AFFINE HECKE ALGEBRAS
FOR LANGLANDS PARAMETERS

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Abstract. It is well-known that affine Hecke algebras are very useful to describe
the smooth representations of any connected reductive $p$-adic group $G$, in terms
of the supercuspidal representations of its Levi subgroups. The goal of this paper
is to create a similar role for affine Hecke algebras on the Galois side of the local
Langlands correspondence.

To every Bernstein component of enhanced Langlands parameters for $G$ we
canonically associate an affine Hecke algebra (possibly extended with a finite $R$-
group). We prove that the irreducible representations of this algebra are naturally
in bijection with the members of the Bernstein component, and that the set of
central characters of the algebra is naturally in bijection with the collection of
cuspidal supports of these enhanced Langlands parameters. These bijections send
tempered or (essentially) square-integrable representations to the expected kind
of Langlands parameters.

Furthermore we check that for many reductive $p$-adic groups, if a Bernstein
component $\mathcal{B}$ for $G$ corresponds to a Bernstein component $\mathcal{B}^\vee$ of enhanced Lang-
lands parameters via the local Langlands correspondence, then the affine Hecke
algebra that we associate to $\mathcal{B}^\vee$ is Morita equivalent with the Hecke algebra as-
associated to $\mathcal{B}$. This constitutes a generalization of Lusztig’s work on unipotent
representations. It might be useful to establish a local Langlands correspondence
for more classes of irreducible smooth representations.

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**Introduction**

Let $F$ be a non-archimedean local field and let $G$ be a connected reductive algebraic group defined over $F$. The conjectural local Langlands correspondence (LLC) provides a bijection between the set of irreducible smooth $G(F)$-representations $\text{Irr}(G(F))$ and the set of enhanced L-parameters $\Phi_e(G(F))$, see [Bor, Vog, ABPS5].

Let $s$ be an inertial equivalence class for $G(F)$ and let $\text{Irr}(G(F))^s$ be the associated Bernstein component. Similarly, inertial equivalence classes $s^\vee$ and Bernstein components $\Phi_e(G(F))^{s^\vee}$ for enhanced L-parameters were developed in [AMS1]. It can be expected that every $s$ corresponds to a unique $s^\vee$ (an "inertial Langlands correspondence"), such that the LLC restricts to a bijection

$$\text{Irr}(G(F))^s \longleftrightarrow \Phi_e(G(F))^{s^\vee}.$$  

The left hand side can be identified with the space of irreducible representations of a direct summand $\mathcal{H}(G(F))^s$ of the full Hecke algebra of $G(F)$. It is known that in many cases $\mathcal{H}(G(F))^s$ is Morita equivalent to an affine Hecke algebra, see [ABPS5 §2.4] and the references therein for an overview.

To improve our understanding of the LLC, we would like to canonically associate $s^\vee$ an affine Hecke algebra $\mathcal{H}(s^\vee)$ whose irreducible representations are naturally parametrized by $\Phi_e(G(F))^{s^\vee}$. Then (1) could be written as

$$\text{Irr}(G(F))^s \cong \text{Irr}(\mathcal{H}(G(F))^s) \longleftrightarrow \text{Irr}(\mathcal{H}(s^\vee)) \cong \Phi_e(G(F))^{s^\vee},$$

and the LLC for this Bernstein component would become a comparison between two algebras of the same kind. If moreover $\mathcal{H}(s^\vee)$ were Morita equivalent to $\mathcal{H}(G(F))^s$, then (1) could even be categorified to

$$\text{Rep}(G(F))^s \cong \text{Mod}(\mathcal{H}(s^\vee)).$$

Such algebras $\mathcal{H}(s^\vee)$ would also be useful to establish the LLC in new cases. Suppose one would like to match $s^\vee$ (essentially a set of cuspidal enhanced Langlands parameters for a Levi subgroup $\mathcal{L}(F)$) with a yet unknown supercuspidal Bernstein block for $\mathcal{L}(F)$. Motivated by some examples, we increase the scope of (3) by considering it only for the full subcategories of finite length objects:

$$\text{Rep}_\mathbb{H}(G(F))^s \cong \text{Mod}_\mathbb{H}(\mathcal{H}(s^\vee)).$$

One could compare $\mathcal{H}(s^\vee)$ with the algebras $\mathcal{H}(G(F))^s$ for various $s = [\mathcal{L}(F), \sigma]$, and only the Bernstein components $\text{Irr}(G(F))^s$ for which (1) holds would be good candidates for the image of $\Phi_e(G(F))^{s^\vee}$ under the LLC. If one would know a lot about $\mathcal{H}(s^\vee)$, this could substantially reduce the number of possibilities for a LLC for both $\Phi_e(\mathcal{L}(F))^{s^\vee}$ and $\Phi_e(G(F))^{s^\vee}$.

This strategy was already employed by Lusztig, for unipotent representations [Lus5, Lus7]. Bernstein components of enhanced L-parameters had not yet been defined when the papers [Lus5, Lus7] were written, but the constructions in them can be interpreted in that way. Lusztig found a bijection between:

- the set of ("arithmetic") affine Hecke algebras associated to unipotent Bernstein blocks of adjoint, unramified groups;
- the set of ("geometric") affine Hecke algebras associated to unramified enhanced L-parameters for such groups.

However, the comparison of these two families of Hecke algebras is not enough to specify a canonical bijection between Bernstein components on the $p$-adic and the
Galois sides. The problem is that one affine Hecke algebra can appear (up to isomorphism) several times on either side. This already happens in the unipotent case for exceptional groups, and the issue seems to be outside the scope of these techniques. In [Lus5, 6.6–6.8] Lusztig wrote down some remarks about this problem, but he does not work it out completely.

The main goal of this paper is the construction of an affine Hecke algebra for any Bernstein component of enhanced L-parameters, for any $G$. But it quickly turns out that this is not exactly the right kind of algebra. Firstly, our geometric construction, which relies on [Lus2, AMS2], naturally includes some complex parameters $z_i$, which we abbreviate to $\vec{z}$. Secondly, an affine Hecke algebra with (indeterminate) parameters is still too simple. In general one must consider the crossed product of such an object with a twisted group algebra (of some finite “R-group”). We call this a twisted affine Hecke algebra, see Proposition 2.2 for a precise definition. Like for reductive groups, there are good notions of tempered representations and of (essentially) discrete series representations of such algebras (Definition 2.6).

**Theorem 1.** [see Theorem 3.18]

(a) To every Bernstein component of enhanced L-parameters $s^\vee$ one can canonically associate a twisted affine Hecke algebra $\mathcal{H}(s^\vee, \vec{z})$.

(b) For every choice of parameters $z_i \in \mathbb{R}_{>0}$ there exists a natural bijection

$$\Phi_e : (G(F))^s \longrightarrow \text{Irr}(\mathcal{H}(s^\vee, \vec{z})/(\{z_i - q_{F}^{1/2}\}_i))$$

(c) For every choice of parameters $z_i \in \mathbb{R}_{\geq 1}$ the bijection from part (b) matches enhanced bounded L-parameters with tempered irreducible representations.

(d) Suppose that $\Phi_e : (G(F))^s$ contains enhanced discrete L-parameters, and that $z_i \in \mathbb{R}_{>1}$ for all $i$. Then the bijection from part (b) matches enhanced discrete L-parameters with irreducible essentially discrete series representations.

(e) The bijection in part (b) is equivariant with respect to the canonical actions of the group of unramified characters of $G(F)$.

This can be regarded as a far-reaching generalization of parts of [Lus5, Lus7]: we allow any reductive group over a non-archimedean local field, and all enhanced L-parameters for that group. We check (see Section 5) that in several cases where the LLC is known, indeed

$$\mathcal{H}(G(F))^s \text{ is Morita equivalent to } \mathcal{H}(s^\vee, \vec{z})/(\{z_i - q_{F}^{1/2}\}_i)$$

for suitable $z_i \in \mathbb{R}_{>1}$, obtaining (3). Notice that on the $p$-adic side the parameters $z_i$ are determined by $\mathcal{H}(G(F))^s$, whereas on the Galois side we specify them manually. In fact, in all our examples we can take $z_i = q_F^{1/2}$. That is a good sign, which indicates that in general $z_i = q_F^{1/2}$ could be the best specialization of the parameters to compare with an affine Hecke algebra coming from a $p$-adic group.

Yet in general the categorification (3) is asking for too much. We discovered that for inner twists of $SL_n(F)$ (5) does not always hold. Rather, these algebras are equivalent in a weaker sense: the category of finite length modules of $\mathcal{H}(G(F))^s$ (i.e. the finite length objects in $\text{Rep}(G(F))^s$) is equivalent to the category of finite dimensional representations of $\mathcal{H}(s^\vee, \vec{z})/(\{z_i - q_{F}^{1/2}\}_i)$.
Let us describe the contents of the paper more concretely. Our starting point is a triple \((G, M, q\mathcal{E})\) where

- \(G\) is a possibly disconnected complex reductive group,
- \(M\) is a quasi-Levi subgroup of \(G\) (the \(G\)-centralizer of the connected centre of a Levi subgroup of \(G^0\)),
- \(q\mathcal{E}\) is a \(M\)-equivariant cuspidal local system on a unipotent orbit \(C_u^M\) in \(M\).

To these data we attach a twisted affine Hecke algebra \(\mathcal{H}(G, M, q\mathcal{E}, \bar{z})\). This algebra can be specialized by setting \(\bar{z}\) equal to some \(\bar{z} \in (\mathbb{C}^\times)^d\). Of particular interest is the specialization at \(\bar{z} = \bar{1}\):

\[
\mathcal{H}(G, M, q\mathcal{E}, \bar{z})/(\{z_i - 1\} ) = \mathcal{O}(T) \times \mathbb{C}[W_{q\mathcal{E}, \bar{z}}],
\]

where \(T = Z(M)^0\), while the subgroup \(W_{q\mathcal{E}} \subset N_G(M)/M\) and the 2-cocycle \(\bar{z} : W_{q\mathcal{E}} \to \mathbb{C}^\times\) also come from the data.

The goal of Section 2 is to understand and parametrize representations of \(\mathcal{H}(G, M, q\mathcal{E}, \bar{z})\). We follow a strategy similar to that in \([Lus3]\). The centre naturally contains \(\mathcal{O}(T)^{W_{q\mathcal{E}}} = \mathcal{O}(T/W_{q\mathcal{E}})\), so we can study \(\text{Mod}(\mathcal{H}(G, M, q\mathcal{E}, \bar{z}))\) via localization at suitable subsets of \(T/W_{q\mathcal{E}}\). In Paragraph 2.1 we reduce to representations with \(\mathcal{O}(T)^{W_{q\mathcal{E}}}\)-character in \(W_{q\mathcal{E}}T_{rs}\), where \(T_{rs}\) denotes the maximal real split torus of \(T\). This involves replacing \(\mathcal{H}(G, M, q\mathcal{E}, \bar{z})\) by an algebra of the same kind, but for a smaller \(G\).

In Paragraph 2.2 we reduce further, to representations of a (twisted) graded Hecke algebra \(\mathbb{H}(G, M, q\mathcal{E}, \bar{r})\). We defined and studied such algebras in our previous paper \([AMS2]\). But there we only considered the case with a single parameter \(r\), here we need \(\bar{r} = (r_1, \ldots, r_d)\). The generalization of the results of \([AMS2]\) to a multi-parameter setting is carried out in Section 4. With that at hand we can use the construction of “standard” \(\mathbb{H}(G, M, q\mathcal{E}, \bar{r})\)-modules and the classification of irreducible \(\mathbb{H}(G, M, q\mathcal{E}, \bar{r})\)-modules from \([AMS2]\) to achieve the same for \(\mathcal{H}(G, M, q\mathcal{E}, \bar{z})\).

For the parametrization we use triples \((s, u, \rho)\) where:

- \(s \in \mathcal{G}^0\) is semisimple,
- \(u \in Z_G(s)^0\) is unipotent,
- \(\rho \in \text{Irr}(\pi_0(Z_G(s, u)))\) such that the quasi-cuspidal support of \((u, \rho)\), as defined in \([AMS1\, \text{§5}]\), is \(G\)-conjugate to \((M, C_u^M, q\mathcal{E})\).

**Theorem 2.** [see Theorem 2.13] 

\(a)\) Let \(\bar{z} \in \mathbb{R}_{\geq 0}^d\). There exists a canonical bijection, say \((s, u, \rho) \mapsto \bar{M}_{s,u,\rho,\bar{z}},\) between:

- \(G\)-conjugacy classes of triples \((s, u, \rho)\) as above,
- \(\text{Irr}(\mathcal{H}(G, M, q\mathcal{E}, \bar{z})/(\{z_i - 1\} ))\).

\(b)\) Let \(\bar{z} \in \mathbb{R}_{> 1}^d\). The module \(\bar{M}_{s,u,\rho,\bar{z}}\) is tempered if and only if \(s\) is contained in a compact subgroup of \(\mathcal{G}^0\).

\(c)\) Let \(\bar{z} \in \mathbb{R}_{> 1}^d\). The module \(\bar{M}_{s,u,\rho,\bar{z}}\) is essentially discrete series if and only if \(u\) is distinguished unipotent in \(\mathcal{G}^0\) (i.e. does not lie in a proper Levi subgroup).

In the case \(M = T, C_u^M = \{1\}\) and \(q\mathcal{E}\) trivial, the irreducible representations in \(\mathcal{H}(G^0, T, q\mathcal{E} = \text{triv})\) were already classified in the landmark paper \([KLu]\), in terms of similar triples. In Paragraph 2.3 we check that the parametrization from Theorem 2 agrees with the Kazhdan–Lusztig parametrization for these algebras.

Remarkably, our analysis also reveals that \([KLu]\) does not agree with the classification of irreducible representations in \([Lus5]\). To be precise, the difference consists
of a twist with a version of the Iwahori–Matsumoto involution. Since \[\text{[KaLu]}\] is widely regarded (see for example \[\text{[Ree, Vog]}\]) as the correct local Langlands correspondence for Iwahori-spherical representations, this entails that the parametrizations obtained by Lusztig in \[\text{[Lus5, Lus7]}\] can be improved by composition with a suitable involution. In the special case \(G = \text{Sp}_{2n}(\mathbb{C})\), that already transpired from work of Mœglin and Waldspurger \[\text{[Wal]}\].

Having obtained a good understanding of affine Hecke algebras attached to disconnected reductive groups, we turn to Langlands parameters. Let \(\phi : W_F \times \text{SL}_2(\mathbb{C}) \to L^G\) be a L-parameter and let \(\rho\) be an enhancement of \(\phi\). (See Section 3 for the precise notions.) Let \(G^\vee\) be the adjoint group of the complex dual group \(G\) and let \(G^\vee_{\text{sc}}\) be the simply connected cover of \(G^\vee\). Let \(Z_{G^\vee}(\phi(I_F))\) be the centralizer of \(\phi(I_F)\) in \(G^\vee\), and let \(J_\phi = Z_{G^\vee_{\text{sc}}}(\phi(I_F))\) denote its inverse image in \(G^\vee_{\text{sc}}\). Similarly, we consider the group \(G^\phi\) defined to be inverse image in \(G^\vee_{\text{sc}}\) of the centralizer of \(\phi(W_F)\) in \(G^\vee\).

We emphasize that the complex groups \(J_\phi\) and \(G^\phi\) can be disconnected – this is the main reason why we have to investigate Hecke algebras for disconnected reductive groups.

Recall that \(\phi\) is determined up to \(G^\vee\)-conjugacy by \(\phi|_{W_F}\) and the unipotent element \(u_\phi = \phi(1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix})\). As the image of a Frobenius element is allowed to vary within one Bernstein component, \((\phi|_{I_F}, u_\phi)\) contains almost all information about such a Bernstein component.

The cuspidal support of \((u_\phi, \rho)\) for \(G = G^\phi\) is a triple \((M, C^M, qE)\) as before. Thus we can associate to \((\phi, \rho)\) the twisted affine Hecke algebra \(H(G, M, qE, \mathbf{r})\). This works quite well in several cases, but in general it is too simple, we encounter various technical difficulties. The main problem is that the torus \(T = Z(M)^\circ\) will not always match up with the torus from which the Bernstein component of \(\Phi_e(G(F))\) containing \((\phi, \rho)\) is built.

Instead we consider the twisted graded Hecke algebra \(\mathbb{H}(G, M, qE, \mathbf{r})\), and we tensor it with the coordinate ring of a suitable vector space to compensate for the difference between \(G^\vee_{\text{sc}}\) and \(G^\vee\). In Paragraph 3.1 we prove that the irreducible representations of the ensuing algebra are naturally parametrized by a subset of the Bernstein component \(\Phi_e(G(F))^\circ\) containing \((\phi, \rho)\). In Paragraph 3.3 we glue families of such algebras together, to obtain the twisted affine Hecke algebras \(H(s^\vee, \mathbf{z})\) featuring in Theorem 1. This requires careful analysis of the involved tori and root systems, which we perform in Paragraph 3.2.

We discuss then, in Section 4, the relation of the above theory with the stable Bernstein center on the Galois side of the LLC. In Section 5 we explain and work out the examples of general linear, special linear and classical groups. It turns out that, for general linear groups (and their inner twists) and classical groups, the extended affine Hecke algebras for enhanced Langlands parameters (with a suitable specialization of the parameters) are Morita equivalent to those obtained from representations of reductive \(p\)-adic groups. In the case of inner twists of special linear groups we establish a slightly weaker result.

Let us compare our paper with similar work by other authors. Several mathematicians have noted that, when two Bernstein components give rise to isomorphic
affine Hecke algebras, this often has to do with the centralizers of the corresponding Langlands parameters. It is known from the work of Bushnell–Kutzko (see in particular [BuKu2]) that every affine Hecke algebra associated to a semisimple type for $\text{GL}_n(F)$ is isomorphic to the Iwahori–spherical Hecke algebra of some $\prod_i \text{GL}_{n_i}(F)$, where $\sum_i n_i \leq n$ and $F_i$ is a finite extension of the field $F$. A similar statement holds for Bernstein components in the principal series of $F$-split reductive groups [Roec Lemma 9.3].

Dat [Dat Corollary 1.1.4] has generalized this to groups of “GL-type”, and in [Dat Theorem 1.1.2] he proves that for such a group $Z_{G^\circ}(\phi(I_F))$ determines $\prod_s \text{Rep}(G(F))^s$, where $s$ runs over all Bernstein components that correspond to extensions of $\phi|_{I_p}$ to $W_F \times \text{SL}_2(\mathbb{C})$. In [Dat §1.3] Dat discusses possible generalizations of these results to other reductive groups, but he did not fully handle the cases where $Z_{G^\circ}(\phi(I_F))$ is disconnected. (It is always connected for groups of GL-type.) Theorem in combination with the considerations about inner twists of $\text{GL}_n(F)$ in Paragraph 5.1 provide explanations for all the equivalences between Hecke algebras and between categories found by Dat.

Heiermann [Hei2 §1] has associated affine Hecke algebras (possibly extended with a finite $R$-group) to certain collections of enhanced L-parameters for classical groups (essentially these sets constitute unions of Bernstein components). Unlike Lusztig he does not base this on geometric constructions in complex groups, rather on affine Hecke algebras previously found on the $p$-adic side in [Hei1]. In his setup [2] holds true by construction, but the Hecke algebras are only related to L-parameters via the LLC, so not in an explicit way.

In [Hei2 §2] it is shown that every Bernstein component of enhanced L-parameters for a classical group is in bijection with a Bernstein component of enhanced unramified L-parameters for a product of classical groups of smaller rank. (Some cases require extending the relevant notions to full orthogonal groups, which is straightforward.) So in the context of [Hei2] the data that we use for affine Hecke algebras are present, and the algebras appear as well (at least up to Morita equivalence), but the link between them is not yet explicit. In Paragraph 5.3 we discuss how our results clarify this.

### 1. Twisted graded Hecke algebras

We will recall some aspects of the (twisted) graded Hecke algebras studied in [AMS2]. Let $G$ be a complex reductive group, possibly disconnected. Let $M$ be a quasi-Levi subgroup of $G$, that is, a group of the form $M = Z_G(Z(L)^\circ)$ where $L$ is a Levi subgroup of $G^\circ$. Notice that $M^\circ = L$ in this case.

We write $T = Z(M)^\circ = Z(M^\circ)^\circ$, a torus in $G^\circ$. Let $P^\circ = M^\circ U$ be a parabolic subgroup of $G^\circ$ with Levi factor $M^\circ$ and unipotent radical $U$. We put $P = MU$. Let $t^\ast$ be the dual space of the Lie algebra $t = \text{Lie}(T)$.

Let $v \in m = \text{Lie}(M)$ be nilpotent, and denote its adjoint orbit by $c_v^M$. Let $qE$ be an irreducible $M$-equivariant cuspidal local system on $c_v^M$. Then the stalk $q\epsilon = qE|_v$ is an irreducible representation of $A_M(v) = \pi_0(Z_M(v))$. Conversely, $v$ and $q\epsilon$ determine $c_v^M$ and $qE$. By definition the cuspidality means that $\text{Res}_{A_M^\circ(v)}^{A_M(v)} q\epsilon$ is a direct sum of irreducible cuspidal $A_M^\circ(v)$-representations. Let $\epsilon \in \text{Irr}(A_M^\circ(v))$ be one of them, and let $E$ be the corresponding $M^\circ$-equivariant cuspidal local system on $c_v^{M^\circ}$. Then $E$ is a subsheaf of $qE$. See [AMS1 §5] for more background.
The triple \((M, C_v^M, qE)\) (or \((M, v, qE)\)) is called a cuspidal quasi-support for \(G\). We denote its \(G\)-conjugacy class by \([M, C_v^M, qE]_G\). To these data we associate the groups

\[
\begin{align*}
W_{gE} &= N_G(qE)/M, \\
W_{qE}^0 &= N_{G^0}(M)/M \cong N_{G^0}(M)/M^0 = W_E, \\
R_{qE} &= N_G(P, qE)/M, \\
N_G(qE) &= \text{Stab}_{N_G(M)}(qE),
\end{align*}
\]

(6)

The group \(W_{qE}\) acts naturally on the set

\[
R(G^0, T) := \{ \alpha \in X^*(T) \setminus \{ 0 \} : \alpha \text{ appears in the adjoint action of } T \text{ on } g \}.
\]

By [Lus1, Theorem 9.2] (see also [AMS2, Lemma 2.1]) \(R(G^0, T)\) is a root system with Weyl group \(W_{qE}^0\). The group \(\mathfrak{R}_{qE}\) is the stabilizer of the set of positive roots determined by \(P\) and

\[
W_{qE} = W_{qE}^0 \times \mathfrak{R}_{qE}.
\]

We choose semisimple subgroups \(G_j \subset G^0\), normalized by \(N_G(qE)\), such that the derived group \(G_{der}^0\) is the almost direct product of the \(G_j\). In other words, every \(G_j\) is semisimple, normal in \(G^0M\), normalized by \(W_{qE}\) (which makes sense because it is already normalized by \(M\)), and the multiplication map

\[
m_{G^0} : Z(G^0)^d \times G_1 \times \cdots \times G_d \to G^0
\]

is a surjective group homomorphism with finite central kernel. The number \(d\) is not specified in advance, it indicates the number of independent variables in our upcoming Hecke algebras. Of course there are in general many ways to achieve (7).

Two choices are always canonical:

- \(G_1 = G_{der}^0\), with \(d = 1\);
- every \(G_j\) is of the form \(N_1N_2\cdots N_k\), where \(\{N_1, \ldots, N_k\}\) is a \(N_G(qE)\)-orbit of simple normal subgroups of \(G^0\).

In any case, (7) gives a decomposition

\[
\begin{align*}
g &= Z(g) \oplus g_1 \oplus \cdots \oplus g_d \\
&\text{where } Z(g) = \text{Lie}(Z(G^0)), g_j = \text{Lie}(G_j).
\end{align*}
\]

Each root system

\[
R_j := R(G_jT, T) = R(G_j, G_j \cap T)
\]

is a \(W_{qE}\)-stable union of irreducible components of \(R(G^0, T)\). Thus we obtain an orthogonal, \(W_{qE}\)-stable decomposition

\[
R(G^0, T) = R_1 \sqcup \cdots \sqcup R_d.
\]

(10)

We let \(\vec{r} = (r_1, \ldots, r_d)\) be an array of variables, corresponding to (7) and (10) in the sense that \(r_j\) is relevant for \(G_j\) and \(R_j\) only. We abbreviate

\[
\mathbb{C}[\vec{r}] = \mathbb{C}[r_1, \ldots, r_d].
\]

Let \(\sharp : (W_{qE}/W_{qE}^0)^2 \to \mathbb{C}^\times\) be a 2-cocycle. Recall that the twisted group algebra \(\mathbb{C}[W_{qE}, \sharp]\) has a \(\mathbb{C}\)-basis \(\{N_w : w \in W_{qE}\}\) and multiplication rules

\[
N_w \cdot N_{w'} = \sharp(w, w')N_{ww'}.
\]

In particular it contains the group algebra of \(W_{qE}^0\).

