \mathrm{Sp}_N$ or $\mathcal{G} = \mathrm{SO}_N$ with N odd, then
$$\mathfrak{R}_{\mathfrak{s}^\vee} \cong \prod\nolimits_{\tau \in C} \langle r_\tau \rangle;$$
- if $\mathcal{G} = \mathrm{SO}_N$ and $\mathcal{L} = \mathrm{GL}_{d_1}^{\ell_1} \times \ldots \times \mathrm{GL}_{d_r}^{\ell_r} \times \mathrm{SO}_{N'}$ with N even and $N' \geqslant 4$, then
$$\mathfrak{R}_{\mathfrak{s}^\vee} \cong \prod\nolimits_{\tau \in C} \langle r_\tau \rangle;$$

- if $\mathcal{G} = \mathrm{SO}_N$ and $\mathcal{L} = \mathrm{GL}_{d_1}^{\ell_1} \times \ldots \times \mathrm{GL}_{d_r}^{\ell_r}$ with N even, then
$$\mathfrak{R}_{\mathfrak{s}^\vee} \cong \prod\nolimits_{\tau \in C_{\mathrm{even}}} \langle r_\tau \rangle \times \langle r_\tau r_{\tau'} \mid \tau, \tau' \in C_{\mathrm{odd}} \rangle.$$

From the shape of M_τ° we can describe the unipotent element v_τ:

M_τ°	v_τ	ℓ
$(\mathbb{C}^\times)^e \times \mathrm{Sp}_{\ell_\tau}(\mathbb{C})$	$(1^e) \times (2, 4, \ldots, 2d-2, 2d)$	$\ell_\tau = d(d+1)$
$(\mathbb{C}^\times)^e \times \mathrm{SO}_{\ell_\tau}(\mathbb{C})$	$(1^e) \times (1, 3, \ldots, 2d-3, 2d-1)$	$\ell_\tau = d^2$
$(\mathbb{C}^\times)^e$	(1^e)	

To be complete, let us describe the cuspidal representations of $A_{M_\tau^\circ}(v_\tau)$. We have
$$A_{M_\tau^\circ}(v_\tau) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^d = \langle z_{\tau,2a}, a \in [\![1,d]\!] \rangle & \text{if } \tau \in I_\phi^- \\ (\mathbb{Z}/2\mathbb{Z})^{d-1} = \langle z_{\tau,2a-1} z_{\tau,2a+1}, a \in [\![1,d-1]\!] \rangle & \text{if } \tau \in I_\phi^+ \end{cases}.$$

Moreover, the cuspidal irreducible representation ϵ_τ of $A_{M_\tau^\circ}(v_\tau)$ satisfies
$$\epsilon_\tau(z_{\tau,2a}) = (-1)^a \text{ if } \tau \in I_\phi^- \quad \text{and} \quad \epsilon_\tau(z_{\tau,2a-1} z_{\tau,2a+1}) = -1 \text{ if } \tau \in I_\phi^+.$$

For all $\tau \in I_\phi^+ \sqcup I_\phi^-$, denote by a_τ the biggest part of the partition of v_τ and by a'_τ the biggest part of the partition of $v_{\tau'}$. In case $v_{\tau'} = 1$, we will assume that $a'_\tau = 0$ if $\tau \in I_\phi^-$ and $a'_\tau = -1$ if $\tau \in I_\phi^+$ (this is compatible with Proposition 3.14).

Finally, we consider the parameter functions. The number m_α from Definition 3.11 equals t_τ unless $\tau \in \mathrm{Irr}(\mathbf{W}_F)_\phi^{+-}, \ell_\tau = 0$ and α is a long root in a type C root system, then $m_\alpha = t_\tau/2$. Recall that $R_{\mathfrak{s}^\vee}$ consists of the roots $m_\alpha \alpha$ with $\alpha \in R(J^\circ, T)_{\mathrm{red}}$. Multiplication by m_α does not change the type of $R(J^\circ, T)_\tau$, only in the exceptional case, there C_{e_τ} is turned into B_{e_τ}.