Let \(c : R(G^0, T)_{\text{red}} \to \mathbb{C}\) be a \(W_{qE}\)-invariant function.

**Proposition 1.1.** There exists a unique associative algebra structure on \(\mathbb{C}[W_{qE}, \sharp] \otimes S(t^*) \otimes \mathbb{C}[\vec{r}]\) such that:

- the twisted group algebra \(\mathbb{C}[W_{qE}, \sharp]\) is embedded as subalgebra;
• the algebra $S(t^*) \otimes \mathbb{C}[\bar{r}]$ of polynomial functions on $t \oplus \mathbb{C}^d$ is embedded as a subalgebra;
• $\mathbb{C}[\bar{r}]$ is central;
• the braid relation $N_{s_\alpha} \xi - s_\alpha \xi N_{s_\alpha} = c(\alpha) r_j (\xi - s_\alpha \xi) / \alpha$ holds for all $\xi \in S(t^*)$ and all simple roots $\alpha \in R_j$.

\[ \begin{align*}
N_{a_\alpha} \xi N_{w^{-1}} = \xi & \text{ for all } \xi \in S(t^*) \text{ and } w \in \mathfrak{N}_{q\xi}. 
\end{align*} \]

Proof. For $d = 1$, $G_1 = G_{\text{der}}^0$ this is [AMS2] Proposition 2.2. The general case can be shown in the same way.

We denote the algebra just constructed by $\mathbb{H}(t, W_{q\xi}, c\bar{r}, \bar{z})$. When $W_{q\xi}^\circ = W_{q\xi}$, there is no 2-cocycle, and write simply $\mathbb{H}(t, W_{q\xi}, c\bar{r})$. It is clear from the defining relations that

\[ \begin{align*}
S(t^*) W_{q\xi} \otimes \mathbb{C}[\bar{r}] = \mathcal{O}(t \times \mathbb{C}^d) W_{q\xi} \text{ is a central subalgebra of } \mathbb{H}(t, W_{q\xi}, c\bar{r}, \bar{z}).
\end{align*} \]

By a central character of an $\mathbb{H}(t, W_{q\xi}, c\bar{r}, \bar{z})$-module we shall mean an element of $t/W_{q\xi} \times \mathbb{C}^d$ by which $\mathcal{O}(t \times \mathbb{C}^d) W_{q\xi}$ acts on that module. For $\xi \in t/W_{q\xi} = Z(\mathfrak{g})^2 W_{q\xi}$ and $(\pi, V) \in \text{Mod}(\mathbb{H}(t, W_{q\xi}, c\bar{r}, \bar{z}))$ we define $(\xi \otimes \pi, V) \in \text{Mod}(\mathbb{H}(t, W_{q\xi}, c\bar{r}, \bar{z}))$ by

\[ (\xi \otimes \pi)(f_1 f_2 N_{w}) = f_1(\xi)(f_1 f_2 N_{w}) \quad f_1 \in S(t^*), f_2 \in \mathbb{C}[\bar{r}], w \in W_{q\xi}. \]

To the cuspidal quasi-support $[M, C^M_{v\xi}, q\xi]_G$ we associated a particular 2-cocycle

$\iota_{q\xi} : (W_{q\xi}/W_{q\xi}^\circ)^2 \to \mathbb{C}^\times,$

see [AMS1] Lemma 5.3]. The pair $(M^\circ, v)$ also gives rise to a $W_{q\xi}^\circ$-invariant function $c : R(G^\circ, T)_{\text{red}} \to \mathbb{Z}$, see [Lus2] Proposition 2.10 or [AMS2] (12)]. We denote the algebra $\mathcal{H}(t, W_{q\xi}, c\bar{r}, \bar{z})$, with this particular $c$, by $\mathbb{H}(G, M, q\xi, \bar{r})$. In [AMS2] we only studied the case $d = 1$, $R_1 = R(G^\circ, T)$, and we denoted that algebra by $\mathbb{H}(G, M, q\xi)$. Fortunately the difference with $\mathbb{H}(G, M, q\xi, \bar{r})$ is so small that almost all properties of $\mathbb{H}(G, M, q\xi)$ discussed in [AMS2] remain valid for $\mathbb{H}(t, W_{q\xi}, c\bar{r}, \bar{z})$. We will proceed to make this precise.

Write $v = v_1 + \cdots + v_d$ with $v_j \in \mathfrak{g}_j = \text{Lie}(G_j)$. Then

\[ \mathcal{C}^M_{v\xi} = \mathcal{C}^{M_1}_{v_1} + \cdots + \mathcal{C}^{M_d}_{v_d}, \text{ where } M_j = M^\circ \cap G_j. \]

The $M^\circ$-action on $(\mathcal{C}^{M_1}_{v_1}, \mathcal{E})$ can be inflated to $Z(G^\circ)^0 \times M_1 \times \cdots \times M_d$, and the pullback of $\mathcal{E}$ becomes trivial on $Z(G^\circ)^0 \mathfrak{g}$ and decomposes uniquely as

\[ \begin{align*}
m^{\circ}_{G^\circ} \mathcal{E} = \mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_d \end{align*} \]

with $\mathcal{E}_j$ a $M_j$-equivariant cuspidal local system on $\mathcal{C}^{M_j}_{v_j}$. From Proposition 1.1 and [AMS2] Proposition 2.2] we see that

\[ \begin{align*}
\mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \bar{r}) = \mathbb{H}(G_1, M_1, \mathcal{E}_1) \otimes \cdots \otimes \mathbb{H}(G_d, M_d, \mathcal{E}_d).
\end{align*} \]

Furthermore the proof of [AMS2] Proposition 2.2] shows that

\[ \begin{align*}
\mathbb{H}(G, M, q\xi, \bar{r}) = \mathbb{H}(G^\circ, M^\circ, \mathcal{E}, \bar{r}) \times \mathbb{C}[\mathfrak{N}_{q\xi}, \iota_{q\xi}].
\end{align*} \]

To parametrize the irreducible representations of these algebras we use some elements of the Lie algebras of the involved algebraic groups. Let $\sigma_0 \in \mathfrak{g}$ be semisimple and $y \in Z(\mathfrak{g})$ nilpotent. We decompose them along [9]:

\[ \begin{align*}
\sigma_0 = \sigma_z + \sigma_{0,1} + \cdots + \sigma_{0,d} \quad \text{with } \sigma_{0,j} \in \mathfrak{g}_j, \sigma_z \in Z(\mathfrak{g}), \\
y = y_1 + \cdots + y_d \quad \text{with } y_j \in \mathfrak{g}_j.
\end{align*} \]
Choose algebraic homomorphisms \( \gamma_j : \text{SL}_2(\mathbb{C}) \to Z_{G_j}(\sigma_{0,j}) \) with \( d\gamma_j \left( \frac{0}{1} \right) = y_j \).

Given \( \vec{r} \in \mathbb{C}^d \), we write \( \sigma_j = \sigma_{0,j} + d\gamma_j \left( \frac{r_0}{0} - r_1 \right) \) and
\[
\begin{align*}
\sigma & = \sigma_0 + d\gamma_j \left( \frac{r}{0} - r_1 \right) .
\end{align*}
\]

(15)

Notice that \( [\sigma, y_j] = [\sigma_j, y_j] = 2y_j r_j y_j \). Let us recall the construction of the standard modules from [Lus2] and [AMS2]. We need the groups
\[
M_j(y_j) = \{(g_j, \lambda_j) \in G_j \times \mathbb{C}^\times : \text{Ad}(g_j)y_j = \lambda_j^2y_j \},
\]
\[
\bar{M}^o(y) = \{(g, \tilde{\lambda}) \in G^o \times (\mathbb{C}^\times)^d : \text{Ad}(g)y_j = \lambda_j^2y_j \forall j = 1, \ldots, d \},
\]
\[
\bar{M}(y) = \{(g, \tilde{\lambda}) \in G^o N_G(qE) \times (\mathbb{C}^\times)^d : \text{Ad}(g)y_j = \lambda_j^2y_j \forall j = 1, \ldots, d \},
\]
and the varieties
\[
P_{y_j} = \{g(P^o \cap G_j) \in G_j/(P^o \cap G_j) : \text{Ad}(g^{-1})y_j \in \mathfrak{c}_{y_j}^M + \text{Lie}(U \cap G_j) \},
\]
\[
P_y = \{gP \in G^o \times P^o : \text{Ad}(g^{-1})y \in \mathfrak{c}_y^{M^o} + \text{Lie}(U) \},
\]
\[
P_y = \{gP \in G^o N_G(qE) \times P^o : \text{Ad}(g^{-1})y \in \mathfrak{c}_y^M + \text{Lie}(U) \}.
\]

The local systems \( \mathcal{E}_j, \mathcal{E} \) and \( qE \) give rise to local systems \( \hat{\mathcal{E}}_j, \hat{\mathcal{E}} \) and \( \hat{qE} \) on \( P_{y_j}, P^o_y \) and \( P_y \), respectively. The groups \( M_j(y_j), \bar{M}^o(y) \) and \( \bar{M}(y) \) act naturally on, respectively, \( (P_{y_j}, \hat{\mathcal{E}}_j), (P^o_y, \hat{\mathcal{E}}) \) and \( (P_y, \hat{qE}) \). With the method from [Lus2] and [AMS2]§3.1 we can define an action of \( \mathbb{H}(G, M, qE, \vec{r}) \times \bar{M}(y) \) on the equivariant homology \( H_*(\bar{M}^o(y)^o(P_{y_j}, \hat{\mathcal{E}})) \), and similarly for \( H_*(\bar{M}^o(y)^o(P_y, \hat{\mathcal{E}})) \) and \( H_*(M_j(y)^o(P_{y_j}, \hat{\mathcal{E}})) \). As in [Lus2] we build
\[
E^o_{y_j,\sigma, r_j} = \mathbb{C}_{\sigma, r_j} \otimes H_*(M_j(y_j)^o(P_{y_j}, \hat{\mathcal{E}})).
\]

Similarly we introduce
\[
E^o_{y,\sigma, \vec{r}} = \mathbb{C}_{\sigma, \vec{r}} \otimes H_*(\bar{M}^o(y)^o(P_y, \hat{\mathcal{E}})),
\]
\[
E^o_{y,\sigma, \vec{r}} = \mathbb{C}_{\sigma, \vec{r}} \otimes H^o_*(\bar{M}(y)^o(P_y, \hat{qE})).
\]

By [AMS2] Theorem 3.2 and Lemma 3.6 these are modules over, respectively, \( \mathbb{H}(G_j, M_j, \mathcal{E}_j) \times \pi_0(Z_{G_j}(\sigma_{0,j}, y_j)) \), \( \mathbb{H}(G^o, M^o, \mathcal{E}, \vec{r}) \times \pi_0(Z_G(qE)(\sigma_0, y)) \) and \( \mathbb{H}(G, M, qE, \vec{r}) \times \pi_0(Z_G N_G(qE)(\sigma_0, y)). \) This last action is the reason to use \( G^o N_G(qE) \) instead of \( G \) in the definition of \( P_y \).

In terms of (14), there is a natural module isomorphism
\[
(16)
\]
\[
E^o_{y,\sigma, \vec{r}} \cong \text{ind}_{\mathbb{H}(G, M, qE, \vec{r}) \to \mathbb{H}(G^o, M^o, \mathcal{E}, \vec{r})} E^o_{y_{\vec{r}}},
\]
It can be proven in the same way as the analogous statement with only one variable \( r \), which is [AMS2] Lemma 3.3.

**Lemma 1.2.** With the identifications [13] there is a natural isomorphism of \( \mathbb{H}(G^o, M^o, \mathcal{E}, \vec{r}) \)-modules
\[
E^o_{y,\sigma, \vec{r}} \cong \mathbb{C}_{\sigma} \otimes \mathbb{C}_{\sigma_1} \otimes \mathbb{C}_{\sigma_1, r_1} \otimes \cdots \otimes \mathbb{C}_{\sigma_d, r_d},
\]
which is equivariant for the actions of the appropriate subquotients of $\tilde{M}^0(y)$.  

**Proof.** From (7) and $Z(G^o)Z(G_j) \subset P^o$ we get natural isomorphisms

$$\mathcal{P}_{y_1} \times \cdots \times \mathcal{P}_{y_d} \rightarrow \mathcal{P}_{y}^o.$$  

Looking at (12) and the construction of $\hat{\mathcal{E}}$ in [Lus2] §3.4, we deduce that

$$\hat{\mathcal{E}} \cong \hat{\mathcal{E}}_1 \otimes \cdots \otimes \hat{\mathcal{E}}_d$$  

as sheaves on $\mathcal{P}_{y}^o$.

From (7) we also get a central extension

$$1 \rightarrow \ker m_{G^o} \rightarrow Z(G^o)^o \times M_1(y_1) \times \cdots \times M_d(y_d) \rightarrow \tilde{M}^0(y) \rightarrow 1.$$  

Here $\ker m_{G^o}$ refers to the kernel of (7), a finite central subgroup which acts trivially on the sheaf $\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_d \cong m_{G^o}^* \mathcal{E}$. Restricting to connected components, we obtain a central extension of $\tilde{M}^0(y)^o$ by

$$\tilde{M} := Z(G^o)^o \times M_1(y_1)^o \times \cdots \times M_d(y_d)^o$$  

In fact, equivariant (co)homology is inert under finite central extensions, for all groups and all varieties. We sketch how this can be deduced from [Lus2] §1. By definition

$$H^*_{\tilde{M}^0(y)^o}(\mathcal{P}_{y}^o, \hat{\mathcal{E}}) = H^*\left(\tilde{M}^0(y)^o \backslash (\Gamma \times \mathcal{P}_{y}^o), r\hat{\mathcal{E}}\right)$$  

for a suitable (in particular free) $\tilde{M}^0(y)^o$-variety $\Gamma$ and a local system derived from $\hat{\mathcal{E}}$. On the right hand side we can replace $\tilde{M}^0(y)^o$ by $\tilde{M}$ without changing anything. If $\tilde{\Gamma}$ is a suitable variety for $\tilde{M}$, then $\tilde{\Gamma} \times \Gamma$ is also one. (The freeness is preserved because (19) is an extension of finite index.) The argument in [Lus2] p. 149 shows that

$$H^*(\tilde{M} \backslash (\Gamma \times \mathcal{P}_{y}^o), r\hat{\mathcal{E}}) \cong H^*\left(\tilde{M} \backslash (\tilde{\Gamma} \times \Gamma \times \mathcal{P}_{y}^o), \tilde{r}\hat{\mathcal{E}}\right) = H^*_M(\mathcal{P}_{y}^o, \hat{\mathcal{E}}).$$  

In a similar way, using [Lus2] Lemma 1.2, one can prove that

$$H^*_{\tilde{M}^0(y)^o}(\mathcal{P}_{y}^o, \hat{\mathcal{E}}) \cong H^*_M(\mathcal{P}_{y}^o, \hat{\mathcal{E}}).$$  

The upshot of (17), (18) and (20) is that we can factorize the entire setting along (13), which gives

$$H^*_{M_1(y)^o}(\mathcal{P}_{y_1}, \hat{\mathcal{E}}_1) \otimes \cdots \otimes H^*_{M_d(y)^o}(\mathcal{P}_{y_d}, \hat{\mathcal{E}}_d) \cong H^*_{\tilde{M}^0(y)^o}(\mathcal{P}_{y}^o, \hat{\mathcal{E}}).$$  

The equivariant cohomology of a point with respect to a connected group depends only on the Lie algebra [Lus2] §1.11, so (19) implies a natural isomorphism

$$H^*_{Z(G^o)}(\{1\}) \times H^*_{M_1(y)^o}(\{y_1\}) \times \cdots \times H^*_{M_d(y)^o}(\{y_d\}) \cong H^*_{\tilde{M}^0(y)^o}(\{y\}).$$  

Thus we can tensor both sides of (21) with $\mathbb{C}_{\sigma, \rho}$ and preserve the isomorphism. □

Given $\rho_j \in \text{Irr}(\pi_0(Z_{G_j}(\sigma_{0,j}, y_j)))$, we can form the standard $\mathbb{H}(G_j, M_j, \mathcal{E}_j)$-module

$$E_{y_j, \sigma_j, r_j, \rho_j}^o := \text{Hom}_{\pi_0(Z_{G_j}(\sigma_{0,j}, y_j))}(\rho_j, E_{y_j, \sigma_j, r_j}^o).$$  

Similarly $\rho^o \in \text{Irr}(\pi_0(Z_{G^o}(\sigma_{0,y}, y)))$ and $\rho \in \text{Irr}(\pi_0(Z_{G^o N_{G}E}(\sigma_{0,y}, y)))$ give rise to

$$E_{y, \sigma, \tilde{\rho}, \rho^o}^o := \text{Hom}_{\pi_0(Z_{G^o N_{G}E}(\sigma_{0,y}, y))}(\rho^o, E_{y, \sigma, \tilde{\rho}, \rho^o}^o),$$  

$$E_{y, \sigma, \rho, \rho} := \text{Hom}_{\pi_0(Z_{G^o N_{G}E}(\sigma_{0,y}, y))}(\rho, E_{y, \sigma, \rho, \rho}).$$  

We call these standard modules for respectively $\mathbb{H}(G^o, M^o, \mathcal{E}, \tilde{\rho})$ and $\mathbb{H}(G, M, q\mathcal{E}, \tilde{\rho})$. 


The canonical map (7) induces a surjection
\[ \pi_0(Z_G(\sigma_{0,1}, y_1)) \times \cdots \times \pi_0(Z_G(\sigma_{0,d}, y_d)) \to \pi_0(Z_{G^0}(\sigma_{0,y})). \]

**Lemma 1.3.** Let \( \rho^j \in \text{Irr}(\pi_0(Z_{G^0}(\sigma_0, y))) \) and let \( \bigotimes^d_{j=1} \rho_j \) be its inflation to \( \prod^d_{j=1} \pi_0(Z_{G_j}(\sigma_{0,j}, y_j)) \) via (23). There is a natural isomorphism of \( \mathbb{H}(G^0, M^0, E, \tilde{r}) \)-modules
\[ E^{\circ}_{g, \sigma, \tilde{r}, \rho} \cong C_{\sigma^0} \otimes E^{\circ}_{y_1, \sigma, \tilde{r}_1, \rho_1} \otimes \cdots \otimes E^{\circ}_{y_d, \sigma, \tilde{r}_d, \rho_d}. \]
Every \( \bigotimes^d_{j=1} \rho_j \in \text{Irr}( \prod^d_{j=1} \pi_0(Z_{G_j}(\sigma_{0,j}, y_j))) \) for which \( \bigotimes^d_{j=1} E^{\circ}_{y_j, \sigma, \tilde{r}_j, \rho_j} \) is nonzero comes from \( \pi_0(Z_{G^0}(\sigma_0, y)) \) via (23).

**Proof.** The module isomorphism follows from the naturality and the equivariance in Lemma 1.2. Suppose that \( \bigotimes^d_{j=1} \rho_j \in \text{Irr}( \prod^d_{j=1} \pi_0(Z_{G_j}(\sigma_{0,j}, y_j))) \) appears in \( \bigotimes^d_{j=1} E^{\circ}_{y_j, \sigma, \tilde{r}_j, \rho_j} \).

By [AMS2 Proposition 3.7] the cuspidal support \( \Psi_{Z_{G^0}(\sigma_0,y)}(y_j, \rho_j) \) is \( G_j \)-conjugate to \( (M_j, C_{y_j}^M, E_j) \). In particular \( \rho_j \) has the same \( Z(G_j) \)-character as \( E_j \), see [Lus1 Theorem 6.5.a]. Hence \( \otimes \rho_j \) has the same central character as \( m_{\sigma_0}^{E_0} \). That central character factors through the multiplication map (7) whose kernel is central, so \( \otimes \rho_j \) also factors through (7). That is, the map (23) induces a bijection between the relevant irreducible representations on both sides.

For some choices of \( \rho \) the standard module \( E_{y, \sigma, \tilde{r}, \rho} \) is zero. To avoid that, we consider triples \( (\sigma_0, y, \rho) \) with:
- \( \sigma_0 \in \mathfrak{g} \) is semisimple,
- \( y \in Z_\mathfrak{g}(\sigma_0) \) is nilpotent,
- \( \rho \in \text{Irr}(\pi_0(Z_G(\sigma_0, y))) \) is such that the cuspidal quasi-support \( q\Psi_{Z_G(\sigma_0)}(y, \rho) \) from [AMS1 §5] is \( G \)-conjugate to \( (M, C_{y}^M, qE) \).

Given in addition \( \tilde{r} \in \mathbb{C}^d \), we construct \( \sigma = \sigma_0 + d\tilde{r} (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \in \mathfrak{g} \) as in (15). Although this depends on the choice of \( \tilde{r} \), the conjugacy class of \( \sigma \) does not.

By definition
\[ \mathbb{H}(G^0 N_G(qE), M, qE, \tilde{r}) = \mathbb{H}(G, M, qE, \tilde{r}), \]
but of course \( \pi_0(Z_{G^0 N_G(qE)}(\sigma_0, y)) \) can be a proper subgroup of \( \pi_0(Z_{G^0}(\sigma_0, y)) \). As shown in the proof of [AMS2 Lemma 3.21], the functor \( \text{ind}^{\pi_0(Z_{G^0}(\sigma_0, y))}_{\pi_0(Z_{G^0 N_G(qE)}(\sigma_0, y))} \) provides a bijection between the \( \tilde{r} \) in the triples for \( G^0 N_G(qE) \) and the \( \rho \) in the triples for \( G \).

For \( \rho = \text{ind}^{\pi_0(Z_{G^0}(\sigma_0, y))}_{\pi_0(Z_{G^0 N_G(qE)}(\sigma_0, y))} \tilde{r} \) we define, in terms of (22),
\[ E_{y, \sigma, \tilde{r}, \rho} = E_{y, \sigma, \tilde{r}, \tilde{r}}. \]

The next result generalizes [AMS2 Theorem 3.20] to several variables \( r_j \). We define \( \text{Irr}_\tilde{r}(\mathbb{H}(G, M, qE, \tilde{r})) \) to be the set of equivalence classes of those irreducible representations of \( \mathbb{H}(G, M, qE, \tilde{r}) \) on which \( r_j \) acts as \( r_j \).

**Theorem 1.4.** Fix \( \tilde{r} \in \mathbb{C}^d \). The standard \( \mathbb{H}(G, M, qE, \tilde{r}) \)-module \( E_{y, \sigma, \tilde{r}, \rho} \) is nonzero if and only if \( q\Psi_{Z_G(\sigma_0)}(y, \rho) = (M, C_{y}^M, qE) \) up to \( G \)-conjugacy. In that case it has a distinguished irreducible quotient \( M_{y, \sigma, \tilde{r}, \rho} \), which appears with multiplicity one in \( E_{y, \sigma, \tilde{r}, \rho} \).

The map \( M_{y, \sigma, \tilde{r}, \rho} \mapsto (\sigma_0, y, \rho) \) sets up a canonical bijection between \( \text{Irr}_\tilde{r}(\mathbb{H}(G, M, qE, \tilde{r})) \) and \( G \)-conjugacy classes of triples as above.
Proof. For \( \mathbb{H}(G_j, M_j, E_j) \) this is [AMS2, Proposition 3.7 and Theorem 3.11]. With [13] and Lemma 1.3 we can generalize that to \( \mathbb{H}(G^0, M^0, qE, \tilde{r}) \). The method to go from there to \( \mathbb{H}(G^0 N_G(qE), M, qE, \tilde{r}) \) is exactly the same as in [AMS2, §3.3–3.4] (for \( \mathbb{H}(G^0, M^0, E) \) and \( \mathbb{H}(G^0 N_G(qE), M, qE) \)). That is, the proof of [AMS2, Theorem 3.20] applies and establishes the theorem for \( \mathbb{H}(G^0 N_G(qE), M, qE, \tilde{r}) \). In view of [24] we can replace \( G^0 N_G(qE) \) by \( G \).

The above modules are compatible with parabolic induction, in a suitable sense and under a certain condition. Let \( Q \subset G \) be an algebraic subgroup containing \( M \), such that \( Q^0 \) is a Levi subgroup of \( G^0 \). Let \( y, \sigma, \tilde{r}, \rho \) be as in Theorem 1.4 with \( \sigma, y \in q = \text{Lie}(Q) \). By [Rec, §3.2] the natural map

\[
\pi_0(Z_Q(\sigma, y)) = \pi_0(Z_{Q^0}(\sigma_0)(y)) \rightarrow \pi_0(Z_{G^0}(\sigma_0)(y)) = \pi_0(Z_G(\sigma, y))
\]

is injective, so we can consider the left hand side as a subgroup of the right hand side. Let \( \rho^Q \in \text{Irr}(\pi_0(Z_Q(\sigma, y))) \) be such that \( q\Psi_{Z_Q(\sigma_0)}(\rho, \rho^Q) = (M, C_v^M, qE) \). Then \( E_{\sigma, \tau, \rho}^Q \) and \( M_{y, \sigma, \tau, \rho}^Q \) are defined.

Further, \( PQ^0 \) is a parabolic subgroup of \( G^0 \) with \( Q^0 \) as Levi factor. The unipotent radical \( R_u(PQ^0) \) is normalized by \( Q^0 \), so its Lie algebra \( u_Q = \text{Lie}(R_u(PQ^0)) \) is stable under the adjoint actions of \( Q^0 \) and \( q \). By [9] \( u_Q \) decomposes as the direct sum of the subspaces \( u_{Q, j} = u_Q \cap q_j \). In particular \( \text{ad}(y) \) acts on \( u_{Q, j} \). We denote the cokernel of \( \text{ad}(y_j) : u_{Q, j} \rightarrow u_{Q, j} \) by \( y\rho_{Q, j} \). From \( |\sigma_j, y_j| = 2r_j y_j \) we see that \( \text{ad}(\sigma_j) \) descends to a linear map \( y\rho_{Q, j} \rightarrow y\rho_{Q, j} \).

Following Lusztig [Lus6, §1.16], we define

\[
e_{y, j} : \text{Lie}(M^0(y^0)) \rightarrow \mathbb{C}
\]

\[
(\sigma, r) \mapsto \det(\text{ad}(\sigma_j) - 2r_j y_{\rho_{Q, j}}) \rightarrow y_{\rho_{Q, j}}
\]

All parameters for which parabolic induction could behave problematically are zeros of a function \( e_{y, j} \).

Proposition 1.5. Let \( y, \sigma, \tilde{r}, \rho \) be as in Theorem 1.4 and assume that, for each \( j = 1, \ldots, d, e_{y, j}(\sigma, r) \neq 0 \) or \( r_j = 0 \).

(a) There is a natural isomorphism of \( \mathbb{H}(G, M, qE, \tilde{r}) \)-modules

\[
\mathbb{H}(G, M, qE, \tilde{r}) \otimes_{(Q, M, qE, \tilde{r})} E_{y, \sigma, \tilde{r}, \rho}^Q \cong 
\bigoplus_{\rho} \text{Hom}_{\pi_0(Z_G(\sigma, y))}(\rho^Q, \rho) \otimes \text{Lie}(\mathbb{C}^M, qE)
\]

where the sum runs over all \( \rho \in \text{Irr}(\pi_0(Z_G(\sigma, y))) \) with \( q\Psi_{Z_G(\sigma_0)}(\rho, \rho^Q) = (M, C_v^M, qE) \).

(b) For \( \tilde{r} = \tilde{0} \) part (a) contains an isomorphism of \( S(t^*) \) modules

\[
\mathbb{H}(G, M, qE, \tilde{r}) \otimes_{(Q, M, qE, \tilde{r})} M_{y, \sigma, \tilde{r}, \rho}^Q \cong 
\bigoplus_{\rho} \text{Hom}_{\pi_0(Z_G(\sigma, y))}(\rho^Q, \rho) \otimes \text{Lie}(\mathbb{C}^M, qE)
\]

(c) The multiplicity of \( M_{y, \sigma, \tilde{r}, \rho} \) in \( \mathbb{H}(G, M, qE, \tilde{r}) \otimes_{(Q, M, qE, \tilde{r})} E_{y, \sigma, \tilde{r}, \rho}^Q \) is

\[
[\rho^Q : \rho]_{\pi_0(Z_G(\sigma, y))}. \quad \text{It already appears that many times as a quotient, via}
\]

\[
E_{y, \sigma, \tilde{r}, \rho}^Q \rightarrow M_{y, \sigma, \tilde{r}, \rho}^Q. \quad \text{More precisely, there is a natural isomorphism}
\]

\[
\text{Hom}_{\mathbb{H}(Q, M, qE, \tilde{r})}(M_{y, \sigma, \tilde{r}, \rho}^Q, M_{y, \sigma, \tilde{r}, \rho}) \cong \text{Hom}_{\pi_0(Z_G(\sigma, y))}(\rho^Q, \rho).
\]
Theorem 1.6. (a) Fix \( r \in \mathbb{C}^d \). There exists a canonical bijection
\[
(\sigma_0, y, \rho) \leftrightarrow IM^* M_{y,d|\gamma|^{\mathbf{r}}_{\eta=0}} - \sigma_0, r, \rho
\]
between conjugacy classes triples as in Theorem 1.4 and \( \text{Irr}(H, \mathbb{C}) \).

(b) Suppose that \( \mathcal{R}(r) \in \mathbb{R}^d_{\geq 0} \). Then \( IM^* M_{y,d|\gamma|^{\mathbf{r}}_{\eta=0}} - \sigma_0, r, \rho \) is tempered if and only if \( \sigma_0 \in i\mathbb{R} = i\mathbb{R} \otimes \mathbb{R} X_*(T) \).

(c) Suppose that \( \mathcal{R}(r) \in \mathbb{R}^d_{\geq 0} \). Then \( IM^* M_{y,d|\gamma|^{\mathbf{r}}_{\eta=0}} - \sigma_0, r, \rho \) is essentially discrete series if and only if \( y \) is distinguished in \( g \). In this case \( \sigma_0 \in Z(g) \).

(d) Let \( \zeta \in g^G = Z(g)^{G_0} \). Then part (a) maps \( (\zeta + \sigma_0, y, \rho) \) to
\[
\zeta \otimes IM^* M_{y,d|\gamma|^{\mathbf{r}}_{\eta=0}} - \sigma_0, r, \rho.
\]

(e) Suppose that \( \mathcal{R}(r) \in \mathbb{R}^d_{\geq 0} \) and that \( \sigma_0 \in i\mathbb{R} + Z(g) \). Then
\[
IM^* M_{y,d|\gamma|^{\mathbf{r}}_{\eta=0}} - \sigma_0, r, \rho = IM^* E_{y,d|\gamma|^{\mathbf{r}}_{\eta=0}} - \sigma_0, r, \rho.
\]

Proof. Part (a) follows immediately from Theorem 1.4. Parts (b) and (c) are consequences of [AMS2] §3.5, see in particular (82) and (83) therein.

(d) From (22) and Lemma 1.3 we see that
\[
E_{y,\sigma^\prime - \zeta, \mathbf{r}, \rho} = -\zeta \otimes E_{y, \sigma^\prime, \mathbf{r}, \rho},
\]
whenever both sides are defined. By Theorem 1.4 the analogous equation for \( M_{y,\sigma^\prime, \mathbf{r}, \rho} \) holds. Apply this with \( \sigma^\prime = d|\gamma|^{\mathbf{r}}_{\eta=0} - \sigma_0 \) and use that \( IM^* \) turns \( -\zeta \) into \( \zeta \).

(e) Notice that \( \sigma_0 - \sigma_z \in i\mathbb{R} \). Write \( \rho = \tau^\mathbf{r} \times \rho^\mathbf{r} \) as in [AMS2] Lemma 3.13. By Lemma 1.3 and [Lus6], Theorem 1.21](for the simple factors of \( G^0_{\text{det}} \))
\[
M^\mathbf{r}_{y,d|\gamma|^{\mathbf{r}}_{\eta=0}} - \sigma_0, r, \rho^\mathbf{r} = M_{G^0_{\text{det}} (y,d|\gamma|^{\mathbf{r}}_{\eta=0} + (\sigma_z - \sigma_0), r, \rho^\mathbf{r})} \otimes \mathbb{C}_{-\sigma_z} =
E_{G^0_{\text{det}} (y,d|\gamma|^{\mathbf{r}}_{\eta=0} + (\sigma_z - \sigma_0), r, \rho^\mathbf{r})} \otimes \mathbb{C}_{-\sigma_z} = E^\mathbf{r}_{y,d|\gamma|^{\mathbf{r}}_{\eta=0}} - \sigma_0, r, \rho^\mathbf{r}.
\]
By [AMS2] Lemma 3.16
\[
M_{y,d|\gamma|^{\mathbf{r}}_{\eta=0}} - \sigma_0, r, \rho = \tau \otimes M^\mathbf{r}_{y,d|\gamma|^{\mathbf{r}}_{\eta=0}} - \sigma_0, r, \rho^\mathbf{r},
\]
while \[\text{AMS2}\] Lemma 3.18] says that
\[
E_{y,d\tilde{\gamma}}(\begin{smallmatrix} r & 0 \\ 0 & r \end{smallmatrix}) - \sigma_0, r, \rho = \tau \times E_{y,d\tilde{\gamma}}(\begin{smallmatrix} r & 0 \\ 0 & r \end{smallmatrix}) - \sigma_0, r, \rho^*.
\]
Applying IM* to both these modules, we obtain the desired statement.

We would like to exhibit the central characters of the \(\mathbb{H}(G, M, qE, r)\)-modules constructed in this section. It has turned out that the treatment of this aspect in \[\text{AMS2}\] was flawed, we also correct that here. We fix a homomorphism of algebraic groups
\[
\gamma_v : SL_2(\mathbb{C}) \to M \quad \text{with} \quad d\gamma_v(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}) = v.
\]
We write
\[
d\gamma_v(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) = \sigma_v = \sigma_{v,1} + \cdots + \sigma_{v,d} \quad \text{where} \quad \sigma_{v,j} \in \text{Lie}(M \cap G_j).
\]
For \(r \in \mathbb{C}^d\) we put
\[
\bar{r}\sigma_v = r_1\sigma_{v,1} + \cdots + r_d\sigma_{v,d} \in m.
\]
We record the linear bijection
\[
\Sigma_v : \ t \oplus \mathbb{C}^d \to t \oplus \mathbb{C}^d(\sigma_v, 1) \quad (\sigma_0, \bar{r}) \mapsto (\sigma_0 + \bar{r}\sigma_v, \bar{r}).
\]
Here the target is a linear subspace of \(m \oplus \mathbb{C}^d\) and the inverse map is
\[
\Sigma_v^{-1} : (\sigma, \bar{r}) \mapsto (\sigma - \bar{r}\sigma_v, \bar{r}).
\]
The next result is a correction of \[\text{AMS2}\] Proposition 3.5], which was based on a wrong interpretation of \[\text{Lus4}\] \(8.13\). Our improvement consists mainly of adding \(\Sigma_v^\pm 1\) at the right places.

**Proposition 1.7.** Let \((y, \sigma, \bar{r})\) be as in \([15]\) and assume that \(P_y\) is nonempty.
(a) \((\text{Ad}(N_G(P, qE)G^0)\sigma - \bar{r}\sigma_v) \cap t\) is a single \(W_{qE}\)-orbit in \(t\).
(b) The \(\mathbb{H}(G, M, qE, r)\)-module \(E_{y,\sigma, r}\) admits the central character
\[
((\text{Ad}(N_G(P, qE)G^0)\sigma - \bar{r}\sigma_v) \cap t, \bar{r}).
\]
(c) The pair \((y, \sigma)\) is \(G^0\)-conjugate to one with \(\sigma_0\) and \(\bar{r}\sigma_v + d\tilde{\gamma}(\begin{smallmatrix} -r & 0 \\ 0 & r \end{smallmatrix})\) in \(t\).
(d) Suppose \((y, \sigma)\) has the properties as in (c). Then \(\sigma_0, \sigma_v\) and \(d\tilde{\gamma}(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix})\) commute, and \(\sigma_v + d\tilde{\gamma}(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})\) \(\in t_{\mathbb{R}}\).

**Proof.** (a) By \[\text{Lus4}\] Theorem 8.11], \(\mathbb{H}(G_j, M_j, \mathcal{E}_j, r_j)\) is canonically isomorphic to the endomorphism algebra of a certain perverse sheaf \(K_j^*\), in the \(G_j \times \mathbb{C}^\times\)-equivariant bounded derived category of constructible sheaves on \(g_j\). According to \[\text{Lus4}\] \(8.13.a.j\), there exists a canonical surjection
\[
(28) \quad H^*_G \times \mathbb{C}^\times(\text{point}) \cong \mathcal{O}(g_j \oplus \mathbb{C})^{G_j \times \mathbb{C}^\times} = \mathcal{O}(g_j)^{G_j} \otimes \mathbb{C}[r_j] \to Z(\text{End}(K_j^*)�).
\]
By \[\text{AMS2}\] Lemma 2.3] the right hand side is
\[
(29) \quad Z(\text{End}(K_j^*)) \cong Z(\mathbb{H}(G_j, M_j, \mathcal{E}_j, r_j)) \cong \mathcal{O}(t \cap g_j)^{W_{\mathcal{E}_j}} \otimes \mathbb{C}[r_j].
\]
By \[\text{Lus4}\] \(8.13.b\], the composition of \(28\) and \(29\) corresponds to an injection like \(\Sigma_v\), namely
\[
(30) \quad (t \cap g_j)/W_{\mathcal{E}_j} \oplus \mathbb{C} \to \text{Irr}(\mathcal{O}(g_j \oplus \mathbb{C})^{G_j \times \mathbb{C}^\times}) \quad (\sigma_{0,j}, r_j) \mapsto (\sigma_{0,j} + r_j\sigma_{v,j}, r_j)�,
\]
where the right hand side is the variety of semisimple adjoint orbits in $g_j \oplus \mathbb{C}$. Hence

$$(\mathrm{Ad}(G_j)\sigma_j - r_j\sigma_{v,j}) \cap t \cap g_j = \mathrm{Ad}(G_j)\sigma_{0,j} \cap t \cap g_j$$

is either empty or a single $W_{E'}$-orbit. We will see in the proof of part (b) that it is nonempty. Combining these statements for all $j = 1, \ldots, d$, we find that $(\mathrm{Ad}(G^0)\sigma - \bar{r}\sigma_v) \cap t$ is a single $W_{E'}$-orbit $W_{E'}\sigma' \subset t$. As $M$ stabilizes $m$ and centralizes $t$:

$$\begin{align*}
\mathrm{Ad}(G^0)M\sigma \cap (t + \bar{r}\sigma_v) &= \mathrm{Ad}(M)(W_{E'}\sigma' + \bar{r}\sigma_v) \cap (t + \bar{r}\sigma_v) \\
&= (W_{E'}\sigma' + \mathrm{Ad}(M)\bar{r}\sigma_v) \cap (t + \bar{r}\sigma_v).
\end{align*}$$

(31)

Here $\mathrm{Ad}(M)(\bar{r}\sigma_v)$ lies in the derived subalgebra of $m$, so the right hand side of (31) equals $W_{E'}\sigma' + \bar{r}\sigma_v$. In other words,

$$(\mathrm{Ad}(G^0)M\sigma - \bar{r}\sigma_v) \cap t = W_{E'}\sigma'.$$

As $N_G(P,qE)G^0/G^0M \cong W_{qE}/W_E$, we can pass from $\mathrm{Ad}(G^0)M$-orbits to $\mathrm{Ad}(N_G(P,qE)G^0)$-orbits in the required way.

(b) The assumption $P_y \neq \emptyset$ implies that $H^*_s(M_y)^\sigma(P_y,qE)$ is nonzero. By [Lus2 Proposition 8.6.c] and because the semisimple adjoint orbits in $\mathrm{Lie}(M(y)^\circ)$ form an irreducible variety, $E^\circ_{y,\sigma,\bar{r}}$ is nonzero for all eligible $(\sigma,\bar{r})/\sim$.

The adjoint action of $O(t \cap g_j)^{W_{E'}} \otimes C[r_j]$ on $E_{y_j,\sigma_j,\tau_j}$ can be realized as

$$Z(\overline{H}(G_j, M_j, \mathcal{E}_j, r_j)) \leftarrow H^*_{G_j \times \mathbb{C}^\times}(\text{point}) \to H^*_{M_j(y_j)}(y_j) \to H^*_{M_j(y_j)^\circ}(P(y_j))$$

and then the product in equivariant homology. By construction $H^*_{M_j(y_j)^\circ}(y_j)$ acts on $E_{y_j,\sigma_j,\tau_j}$ via the character $(\sigma_j,\tau_j)/\sim$. Hence $H^*_{G_j \times \mathbb{C}^\times}(\text{point})$ acts via the character $\mathrm{Ad}(G_j \times \mathbb{C}^\times)(\sigma_j,\tau_j)$. In view of (28)–(30), $Z(\overline{H}(G_j, M_j, \mathcal{E}_j, r_j))$ acts via

$$((\mathrm{Ad}(G_j)\sigma_j - r_j\sigma_{v,j}) \cap g_j \cap t, r_j).$$

For all $j = 1, \ldots, d$ together, this shows that $Z(\overline{H}(G^0, M^0, \mathcal{E}, \bar{r}))$ acts on $E^\circ_{y,\sigma,\bar{r}}$ as $((\mathrm{Ad}(G^0)\sigma - \bar{r}\sigma_v) \cap t, \bar{r})$. Now we use that $N_G(P,qE)G^0/G^0 \cong W_{qE}/W_E$ and

$$Z(\overline{H}(G, M, qE, r)) = Z(\overline{H}(G^0, M^0, \mathcal{E}, \bar{r}))^{W_{qE}/W_E},$$

and we conclude with (10).

c By part (b) with $r = 0$ we may assume that $\sigma_0 \in t$. Then $\exp(y)$ is contained in the reductive group $Z_G(\sigma_0)^\circ$, so we can arrange that the image of $\gamma$ lies in there. Applying part (b) to this group, we find $g \in Z_G(\sigma_0)^\circ$ such that

$$\mathrm{Ad}(g)\sigma = \sigma_0 + \mathrm{Ad}(g)d\gamma(\begin{pmatrix} 0 & \bar{r} \\ \bar{r} & 0 \end{pmatrix}) \quad \text{lies in} \quad t + \bar{r}\sigma_v.$$

Then $\mathrm{Ad}(g)d\gamma(\begin{pmatrix} 0 & \bar{r} \\ \bar{r} & 0 \end{pmatrix}) - \bar{r}\sigma_v \in t$, so $(\mathrm{Ad}(g)y, \mathrm{Ad}(g)\sigma)$ has the required properties.

d The assumption and $\sigma_v \in m$ imply that $d\gamma(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) \in m$. As $\sigma_0 \in t = Z(m)$, it commutes with both $\sigma_v$ and $d\gamma(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})$. The latter two differ by an element of $t = Z(m)$, so they commute as well. It follows that

$$\chi_{y,v} : z \mapsto \gamma(\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}) \gamma_v(\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix})$$

is an algebraic cocharacter of $T$. By definition of $t_k$, the derivative

$$d\chi_{y,v} : r \mapsto d\gamma(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) - r\sigma_v$$

evaluates to an element of $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R} = t_k$ for every $r \in \mathbb{R}$.

In the context of Proposition 1.6 the central characters are as follows. 

$\square$
Lemma 1.8. Suppose that $\sigma_0, \sigma_v + d\gamma \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathfrak{t}$ (which can always be arranged by Proposition 1.7c).

(a) The modules $E_{y,\sigma,\bar{r},\rho}$ and $M_{y,\sigma,\bar{r},\rho}$ admit the central character $W_{qE}(\sigma_0 \pm (\bar{r}\sigma_v + d\gamma \begin{pmatrix} -r^0 \\ 0 \end{pmatrix}), \bar{r})$.

(b) Both $IM^* M_{y,\sigma,\bar{r},\rho}$ and $IM^* E_{y,\sigma,\bar{r},\rho}$ admit the central character $W_{qE}(\sigma_0 \pm (\bar{r}\sigma_v + d\gamma \begin{pmatrix} -r^0 \\ 0 \end{pmatrix}), \bar{r})$.

Proof. In both cases the irreducible module is a quotient of the standard module, so it suffices to consider the latter.

(a) From Proposition 1.7c we know that $W_{qE}(\sigma_0 + d\gamma \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \bar{r})$ is the central character of $E_{y,\sigma,\bar{r},\rho}$.

As in the proof of Proposition 1.7c, we may assume that $\text{im}(\bar{r}) \subset Z_G(\sigma_0)$. Put $s_y = \gamma \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and consider the parameter $(\text{Ad}(s_y)\sigma, \text{Ad}(s_y)y, \bar{r})$. We have
\[\text{Ad}(s_y)\sigma + \bar{r}s_v = \sigma_0 + d\gamma \begin{pmatrix} -r^0 \\ 0 \end{pmatrix} - \bar{r}(-s_v) \in \mathfrak{t}.\]

Here $-s_v$ is the semisimple element in the $\mathfrak{sl}_2$-triple $d\gamma_v \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $-s_v, -v$, which is conjugate to $v, s_v, d\gamma_v \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ by $\gamma_v \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in M$. Thus Proposition 1.7c says that $W_{qE}(\sigma_0 + \bar{r}\sigma_v + d\gamma \begin{pmatrix} -r^0 \\ 0 \end{pmatrix}, \bar{r})$ is also the central character of $E_{y,\sigma,\bar{r},\rho}$.

(b) From [26] we see that the effect of $IM^*$ on central characters is $W_{qE}(\sigma, \bar{r}) \mapsto W_{qE}(-\sigma, \bar{r})$. Combine that with part (a). \qed

2. Twisted affine Hecke algebras

We would like to push the results of [AMS2] and the previous section to affine Hecke algebras, because these appear more directly in the representation theory of reductive $p$-adic groups. This can be achieved with Lusztig’s reduction theorems [Lus3]. The first reduces to representations with a “real” central character (to be made precise later), and the second reduction theorem relates representations of affine Hecke algebras with representations of graded Hecke algebras.

Our goal is a little more specific though, we want to consider not just one (twisted) graded Hecke algebra, but a family of those, parametrized by a torus. We want to find a (twisted) affine Hecke algebra which contains all members of this family as some kind of specialization. Let us mention here that, although we phrase this section with quasi-Levi subgroups and cuspidal quasi-supports, all the results are equally valid for Levi subgroups and cuspidal supports.

Let $G$ be a possibly disconnected complex reductive group and let $(M, C^*_v, qE)$ be a cuspidal quasi-support for $G$. For any $t \in T = Z(M) \circ E_{\mathfrak{g},\mathfrak{t}}$ the reductive group $G_t = Z_G(t)$ contains $M$, and we can consider the twisted graded Hecke algebra
\[\mathbb{H}(G_t, M, qE, \bar{r}) = \mathbb{H}(t, N_{G_t}(qE)/M, c_t\bar{r}, z_{qE,t}).\]

Here $\bar{r} = (r_1, \ldots, r_d)$ refers to the almost direct factorization of $G_t^\circ$ induced by $\mathfrak{g}_{\mathfrak{t}}$. Let us investigate how these algebras depend on $t$. For any $t \in T$, the 2-cocycle $z_{qE,t}$ of $N_{G_t}(qE)/M$ is just the restriction of $z_{qE} : W_{qE}^2 \to \mathbb{C}^*$. This can be seen from [Lus1, §3] and the proofs of [AMS1, Proposition 4.5 and Lemma 5.4]. More concretely, the perverse sheaves $(pr_1)_! q_{E,\mathfrak{t}}^E$ and $(pr_1)_! q_{E,\mathfrak{t}}^E$ on $\text{Lie}(G)$ from [AMS2 (90)] and [Lus2, §3.4] extend the perverse sheaves $q_{\pi_*, qE}$ and $q_{\pi_*, qE}$ on $\text{Lie}(G)_{RS}$ from [AMS1, §5]. The latter naturally contain the corresponding objects $q_{\pi_{t,*}}(qE)$ and
\(q_{\pi_\ast}(\bar{q}\hat{E})\) for \(G_t\). We denote the category of \(G\)-equivariant perverse sheaves on a \(G\)-variety \(X\) by \(\mathcal{P}_G(X)\). The algebra
\[
\mathbb{C}[N_{G_t}(q\mathcal{E})/M, \mathbb{Z}_{q\mathcal{E}, t}] \cong \text{End}_{\mathcal{P}_G, \text{Lie}(G_t)_{RS}}(q_{\pi_\ast}(\bar{q}\hat{E}))
\]
from [AMS1] Proposition 4.5 and Lemma 5.4] is canonically embedded in
\[
\mathbb{C}[W_{q\mathcal{E}}, \mathbb{Z}_{q\mathcal{E}}] \cong \text{End}_{\mathcal{P}_G, \text{Lie}(G)_{RS}}(q_{\pi_\ast}(\bar{q}\hat{E})).
\]
We will simply write \(W_{q\mathcal{E}, t}\) for \(N_{G_t}(q\mathcal{E})/M\), and \(\mathbb{Z}_{q\mathcal{E}}\) for \(\mathbb{Z}_{q\mathcal{E}, t}\).

On the other hand, the parameter function \(c_t : R(Z_G(t)^0, T)_{\text{red}} \to \mathbb{C}\) could depend on \(t\), we have to specify which \(t\) we use for a given root \(\alpha\). Recall that \(c_t(\alpha)\) was defined in [Lus2] §2. For any root \(\alpha \in R(G^0, T)\):
\[
g_\alpha \subset \text{Lie}(G_t) \iff \alpha(t) = 1.
\]
From [Lus2] Proposition 2.2 we know that \(R(G^0, T)\) is a root system, so \(R(G^0, T) \cap \mathbb{R}\alpha \subseteq \{\alpha, 2\alpha, -\alpha, -2\alpha\}\) for every nondivisible root \(\alpha\).