If $\alpha \in R_{\tau,\mathrm{red}}$ is not a short root in a type B root system, then by [Lus2, 2.13] $c(\alpha) = 2$, so $\lambda(\alpha) = m_\alpha$. For the simple short root $\alpha_\tau \in R_{\tau,\mathrm{red}}$ we have $c(\alpha_\tau) = a_\tau + 1, c^*(\alpha_\tau) = a'_\tau + 1$ and $m_\alpha = t_\tau$. Hence
$$\lambda(\alpha_\tau) = (a_\tau + a'_\tau + 2) t_\tau / 2 \quad \text{and} \quad \lambda^*(\alpha_\tau) = |a_\tau - a'_\tau| t_\tau / 2.$$

We conclude that
$$\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}}) = \mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}}) \rtimes \mathbb{C}[\mathfrak{R}_{\mathfrak{s}^\vee}]. \tag{5.24}$$

Via the specialization of $\mathbf{z}_\tau$ at $q_F^{1/2}$, (5.24) becomes the extended affine Hecke algebra given in [Hei2]. Moreover, it was shown in [Hei2] that there is an equivalence of categories between $\mathrm{Rep}(\mathcal{G}(F))^{\mathfrak{s}}$ and the right modules over $\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}}) / (\{\mathbf{z}_\tau - q_F^{1/2}\}_\tau)$. Together with the LLC for $\mathcal{G}(F)$ we get bijections
$$\mathrm{Irr}\Big(\mathcal{H}(\mathfrak{s}^\vee, \vec{\mathbf{z}}) / \big(\{\mathbf{z}_\tau - q_F^{1/2}\}_\tau\big)\Big) \longleftrightarrow \mathrm{Irr}(\mathcal{G}(F))^{\mathfrak{s}} \longleftrightarrow \Phi_{\mathrm{e}}(\mathcal{G}(F))^{\mathfrak{s}^\vee}. \tag{5.25}$$

It does not seem unlikely that this works out to the same bijection as in Theorem 3.18.a. But at present that is hard to check, because the LLC is not really explicit.

Example 5.3. We consider an example that illustrates many of the above aspects. Let $\tau \colon \mathbf{W}_F \to \mathrm{GL}_4(\mathbb{C})$ be an irreducible representation of $\mathbf{W}_F$, self-dual of symplectic type and let $\varphi \colon \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{37}(\mathbb{C})$ be defined by
$$\mathrm{Std}_{GL_{37}} \circ \varphi = 1 \boxtimes (S_5 \oplus S_3 \oplus S_1) \oplus \xi \boxtimes (S_3 \oplus S_1) \oplus \tau \boxtimes (S_4 \oplus S_2),$$

with $\xi\colon \mathbf{W}_F \to \mathbb{C}^\times$ an unramified quadratic character. We have

$$\mathrm{Z}_{\mathrm{SO}_{37}(\mathbb{C})}(\varphi|_{\mathbf{W}_F})^\circ \cong \mathrm{SO}_9(\mathbb{C}) \times \mathrm{SO}_4(\mathbb{C}) \times \mathrm{Sp}_6(\mathbb{C}),$$

and φ defines a L-packet $\Pi_\varphi(\mathrm{Sp}_{36}(F))$ with 2^5 elements, of which two are supercuspidal. Let $\sigma \in \Pi_\varphi(\mathrm{Sp}_{36}(F))$ be supercuspidal, corresponding to an enhanced Langlands parameter (φ, ε) with ε cuspidal. Consider $\mathcal{G}(F) = \mathrm{Sp}_{58}(F)$, the Levi subgroup

$$\mathcal{L}(F) = \mathrm{GL}_4(F)^2 \times \mathrm{GL}_1(F)^3 \times \mathrm{Sp}_{36}(F)$$

and an irreducible supercuspidal representation $\pi_\tau^{\otimes 2} \boxtimes 1^{\otimes 3} \boxtimes \sigma$ of $\mathcal{L}(F)$. The cuspidal pair $\mathfrak{s} = [\mathcal{L}(F), \pi_\tau^{\otimes 2} \boxtimes 1^{\otimes 3} \boxtimes \sigma]$ of $\mathcal{G}(F)$ admits $\mathfrak{s}^\vee = [\mathcal{L}^\vee, \phi, \varepsilon]$ as dual inertial equivalence class, where $\phi\colon \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathcal{L}^\vee$,

$$\mathcal{L}^\vee = \mathrm{GL}_4(\mathbb{C})^2 \times \mathrm{GL}_1(\mathbb{C})^3 \times \mathrm{SO}_{37}(\mathbb{C}) \quad \text{and} \quad \mathrm{Std}_{\mathcal{L}^\vee} \circ \phi = (\tau \oplus \tau^\vee)^{\oplus 2} \oplus (1 \oplus 1^\vee)^{\oplus 3} \oplus \varphi.$$