**Proposition 2.1.** [Lus2] Propositions 2.8, 2.10 and 2.12]

Let \(y \in \mathfrak{m}\) is an element of the nilpotent orbit defined by the cuspidal quasi-support \((M, C^M, q\mathcal{E})\).

(a) Suppose that \(R(G^0, T) \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}\). Then \(c_t(\alpha)\) satisfies
\[
0 = \text{ad}(y)^{c(\alpha)-1} : g_\alpha \to g_\alpha \quad \text{and} \quad 0 \neq \text{ad}(y)^{c(\alpha)-2} : g_\alpha \to g_\alpha.
\]
This condition is independent of \(t\), as long as \(g_\alpha \subseteq \text{Lie}(G_t)\). So we can unambiguously write \(c(\alpha)\) for \(c_t(\alpha)\) in this case. Moreover \(c(\alpha) \in \mathbb{N}\) is even.

(b) Suppose that \(R(G^0, T) \cap \mathbb{R}\alpha = \{\alpha, 2\alpha, -\alpha, -2\alpha\}\).

When \(\alpha(t) = 1\), \(\{\alpha, 2\alpha\} \subset R(Z_G(t)^0, T)\). Then \(c_t(\alpha)\) is again given by (33), and it is odd. We write \(c(\alpha) = c_t(\alpha)\) for such a \(t \in T\). Furthermore \(c_t(2\alpha)\) is given by (33) with \(2\alpha\) instead of \(\alpha\), and it equals 2.

When \(\alpha(t) = -1\), still \(2\alpha \in R(Z_G(t)^0, T)\), and \(c_t(2\alpha)\) is given by (33) with \(2\alpha\) instead of \(\alpha\). It equals 2, and we write \(c(2\alpha) = 2\).

With the conventions from Proposition 2.1, \(c_t\) is always the restriction of \(c : R(G^0, T) \to \mathbb{C}\) to \(R(Z_G(t)^0, T)_{\text{red}}\).

Now we construct the algebras that we need.

**Proposition 2.2.** Consider the following data:

- the root datum \(\mathcal{R} = (R(G^0, T), X^*(T), R(G^0, T)^\vee, X_*(T))\), with simple roots determined by \(P\);
- the group \(W_{q\mathcal{E}} = W_{q\mathcal{E}} \rtimes \mathfrak{R}_{q\mathcal{E}}\);
- a 2-cocycle \(\mathfrak{z} : (W_{q\mathcal{E}}/W_{q\mathcal{E}}) \to \mathbb{C}^\times\);
- \(W_{q\mathcal{E}}\)-invariant functions \(\lambda : R(G^0, T)_{\text{red}} \to \mathbb{Z}_{\geq 0}\) and \(\lambda^\vee : \{\alpha \in R(G^0, T)_{\text{red}} : \alpha^\vee \in 2X_*(T)\} \to \mathbb{Z}_{\geq 0}\);
- an array of invertible variables \(\mathbf{z} = (z_1, \ldots, z_d)\), corresponding to the decomposition (9) of \(\mathfrak{g}\).

The vector space
\[
\mathcal{O}(T \times (\mathbb{C}^\times)^d) \otimes \mathbb{C}[W_{q\mathcal{E}}] = \mathbb{C}[X^*(T)] \otimes \mathbb{C}[\mathbf{z}, \mathbf{z}^{-1}] \otimes \mathbb{C}[W_{q\mathcal{E}}] \otimes \mathbb{C}[\mathfrak{R}_{q\mathcal{E}}, \mathfrak{z}]
\]
admits a unique algebra structure such that:

- \(\mathbb{C}[X^*(T)], \mathbb{C}[\mathbf{z}, \mathbf{z}^{-1}]\) and \(\mathbb{C}[\mathfrak{R}_{q\mathcal{E}}, \mathfrak{z}]\) are embedded as subalgebras;
• \( C(\bar{z}, \bar{z}^{-1}) = C(z_1, z_1^{-1}, \ldots, z_d, z_d^{-1}) \) is central;

• the span of \( W_q^\circ \) is the Iwahori–Hecke algebra \( \mathcal{H}(W_q^\circ, \bar{z}^{2\lambda}) \) of \( W_q^\circ \) with parameters \( \bar{z}^{2\lambda} \). That is, it has a basis \( \{ N_w : w \in W_q^\circ \} \) such that

\[
N_w N_v = N_{wv} \quad \text{if } \ell(w) + \ell(v) = \ell(wv),
\]

\[
(N_{sa} + z_j^{-\lambda(\alpha)})(N_{sa} - z_j^{\lambda(\alpha)}) = 0 \quad \text{if } \alpha \in R(G_j T, T)_{\text{red}} \text{ is a simple root}.
\]

• for \( \gamma \in R_q^\circ \), \( w \in W_q^\circ \) and \( x \in X^*(T) \):

\[
N_\gamma N_w \theta_x N_\gamma^{-1} = N_{\gamma w \gamma^{-1}} \theta_\gamma(x).
\]

• for a simple root \( \alpha \in R(G_j T, T) \) and \( x \in X^*(T) \), corresponding to \( \theta_x \in \mathcal{O}(T) \):

\[
\theta_x N_{sa} - N_{sa} \theta_{sa(x)} = \begin{cases} (z_j^{\lambda(\alpha)} - z_j^{-\lambda(\alpha)})(\theta_x - \theta_{sa(x)})/\theta_0 - \theta_\alpha & \alpha \not\in 2X_\ast(T) \\ (z_j^{\lambda(\alpha)} - z_j^{-\lambda(\alpha)} + \theta_\alpha(z_j^{\lambda(\alpha)} - z_j^{-\lambda(\alpha)}))(\theta_x - \theta_{sa(x)})/\theta_0 - \theta_{-2\alpha} & \alpha \in 2X_\ast(T) \end{cases}
\]

**Proof.** In the case \( R_q^\circ = 1 \), the existence and uniqueness of such an algebra is well-known. It follows for instance from [Lus3, §3], once we identify \( T_{sa} \) from [Lus3] with \( z_j^{\lambda(\alpha)} N_{sa} \). It is called an affine Hecke algebra and denoted by \( \mathcal{H}(R, \lambda, \lambda^\ast, \bar{z}) \).

Since \( \lambda \) and \( \lambda^\ast \) are \( W_q^\circ \)-invariant,

\[
A_\gamma : N_w \theta_x \mapsto N_{\gamma w \gamma^{-1}} \theta_\gamma(x)
\]

defines an automorphism of \( \mathcal{H}(R, \lambda, \lambda^\ast, \bar{z}) \). Clearly

\[
R_q^\circ \rightarrow \text{Aut}(\mathcal{H}(R, \lambda, \lambda^\ast, \bar{z})) : \gamma \mapsto A_\gamma
\]

is a group homomorphism. Pick a central extension \( R_q^{\circ +} \) of \( R_q^\circ \) and a central idempotent \( p_{\text{red}} \) such that \( C[R_q^\circ, \bar{z}] \cong p_{\text{red}} C[R_q^{\circ +}] \). Now the same argument as in the proof of [AMS2, Proposition 2.2] shows that the algebra

\[
(34) \quad C[R_q^\circ, \bar{z}] \rtimes \mathcal{H}(R, \lambda, \lambda^\ast, \bar{z}) \cong p_{\text{red}} C[R_q^{\circ +} \rtimes \mathcal{H}(R, \lambda, \lambda^\ast, \bar{z}) \subset R_q^{\circ +} \rtimes \mathcal{H}(R, \lambda, \lambda^\ast, \bar{z})
\]

has the required properties. \( \square \)

When \( R_q^\circ = 1 \), specializations of \( \mathcal{H}(R, \lambda, \lambda^\ast, \bar{z}) \) at \( \bar{r} = \bar{r}^\circ \in \mathbb{R}_0^d \) figure for example in [Opd1]. In relation with \( p_{\text{red}} \)-adic groups one should think of the variables \( \bar{z} \) as as \( (q_{j_1^{1/2}})_{j_1=1}^d \), where \( q_j \) is the cardinality of some finite field.

We define, for \( \alpha \in R(G_0, T)_{\text{red}} \):

\[
\begin{align*}
\lambda(\alpha) &= c(\alpha)/2 & 2\alpha &\not\in R(G_0, T) \\
\lambda^\ast(\alpha) &= c(\alpha)/2 & 2\alpha &\not\in R(G_0, T), \alpha \not\in 2X_\ast(T) \\
\lambda(\alpha) &= c(\alpha)/2 + c(2\alpha)/4 & 2\alpha &\in R(G_0, T) \\
\lambda^\ast(\alpha) &= c(\alpha)/2 - c(2\alpha)/4 & 2\alpha &\in R(G_0, T).
\end{align*}
\]

By Proposition 2.1, \( \lambda(\alpha) \in \mathbb{Z}_{\geq 0} \) in all cases.

For \( \bar{z} = \bar{z}^\circ \) [AMS2, (91)] says that

\[
C[R_q^\circ, z^\circ_q] \cong \text{End}_{P_{\text{red}} \text{Lie}(G)^{\text{RS}}} \left( q\pi^+(q^\circ_q) \right).
\]

We denote the algebra constructed in Proposition 2.1 with these extra data, by \( \mathcal{H}(G, M, q^\circ_q, \bar{z}) \). Since it is built from an affine Hecke algebra \( \mathcal{H}(R, \lambda, \lambda^\ast, \bar{z}) \) and a
twisted group algebra \( \mathbb{C}[W_{q\xi}, \bar{z}_{q\xi}] \), we refer to it as a twisted affine Hecke algebra. When \( d = 1 \) we simply write \( \mathcal{H}(G, M, q\xi) \). We record that

\[
\mathcal{H}(G, M, q\xi) = \mathcal{H}(G, M, q\xi, \bar{z})/\langle \{ z_i - z_j : 1 \leq i, j \leq d \} \rangle.
\]

The same argument as for [AMS2] Lemma 2.8 shows that

\[
\mathcal{H}(G, M, q\xi, \bar{z}) = \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \bar{z}) \times \text{End}_{\mathfrak{D}_{\text{res}}}^+(q\pi_\xi(q\bar{E})).
\]

If we are in one of the cases [9], then with this interpretation \( \mathcal{H}(G, M, q\xi, \bar{z}) \) depends canonically on \( (G, M, q\xi) \). In general the algebra \( \mathcal{H}(G, M, q\xi, \bar{z}) \) is not entirely canonical, since it involves the choice of a decomposition [9].

**Lemma 2.3.** \( O(T \times (\mathbb{C}^*)^d)_{W_{q\xi}} = O(T)_{W_{q\xi}} \otimes \mathbb{C}[\bar{z}, \bar{z}^{-1}] \) is a central subalgebra of \( \mathcal{H}(G, M, q\xi, \bar{z}) \). It equals \( Z(\mathcal{H}(G, M, q\xi, \bar{z})) \) if \( W_{q\xi} \) acts faithfully on \( T \).

**Proof.** The case \( W_{q\xi} = 1, d = 1 \) is [Lus3] Proposition 3.11. The general case from readily from that, as observed in [Sol3] §1.2].

For \( \zeta \in Z(G) \cap G^o \) and \( (\pi, V) \in \text{Mod}(\mathcal{H}(G, M, q\xi, \bar{z})) \) we define \( (\zeta \otimes \pi, V) \in \text{Mod}(\mathcal{H}(G, M, q\xi, \bar{z})) \) by

\[
(\zeta \otimes \pi)(f_1 f_2 N_w) = f_1(\zeta \pi)(f_1 f_2 N_w) \quad f_1 \in O(T), f_2 \in \mathbb{C}[\bar{z}, \bar{z}^{-1}], w \in W_{q\xi}.
\]

### 2.1. Reduction to real central character.

Let \( T = T_{\text{un}} \times T_{\text{rs}} \) be the polar decomposition of the complex torus \( T \), in a unitary and a real split part:

\[
T_{\text{un}} = \text{Hom}(X^*(T), S^1) = \exp(it_{\mathbb{R}}),
\]

\[
T_{\text{rs}} = \text{Hom}(X^*(T), \mathbb{R}_{>0}) = \exp(t_{\mathbb{R}}).
\]

Let \( t = (t|t|^{-1})|t| \in T_{\text{un}} \times T_{\text{rs}} \) denote the polar decomposition of an arbitrary element \( t \in T \).

By Lemma 2.3 every irreducible representation of \( \mathcal{H}(G, M, q\xi, \bar{z}) \) admits a \( O(T \times (\mathbb{C}^*)^d)_{W_{q\xi}} \)-character, an element of \( T/W_{q\xi} \otimes (\mathbb{C}^*)^d \). We will refer to this as the central character. Following [BaMo] Definition 2.2 we say that a central character \( (W_{q\xi}, \bar{z}) \) is "real" if \( \bar{z} \in \mathbb{R}^d \) and the unitary part \( t|t|^{-1} \) is fixed by \( W_{q\xi}^0 \).

For \( t \in T \) we define \( \tilde{Z}_G(t) \) to be the subgroup of \( G \) generated by \( Z_G(t) \) and the root subgroups for \( \alpha \in R(G^o, T) \) with \( \alpha^\vee \in 2X_+(T) \) and \( \alpha(t) = -1 \). Thus \( R(\tilde{Z}_G(t)^o, T) \) consists of the roots \( \alpha \in R(G^o, T) \) with \( s_\alpha(t) = t \). The analogue of \( \mathfrak{f}_{q\xi} \) for \( \tilde{Z}_G(t) \) is \( \mathfrak{f}_{q\xi,t} \), the stabilizer of \( R(\tilde{Z}_G(t)^o, T) \cap R(P, T) \) in \( W_{q\xi,t} \).

Our first reduction theorem will relate modules of \( \mathcal{H}(G, M, q\xi, \bar{z}) \) and of \( \mathcal{H}(\tilde{Z}_G(t), M, q\xi, \bar{z}) \). Assuming that every \( z_j \) acts via a positive real number, we end up with representations admitting a real central character. To describe the effect on \( O(T \times (\mathbb{C}^*)^d) \)-weights, we need some preparations. Consider the set

\[
W_{q\xi}^o = \{ w \in W_{q\xi} : w(R(\tilde{Z}_G(t)^o, T) \cap R(P, T)) \subset R(P, T) \}.
\]

Recall that the parabolic subgroup \( P \subset G^o \) determines a set of simple reflections and a length function on the Weyl group \( W_{q\xi}^o = W_{q\xi} \). We use this to define two cones in \( t_{\mathbb{R}} = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R} \):

\[
t_{\mathbb{R}}^+ := \{ x \in t_{\mathbb{R}} : \langle x, \alpha \rangle \geq 0 \forall \alpha \in R(P, T) \},
\]

\[
t_{\mathbb{R}}^- := \{ \sum_{\alpha \in R(P, T)} x_\alpha \alpha^\vee : x_\alpha \leq 0 \}.
\]
**Lemma 2.4.** (a) $W^+_E$ is the unique set of shortest length representatives for $W_E/W(\tilde{Z}_G(t)^0, T)$ in $W_E$.

(b) $\bigcup_{w\in W^+_E} w w^{-1} t^+_{t_R}$ equals $t^+_{t_R}$, the analogue of $t^+_{t_R}$ for the group $\tilde{Z}_G(t)^0$. The same holds for $t^-_{t_R}$.

(c) $\{x \in t_R : W^+_E x \subseteq t^-_{t_R}\}$ equals $t^-_{t_R}$, the analogue of $t^-_{t_R}$ for $\tilde{Z}_G(t)^0$.

**Proof.** (a) This well-known when section 1.10.c and \S 2.5. By [Hum, Theorem 1.8] there exists a unique $v$.

(b) Suppose that $x$.

(c) The definition of $v$.

Hence $W^+_E$ is a set of representatives for $W_E/W(\tilde{Z}_G(t)^0, T)$.

Consider $w \in W_E$ of minimal length in $w W(\tilde{Z}_G(t)^0, T)$. By [Hum] Proposition 5.7, $w(\alpha) \in R(P, T)$ for all $\alpha \in R(\tilde{Z}_G(t)^0, T) \cap R(P, T)$, so $w \in W^+_E$. We deduce that every left coset of $W(\tilde{Z}_G(t)^0, T)$ contains a unique element of minimal length, namely its representative in $W^+_E$.

(b) Suppose that $x \in t^+_R$ and $\alpha \in R(\tilde{Z}_G(t)^0, T) \cap R(P, T)$. For all $w \in W^+_E$ we have $w \alpha \in R(P, T)$, so

$$\langle \alpha, w^{-1} x \rangle = \langle w \alpha, x \rangle \geq 0.$$

Hence $\bigcup_{w \in W^+_E} w^{-1} t^+_{t_R} \subseteq t^+_{t_R}$. Let $S$ be a sphere in $t_R$ centred in 0. Then

$$\text{vol}(S)/\text{vol}(S \cap t^+_{t_R}) = |W_E| \quad \text{and} \quad \text{vol}(S)/\text{vol}(S \cap t^+_{t_R}) = |W(\tilde{Z}_G(t)^0, T)|.$$

With part (a) it follows that

$$|W^+_E| \text{vol}(S \cap t^+_{t_R}) = |W_E| \text{vol}(S \cap t^+_{t_R})/|W(\tilde{Z}_G(t)^0, T)| = \text{vol}(S \cap t^+_{t_R}).$$

Since $t^+_{t_R}$ is a Weyl chamber for $W_E$, the translates $wt^+_{t_R}$ intersect $t^+_{t_R}$ only in a set of measure zero. Hence the left hand side of (39) is the volume of $S \cap \bigcup_{w \in W^+_E} w^{-1} t^+_{t_R}$. As $\bigcup_{w \in W^+_E} w^{-1} t^+_{t_R}$ contains $t^+_{t_R}$ and both are cones defined by linear equations coming from roots, the equality (39) shows that they coincide.

The same reasoning applies to $t^-_{t_R}$ and the dual root systems.

(c) The definition of $W^+_E$ entails $W^+_E t^-_{t_R} \subseteq t^-_{t_R}$. Conversely, suppose that $x \in t_R$ and that $W^+_E x \subseteq t^-_{t_R}$. For every $w \in W^+_E$ and every $\lambda \in t^+_{t_R}$:

$$\langle x, w^{-1} \lambda \rangle = \langle w x, \lambda \rangle \leq 0.$$

In view of part (b) for $t^+_{t_R}$, this means that $x \in t^-_{t_R}$.

**Theorem 2.5.** Let $t \in T_{un}$.

(a) There is a canonical equivalence between the following categories:

- finite dimensional $\mathcal{H}(\tilde{Z}_G(t), M, qE, \mathbb{Z})$-modules with $O(T \times (\mathbb{C}^\times)^d)$-weights in $t_{t_R} \times \mathbb{R}^d_{>0}$;
- finite dimensional $\mathcal{H}(G, M, qE, \mathbb{Z})$-modules with $O(T \times (\mathbb{C}^\times)^d)$-weights in $W_qET_{t_R} \times \mathbb{R}^d_{>0}$. 

□
It is given by localization of the centre and induction, and we denote it (suggestionly) by $\text{ind}^G_{\mathcal{H}(G,M,q\mathcal{E},\mathcal{Z})}$.

(b) The above equivalences are compatible with parabolic induction, in the following sense. Let $Q \subset G$ be an algebraic subgroup such that $Q \cap G^0$ is a Levi subgroup of $G^0$ and $Q \supset M$. Then

$$\text{ind}^G_{\mathcal{H}(G,M,q\mathcal{E},\mathcal{Z})} \circ \text{ind}^G_{\mathcal{H}(Z_G(t),M,q\mathcal{E},\mathcal{Z})} = \text{ind}^G_{\mathcal{H}(G,M,q\mathcal{E},\mathcal{Z})} \circ \text{ind}^G_{\mathcal{H}(Z_Q(t),M,q\mathcal{E},\mathcal{Z})}$$

(c) The set of $\mathcal{O}(T \times (\mathbb{C}^*)^d)$-weights of $\text{ind}^G_{\mathcal{H}(G,M,q\mathcal{E},\mathcal{Z})}(V)$ is

$$\{ (wx,\mathcal{Z}) : w \in \mathfrak{r}_{q\mathcal{E}}W_{q\mathcal{E}}^{\circ}, (x,\mathcal{Z}) \text{ is a } \mathcal{O}(T \times (\mathbb{C}^*)^d)\text{-weight of } V \}.$$ 

Proof. (a) The case $d = 1$, $\mathfrak{r}_{q\mathcal{E}} = 1$ was proven in [Lus3] Theorem 8.6.

Let $\mathfrak{r}_{q\mathcal{E}}^+ \to \mathfrak{r}_{q\mathcal{E}}$ be a central extension as in (34). Extend it trivially to a central extension $\mathfrak{r}_{q\mathcal{E}}^+W_{q\mathcal{E}}^0 \to W_{q\mathcal{E}}$ and let $\mathfrak{r}_{q\mathcal{E},t}^+$ be the inverse image of $\mathfrak{r}_{q\mathcal{E},t}^+ \subset W_{q\mathcal{E},t}$ in $\mathfrak{r}_{q\mathcal{E}}^+W_{q\mathcal{E}}^0$. Then

$$\mathcal{H}(G,M,q\mathcal{E},\mathcal{Z}) = \mathcal{H}(G^0,M^0,\mathcal{E},\mathcal{Z}) \times p_\mathcal{Z} \mathbb{C}[\mathfrak{r}_{q\mathcal{E}}^+]$$

As $p_\mathcal{Z} \subset \mathbb{C}[\ker(\mathfrak{r}_{q\mathcal{E}}^+ \to \mathfrak{r}_{q\mathcal{E}})]$ is a central idempotent, we may just as well establish the analogous result for the algebras

$$\mathcal{H}(G^0,M^0,\mathcal{E},\mathcal{Z}) \times \mathfrak{r}_{q\mathcal{E},t}^+ \text{ and } \mathcal{H}(Z_{G^0}(t),M^0,\mathcal{E},\mathcal{Z}) \times \mathfrak{r}_{q\mathcal{E},t}^+.$$ 

Since we are dealing with finite dimensional representations only, we can decompose them according to the (generalized) weights for the action of the centre. Fix $(x,\mathcal{Z}) \in T_{ir} \times \mathbb{R}^d$. Denote the category of finite dimensional $\mathcal{A}$-modules with weights in $U$ by $\text{Mod}_{f,U}(\mathcal{A})$. We compare the categories

$$\text{Mod}_{f,W_{q\mathcal{E},t},\mathfrak{A} \times \{ \mathcal{Z} \}}(\mathcal{H}(Z_{G^0}(t),M^0,\mathcal{E},\mathcal{Z}) \times \mathfrak{r}_{q\mathcal{E},t}^+),$$

The most appropriate technique to handle the general case is analytic localization, as in [Opd1] §4 (but there with fixed parameters $z_1, \ldots, z_d$). For a submanifold $U \subset T \times (\mathbb{C}^*)^d$, let $\mathcal{C}^\text{an}(U)$ be the algebra of complex analytic functions on $U$. We assume that $U$ is $W_{q\mathcal{E}}$-stable and Zariski-dense. Then the restriction map $\mathcal{O}(T \times (\mathbb{C}^*)^d) \to \mathcal{C}^\text{an}(U)$ is injective, and we can form the algebra

$$\mathcal{H}^\text{an}(U) := \mathcal{C}^\text{an}(U)W_{q\mathcal{E}} \otimes_{\mathcal{O}(T \times (\mathbb{C}^*)^d)W_{q\mathcal{E}}} \mathcal{H}(G^0,M^0,\mathcal{E},\mathcal{Z}) \times \mathfrak{r}_{q\mathcal{E},t}^+.$$ 

As observed in [Opd1] Proposition 4.3], the finite dimensional modules of $\mathcal{H}^\text{an}(U)$ can be identified with the finite dimensional modules of $\mathcal{H}(G^0,M^0,\mathcal{E},\mathcal{Z}) \times \mathfrak{r}_{q\mathcal{E},t}^+$ with $\mathcal{O}(T \times (\mathbb{C}^*)^d)$-weights in $U$.

In [Sol3] Conditions 2.1 it is described how one can find an open neighborhood $U_0 \subset T \times (\mathbb{C}^*)^d$ of $(x,\mathcal{Z})$, which is so small that localization to $U_0$ is more or less equivalent to localization at $(x,\mathcal{Z})$. We take $U = W_{q\mathcal{E}}U_0$ and $\tilde{U} = W_{q\mathcal{E},t}U_0$. By Lusztig’s first reduction theorem, in the version [Sol3] Theorem 2.1.2), there is a natural inclusion of

$$\mathcal{H}^\text{an}_t(\tilde{U}) := \mathcal{C}^\text{an}(\tilde{U})W_{q\mathcal{E},t} \otimes_{\mathcal{O}(T \times (\mathbb{C}^*)^d)W_{q\mathcal{E},t}} \mathcal{H}(Z_{G^0}(t),M^0,\mathcal{E},\mathcal{Z}) \times \mathfrak{r}_{q\mathcal{E},t}^+,$$
in $\mathcal{H}^\infty(U)$, which moreover is a Morita equivalence. Hence the composed functor
\[
\text{ind}_{\mathcal{H}(G^\circ, M^\circ, \mathcal{E}, \bar{z})}^{\mathcal{H}(Z_{G^\circ}(t), M^\circ, \mathcal{E}, \bar{z}) \times \mathcal{R}_{q,t}^+} \cdot \text{Mod}_{f,U} \left( \mathcal{H}(Z_{G^\circ}(t), M^\circ, \mathcal{E}, \bar{z}) \times \mathcal{R}_{q,t}^+ \right) \to \text{Mod}_{f} \left( \mathcal{H}(G^\circ, M^\circ, \mathcal{E}, \bar{z}) \times \mathcal{R}_{q,t}^+ \right)
\]
is an equivalence of categories. We specialize this at $W_{q,t}x \times \{z\} \subset U$ and we restrict to modules on which $p_z$ acts as the identity. Via (41) and (40) this gives the required equivalence of categories $\text{ind}_{\mathcal{H}(Z_{G^\circ}(t), M^\circ, \mathcal{E}, \bar{z})}^{\mathcal{H}(G^\circ, M^\circ, \mathcal{E}, \bar{z})} \to \text{ind}_{\mathcal{H}(Z_{G^\circ}(t), M^\circ, \mathcal{E}, \bar{z})}^{\mathcal{H}(G^\circ, M^\circ, \mathcal{E}, \bar{z})}$.

(b) We just showed that the above functor is really induction between localizations of the indicated algebras. Similar remarks apply to the functor $\text{ind}_{\mathcal{H}(G, M^\circ, \mathcal{E}, \bar{z})}^{\mathcal{H}(G, M^\circ, \mathcal{E}, \bar{z})}$. Thus the acclaimed compatibility with parabolic induction is just an instance of the transitivity of induction.

(c) Lemma 2.4 and the constructions in [Sol3] §2.1 entail that
\[
\text{ind}_{\mathcal{H}(Z_{G^\circ}(t), M, q\mathcal{E}, \bar{z})}^{\mathcal{H}(G^\circ, M, q\mathcal{E}, \bar{z})} (V) \cong \mathbb{C}[\mathcal{R}_{q,t} W^t_{\mathcal{E}}, \bar{z}] \otimes \mathbb{C}[\mathcal{R}_{q,t}, \bar{z}] \quad V
\]
as $\mathcal{O}(T \times (\mathbb{C}^\circ)^d)$-modules. Notice that the group $\mathcal{R}_{q,t}$ acts from the right on $\mathcal{R}_{q,t} W^t_{\mathcal{E}}$, because it stabilizes $R(Z_{G^\circ}(t)^0, T) \cap R(P, T)$. Since
\[
\mathcal{H}(Z_{G^\circ}(t), M, q\mathcal{E}, \bar{z}) \cong \mathcal{H}(Z_{G^\circ}(t)^0, M, M, q\mathcal{E}, \bar{z}) \times \mathbb{C}[\mathcal{R}_{q,t}, \bar{z}]
\]
the $\mathcal{O}(T \times (\mathbb{C}^\circ)^d)$-weights of $V$ come in full $\mathcal{R}_{q,t}$-orbits. It was observed in the proof of [Opd1] Proposition 4.20] that the $\mathcal{O}(T \times (\mathbb{C}^\circ)^d)$-weights of $\mathbb{C}w \otimes V$ ($w \in W^t_{\mathcal{E}}$) are precisely $(wx, \bar{z})$ with $(x, \bar{z})$ a $\mathcal{O}(T \times (\mathbb{C}^\circ)^d)$-weight of $V$. Multiplication by $N_\gamma$ ($\gamma \in \mathcal{R}_{q,t}$) just changes a weight $(x, \bar{z})$ to $(\gamma x, \bar{z})$. These observations and (43) prove that the $\mathcal{O}(T \times (\mathbb{C}^\circ)^d)$-weights of $\text{ind}_{\mathcal{H}(Z_{G^\circ}(t), M, q\mathcal{E}, \bar{z})}^{\mathcal{H}(G^\circ, M, q\mathcal{E}, \bar{z})} (V)$ are as stated.

In our reduction process we would like to preserve the analytic properties from [AMS2] §3.5. Just as in [AMS2] (79)], we can define $\mathcal{O}(T)$-weights for modules of affine Hecke algebras or extended versions such as $\mathcal{H}(Z_{G^\circ}(t), M, q\mathcal{E}, \bar{z})$. We denote the set of $\mathcal{O}(T)$-weights of a module $V$ for such an algebra by $W(t)(V)$. We can apply the polar decomposition (38) to it, which gives a set $\{W(t)(V)\} \subset T_{\text{rs}}$.

Let us recall the definitions of temperedness and discrete series from [Opd1] §2.7.

Definition 2.6. Let $V$ be a finite dimensional $\mathcal{H}(G, M, q\mathcal{E}, \bar{z})$-module. We say that $V$ is tempered (respectively anti-tempered) if $|W(t)(V)| \subset \exp(t^-_{\mathcal{R}})$, respectively $\subset \exp(-t^-_{\mathcal{R}})$.

Let $t^-_{\mathcal{R}}$ be the interior of $t_{\mathcal{R}}$ in $t_{\mathcal{R}}$. We call $V$ discrete series (resp. anti-discrete series) if $|W(t)(V)| \subset \exp(t^-_{\mathcal{R}})$, respectively $\subset \exp(-t^-_{\mathcal{R}})$. The module $V$ is essentially discrete series if its restriction to $\mathcal{H}(G/Z(G)^0 \circ, M/Z(G)^0 \circ, q\mathcal{E}, \bar{z})$ is discrete series, or equivalently if $|W(t)(V)| \subset \exp(Z(g) \oplus t^-_{\mathcal{R}})$.

The next result fills a gap in [Sol3] Theorem 2.3.1], where it was used between the lines. Similar results, for $G^\circ_{\text{det}}$ only and with somewhat different notions of temperedness and discrete series, were proven in [Lus7] Lemmas 3.4 and 3.5.

Proposition 2.7. The equivalence from Theorem 2.5 a, and its inverse, preserve:
(a) (anti-)temperedness,
(b) the discrete series.
(c) The $\mathcal{H}(\tilde{Z}_G(t), M, qE, \tilde{z})$-module $\text{ind}^{H(G, M, qE, \tilde{z})}_{\mathcal{H}(\tilde{Z}_G(t), M, qE, \tilde{z})}(V)$ is essentially discrete series if and only if $V$ is essentially discrete series and $R(\tilde{Z}_G(t)^0, T)$ has full rank in $R(G^0, T)$.

Remark 2.8. The extra condition for essentially discrete series representations is necessary, for the centre of $\tilde{Z}_G(t)^0$ can be of higher dimension than that of $G^0$.

Proof. Let $V$ be a finite dimensional $\mathcal{H}(\tilde{Z}_G(t), M, qE, \tilde{z})$-module with $\mathcal{O}(T \times (\mathbb{C}^\times)^d)$-weights in $tT_{ts} \times \mathbb{R}_d$.

(a) The $\mathcal{O}(T)$-weights of $\text{ind}^{H(G, M, qE, \tilde{z})}_{\mathcal{H}(\tilde{Z}_G(t), M, qE, \tilde{z})}(V)$ were given in Theorem 2.5(c). As $\log = \exp^{-1} : T_{ts} \to tR$ is $W_qE$-equivariant, it entails that

$$\log |\text{Wt}(\text{ind}^{H(G, M, qE, \tilde{z})}_{\mathcal{H}(\tilde{Z}_G(t), M, qE, \tilde{z})}(V))| = \mathfrak{t}_{qE}W_q^t \log |\text{Wt}(V)|.$$  

Recall from Lemma 2.4(c) that

$$t_{\mathbb{R}}^{-t} = \{ x \in tR : W_{\mathbb{R}}^t x \subset t_{\mathbb{R}} \} = \{ x \in tR : \mathfrak{t}_{qE}W_q^t x \subset t_{\mathbb{R}} \}.$$  

Comparing these with the definition of (anti-)temperedness for $G$ and for $\tilde{Z}_G(t)$, we see that $V$ is (anti-)tempered if and only if $\text{ind}^{H(G, M, qE, \tilde{z})}_{\mathcal{H}(\tilde{Z}_G(t), M, qE, \tilde{z})}(V)$ is so.

(b) We have to assume that $Z(G^0)$ is finite, for otherwise $\exp(t_{\mathbb{R}}^{-t})$ is empty and there are no discrete series representations on any side of the equivalences.

Suppose that $V$ is discrete series. Then $\tilde{Z}_G(t)^0$ is semisimple, so $R(\tilde{Z}_G(t)^0, T)$ is of full rank in $R(G^0, T)$. This implies that $t_{\mathbb{R}}^{-t}$ is an open subset of $t_{\mathbb{R}}^{-t}$. The same argument as for part (a) shows that $\text{ind}^{H(G, M, qE, \tilde{z})}_{\mathcal{H}(\tilde{Z}_G(t), M, qE, \tilde{z})}(V)$ is discrete series.

Conversely, suppose that $\text{ind}^{H(G, M, qE, \tilde{z})}_{\mathcal{H}(\tilde{Z}_G(t), M, qE, \tilde{z})}(V)$ is discrete series. It is tempered, so $V$ is tempered and $|\text{Wt}(V)| \subset \exp(t_{\mathbb{R}}^{-t})$. Suppose that $\tilde{Z}_G(t)^0$ is not semisimple. Then

$$t_{\mathbb{Z}} := \text{Lie}(Z(\tilde{Z}_G(t)^0)) = \bigcap_{\alpha \in R(\tilde{Z}_G(t)^0, T)} \ker \alpha$$

has positive dimension. In particular $t_{\mathbb{Z}}$ contains nonzero elements $\lambda \in t_{\mathbb{R}}^{t_{\mathbb{R}}} + t_{\mathbb{R}}$, for example the sum of the fundamental weights for simple roots not in $R(\tilde{Z}_G(t)^0, T)$. Let $t' \in T$ be any weight of $V$. Then $\log |t'| \in t_{\mathbb{R}}^{t_{\mathbb{R}}} \subset \text{Lie}(\tilde{Z}_G(t)^0)$. Hence $\langle \log |t'|, \lambda \rangle = 0$, which means that $\log |t'| \in t_{\mathbb{R}} \setminus t_{\mathbb{R}}^{-t}$. But $t'$ is also a weight of $\text{ind}^{H(G, M, qE, \tilde{z})}_{\mathcal{H}(\tilde{Z}_G(t), M, qE)}(V)$, and that is a discrete series representation, so $\log |t'| \in t_{\mathbb{R}}^{-t}$. This contraction shows that $\tilde{Z}_G(t)^0$ is semisimple.

Suppose now that $\log |t'|$ does not lie in the interior of $t_{\mathbb{R}}^{-t}$. Then it is orthogonal to a nonzero element $\lambda'$ in the boundary of $t_{\mathbb{R}}^{t_{\mathbb{R}}} + t_{\mathbb{R}}$. By Lemma 2.4(b) we can choose a $w \in W_{\mathbb{R}}^t$ such that $w\lambda' \in t_{\mathbb{R}}^{t_{\mathbb{R}}}$. Theorem 2.5(c) $w^t$ is a weight of $\text{ind}^{H(G, M, qE, \tilde{z})}_{\mathcal{H}(\tilde{Z}_G(t), M, qE, \tilde{z})}(V)$, and it satisfies

$$\langle \log |w^t|, \lambda' \rangle = \langle \log |t'|, \lambda' \rangle = 0.$$  

This shows that $\log |w^t| \notin t_{\mathbb{R}}^{-t}$, which contradicts that $\text{ind}^{H(G, M, qE, \tilde{z})}_{\mathcal{H}(\tilde{Z}_G(t), M, qE, \tilde{z})}(V)$ is discrete series. Therefore $\log |t'|$ belongs to $t_{\mathbb{R}}^{-t}$. As $t'$ was an arbitrary weight of $V$, this proves that $V$ is discrete series.

(c) Suppose that $\text{ind}^{H(G, M, qE, \tilde{z})}_{\mathcal{H}(\tilde{Z}_G(t), M, qE, \tilde{z})}(V)$ is essentially discrete series. Its restriction to
\( \mathcal{H}(G/Z(G^o), M/Z(G^o), qE, \overline{Z}) \) is discrete series, so by what we have just proven \( V \) is discrete series as a module for \( \mathcal{H}(ZG/Z(G^o)t, M/Z(G^o), qE, \overline{Z}) \), and \( ZG/Z(G^o)t \) is semisimple. Then \( R(ZG(t)^o, T) \) has full rank in \( R(G^o, T) \) and the restriction of \( V \) to the smaller algebra \( \mathcal{H}(Z_G(t)^o / Z(G^o)^o, M/Z(G^o)^o, qE, \overline{Z}) \) is also discrete series, so \( V \) is essentially discrete series.

Conversely, suppose that \( V \) is essentially discrete series and that \( R(\tilde{Z}_G(t)^o, T) \) has full rank in \( R(G^o, T) \). The second assumption implies that \( Z(G^o)^o \) is also the connected centre of \( \tilde{Z}_G(t)^o \). The same argument as in the tempered and the discrete case shows that

\[
|\text{Wt}(\text{ind}_{H(\tilde{Z}_G(t), M, qE, \overline{Z})}^H(G, M, qE, \overline{Z}) V)| \subset \exp(t_{\mathbb{R}}^- \oplus Z(g)).
\]

This means that \( \text{ind}_{H(\tilde{Z}_G(t), M, qE, \overline{Z})}^H(G, M, qE, \overline{Z}) (V) \) is essentially discrete series. \qed

Suppose that \( t' \in W_{qE}t \). Then we can apply Theorem 2.5.a also with \( t' \) instead of \( t \), and that should give essentially the same equivalence of categories. We check this in a slightly more general setting, which covers all \( t' \in T \cap \text{Ad}(G)t \). (Recall that for \( g, h \in G \) we write \( \text{Ad}(g)(h) = ghg^{-1} \).) By [Lus4] §8.13.b and [AMS2] Proposition 3.5.a

\[
T \cap \text{Ad}(G)t \text{ equals } T \cap \text{Ad}(N_G(T))t \supset W_{qE}t.
\]

Let \( g \in N_G(M) = N_G(T) \), with image \( \bar{g} \) in \( N_G(M)/M \). Conjugation with \( g \) yields an algebra isomorphism

\[
\begin{align*}
\text{Ad}(g) : \mathcal{H}(\tilde{Z}_G(t), M, qE, \overline{Z}) &\to \mathcal{H}(\tilde{Z}_G(gtg^{-1}), M, \text{Ad}(g^{-1})^*qE, \overline{Z}), \\
\text{Ad}(g)(N_w) &= N_{gw\bar{g}^{-1}}, \quad \text{Ad}(g)\theta_x = \theta_{x\circ\text{Ad}(g^{-1})} = \theta_{g^{-1}x}, \quad \text{Ad}(g)\overline{z}_j = \overline{z}_j,
\end{align*}
\]

where \( w \in W_{qE} \) and \( x \in X^*(T) \). Notice that this depends only on \( g \) through its class in \( N_G(M)/M \).

**Lemma 2.9.** Let \( t \in T_{\text{un}} \) and \( g \in N_G(M) \). Then

\[
\text{ind}_{H(\tilde{Z}_G(t), M, qE, \overline{Z})}^H(G, M, qE, \overline{Z}) = \text{Ad}(g)^* \circ \text{ind}_{H(\tilde{Z}_G(gtg^{-1}), M, \text{Ad}(g^{-1})^*qE, \overline{Z})}^H(G, M, \text{Ad}(g^{-1})^*qE, \overline{Z}) \circ \text{Ad}(g^{-1})^*
\]

as functors between the appropriate categories of modules of these algebras (as specified in Theorem 2.5).\]

**Remark 2.10.** This result was used, but not proven, in [Lus5] §4.9 and §5.20 and [Sol3, Theorem 2.3.1].

**Proof.** Our argument for Theorem 2.5.a, with (34), shows how several relevant results can be extended from \( \mathcal{H}(G^o M, M, qE, \overline{Z}) \) to \( \mathcal{H}(G, M, qE, \overline{Z}) \). This justifies the below use of some results from [Lus3], which were formulated only for \( \mathcal{H}(G^o M, M, qE) \).

Let \( (\pi, V) \) be a finite dimensional \( \mathcal{H}(G, M, qE) \)-module with \( \mathcal{O}(T \times \mathbb{C}^x) \)-weights in \( W_{qE}tT_{\text{rs}} \times \mathbb{R}_{>0} \). In [Lus3] §8 \( V \) is decomposed canonically as \( \bigoplus_{t' \in W_{qE}t} V_{t'T_{\text{rs}}} \), where \( V_{t'T_{\text{rs}}} \) is the sum of all generalized \( \mathcal{O}(T) \)-weight spaces with weights in \( t'T_{\text{rs}} \). Then \( V_{t'T_{\text{rs}}} \) is a module for \( \mathcal{H}(Z_G(t'), M, qE) \) and

\[
V = \text{ind}_{H(Z_G(t'), M, qE, \overline{Z})}^H(G, M, qE, \overline{Z}) (V_{t'T_{\text{rs}}}).
\]
Assume that $g \in N_G(M, qE)$, so $\bar{g} \in W_qE$. Then $V_tT_n$ and $V_{tg^{-1}T_n}$ are related via multiplication with an element $\tau_{\bar{g}}$, which lives in a suitable localization of $\mathcal{H}(G, M, qE, \bar{z})$ [Lus3 §5]. We can rewrite the right hand side of (46) as

$$\text{ind}_{\mathcal{H}(\mathcal{Z}_G(t), M, qE, \bar{z})}^{\mathcal{H}(G, M, qE, \bar{z})} (\tau_{\bar{g}} V_{g^{-1}tgT_n}) = \tau_{\bar{g}} (\text{ind}_{\mathcal{H}(\mathcal{Z}_G(tg^{-1}), M, qE, \bar{z})}^{\mathcal{H}(G, M, qE, \bar{z})} (V_{g^{-1}tgT_n})).$$

From [Lus3 §8.8] and [Sol1, Lemma 4.2] we see that the effect of conjugation by $\tau_{\bar{g}}$ on $\mathcal{H}(G, M, qE, \bar{z})$ and $\mathcal{H}(\mathcal{Z}_G(t), M, qE, \bar{z})$ boils down to the algebra isomorphism (45). The right hand side of (47) becomes

$$\text{Ad}(g)^* \circ \text{ind}_{\mathcal{H}(\mathcal{Z}_G(tg^{-1}), M, qE, \bar{z})}^{\mathcal{H}(G, M, qE, \bar{z})} \circ \text{Ad}(g^{-1})^* \left(V_{tT_n}\right),$$

which proves the lemma for such $g$.

Now we consider a general $g \in N_G(M)$. We will analyse

$$\text{Ad}(g)^* \circ \text{ind}_{\mathcal{H}(G, M, Ad(g^{-1})^*qE, \bar{z})}^{\mathcal{H}(G, M, Ad(g^{-1})^*qE, \bar{z})} \circ \text{Ad}(g^{-1})^* \left(V_{tT_n}\right).$$

From the above we see that the underlying vector space is

$$\bigoplus_{w \in gW_qEg^{-1}/gW_qEt} \tau_{w}(\text{Ad}(g^{-1})^*V_{tT_n}) = \bigoplus_{w \in W_qE/W_qEt} \text{Ad}(g^{-1})^*\tau_{w}V_{tT_n} = \text{Ad}(g^{-1})^* V.$$

The action of $\mathcal{H}(\mathcal{Z}_G(tg^{-1}), M, Ad(g^{-1})^*qE, \bar{z}) = \text{Ad}(g)\mathcal{H}(\mathcal{Z}_G(t), M, qE, \bar{z})$ works out to

$$(\text{Ad}(g)h) \cdot (\text{Ad}(g^{-1})^*v) = \text{Ad}(g^{-1})^*(h \cdot v).$$

Thus (48) can be identified with $V$. □

2.2. Parametrization of irreducible representations.

Next we want to reduce from $\mathcal{H}(\mathcal{Z}_G(t), M, qE, \bar{z})$-modules to modules over $\mathbb{H}(G_t, M, qE, \bar{f})$. The exponential map for $T \times \mathbb{C}^d$ gives a $W_qE, t$-equivariant map

$$\exp_t : t \oplus \mathbb{C}^d \rightarrow T \times (\mathbb{C}^d)^d, \quad \exp_t(x, r_1, \ldots, r_d) = (t \exp(x), \exp r_1, \ldots, \exp r_d).$$

Notice that the restriction $\exp_t : t_R \oplus \mathbb{R}^d \rightarrow tT_n \times \mathbb{R}^d_{>0}$ is a diffeomorphism.

**Theorem 2.11.** Let $t \in T_{un}$.

(a) There is a canonical equivalence between the following categories:

- finite dimensional $\mathbb{H}(G_t, M, qE, \bar{f})$-modules with $O(t \oplus \mathbb{C}^d)$-weights in $t_R \oplus \mathbb{R}^d$;
- finite dimensional $\mathcal{H}(\mathcal{Z}_G(t), M, qE, \bar{z})$-modules with $O(T \times (\mathbb{C}^d)^d)$-weights in $tT_n \times \mathbb{R}^d_{>0}$.

It is given by localization with respect to central ideals in combination with the map $\exp_t$. We denote this equivalence by $(\exp_t)_*$. (b) The functor $(\exp_t)_*$ is compatible with parabolic induction, in the following sense. Let $Q \subset G$ be an algebraic subgroup such that $Q \cap G^0$ is a Levi subgroup of $G^0$ and $Q \supset M$. Then

$$\text{ind}_{\mathcal{H}(\mathcal{Z}_Q(t), M, qE, \bar{z})}^{\mathcal{H}(\mathcal{Z}_G(t), M, qE, \bar{z})} \circ (\exp_t)_* = (\exp_t)_* \circ \text{ind}_{\mathbb{H}(Q_t, M, qE, \bar{f})}^{\mathbb{H}(G_t, M, qE, \bar{f})}.$$  

(c) The functor $(\exp_t)_*$ preserves the underlying vector space of a representation, and it transforms a $S(t \oplus \mathbb{C}^d)$-weight $(x, \bar{r})$ into a $O(T \times (\mathbb{C}^d)^d)$-weight $\exp_t(x, \bar{r})$.

(d) The functors $(\exp_t)_*$ and $(\exp_t)^{-1}_*$ preserve (anti-)temperedness and (essentially) discrete series.
Proof. (a) The case \( d = 1, \mathfrak{R}_{q^E} = 1 \) was proven in [Lus3, Theorem 9.3].