We assume that $\tau|_{\mathbf{I}_F} = \theta \oplus \theta^{\mathrm{Frob}_F}$ with $\theta^\vee \cong \theta$, so $t_\tau = 2$. We first compute $W_{\mathfrak{s}^\vee}^\circ$:

$$\begin{aligned} \phi|_{\mathbf{I}_F} &= \tau|_{\mathbf{I}_F}^{\oplus 4} \oplus 1|_{\mathbf{I}_F}^{\oplus 6} \oplus 1|_{\mathbf{I}_F}^{\oplus 9} \oplus \xi|_{\mathbf{I}_F}^{\oplus 4} \oplus \tau|_{\mathbf{I}_F}^{\oplus 6} = \theta^{\oplus 10} \oplus \theta^{\mathrm{Frob}_F \oplus 10} \oplus 1^{\oplus 19}, \\ J^\circ &= \mathrm{Z}_{\mathcal{G}^\vee}(\phi|_{\mathbf{I}_F})^\circ \cong \mathrm{Sp}_{10}(\mathbb{C})^2 \times \mathrm{SO}_{19}(\mathbb{C}). \end{aligned}$$

The torus T is decomposed as $T = (\mathbb{C}^\times)^2 \times (\mathbb{C}^\times)^3$. The first part $(\mathbb{C}^\times)^2$ is embedded in an obvious way in $(\mathbb{C}^\times)^2 \times \mathrm{Sp}_6(\mathbb{C})$ and then in $\mathrm{Sp}_{10}(\mathbb{C})^2$ diagonally as Levi subgroup. The second part $(\mathbb{C}^\times)^3$ is embedded in $(\mathbb{C}^\times)^3 \times \mathrm{SO}_{13}(\mathbb{C})$ and then in $\mathrm{SO}_{19}(\mathbb{C})$ as Levi subgroup as well. The root system $R(J^\circ, T)$ (resp. $R(J^\circ, T)_{\mathrm{red}}$) is $BC_2 \times B_3$ (resp. $B_2 \times B_3$), so $W_{\mathfrak{s}^\vee}^\circ = W_{B_2} \times W_{B_3}$.

From the above discussion, we can see that ϕ is already a basepoint. If we denote by ϕ' the parameter defined by $\phi' = (\tau' \oplus \tau'^\vee)^{\oplus 2} \oplus (\xi \oplus \xi^\vee)^{\oplus 3} \oplus \varphi$, then ϕ' is another basepoint. Indeed, we have:

$$\begin{aligned} \phi|_{\mathbf{W}_F} &= \tau^{\oplus 10} \oplus 1^{\oplus 15} \oplus \xi^{\oplus 4} \\ G_\phi^\circ = \mathrm{Z}_{\mathcal{G}^\vee}(\phi|_{\mathbf{W}_F})^\circ &\cong \mathrm{Sp}_{10}(\mathbb{C}) \times \mathrm{SO}_{15}(\mathbb{C}) \times \mathrm{SO}_4(\mathbb{C}) \\ M_\phi^\circ = \mathrm{Z}_{\mathcal{L}^\vee}(\phi|_{\mathbf{W}_F})^\circ &\cong \big((\mathbb{C}^\times)^2 \times \mathrm{Sp}_6(\mathbb{C})\big) \times \big((\mathbb{C}^\times)^3 \times \mathrm{SO}_9(\mathbb{C})\big) \times \mathrm{SO}_4(\mathbb{C}) \\ \phi'|_{\mathbf{W}_F} &= \tau'^{\oplus 4} \oplus \tau^{\oplus 6} \oplus 1^{\oplus 9} \oplus \xi^{\oplus 10} \\ G_{\phi'}^\circ = \mathrm{Z}_{\mathcal{G}^\vee}(\phi'|_{\mathbf{W}_F})^\circ &\cong \mathrm{Sp}_4(\mathbb{C}) \times \mathrm{Sp}_6(\mathbb{C}) \times \mathrm{SO}_9(\mathbb{C}) \times \mathrm{SO}_{10}(\mathbb{C}) \\ M_{\phi'}^\circ = \mathrm{Z}_{\mathcal{L}^\vee}(\phi'|_{\mathbf{W}_F})^\circ &\cong (\mathbb{C}^\times)^2 \times \mathrm{Sp}_6(\mathbb{C}) \times \mathrm{SO}_9(\mathbb{C}) \times \big((\mathbb{C}^\times)^3 \times \mathrm{SO}_4(\mathbb{C})\big). \end{aligned}$$