For the general case we use the similar techniques and notations as in the proof of Theorem 2.5.a. By the same argument as over there, it suffices to compare the categories

\[
\text{Mod}_{f,W_{q^E,t x \times \{z\}}}(\mathcal{H}(\hat{Z}_G(t)^{\circ}, M^\circ, \mathcal{E}, \bar{z}) \times \mathfrak{M}^+_{q^E,t}),
\]

\[
\text{Mod}_{f,W_{q^E,t \log(x) \times \{\log(z)\}}}(\mathbb{H}(Z_G(t)^{\circ}, M^\circ, \mathcal{E}, \bar{r}) \times \mathfrak{M}^+_{q^E,t}).
\]

Recall from (35) that the parameter functions for these algebras are related by

\[
\begin{align*}
    c_t(\alpha) &= 2\lambda(\alpha) & 2\alpha &\notin R(\hat{Z}_G(t)^{\circ}, T), \\
    c_t(\alpha) &= \lambda(\alpha) + \lambda^*(\alpha) & 2\alpha &\in R(\hat{Z}_G(t)^{\circ}, T), \\
    c_t(2\alpha)/2 &= \lambda(\alpha) - \lambda^*(\alpha) & 2\alpha &\in R(\hat{Z}_G(t)^{\circ}, T).
\end{align*}
\]

Let us define \( k : R(\hat{Z}_G(t)^{\circ}, T)_{\text{red}} \to \mathbb{R} \) by

\[
\begin{align*}
    k(\alpha) &= 2\lambda(\alpha) & 2\alpha &\notin R(\hat{Z}_G(t)^{\circ}, T), \\
    k(\alpha) &= \lambda(\alpha) + \alpha(t)\lambda^*(\alpha) & 2\alpha &\in R(\hat{Z}_G(t)^{\circ}, T).
\end{align*}
\]

The only difference between \( \mathbb{H}(t, W(\hat{Z}_G(t)^{\circ}, T), k\bar{r}) \) and \( \mathbb{H}(Z_G(t)^{\circ}, M^\circ, \mathcal{E}, \bar{r}) \) arises from roots \( \alpha \in R(\hat{Z}_G(t)^{\circ}, T) \setminus R(Z_G(t)^{\circ}, T) \) with \( \alpha(t) = -1 \). The corresponding braid relations are

\[
\begin{align*}
    N_{s_{\alpha}}\xi - s_{\alpha}\xi N_{s_{\alpha}} &= (\lambda(\alpha) - \lambda^*(\alpha)) r_j\xi / \alpha, \\
    N_{s_{2\alpha}}\xi - s_{2\alpha}\xi N_{s_{2\alpha}} &= c_t(2\alpha) r_j\xi / (2\alpha),
\end{align*}
\]

Since \( s_{\alpha} = s_{2\alpha} \) and \( c_t(2\alpha) = 2(\lambda(\alpha) - \lambda^*(\alpha)) \), these two braid relations are equivalent, and we may identify

\[
\mathbb{H}(t, W(\hat{Z}_G(t)^{\circ}, T), k\bar{r}) \times \mathfrak{M}^+_{q^E,t} = \mathbb{H}(Z_G(t)^{\circ}, M^\circ, \mathcal{E}, \bar{r}) \times \mathfrak{M}^+_{q^E,t}.
\]

Let \( V \subset t \times \mathbb{C}^d \) be a \( W_{q^E,t} \)-stable, Zariski-dense submanifold. Like in (42), we can form the algebra

\[
\mathbb{H}^n_t(V) := C^{an}(V)^{W_{q^E,t}} \otimes_{O(t \oplus \mathbb{C}^d)^{W_{q^E,t}}} \mathbb{H}(t, W(\hat{Z}_G(t)^{\circ}, T), k\bar{r}) \times \mathfrak{M}^+_{q^E,t}.
\]

The argument for [Opd1, Proposition 4.3] shows that its finite dimensional modules are precisely the finite dimensional \( \mathbb{H}(t, W(\hat{Z}_G(t)^{\circ}, T), k\bar{r}) \times \mathfrak{M}^+_{q^E,t} \)-modules with \( O(t \oplus \mathbb{C}^d) \)-weights in \( V \). If \( \exp_{q^E,t} \) is injective on \( V \), it induces an algebra isomorphism

\[
\exp_{q^E,t}^* : C^{an}(\exp_{q^E,t}(V))^{W_{q^E,t}} \to C^{an}(V)^{W_{q^E,t}}.
\]

We suppose in addition that \( V \) is contained in a sufficiently small open neighborhood of \( t_{\mathbb{R}} \oplus \mathbb{R}^d \). In view of the relations between the parameters (50) and (51), we can apply [Sol3, Theorem 2.1.4.b]. It shows that (53) extends to an isomorphism of \( C^{an}(V)^{W_{q^E,t}} \)-algebras

\[
\Phi_t : C^{an}(\exp_{q^E,t}(V))^{W_{q^E,t}} \otimes_{O(T \times (\mathbb{C}^d)^{W_{q^E,t}}} \mathcal{H}(\hat{Z}_G(t)^{\circ}, M^\circ, \mathcal{E}, \bar{z}) \times \mathfrak{M}^+_{q^E,t} \to \mathbb{H}^n_t(V),
\]

which is the identity on \( \mathbb{C}[\mathfrak{M}^+_{q^E,t}] \).

Choosing for \( V \) a small neighborhood of \( W_{q^E,t} \log(x) \times \{\log(z)\} \) in \( t \oplus \mathbb{C}^d \), \( \Phi_t \) induces an equivalence between the categories of modules with weights in, respectively, \( W_{q^E,t} t x \times \{z\} \) and \( W_{q^E,t} \log(x) \times \{\log(z)\} \). In view of [Opd1, Proposition 4.3] and (52), this provides the equivalence between the categories (49).
Since $\Phi_t$ fixes $p_\gamma \in \mathbb{C}[\mathfrak{g}_q^{+,t}]$, we can restrict that equivalence to modules on which $p_\gamma$ acts as the identity.

(b) For $G^\circ$ this is shown in [BaMo] Theorem 6.2 and [Sol2] Proposition 6.4. Extending $G^\circ$ to a disconnected group boils down to extending the involved algebras by $\mathbb{C}[\mathfrak{g}_q^{+,t}, \mathfrak{z}_q^e]$ or $\mathbb{C}[\mathfrak{g}_Q, \mathfrak{z}_q^e]$. As we noted in proof of part (a), the algebra homomorphism $\Phi_t$ used to define $(\exp_t)_*$ is the identity on $\mathbb{C}[\mathfrak{g}_q^{+,t}, \mathfrak{z}_q^e] \subset \mathbb{C}[\mathfrak{g}_q^{+,t}]$. Hence this extension works the same on both sides of the equivalence, and the argument given in [Sol2] §6 generalizes to the current setting.

(c) By construction [Sol3] §2.1 $(\exp_t)_* \pi = \pi \circ \exp_t^*$ as $\mathcal{O}(T \times (C^\times)^d)$-representations.

(For $f \in \mathcal{O}(T \times (C^\times)^d)$ the action of $f \circ \exp_t$ on the vector space underlying $\pi$ is defined via a suitable localization.) This immediately implies that $(\exp_t)_*$ has the effect of $\exp_t$ on weights.

(d) This result generalizes the observations made in [Slo] (2.11). Let $V$ be a finite dimensional $\mathcal{H}(\tilde{G}_t(t), M, q\mathcal{E})$-module with $\mathcal{O}(T \times (C^\times)^d)$-weights in $tT_{rs} \times \mathbb{R}_{>0}$. By part (b)

$$Wt((\exp_t)_*^{-1}V) = \exp_t^{-1}(Wt(V)) \subset tR.$$ 

By assumption $t \in T_{un}$, so we get

$$|Wt(V)| = \exp \left( R \left( Wt((\exp_t)_*^{-1}V) \right) \right).$$

Comparing [AMS2] Definition 3.24] and Definition 2.6 we see that $(\exp_t)_*$ and $(\exp_t)_*^{-1}$ preserve (anti-)temperedness and the discrete series. With [AMS2] Definition 3.27 we see that "essentially discrete series" is also respected.

Theorems 2.5 and 2.11 together provide an equivalence between $\mathbb{H}(G_t, M, q\mathcal{E}, \vec{r})$-modules with central character in $tR/W_{\mathfrak{q}_e,t} \times \mathbb{R}_0$ and $\mathcal{H}(G, M, q\mathcal{E}, \vec{z})$-modules with central character in $W_{\mathfrak{q}_e}tT_{rs}/W_{\mathfrak{q}_e} \times \mathbb{R}_0$, where $t \in T_{un}$.

Recall from [AMS2] Corollary 3.23 and Theorem 1.4 that we can parametrize $\text{Irr}_F(\mathbb{H}(G_t, M, q\mathcal{E}, \vec{r}))$ with $N_{G_t}(M)/M$-orbits of triples $(\sigma_0, \mathcal{C}, \mathcal{F})$, where $\sigma_0 \in t, \mathcal{C}$ is a nilpotent $Z_{G_t}(\sigma_0)$-orbit in $Z_0(\sigma_0)$ and $\mathcal{F}$ is an irreducible $Z_{G_t}(\sigma_0)$-equivariant local system on $\mathcal{C}$ such that $\Psi_{Z_{G_t}(\sigma_0)}(\mathcal{C}, \mathcal{F}) = (M, \mathcal{C}_v, q\mathcal{E})$, up to $Z_{G_t}(\sigma_0)$-conjugacy.

To find all irreducible representations with $S(t^*)_{W_{\mathfrak{e}}}^\phi$-character in $tR$ (those are all we need for the relation with affine Hecke algebras) it suffices to consider such triples $(\sigma_0, \mathcal{C}, \mathcal{F})$ with $\sigma_0 \in tR$. To phrase things more directly in terms of the group $G$, we allow $t$ to vary in $T_{un}$ and we replace $\sigma_0$ by $t' = t \exp(\sigma_0) \in tT_{rs}$. In other words, we consider triples $(t', \mathcal{C}, \mathcal{F})$ such that:

- $t' \in T$ with unitary part $t = t'|t'|^{-1}$;
- $\mathcal{C}$ is a nilpotent $Z_{G}(t')$-orbit in $Z_0(t') = \text{Lie}(G_{t'})$;
- $\mathcal{F}$ is an irreducible $Z_{G}(t')$-equivariant local system on $\mathcal{C}$ with $q\Psi_{Z_{G}(t')} (\mathcal{C}, \mathcal{F}) = (M, \mathcal{C}_v, q\mathcal{E})$, up to $Z_{G}(t')$-conjugacy.

To such a triple we can associate the standard $\mathbb{H}(G_t, M, q\mathcal{E}, \vec{r})$-modules

$$E_{y, \text{log } |t'|+d\gamma \left( \begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix} \right), \vec{r}, \rho} \quad \text{and} \quad IM^*E_{y, -\text{log } |t'|-d\gamma \left( \begin{smallmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{smallmatrix} \right), \vec{r}, \rho},$$

where $y \in \mathcal{C}$ and $\rho$ is the representation of $\pi_0(Z_{G}(t', y))$ on $\mathcal{F}_y$. Furthermore $\gamma : \text{SL}_2(\mathcal{C}) \to Z_{G}(t')^\bullet$ is an algebraic homomorphism with

$$d\gamma \left( \begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix} \right) = y \quad \text{and} \quad d\gamma \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) \in t + \sigma_v,$$
Proof. For irreducible $\text{Irr}_N$ we note that at this point we still have to consider $\vec{z}$ using the decomposition (9) of $g$ of parameters $(57)$ and $(58)$.

By [AMS2] Corollary 3.23 all these representations depend only on the $N_G(M)/M$-orbit of $(t', C, F)$, not on the additional choices.

For $\vec{z} \in \mathbb{R}^d_{>0}$ we consider the irreducible $\mathcal{H}(G, M, qE, \vec{z})$-module

\begin{equation}
\text{ind}^{\mathcal{H}(G, M, qE, \vec{z})}_{\mathcal{H}(Z_G(t), M, qE, \vec{z})} (\exp_t)_* \text{IM}^* M_{y, \log |t'| + d\vec{z} \left( \begin{array}{cc} 0 & 0 \\ \log \vec{z} & -\log \vec{z} \end{array} \right) - \log |s|, \log \vec{z}, \rho}.
\end{equation}

**Lemma 2.12.** Fix $\vec{z} \in \mathbb{R}^d_{>0}$. The representations (56) provide a bijection between $\text{Irr}_{\vec{z}}(\mathcal{H}(G, M, qE, \vec{z}))$ and $N_G(M)/M$-orbits of triples $(t', C, F)$ as above.

*Proof.* For irreducible $\mathcal{H}(\tilde{Z}_G(t), M, qE, \vec{z})$-representations with central character in $W_{qE,t} \times \mathbb{R}_{>0}$ this follows from [AMS2] Corollary 3.23 and Theorems 2.11 and 2.5.

We note that at this point we still have to consider $N_G(M)/M$-conjugacy classes of parameters $(t', C, F)$.

With Theorem 2.5 we extend this to the whole of $\text{Irr}_{\vec{z}}(\mathcal{H}(G, M, qE, \vec{z}))$. In view of (44), this involves the choice of a unitary element $t$ in a $N_G(M)$-orbit in $T$. But by Lemma 2.9 the parametrization does not depend on that choice. Hence the representation (56) depends, up to isomorphism, only on the $N_G(M)/M$-orbit of $(t', C, F)$.

To simplify the parameters, we would like to get rid of the restriction $t' \in T$ – we would rather allow any semisimple element of $G^\circ$. It is also convenient to replace $C$ by a single unipotent element (contained in $\exp C$) in $G^\circ$, and $F$ by the associated representation of the correct component group.

As new parameters we take triples $(s, u, \rho)$ such that:

- $s \in G^\circ$ is semisimple;
- $u \in Z_G(s)^\circ$ is unipotent;
- $\rho \in \text{Irr}(\pi_0(Z_G(s), u)))$ with $q\Psi(Z_G(s), u, \rho) = (M, C_v^M, qE)$ up to $G$-conjugacy.

Assume that $s \in T$ and choose an algebraic homomorphism $\gamma_u : \text{SL}_2(\mathbb{C}) \to Z_G(s)^\circ$ with

\begin{equation}
\gamma_u \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) = u \quad \text{and} \quad d\gamma_u \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \in t + \sigma_v.
\end{equation}

Using the decomposition (9) of $g$ we write, like in (15),

\begin{equation}
\gamma_u \left( \begin{array}{cc} \vec{z} & 0 \\ 0 & \vec{z}^{-1} \end{array} \right) = \exp \left( d\gamma_u \left( \begin{array}{cc} \log \vec{z} & 0 \\ 0 & -\log \vec{z} \end{array} \right) \right) \in M.
\end{equation}

For $\vec{z} \in \mathbb{R}^d_{>0}$ we define the standard $\mathcal{H}(G, M, qE, \vec{z})$-module

\[ E_{s, u, \rho, \vec{z}} = \text{ind}^{\mathcal{H}(G, M, qE, \vec{z})}_{\mathcal{H}(Z_G(s|s|^{-1}), M, qE, \vec{z})} (\exp_s|s|^{-1})_* \text{IM}^* E_{\log u, d\gamma_u \left( \begin{array}{cc} \log \vec{z} & 0 \\ 0 & -\log \vec{z} \end{array} \right) - \log |s|, \log \vec{z}, \rho} \]

and its irreducible quotient

\[ M_{s, u, \rho, \vec{z}} = \text{ind}^{\mathcal{H}(G, M, qE, \vec{z})}_{\mathcal{H}(Z_G(s|s|^{-1}), M, qE, \vec{z})} (\exp_s|s|^{-1})_* \text{IM}^* M_{\log u, d\gamma_u \left( \begin{array}{cc} \log \vec{z} & 0 \\ 0 & -\log \vec{z} \end{array} \right) - \log |s|, \log \vec{z}, \rho} \]
Even when \( s \notin T \), the condition on \( \rho \) and \([\text{AMS}2\text{ Propositions 3.5.a and 3.7}]\) guarantee the existence of a \( g_0 \in G^o \) such that \( g_0sg_0^{-1} \in T \). In this case we put
\[
E_{s,u,\rho,\vec{z}} := E_{g_0sg_0^{-1},g_0ug_0^{-1},g_0u,\rho,\vec{z}} \quad \text{and} \quad M_{s,u,\rho,\vec{z}} := M_{g_0sg_0^{-1},g_0ug_0^{-1},g_0u,\rho,\vec{z}}.
\]

We extend the polar decomposition (38) to this setting by
\[
|s| := g_0^{-1}|g_0sg_0^{-1}|g_0.
\]

With the Jordan decomposition in \( G^o \) it is possible to combine \( s \) and \( u \) in a single element \( g = su \in G^o \). Then \( s \) equals the semisimple part \( gs \), \( u \) becomes the unipotent part \( gu \), and \( \rho \in \operatorname{Irr}(\pi_0(Z_G(g))) \).

Now we come to our main result about affine Hecke algebras. In the case that \( G \) is connected, it is almost the same parametrization as in \([\text{Lus5}, \S 5.20\text{]}\) and \([\text{Lus7 Theorems 10.4}]\). The only difference is that we twist by the Iwahori–Matsumoto unipotent part \( g \) element.

We extend the polar decomposition (38) to this setting by
\[
|s| := g_0^{-1}|g_0sg_0^{-1}|g_0.
\]

There are no essentially discrete series representations on which at least one \( \chi \in \operatorname{Irr}(G) \cap G^o \). In this case \( |s| \in Z(G^o) \).

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Proof. (a) The uniqueness in the Jordan decomposition entails that the first map is a canonical bijection.

We already noted in (59) that, for every eligible triple \( (s, u, \rho) \), \( s \) lies in \( \operatorname{Ad}(G^o)T \). Therefore we may restrict to triples with \( s \in T \). Consider the map
\[
(s, u, \rho) \mapsto (s, c_{\log u}, F),
\]
where \( F \) is determined by \( F_{\log u} = \rho \). As in the proof of \([\text{AMS}2\text{ Corollary 3.23}]\), this gives a canonical bijection between \( G \)-conjugacy classes of triples \( (s, u, \rho) \) and...
the parameters used in Lemma 2.12. Furthermore (57) just reflects (55), so Lemma 2.12 yields the desired canonical bijection with \( \text{Irr}_e(\mathcal{H}(G, M, qE, \bar{z})) \).

(b) By Lemma 1.8 the \( \mathbb{H}(Z_G(s|s|^{-1}), M, qE, \bar{F}) \)-representation

\[
\text{IM}^* E \left( \log u, d\gamma_u \left( \begin{array}{cc} \log z & 0 \\ 0 & -\log z \end{array} \right) \right) \log |s|, \log \bar{z}, \rho
\]

admits the central character \( W_{qE, s|s|^{-1}} \left( \log |s| \pm d\chi_{u,v}(\log \bar{z}), \log \bar{z} \right) \).

By Theorems 2.11c and 2.5c the central character of \( \bar{E}_{s,u,\rho,\bar{z}} \) becomes

\[
(W_{qE}s \, \bar{\chi}_{u,v}(\bar{z}), \bar{z}) = W_{qE}(s \, \bar{\chi}_{u,v}(\bar{z})^{-1}, \bar{z}).
\]

The same holds for the quotient \( \bar{M}_{s,u,\rho,\bar{z}} \).

(c) Suppose that \( s \in T \). By [AMS2] (84) the representation (61) and its quotient

\[
\text{IM}^* M \left( \log u, d\gamma_u \left( \begin{array}{cc} \log z & 0 \\ 0 & -\log z \end{array} \right) \right) \log |s|, \log \bar{z}, \rho
\]

are tempered if and only if \( \log |s| \in i\mathbb{R} \). By definition \( \log |s| \in t_{\mathbb{R}} \), so this condition is equivalent to \( \log |s| = 0 \). This in turn is equivalent to \( |s| = 1 \) and to \( s \in T_{un} \).

By Theorem 2.11d and Proposition 2.7b this is also equivalent to temperedness of \( \bar{E}_{s,u,\rho,\bar{z}} \) or \( \bar{M}_{s,u,\rho,\bar{z}} \).

The proof of part (a) shows that also for general \( s \), temperedness is equivalent to \( |s| = 1 \). This happens if and only if \( s \) lies in the unitary part of a torus conjugate to \( T \), which in turn is equivalent to \( s \) lying in a compact subgroup of \( G^0 \).

(d) As in part (c), it suffices to consider the case \( s \in T \).

Suppose that \( \bar{M}_{s,u,\rho,\bar{z}} \) is essentially discrete series. By Proposition 2.7c and Theorem 2.11d the representation (61) has the same property. Moreover we saw in the proof of Proposition 2.7c that \( Z_{G^0} \) is semisimple. Up to doubling some roots (with respect to \( T \), \( Z_{G^0} \) has the same root system, so that group is semisimple as well.

By assumption \( \log \bar{z} \in \mathbb{R}_{>0} \). Now [AMS2] (85)] says that \( s \) is distinguished in \( \text{Lie}(Z_G(s|s|^{-1})^0) \). In view of the aforementioned semisimplicity, this is the same as distinguished in \( g \). So \( s \) is distinguished in \( G^0 \).

Conversely, suppose that \( u \) is distinguished in \( G^0 \), or equivalently that \( \log u \) is distinguished in \( g \). As \( u \) commutes with \( s \), it also commutes with \( |s| \) and with \( s|s|^{-1} \). This implies that \( R(Z_G(s|s|^{-1})^0, T) \) and \( R(Z_G(s|s|^{-1})^0, T) \) have full rank in \( R(G^0, T) \). By [AMS2] (85)], Theorem 2.11d and Proposition 2.7c \( \bar{M}_{s,u,\rho,\bar{z}} \) is essentially discrete series.

Suppose that either of the above two conditions holds. Then \( |s| \in T_{un} \) commutes with the distinguished unipotent element \( u \in G^0 \). This implies that the semisimple subalgebra \( \mathbb{C}\log |s| \subset g \) is contained in \( Z(g) \). Hence \( |s| \in Z(G^0) \). Moreover [AMS2] Theorem 3.26b] and Lemma 1.3 imply that \( \bar{E}_{s,u,\rho,\bar{z}} = \bar{M}_{s,u,\rho,\bar{z}} \).

Finally, suppose that \( \mathcal{H}(G, M, qE, \bar{z}) \) has an essentially discrete series representation on which \( z_j \) acts as 1. Its dimension is finite, so it has an irreducible subquotient, say \( \bar{M}_{s,u,\rho,\bar{z}} \). Then \( \text{IM}^* \text{IM} \log u, -\log |s|, \log \bar{z}, \rho \) restricts to an essentially discrete series representation of \( \mathbb{H}(Z_G(s|s|^{-1}), M', E) \), which is annihilated by \( r_j \). By [13] and [14] it contains a \( \mathbb{H}(G, M_j, E_j) \)-representation with the same properties. But [AMS2] Theorem 3.26c] says that this is impossible.
(e) By Proposition 1.6.d

\[(\exp_{\zeta s}(\zeta s^{-1}))^* \otimes \text{IM}^* M_{1,0}(d_{\gamma_0}) = (\exp_{\zeta}^{s-1})(\zeta s^{-1})^* \otimes \text{IM}^* M_{1,0}(d_{\gamma_0}) = \log|\zeta| \log z, \rho = \log|\zeta| \log z, \rho.
\]

From Theorem 2.11.a and the definitions of \(\zeta \otimes, \log|\zeta| \otimes\) we see that this equals

\[\zeta \otimes (\exp_{s-1})^* \otimes \text{IM}^* M_{1,0}(d_{\gamma_0}) = \log|\zeta| \log z, \rho = \log|\zeta| \log z, \rho.
\]

Since \(\zeta\) is central in \(G\), \(H(\tilde{Z}_G(s|s|^{-1}), M, qE, \tilde{z})\) does not change upon replacing \(s\) by \(\zeta s\), and \(\zeta \otimes\) is preserved by \(\text{ind}_{H(\tilde{Z}_G(s|s|^{-1}), M, qE, \tilde{z})}^{H(G,M,qE,\tilde{z})}\). This proves the claim for \(\tilde{M}_{s,u,\rho,\tilde{z}}\), while the argument for \(\tilde{E}_{s,u,\rho,\tilde{z}}\) is analogous.

(f) We use Theorems 2.5.a and 2.11.a to the translate the statement to modules over \(\text{Hom}(G,M,qE,\tilde{r})\), with \(\tilde{r}\) acting as \(\log(\tilde{z})\) in \(\mathbb{R}_{\geq 0}\). Then we apply Proposition 1.6.e. □

Let us discuss the relation between the parametrization from Theorem 2.13.a and parabolic induction. Suppose that \(Q \subset G\) is a Levi subgroup of \(G\) and \(M \subset Q\). Let \((s, u, \rho)\) be as above, with \(s \in Q\). Also take \(\rho Q \in \text{Irr}(\pi_0(\tilde{Z}_G(s,u)))\) with \(q\Psi_{\tilde{Z}_G(s,u)}(u, \rho Q) = (M, \text{C}^M, qE)\) up to \(G\)-conjugation.

Recall \(\epsilon_j\) from \([12]\). We extend it to the current setting by defining

\[\epsilon_{u,j}(s, \tilde{z}) = \epsilon_{u,j}\left(d_{\gamma_0}(\log z, 0 - \log z) - \log|s|, \log z\right).
\]

Corollary 2.14. Assume that, for each \(j = 1, \ldots, d\), \(\epsilon_{u,j}(s, \tilde{z}) \neq 0\) or \(z_j = 1\).

(a) There is a natural isomorphism of \(H(G,M,qE,\tilde{z})\)-modules

\[H(\tilde{Z}_G(s|s|^{-1}), M, qE, \tilde{z}) \otimes_{H(\tilde{Q}_{G,(M,qE,\tilde{z})})} \tilde{E}_{s,u,\rho Q,\tilde{z}}^Q \cong \bigoplus_{\rho} \text{Hom}_{\pi_0(\tilde{Z}_G(s,u))}(\rho Q, \rho) \otimes \tilde{E}_{s,u,\rho Q,\tilde{z}}^Q,
\]

where the sum runs over all \(\rho \in \text{Irr}(\pi_0(\tilde{Z}_G(s,u)))\) with \(q\Psi_{\tilde{Z}_G(s,u)}(u, \rho) = (M, \text{C}^M, qE)\) up to \(G\)-conjugation. For \(\tilde{z} = \tilde{1}\) this isomorphism contains

\[H(\tilde{1}, M, qE, \tilde{z}) \otimes_{H(\tilde{Q}_{G,M,qE,\tilde{z}})} \tilde{M}_{s,u,\rho Q,\tilde{z}}^Q \cong \bigoplus_{\rho} \text{Hom}_{\pi_0(\tilde{Z}_G(s,u))}(\rho Q, \rho) \otimes M_{s,u,\rho,\tilde{z}}.
\]

(b) The multiplicity of \(\tilde{M}_{s,u,\rho,\tilde{z}}\) in \(H(\tilde{1}, M, qE, \tilde{z}) \otimes_{H(\tilde{Q}_{G,M,qE,\tilde{z}})} \tilde{E}_{s,u,\rho Q,\tilde{z}}^Q\) is

\[\big[\rho Q : \rho\big]_{\pi_0(\tilde{Z}_G(s,u))}.\]

It already appears that many times as a quotient, via \(\tilde{E}_{s,u,\rho Q,\tilde{z}}^Q \rightarrow \tilde{M}_{s,u,\rho Q,\tilde{z}}^Q\). More precisely, there is a natural isomorphism

\[\text{Hom}_{H(\tilde{Z}_G,M,qE,\tilde{z})}(\tilde{M}_{s,u,\rho Q,\tilde{z}}^Q, \tilde{M}_{s,u,\rho,\tilde{z}}) \cong \text{Hom}_{\pi_0(\tilde{Z}_G(s,u))}(\rho Q, \rho)^*.
\]

Proof. Recall that the analogous statement for twisted graded Hecke algebras is Proposition 1.5. To that we can apply the Iwahori–Matsumoto involution, supported by \([AMS2, (83)]\). Next, part (b) of Theorem 2.11 allows us to apply part (a) while retaining the desired properties. The same goes for Theorem 2.5. Then we have transferred Proposition 1.5 to the representations \(\tilde{E}_{s,u,\rho,\tilde{z}}\) and \(\tilde{M}_{s,u,\rho,\tilde{z}}\). □

Notice that the parameters in Theorem 2.13.a do not depend on \(\tilde{z}\). This enables us to relate \(\text{Irr}_\tilde{z}(H(G, M, qE, \tilde{z}))\) to an extended quotient of \(T\) by \(W_qE\), as in \([ABPS5, \S 2.3]\) and \([AMS2, (87)]\). The 2-cocycle \(\tilde{z}_qE\) of \(W_qE\) gives rise to a twisted version of the extended quotient \(T//W_qE\), see \([ABPS5, \S 2.1]\).
Theorem 2.15. Let \( \vec{z} \in \mathbb{R}^d_{>0} \). There exists a canonical bijection
\[
\mu_{G,M,q\mathcal{E}} : (T//W_{q\mathcal{E}})_{\vec{z}_{q\mathcal{E}}} \to \text{Irr}_\vec{z}(\mathcal{H}(G, M, q\mathcal{E}, \vec{z}))
\]
such that:
- \( \mu_{G,M,q\mathcal{E}}(T_{\text{un}}//W_{q\mathcal{E}})_{\vec{z}_{q\mathcal{E}}} = \text{Irr}_{\vec{z},\text{temp}}(\mathcal{H}(G, M, q\mathcal{E}, \vec{z})) \) when \( \vec{z} \in \mathbb{R}^d_{\geq 1} \);
- the central character of \( \mu_{G,M,q\mathcal{E}}(t, \pi_t) \) is \( (W_{q\mathcal{E}}t \chi(\vec{z}), \vec{z}) \), for some algebraic cocharacter \( \chi \) of \( Z_G(t)^0 \).

Remark 2.16. Together with [Sol3] Theorem 5.4.2 this proves a substantial part of the ABPS conjectures [ABPS1] for the twisted affine Hecke algebra \( \mathcal{H}(G, M, q\mathcal{E}, \vec{z}) \).

For \( \vec{z} \in (0, 1]^d \), \( \mu_{G,M,q\mathcal{E}}(T_{\text{un}}//W_{q\mathcal{E}})_{\vec{z}_{q\mathcal{E}}} \) is the anti-tempered part of \( \text{Irr}_\vec{z}(\mathcal{H}(G, M, q\mathcal{E}, \vec{z})) \), compare with [AMS2] Theorem 3.29.

Proof. From Proposition 2.2 we see that
\[ \mathcal{H}(G, M, q\mathcal{E}, \vec{z})/\langle z_1 - 1, \ldots, z_d - 1 \rangle \cong \mathcal{O}(T) \times \mathbb{C}[W_{q\mathcal{E}}, \vec{z}_{q\mathcal{E}}]. \]
By [ABPS5] Lemma 2.3 there exists a canonical bijection
\[
(T//W_{q\mathcal{E}})_{\vec{z}_{q\mathcal{E}}} \to \text{Irr}(\mathcal{O}(T) \times \mathbb{C}[W_{q\mathcal{E}}, \vec{z}_{q\mathcal{E}}])
\]
\[
(t, \pi_t) \to \mathcal{C}_t \rtimes \pi_t = \text{ind}_{\mathcal{O}(T) \times \mathbb{C}[W_{q\mathcal{E}}, \vec{z}_{q\mathcal{E}}]}^{\mathcal{O}(T) \times \mathbb{C}[W_{q\mathcal{E}}, \vec{z}_{q\mathcal{E}}]}(\mathcal{C}_t \otimes V_{\pi_t}).
\]
We consider \( \mathcal{C}_t \rtimes \pi_t \) as an irreducible \( \mathcal{H}(G, M, q\mathcal{E}, \vec{z}) \)-representation with central character \( (W_{q\mathcal{E}}t, 1) \). By Theorem 2.15 there exist \( u \) and \( \rho \), unique up to \( Z_G(t) \)-conjugation, such that \( \mathcal{C}_t \rtimes \pi_t \cong M_{t,u,\rho,1} \). Now we define
\[
\mu_{G,M,q\mathcal{E}}(t, \pi_t) = M_{t,u,\rho,\vec{z}}.
\]
This is canonical because Theorem 2.13 a is. The properties involving temperedness and the central character follow from parts (c) and (b) of Theorem 2.13 \( \square \)

2.3. Comparison with the Kazhdan–Lusztig parametrization.
Irreducible representations of affine Hecke algebras were also classified in [KaLu] in terms of equivariant K-theory. This concerns the cases with only one complex parameter \( q = z^2 \), which is not a root of unity. In terms of Proposition 2.2 this means that \( \lambda = \lambda^* = 1 \). In view of [35] and [Lus2] Proposition 2.8, this happens if and only if \( T = M^0 \) is a maximal torus of \( G^0 \) and \( v = 1 \). For the upcoming comparison we assume that \( M = Z_G(T) \) equals \( T \). Then \( \pi_0(Z_M(v)) = 1, q\mathcal{E} \) is the trivial representation and
\[
\mathfrak{R}_{q\mathcal{E}} = N_G(T, B)/T \cong G/G^0,
\]
where \( B \) is a Borel subgroup of \( G^0 \) containing \( T \) (called \( P \) before). The Kazhdan–Lusztig parametrization was extended to algebras of the form
\[ \mathcal{H}(G, T, q\mathcal{E} = \text{triv}) = \mathcal{H}(\mathcal{R}(G^0, T), \lambda = 1, \lambda^* = 1, z) \times \mathfrak{R}_{q\mathcal{E}} \]
in [ABPS4] §9. The parameters are triples \( (t_q, u, \rho) \), where
- \( t_q \in T \) is semisimple;
- \( u \in G^0 \) is unipotent and \( t_q u t_q^{-1} = u^q \);
- \( B_{G^0}^{t_q,u} \) is the variety of Borel subgroups of \( G^0 \) containing \( t_q \) and \( u \);
- \( \rho \in \text{Irr}(\pi_0(Z_G(t_q, u))) \) such that every irreducible component of \( \rho|_{\pi_0(Z_{G^0}(t_q, u))} \) appears in \( H_*(B_{G^0}^{t_q,u}, \mathbb{C}) \).
Two triples of this kind are considered equivalent if they are $G$-conjugate. The representation $\tilde{M}(t_q, u, \rho)$ attached to these data is the unique irreducible quotient of the standard module

$$\tilde{E}_{t_q, u, \rho} := \text{Hom}_{\pi_0(Z_G(t_q, u))}(\rho, H_s(B_{G^{\circ}}^{t_q, u} \times \mathcal{R}_q, \mathbb{C})).$$

The classification of $\mathcal{H}(G^\circ, T, \mathcal{E} = \text{triv})$ with $q = z = 1$ goes back to Kato [Kat, Theorem 4.1], see also [ABPS4] [8]. With [ABPS4] Remark 9.2 and the subsequent argument (which underlies the above for $q \neq 1$) it can be extended to $\mathcal{H}(G, T, q\mathcal{E} = \text{triv})$. The parameters are the same as above (only with $q = 1$), and the irreducible module is

$$\tilde{M}(t_1, u, \rho) = \text{Hom}_{\pi_0(Z_G(t_1, u))}(\rho, H_{d(u)}(B_{G^{\circ}}^{t_1, u} \times \mathcal{R}_q, \mathbb{C})),
$$

where $d(u)$ refers to the dimension of $B_{G^{\circ}}^{t_1, u}$ as a real variety. Clearly $\tilde{M}(t_1, u, \rho)$ is again a quotient of $\tilde{E}_{t_1, u, \rho}$, but for $q = 1$ [62] has other irreducible quotients as well, in lower homological degree.

**Lemma 2.17.** The above set of parameters $(t_q, u, \rho)$ is naturally in bijection with the sets of parameters used in Theorem 2.13.a.

**Proof.** By [ABPS4] Lemma 7.1, we obtain the same $G$-conjugacy classes of parameters if we replace the above $t_q$ by a semisimple element $s \in Z_G(u)$. In Theorem 2.13 we also have parameters $(s, u, \rho)$, but with a different condition on $\rho$, namely that

$$q\Psi_{Z_G(s)}(u, \rho) = (T, v = 1, q\epsilon = \text{triv}).$$

By definition this is equivalent to

$$\Psi_{Z_G(s)^\circ}(u, \rho_s) = (T, v = 1, \epsilon = \text{triv}),$$

for any irreducible constituent $\rho_s$ of $\rho|_{\pi_0(Z_{G(s)^\circ})}$. Write $r = \log z \in \mathbb{R}$ and $y = \log(u) \in \text{Lie}(Z_G(s))$. According to [AMS2] Proposition 3.7 for the group $Z_G(s)^\circ$, (64) is equivalent to $\rho_s$ appearing in

$$E_{y, 0, r}^\circ = \mathbb{C}_{0, r} \otimes H^1((y)) \cong H^1(M(y)^\circ) \cong H^1(M(y)^\circ) \cong H_s(P_y^\circ, \mathbb{C}) = H_s(P_y^\circ, \mathbb{C}).$$

To make this more explicit, we assume (as we may) that $s \in T$. Then $Z_B(s) = Z_G(s)^\circ \cap B$ is a Borel subgroup of $Z_G(s)^\circ$ and

$$P_y^\circ = \{gZ_B(s) \in Z_G(s)^\circ/Z_B(s) : \text{Ad}(g^{-1})y \in \text{Lie}(Z_B(s))\} = \{gZ_B(s) \in Z_G(s)^\circ/Z_B(s) : u \in gZ_B(s)g^{-1}\} = B_{Z_G(s)^\circ}^u.
$$

Hence (64) is equivalent to $\rho_s$ appearing in $H_s(B_{Z_G(s)^\circ}^u, \mathbb{C})$. Let $\rho^\circ$ be a $\pi_0(Z_{G(s)}(s, u))$-constituent of $\rho$ containing $\rho_s$. By [ABPS4] Proposition 6.2 there are isomorphisms of $Z_G(s, u)$-varieties

$$B_{G^{\circ}}^{t_q, u} \cong B_{G^{\circ}}^{s, u} \cong B_{Z_G(s)^\circ}^u \times Z_{G(s)}(s, u)/Z_{G(s)}(s)^\circ(u).
$$

With this and Frobenius reciprocity we see that the condition on $\rho_s$ is also equivalent to $\rho^\circ$ appearing in $H_s(B_{Z_G(s)^\circ}^u, \mathbb{C})$. We conclude that the parameters $(s, u, \rho)$ in Theorem 2.13 are equivalent to those in [ABPS4] [9], the only change being $s \leftrightarrow t_q$. □
Proposition 2.18. The parametrization of \( \text{Irr}_0(\mathcal{H}(G, T, qE = \text{triv})) \) obtained in Theorem 2.14 a agrees with the above parametrization by the representations \( M(t_q, u, \rho) \), when we set \( q = z^2 \in \mathbb{R}_{>0} \) and take Lemma 2.17 into account. Moreover the standard modules \( \mathcal{E}_{s,u,\rho,z} \) and \( \mathcal{E}_{t_q,u,\rho} \) are isomorphic.

In other words, our classification of irreducible representations of affine Hecke algebras agrees with that of Kazhdan–Lusztig and the extended versions thereof.

Remark 2.19. Our parametrization differs from the one used by Lusztig in [Lus5, §5.20] and [Lus7, Theorem 10.4], namely by the Iwahori–Matsumoto involution. Thus Proposition 2.18 shows that the classification of unipotent representations of Iwahori–spherical representations in \([\text{KaLu}]\).

Proof. Let \((s, u, \rho)\) be a triple as above, and choose an algebra homomorphism \( \gamma_u : SL_2(\mathbb{C}) \to Z_G(s)^0 \) with \( \gamma_u(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = u \). Then we can take \( t_q = s\gamma_u(\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}) \), where \( z^2 = q \). Recall that \( \bar{M}(t_q, u, \rho) \) is a quotient of \( \bar{E}_{t_q, u, \rho} \) from (62). Write \( \rho = \rho^\circ \ltimes \tau^* \), where

\[
\tau^* \in \text{Irr}(\mathcal{R}_{qE,s,u,\rho^\circ}) \quad \text{with} \quad \mathcal{R}_{qE,s,u,\rho^\circ} = \pi_0(Z_G(s, u))_{\rho^\circ}/\pi_0(Z_G^\circ(s, u)).
\]

From [ABPS4] (72) we see that \( \bar{E}_{t_q, u, \rho} \) equals

\[
(67) \quad \text{Hom}_{\pi_0(Z_G^\circ(s, u))}(\rho^\circ, H_*(\mathcal{B}_{G^\circ}^u, \mathbb{C})) \ltimes \tau.
\]

To the part without \( \ltimes \tau \) we can apply \([\text{EvMi}] \), which compares the two parametrizations. In \([\text{EvMi}] \) both the Iwahori–Matsumoto involution and a related “shift” are mentioned. This involution is necessary to get temperedness for the same parameters in both classifications. Unfortunately, it is not entirely clear what Evens and Mirkovich mean by a “shift”, for signs can be inserted at various places. In any case their argument is based on temperedness and a comparison of weights [EvMi, Theorem 5.5], and it will work once we arrange the modules such that these two aspects match. With this in mind, \([\text{EvMi}] \) Theorem 6.10 says that the \( \mathcal{H}(Z_G^\circ(s)|s|^{-1}), T, \text{triv}) \)-module obtained from \( \text{Hom}_{\pi_0(Z_G^\circ(s, u))}(\rho^\circ, H_*(\mathcal{B}_{G^\circ}^u, \mathbb{C})) \) via Theorems 2.5 and 2.11 is \( \text{IM}^*E_{y,d\gamma_u(\begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix}) \log |s|^2, r, \rho^\circ} \). The extension with the group \( \mathcal{R}_{qE} \) is handled in the same way for all algebras under consideration here, namely with Clifford theory. It follows that applying Theorems 2.5 and 2.11 to (67) yields

\[
(68) \quad \left( \text{IM}^*E_{y,d\gamma_u(\begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix}) \log |s|^2, r, \rho^\circ} \right) \ltimes \tau.
\]

Moreover \( \text{IM} \) is the identity on \( \mathbb{C}[\mathcal{R}_{qE}] \), so the large brackets are actually superfluous here. Notice that the subgroup of \( \Gamma \) appearing in \( Z_G(s)|s|^{-1} \) is \( \Gamma_{\text{Ad}(G^\circ)|s|^{-1}} \), the stabilizer of the \( \text{Ad}(G^\circ) \)-orbit of \( s|s|^{-1} \). The action of \( \mathcal{R}_{qE,s,u,\rho^\circ} \) underlying \( \ltimes \tau \) in (67) comes from the action of \( \pi_0(Z_G(s, u)) \) on \( H_*(\mathcal{B}_{Z_G^\circ(s)|s|^{-1}}^u \times \Gamma_{\text{Ad}(G^\circ)|s|^{-1}}, \mathbb{C}) \). By (65) for the group \( Z_G(s)|s|^{-1} \):

\[
\mathcal{B}_{Z_G^\circ(s)|s|^{-1}}^u \times \Gamma_{\text{Ad}(G^\circ)|s|^{-1}} = \mathcal{P}_y.
\]

Via this equality the \( \pi_0(Z_G(s, u)) \)-action on \( H_*(\mathcal{B}_{Z_G^\circ(s)|s|^{-1}}^u \times \Gamma_{\text{Ad}(G^\circ)|s|^{-1}}, \mathbb{C}) \) agrees with the action on

\[
H_*(\mathcal{P}_y, \mathbb{C}) \cong C_{|s|, r} \otimes H^M(y)^\circ \otimes H^M(y)^\circ(\{y\})
\]
from [AMS2] Theorem 3.2.d. Hence

$$\text{IM}^* E_{y,d}\gamma_0\left(\begin{smallmatrix} 0 & 0 \\ -r & 0 \end{smallmatrix}\right) - \log |s|, r, \rho^\circ \times \tau = \text{IM}^* \left( E_{y,d}\gamma_0\left(\begin{smallmatrix} 0 & 0 \\ -r & 0 \end{smallmatrix}\right) - \log |s|, r, \rho^\circ \times \tau \right)$$

$$= \text{IM}^*\left( E_{y,d}\gamma_0\left(\begin{smallmatrix} 0 & 0 \\ -r & 0 \end{smallmatrix}\right) - \log |s|, r, \rho^\circ \times \tau \right)$$

We see that the standard modules \( \overline{E}_{t_q,u,\rho} \) and \( \overline{E}_{s,u,\rho,\varepsilon} \) give the same module upon applying Theorems 2.5 and 2.11. Hence they are isomorphic.

From here on we have to assume that \( q = z^2 \in \mathbb{R}_{>0} \) is not a root of unity. We recognize the unique irreducible quotient of the right hand side as \( \overline{M} \), a part of the definition of \( M_{s,u,\rho,\varepsilon} \). Using Theorems 2.11 and 2.5 again, but now in the opposite direction, we see that both \( M_{s,u,\rho,\varepsilon} \) and \( \overline{M}(t_q, u, \rho) \) are the unique irreducible quotient of

$$\text{ind}_{\mathcal{H}(\mathcal{G},\mathcal{M},\mathcal{E})}^{\mathcal{H}(\mathcal{Z}_G(s)|^{-1}, \mathcal{M}, \mathcal{E})} \left( \exp s|s|^{-1} \right) \text{IM}^* E_{\log u,d}\gamma_0\left(\begin{smallmatrix} 0 & 0 \\ -r & 0 \end{smallmatrix}\right) - \log |s|, \log z, \rho^\circ$$

Thus the two parametrizations agree when \( q = z^2 \neq 1 \).

For \( q = z = 1 \) a different argument is needed. We note that (67) still applies, which enables us to write

$$\overline{M}(t_1 = s, u, \rho) = \text{Hom}_{\pi_0(\mathcal{Z}_G(s), u)}\left( \rho^\circ, \mathcal{H}_{d(u)}(\mathcal{B}^u_{Z_G(s)}, \mathbb{C}) \right) \times \tau.$$ 

From the definition of the \( X^*(T) \)-action in [Kat, §3] we see that \( H_s(\mathcal{B}^u_{Z_G(s)}, \mathbb{C}) \) is completely reducible as a \( X^*(T) \)-module. With [ABPS4] Theorem 8.2 we deduce that the weight space for \( s \in T \) is, as \( (W_q\mathcal{E})_s^* \)-representation, equal to

$$\text{Hom}_{\pi_0(\mathcal{Z}_G(s), u)}\left( \rho, \mathcal{H}_{d(u)}(\mathcal{B}^u_{Z_G(s)} \times \Gamma_{\text{Ad}(G^s)}, \mathbb{C}) \right) = \text{Hom}_{\pi_0(\mathcal{Z}_G(s), u)}\left( \rho^\circ, \mathcal{H}_{d(u)}(\mathcal{B}^u_{Z_G(s)}, \mathbb{C}) \right) \times \tau.$$ 

From [AMS2] (39) we can also determine the \( X^*(T) \)-weight space for \( s \) in \( M_{s,u,\rho,1} \).

First we look at the \( S(t^*) \)-weight \( -\log |s| \) in \( M_{y,-\log |s|,0,\rho^\circ} \), that gives \( M_{y,-\log |s|,0,\rho^\circ} \).

As in [AMS2] Section 3.2, we denote the underlying \( W(\mathcal{Z}_G(s), T) \)-representation by \( M_{y,\rho^\circ} \). Next we replace \( Z_G(s) \) by \( Z_G(s) \) and \( \rho^\circ \) by \( \rho = \rho^\circ \times \tau^* \), obtaining the \( (W_q\mathcal{E})_s^* \)-representation

$$M_{y,-\log |s|,0,\rho^\circ} \times \tau = M_{y,\rho^\circ} \times \tau.$$ 

Applying the Iwahori–Matsumoto involution and Theorem 2.11 we get

$$\left( \exp s|s|^{-1} \right) \text{IM}^* (M_{y,-\log |s|,0,\rho^\circ} \times \tau).$$

The previous \( S(t^*) \)-weight space (69) for \( -\log |s| \) has now been transformed into the \( X^*(T) \)-weight space for \( s \) in the representation \( M_{s,u,\rho,1} \) with respect to the group \( Z_G(s) \). To land inside \( M_{s,u,\rho,1} \) with respect to \( G \), we must still apply Theorem 2.5. But that does not change the \( X^*(T) \)-weight space for \( s \), so we can stick to (70).

For \( r = 0, z = 1 \) the map \( \left( \exp s|s|^{-1} \right) \) becomes the identity on \( \mathbb{C}[W_q] \), see [Sol3] (2.5) and (1.25)]. It remains to compare the \( \mathbb{C}[W_q] \)-modules

$$\text{IM}^* (M_{y,\rho^\circ} \times \tau) \quad \text{and} \quad \text{Hom}_{\pi_0(\mathcal{Z}_G(s), u)}\left( \rho, \mathcal{H}_{d(u)}(\mathcal{B}^u_{Z_G(s)}, \mathbb{C}) \right) \times \tau.$$ 

By definition [AMS2] Section 3.2 \( M_{y,\rho^\circ} \) is the \( W(\mathcal{Z}_G(s), T) \)-representation associated to \( (y, \rho^\circ) \) by the generalized Springer correspondence from [Lus1]. It differs
from the classical Springer correspondence by the sign representation, so
\[ M_{\nu, \rho} = \text{sign} \otimes \text{Hom}_{\pi_0(Z_{G(s)}^\circ)}(\rho^\circ, H_d(u)(B_{Z_{G(s)}^\circ}^n, \mathbb{C})). \]

On both sides of (71) the actions underlying \( \rtimes \) come from the action of \( Z_G(s, u) \) on \( H_s(B_{Z_{G(s)}^\circ}^n) \times \Gamma_{\text{Ad}(G^\circ)s}, \mathbb{C}) \cong H_s(Fu, \mathbb{C}). \) Moreover \( \text{IM}(w) = \text{sign}(w)w \) for \( w \in W(Z_G(s)^\circ, T) \) and \( \text{IM} \) is the identity on the group \( \mathfrak{M} \) for \( Z_G(s) \). We conclude that the two representations in (71) are equal.

This proves that \( M(t_1 = s, u, \rho) \) and \( M_{s, u, \rho, 1} \) have the same \( X(T) \)-weight space for the weight \( s \). Since both representations are irreducible, that implies that they are isomorphic. \( \square \)

3. Langlands parameters

Let \( F \) be a non-archimedean local field and let \( G \) be a connected reductive group defined over \( F \). In this section we construct a bijection between enhanced Langlands parameters for \( G(F) \) and a certain collection of irreducible representations of twisted Hecke algebras.

To this end we have to collect several notions about \( L \)-parameters, for which we follow [AMS1]. For the background we refer to that paper, here we do little more than recalling the necessary notations.

Let \( W_F \) be the Weil group of \( F \), \( I_F \) the inertia subgroup and \( \text{Frob}_F \in W_F \) a geometric Frobenius element. Let \( G^\vee \) be the complex dual group of \( G \). It is endowed with an action of \( W_F \), which preserves a pinning of \( G^\vee \). The Langlands dual group is \( L^G = G^\vee \rtimes W_F \).

**Definition 3.1.** A Langlands parameter for \( L^G \) is a continuous group homomorphism
\[ \phi : W_F \times SL_2(\mathbb{C}) \to G^\vee \rtimes W_F \]
such that:
- \( \phi(w) \in G^\vee \) for all \( w \in W_F \);
- \( \phi(W_F) \) consists of semisimple elements;
- \( \phi|_{SL_2(\mathbb{C})} \) is algebraic.

We call a \( L \)-parameter:
- bounded, if \( \phi(\text{Frob}_F) = (c, \text{Frob}_F) \) with \( c \) in a compact subgroup of \( G^\vee \);
- discrete, if \( Z_{G^\vee}(\phi)^\circ = Z(G^\vee)W_F.\phi \).

With [Bor, §3] it is easily seen that this definition of discreteness is equivalent to the usual definition with proper Levi subgroups.