Here $\mathfrak{R}_{\mathfrak{s}^\vee}$ is trivial, so $W_{\mathfrak{s}^\vee} = W_{\mathfrak{s}^\vee}^\circ$. Denote by α_1, α_2 (resp. $\beta_1, \beta_2, \beta_3$) the simple roots of B_2 (resp. B_3) with α_2 (resp. β_3) the short root. Then $a_1 = a'_\xi = 5$, $a_\xi = a'_1 = 3$, $a_\tau = 4$ and $a'_\tau = 0$. The parameters are given by $\lambda(\alpha_1) = t_\tau = 2$, $\lambda(\beta_1) = \lambda(\beta_2) = 1$ and

$$\lambda(\alpha_2) = t_\tau \frac{4+2}{2} = 6,\ \lambda(\beta_3) = \frac{5+3+2}{2} = 5,\ \lambda^*(\alpha_2) = t_\tau \frac{4}{2} = 4,\ \lambda^*(\beta_3) = \frac{5-3}{2} = 1.$$

Specializing $\vec{\mathbf{z}}$ to $q_F^{1/2}$, the quadratic relations in the Hecke algebra become

$$\begin{aligned} &(N_{s_{\alpha_1}} - q_F^2)(N_{s_{\alpha_1}} + q_F^{-2}) = 0,\ (N_{s_{\alpha_2}} - q_F^3)(N_{s_{\alpha_2}} + q_F^{-3}) = 0, \\ &(N_{s_{\beta_3}} - q_F^{5/2})(N_{s_{\beta_3}} + q_F^{-5/2}) = 0,\ (N_{s_{\beta_i}} - q_F^{1/2})(N_{s_{\beta_i}} + q_F^{-1/2}) = 0 \quad (i = 1, 2). \end{aligned}$$

Appendix A.

In this appendix we prove a number theoretic result which probably has been known for a long time, but for which we could not find a reference. Let $\mathbf{W}_F$ be the Weil group of the non-archimedean local field F, $\mathbf{I}_F$ the inertia subgroup and $\mathbf{P}_F$ the wild inertia subgroup of $\mathbf{W}_F$. Let $\mathrm{Frob}_F \in \mathbf{W}_F$ a geometric Frobenius element and let q_F be the cardinality of the residue field of F.

Lemma A.1. *$Z(\mathbf{W}_F) = Z(\mathbf{I}_F) = Z(\mathbf{P}_F) = \{\mathrm{id}\}$.*

Proof. According to [Jan, §3], $\mathbf{P}_F$ is a free pro-p group on more than one generator. In particular its centre is trivial.

It follows from [Ser, Corollary 1 to Proposition IV.2.9] that an arbitrary element x of $\mathbf{I}_F \setminus \mathbf{P}_F$ does not commute with some elements of $\mathbf{P}_F$. Namely, we apply [ibid] to the Galois group of some finite Galois extension E/F, which we choose so large that x ends up in the ramification group $\mathrm{Gal}(E/F)_0$ but not in $\mathrm{Gal}(E/F)_1$. Then [ibid] says that x does not commute with most elements of $\mathrm{Gal}(E/F)_1$, and we can lift that noncommutativity back to $\mathbf{I}_F = \mathrm{Gal}(F_s/F)_0$. Hence $Z(\mathbf{I}_F) = \{\mathrm{id}\}$.

The group $\mathbf{I}_F/\mathbf{P}_F$ is isomorphic to $\hat{\mathbb{Z}}/\mathbb{Z}_p$ and the conjugation action of Frob_F^{-1} on it equals raising elements to the power q_F [Iwa]. Hence $\mathrm{Frob}_F^n x$ with $x \in \mathbf{I}_F, n \in \mathbb{Z} \setminus \{0\}$ does not commute (in $\mathbf{W}_F/\mathbf{P}_F$) with most elements of $\mathbf{I}_F/\mathbf{P}_F$. As $\mathbf{W}_F = \cup_{n\in\mathbb{Z}} \mathrm{Frob}_F^n \mathbf{I}_F$, we find that $Z(\mathbf{W}_F) = \{\mathrm{id}\}$. $\square$

Lemma A.1 is used in the definition of $X_{\mathrm{nr}}({}^L\mathcal{G})$ in (3.2), and was already used in the same way in [AMS1].

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