Let \( G^\vee_{sc} \) be the simply connected cover of the derived group \( G^\vee_{\text{der}} \). Let \( Z_{G^\vee}(\phi) \) be the image of \( Z_{G^\vee}(\phi) \) in the adjoint group \( G^\vee_{ad} \). We define
\[ Z_{G^\vee}(\phi) = \text{inverse image of } Z_{G^\vee}(\phi) \text{ under } G^\vee_{sc} \to G^\vee_{ad}. \]

Notice that the conjugation action of \( G^\vee_{sc} \rtimes W_F \) on \( G^\vee_{sc} \) descends to an action of \( G^\vee \rtimes W_F \) on \( G^\vee_{sc} \).

**Definition 3.2.** To \( \phi \) we associate the finite group \( S_{\phi} := \pi_0(Z_{G^\vee_{sc}}(\phi)) \). An enhancement of \( \phi \) is an irreducible representation of \( S_{\phi} \).

The group \( G^\vee \) acts on the collection of enhanced \( L \)-parameters for \( L^G \) by
\[ g \cdot (\phi, \rho) = (g\phi g^{-1}, g \cdot \rho), \quad \text{where } g \cdot \rho(a) = \rho(g^{-1}ag) \text{ for } a \in S_{\phi}. \]
Let $\Phi_e(\mathcal{L})$ be the collection of $\mathcal{G}^\vee$-orbits of enhanced L-parameters.

Let us consider $\mathcal{G}(F)$ as an inner twist of a quasi-split group. Via the Kottwitz isomorphism it is parametrized by a character of $Z(\mathcal{G}_{sc}^\vee)^{W_F}$, say $\zeta_G$. We say that $(\phi, \rho) \in \Phi_e(\mathcal{L})$ is relevant for $\mathcal{G}(F)$ if $Z(\mathcal{G}_{sc}^\vee)^{W_F}$ acts on $\rho$ as $\zeta_G$. The subset of $\Phi_e(\mathcal{L})$ which is relevant for $\mathcal{G}(F)$ is denoted $\Phi_e(\mathcal{G}(F))$.

As is well-known, $(\phi, \rho) \in \Phi_e(\mathcal{L})$ is already determined by $\phi|_{W_F}$ (the restriction to the first factor of $W_F \times SL_2(\mathbb{C})$), the unipotent element $u_\phi := \phi(1, (\frac{1}{0} 1))$ and the enhancement $\rho$. Sometimes we will also consider $G^\vee$-conjugacy classes of such triples $(\phi|_{W_F}, u_\phi, \rho)$ as enhanced L-parameters. An enhanced L-parameter $(\phi|_{W_F}, v, qe)$ will often be abbreviated to $(\phi_v, qe)$. We will study enhanced Langlands parameters via their cuspidal support, as introduced in [AMS1].

**Definition 3.3.** For $(\phi, \rho) \in \Phi_e(\mathcal{L})$ we write $G_{\phi} = Z_{G_{sc}}^1(\phi|_{W_F})$, a complex reductive group. We say that $(\phi, \rho)$ is cuspidal if $\phi$ is discrete and $(u_\phi = \phi(1, (\frac{1}{0} 1)), \rho)$ is a cuspidal pair for $G_{\phi}$ in the sense of [AMS1, §3]. (This means that $\rho = F_{u_\phi}$, for a $G_{\phi}$-equivariant cuspidal local system $F$ on $\mathcal{C}_{\Phi_{\phi}}$.) We denote the collection of cuspidal L-parameters for $\mathcal{L}$ by $\Phi_{\text{cusp}}(\mathcal{L})$, and the subset which is relevant for $\mathcal{G}(F)$ by $\Phi_{\text{cusp}}(\mathcal{G}(F))$.

We denote the cuspidal quasi-support of $(u_\phi, \rho)$, in the sense of [AMS1, §5], by $[M, v, qe]_{G_{\phi}}$. In particular $v \in M \subset G_{\phi} \subset G_{\text{sc}}^\vee$.

**Proposition 3.4.** [AMS1, Proposition 7.3]

Let $(\phi, \rho) \in \Phi_e(\mathcal{G}(F))$. Upon replacing $(\phi, \rho)$ by $G^\vee$-conjugate and replacing $(M, v, qe)$ by a $G_{\phi}$-conjugate, there exists a Levi subgroup $\mathcal{L}(F) \subset \mathcal{G}(F)$ such that $(\phi|_{W_F}, v, qe)$ is a cuspidal L-parameter for $\mathcal{L}(F)$. Moreover

$$\mathcal{L}^\vee \times W_F = Z_{\mathcal{G}^\vee \times W_F}(Z(M)^\circ),$$

and this group is uniquely determined by $(\phi, \rho)$ up to $\mathcal{G}^\vee$-conjugation.

Suppose that $(\phi, \rho)$ is as in Proposition 3.4. We define its modified cuspidal support as

$$L\Psi(\phi, \rho) = (\mathcal{L}^\vee \times W_F, \phi|_{W_F}, v, qe)/\mathcal{G}^\vee$$

The right hand side consists of a Langlands dual group and a cuspidal enhanced L-parameter for that (up to $\mathcal{G}^\vee$-conjugacy). Every enhanced L-parameter for $\mathcal{L}$ is conjugate to one as above, so $L\Psi$ can be considered as a well-defined map from $\Phi_e(\mathcal{L})$ to pairs consisting of a $W_F$-stable Levi subgroup of $\mathcal{G}^\vee$ and a cuspidal L-parameter for the associated L-group. Notice that $L\Psi$ preserves boundedness of enhanced L-parameters.

We also need Bernstein components of enhanced L-parameters. Recall from [Hai §3.3.1] that the group of unramified characters of $\mathcal{L}(F)$ is naturally isomorphic to $Z(\mathcal{L}^\vee \times 1_F)^\vee_{W_F}$. We consider this as an object on the Galois side of the local Langlands correspondence and we write

$$X_{inf}(\mathcal{L}) = Z(\mathcal{L}^\vee \times 1_F)^\vee_{W_F}.$$

Given $(\phi', \rho') \in \Phi_e(\mathcal{L}(F))$ and $z \in Z(\mathcal{L}^\vee \times 1_F)^{W_F}$, we define $(z\phi', \rho') \in \Phi_e(\mathcal{L}(F))$ by

$$z\phi' = \phi'$$

on $1_F \times SL_2(\mathbb{C})$ and $(z\phi')(\text{Frob}_F) = \hat{z}\phi'(\text{Frob}_F)$,
Definition 3.5. An inertial equivalence class for $\Phi_e(\mathcal{G}(F))$ is the $\mathcal{G}^\lor$-conjugacy class $s^\lor$ of a pair $(\mathcal{L}^\lor \ltimes W_F, s^\lor_{\mathcal{L}})$, where $\mathcal{L}(F)$ is a Levi subgroup of $\mathcal{G}(F)$ and $s^\lor_{\mathcal{L}}$ is a $X_{nr}(\mathcal{L})$-orbit in $\Phi_e(\mathcal{L}(F))$.

The Bernstein component of $\Phi_e(\mathcal{G}(F))$ associated to $s^\lor$ is

$$\Phi_e(\mathcal{G}(F))^s_{\mathcal{L}} := L\Psi^{-1}(\mathcal{L}^\lor \ltimes W_F, s^\lor_{\mathcal{L}}).$$

We denote the set of inertial equivalence classes for $\Phi_e(\mathcal{G}(F))$ by $\mathcal{B}^\lor(\mathcal{G}(F))$.

In this way, we obtain a partition of the set $\Phi_e(\mathcal{G}(F))$ analogous to the partition of $\text{Irr}(\mathcal{G}(F))$ induced by its Bernstein decomposition:

$$\Phi_e(\mathcal{G}(F)) = \bigcup_{s^\lor \in \mathcal{B}^\lor(\mathcal{G}(F))} \Phi_e(\mathcal{G}(F))^{s^\lor}.$$  

We note that $\Phi_e(\mathcal{L}(F))^{s^\lor}$ is a torsor for the quotient of the complex torus $X_{nr}(\mathcal{L})$ by a finite subgroup. In particular $\Phi_e(\mathcal{L}(F))^{s^\lor}$ isomorphic to a torus as complex algebraic variety, albeit not a canonical way.

With an inertial equivalence class $s^\lor$ for $\Phi_e(\mathcal{G}(F))$ we associate the finite group

$$W_{s^\lor} := \text{stabilizer of } s^\lor_{\mathcal{L}} \text{ in } N_{\mathcal{G}^\lor(\mathcal{L}^\lor \ltimes W_F)}/L_{s^\lor}.$$  

Let $W_{s^\lor, \phi, qt}$ be the isotropy group of $(\phi, q \epsilon) \in s^\lor_{\mathcal{L}}$. With the generalized Springer correspondence [AMS1, Theorem 5.5] we can attach to any element of $L\Psi^{-1}(\mathcal{L}^\lor \ltimes W_F, \phi, \epsilon)$ an irreducible projective representation of $W_{s^\lor, \phi, qt}$. More precisely, consider the cuspidal quasi-support

$$qt = [G_\phi \cap L_c^\lor, v, \epsilon \mid G_\phi],$$

where $L_c^\lor \subset G^\lor_{sc}$ is the preimage of $L^\lor$ under $G^\lor_{sc} \to G^\lor$. In this setting we write the group $W_{qt}$ from [6] as $W_{qt}$. By [AMS1, Lemma 8.2] $W_{qt}$ is canonically isomorphic to $W_{s^\lor, \phi, q \epsilon}$. According to [AMS1, Proposition 9.1] there exist a 2-cocycle $\kappa_{qt}$ of $W_{qt}$ and a bijection (canonical up to the choice of $\kappa_{qt}$ in its cohomology class)

$$L\Sigma_{qt} : L\Psi^{-1}(\mathcal{L}^\lor \ltimes W_F, \phi, \epsilon) \to \text{Irr}(\mathbb{C}[W_{qt}, \kappa_{qt}]).$$

It is given by applying the generalized Springer correspondence for $(G_\phi, qt)$ to $(u_\phi, \rho)$.

**Theorem 3.6.** [AMS1, Theorem 9.3]  
There exists a bijection

$$\Phi_e(\mathcal{G}(F))^{s^\lor} \leftrightarrow (\Phi_e(\mathcal{L}(F))^{s^\lor_{\mathcal{L}}} / W_{s^\lor})_{\kappa_{qt}}.$$  

It is almost canonical, in the sense that it depends only on the choices of 2-cocycles $\kappa_{qt}$ as above.

### 3.1. Graded Hecke algebras.

In Theorem 2.15 we saw that the irreducible representations of a (twisted) affine Hecke algebra can be parametrized with a (twisted) extended quotient of a torus by a finite group. Motivated by the analogy with Theorem 3.6 we want to associate to any Bernstein component $\Phi_e(\mathcal{G}(F))^{s^\lor}$ a twisted affine Hecke algebra, whose irreducible representations are naturally parametrized by $\Phi_e(\mathcal{G}(F))^{s^\lor}$. As this turns out to be complicated, we first do something similar with twisted graded Hecke algebras. From a Bernstein component we will construct a family of algebras, such that a suitable subset of their irreducible representations is canonically in bijection with...
As before, we abbreviate $T = Z(M)$. Of course this will be based on the cuspidal quasi-support $[M, v, q \epsilon]_{G_\phi}$ for the group

$$G_\phi := Z_{G_{sc}}^1(\phi|_{W_F}).$$

As before, we abbreviate $T = Z(M)$. One problem is that $Z(G_{sc})$ was left out of $G_{sc}$, so we cannot see it when working in $G_\phi$. We resolve this in a crude way, replacing $G_\phi$ by $G_{\phi} \times X_{nr}(L)$. Although that is not a subgroup of $G_{\phi}$ or $G_{sc}$, the next result implies that the real split part of its centre has the desired shape.

**Lemma 3.7.** We use the notations from Proposition 3.4. The natural map

$$T \times X_{nr}(L) \rightarrow X_{nr}(L)$$

is a finite covering of complex tori.

**Proof.** In Proposition 3.4 we saw that

$$L^\vee \times W_F = Z_{G_{sc}} \times W_F(T).$$

Hence the image of $M^\circ$ under the covering $G_{sc} \rightarrow G_{der}$ is contained in $L^\vee$. It also shows that $W_F$ fixes $T$ pointwise, so

$$T = (Z(M)_{I_F})_{W_F}.$$ 

As $L^\vee$ is a Levi subgroup of $G_{sc}$, it contains $Z(G_{sc})^\circ$. Hence there exists a natural map

$$T \times X_{nr}(G) = (Z(M)_{I_F} \times Z(G_{sc})_{I_F})_{W_F} \rightarrow (Z(L^\vee)_{I_F})_{W_F} = X_{nr}(L).$$

The intersection of $Z(G_{sc})^\circ$ and $G_{der}$ is finite and $T$ lands in $G_{der} \cap L^\vee$, so the kernel of (77) is finite.

Recall from Proposition 3.4 that $\phi(W_F) \subset L^\vee \times W_F$. Hence

$$Z(L^\vee \times W_F) \subset Z_{G_{sc}}(\phi(W_F)) \text{ and } Z(L^\vee \times W_F)^\circ \subset Z_{G_{sc}}(\phi(W_F))^\circ.$$ 

Since $M^\circ$ is a Levi subgroup of $Z_{G_{sc}}(\phi(W_F))^\circ$ and by (76), $T$ equals $Z(L^\vee \times W_F)^\circ$. In particular

$$\dim T = \dim Z(L^\vee \times W_F)^\circ = \dim Z(L^\vee \times I_F)_{W_F} =$$

$$\dim Z(L^\vee \times I_F)_{W_F} - \dim Z(G_{sc})^\circ_{W_F},$$

showing that both sides of (77) have the same dimension. As the map is an algebraic homomorphism between complex tori and has finite kernel, it is surjective.

Recall that $s_{\phi}$ came from the cuspidal quasi-support $(M, v, q \epsilon)$. For $(\phi_b|_{W_F}, v, q \epsilon) \in \Phi_e(L(F))^2 \mathbb{C}$ we can consider the group

$$Z_{G_{sc}}(\phi_b|_{W_F}) \times X_{nr}(L) = G_{\phi_b} \times X_{nr}(L),$$

which contains $M \times X_{nr}(L)$ as a quasi-Levi subgroup. We choose an almost direct factorization for $G_{\phi_b}$ as in (77) and we put

$$H(\phi_b, v, q \epsilon, \rho) := H(G_{\phi_b} \times X_{nr}(L), M \times X_{nr}(L), q \epsilon, c \epsilon),$$

where

$$H(G_{\phi_b} \times X_{nr}(L), M \times X_{nr}(L), q \epsilon, c \epsilon).$$
where \( qE \) is the \( M \)-equivariant cuspidal local system on \( C^M_{\log v} \), with \( qE_{\log v} = qe \) as representations of \( \pi_0(Z_M(v)) = \pi_0(Z_M(\log v)) \). From Lemma 3.7 we see that
\[
\mathbb{H}(\phi_b, v, qe, \vec{r}) = \mathbb{H}(Z_{\mathbb{G}_{\mathbb{C}^M}}(\phi_b | W_F), M, qE, \vec{r}) \otimes S(\text{Lie}(X_{nr}(L^G))^*)
\]
\[
= \mathbb{H}(G_{\phi_b}, M, qE, \vec{r}) \otimes S(\text{Lie}(Z^v \times I_{\mathbb{C}^M}^\vee))^*).
\]
We say that a representation of \( \mathbb{H}(\phi_b, v, qe, \vec{r}) \) is essentially discrete series if its restriction to \( \mathbb{H}(G_{\phi_b}, M, qE, \vec{r}) \) is so, in the sense of [AMS2] Definition 3.27. That means that the real parts of its weights (as \( \mathbb{H}(G_{\phi_b}, M, qE, \vec{r}) \)-representation) must lie in \( \text{Lie}(Z(G_{\phi_b})^0) \otimes t^2_{\mathbb{R}} \).

Let \( X_{nr}(L^L) = X_{nr}(L^L)_{\text{un}} \times X_{nr}(L^L)_{\text{nr}} \) be the polar decomposition of the complex torus \( X_{nr}(L^L) \). Let \( (\phi_b | W_F, v, qe) \in \Phi_v(L(F))^\vee \) with \( \phi_b \) bounded. Suppose that \( (\phi, \rho) \in \Phi_v(L(F))^\vee \) with:

- \( \phi \big|_{I_F} = \phi_b | I_F \); 
- \( \phi(\text{Frob}_F)\phi(\text{Frob}_F)^{-1} \in X_{nr}(L^L)^{rs} \); 
- \( d\phi |_{SL_2(C)} \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) \in \text{Lie}(M) \).

For such \( (\phi, \rho) \) and \( \vec{r} \in \mathbb{C}^d \) we define
\[
E(\phi, \rho, \vec{r}) = \text{IM}^* E_{\log(\phi_b) \log(\phi(\text{Frob}_F)^{-1} \phi_b(\text{Frob}_F)) + d\phi(\vec{r} \cdot \vec{0} \cdot \vec{r})}, \vec{r}, \rho) \in \text{Mod}(\mathbb{H}(\phi_b, v, qe, \vec{r})),
\]
\[
M(\phi, \rho, \vec{r}) = \text{IM}^* M_{\log(\phi_b) \log(\phi(\text{Frob}_F)^{-1} \phi_b(\text{Frob}_F)) + d\phi(\vec{r} \cdot \vec{0} \cdot \vec{r})}, \vec{r}, \rho) \in \text{Irr}(\mathbb{H}(\phi_b, v, qe, \vec{r})).
\]
If in addition \( d\phi \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) \in \text{Lie}(T) + \sigma_v \), as can always be arranged by Proposition 1.7c, then we define an algebraic cocharacter \( \chi_{\phi,v} = \chi_{u_{\phi,v}} \) of \( T \) by
\[
\chi_{\phi,v}(z) = \phi(1, \left( \begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix} \right) ) \gamma_v \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right).
\]
We note that \( \chi_{\phi,v} \) stems from [AMS1] Lemma 7.6 and that
\[
d\chi_{\phi,v}(\vec{r}) = d\phi(\vec{r} \cdot \vec{0} \cdot \vec{r}) - \vec{r} \sigma_v.\]

**Theorem 3.8.** Fix \( \vec{r} \in \mathbb{C}^d \) and \( (\phi_b | W_F, v, qe) \in \Phi_v(L(F))^\vee \) with \( \phi_b \) bounded.

(a) The map \( (\phi, \rho) \mapsto M(\phi, \rho, \vec{r}) \) defines a canonical bijection between
- \( L(\Psi^{-1}(L^v \times W_F, X_{nr}(L^L))_{\mathbb{R}}) \)
- the irreducible representations of \( \mathbb{H}(\phi_b, v, qe, \vec{r}) \) with central character in \( \text{Lie}(X_{nr}(L^L)^{rs})/W_{L^v,\phi_b,v,qe} \times \{ \vec{r} \} \).

(b) Assume that \( R(\vec{r}) \in \mathbb{R}_{\geq 0} \). The following are equivalent:
- \( \phi \) is bounded;
- \( L(\Psi(\phi, \rho) = (L^v \times W_F, \phi_b | W_F, v, qe) \);
- \( E(\phi, \rho, \vec{r}) \) is tempered;
- \( M(\phi, \rho, \vec{r}) \) is tempered.

(c) Suppose that \( R(\vec{r}) \in \mathbb{R}_{\geq 0} \). Then \( \phi \) is discrete if and only if \( M(\phi, \rho, \vec{r}) \) is essentially discrete series and the rank of \( R(G_{\phi_b}^0, T) \) equals \( \text{dim}_C(T) \).

In this case \( \phi(\text{Frob}_F)\phi(\text{Frob}_F)^{-1} \) comes from an element of \( Z(G_{\phi_b}^0) \times X_{nr}(L^G) \) via Lemma 3.7.

(d) Let \( \zeta \in X_{nr}(Z^L)^{rs} \). Then
\[
M(\zeta \phi, \rho, \vec{r}) = \log(\zeta) \otimes M(\phi, \rho, \vec{r}) \quad \text{and} \quad E(\zeta \phi, \rho, \vec{r}) = \log(\zeta) \otimes E(\phi, \rho, \vec{r}).
\]

(e) Suppose that \( R(\vec{r}) \in \mathbb{R}_{\geq 0} \) and that \( \phi(\text{Frob}_F)\phi(\text{Frob}_F)^{-1} \) comes from \( Z(G_{\phi_b}^0) \times X_{nr}(L^G) \) via Lemma 3.7. Then \( M(\phi, \rho, \vec{r}) = E(\phi, \rho, \vec{r}) \).
(f) If \( d \phi \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) \in \text{Lie}(T) + \sigma_\rho \), then \( E(\phi, \rho, \bar{r}) \) and \( M(\phi, \rho, \bar{r}) \) admit the central character \( W_{\mathcal{S}_u, (\phi_0), q_0} \).

**Proof.** (a) By Theorem 3.6, every element of \( L^1(\mathcal{L}^\vee \times \mathbf{W}_F, X_{\text{nr}}(\mathcal{L}^\vee)_\text{rs} \phi_b|_W F, v, q_\epsilon) \) has a representative \((\phi, \rho)\) with \( \phi|_\mathbf{W}_F \in X_{\text{nr}}(\mathcal{L}^\vee)_\text{rs} \phi_b|_W F \). Then \( \phi|_{\mathbf{L}} F \) is fixed, so \( \phi|_\mathbf{W}_F \) can be described by the single element \( \phi(F_{\text{Frob}}) \phi_b(F_{\text{Frob}})^{-1} \in X_{\text{nr}}(\mathcal{L}^\vee)_\text{rs}. \)

Since \( X_{\text{nr}}(\mathcal{L}^\vee)_\text{rs} \) is the real split part of a complex torus, there is a unique logarithm \( (81) \)

\[
\sigma_0 = \log(\phi(F_{\text{Frob}}) \phi_b(F_{\text{Frob}})^{-1}) \in \text{Lie}(X_{\text{nr}}(\mathcal{L}^\vee)_\text{rs}).
\]

Clearly \((\phi_b, v)\) is the unique bounded \( L \)-parameter in \( X_{\text{nr}}(\mathcal{L}^\vee)_\text{rs}(\phi_b, v) \). Hence every element of \( G_{\phi}^\vee \) centralizes \( \phi \) also centralizes \( \phi_b \), which implies

\[
G_\phi = Z_{G_{\phi}^\vee}(\phi|_\mathbf{W}_F) \subset Z_{G_{\phi}^\vee}(\phi_b|_W F) = G_{\phi_b}.
\]

In particular \( \phi(\text{SL}_2(\mathbb{C})) \subset G_{\phi_b} \) and

\[
\pi_0(Z_{G_{\phi_b}}(u_\phi)) = \pi_0(Z_{G_{\phi_b}}(\sigma_0, \log(u_\phi))).
\]

By assumption \( q \Psi_{G_{\phi_b}}(u_\phi, \rho) = (v, q_\epsilon) \), and by [AMS2 Proposition 3.7] this cuspidal quasi-support is relevant for

\[
\mathbb{H}(\phi_b, v, q_\epsilon, \bar{r}) = \mathbb{H}(G_{\phi_b} \times X_{\text{nr}}(\mathcal{L}^\gamma), M \times X_{\text{nr}}(\mathcal{L}^\gamma), q_\epsilon, \bar{r}).
\]

By Proposition 1.7, \((\phi, \rho)\) is conjugate to an enhanced \( L \)-parameter with all the above properties, which in addition satisfies

\[
d\phi|_{\text{SL}_2(\mathbb{C})} \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) \in \text{Lie}(M).
\]

Consequently \((\log(u_\phi), \sigma_0, \bar{r}, \rho)\) is a parameter of the kind considered in Section 1 and \( \phi|_{\text{SL}_2(\mathbb{C})} \) can play the role of \( \gamma \) from (15). By reversing the above procedure every parameter \((y, \sigma', \bar{r}, \rho')\) for \( \mathbb{H}(\phi_b, v, q_\epsilon, \bar{r}) \) gives rise to an element of

\[
L^1(\mathcal{L}^\vee \times \mathbf{W}_F, X_{\text{nr}}(\mathcal{L}^\vee)_\text{rs} \phi_b|_W F, v, q_\epsilon).
\]

The equivalence relations on these two sets of parameters agree, for both come from conjugation by \( G_{\phi_b} \).

Now it follows from Theorem 1.4, Proposition 1.7 and Lemma 1.8 that

\[
L^1(\mathcal{L}^\vee \times \mathbf{W}_F, X_{\text{nr}}(\mathcal{L}^\vee)_\text{rs} \phi_b|_W F, v, q_\epsilon)
\]

parametrizes the part of \( \text{Irr}_r(\mathbb{H}(\phi_b, v, q_\epsilon)) \) with central character in

\[
\text{Lie}(X_{\text{nr}}(\mathcal{L}_\text{rs})/W_{\mathcal{S}_u, \phi_b, v, q_\epsilon} \times \{ \bar{r} \}).
\]

As in [AMS2 Theorem 3.29] and Proposition 1.6, we compose this parametrization with the Iwahori–Matsumoto involution from (26). Then the representation associated to \((\phi, \rho)\) becomes \( \pi(\phi, \rho, r) \).

(b) By [AMS2 Theorem 3.25] and [AMS2 (84)] the third and the fourth statements are both equivalent to

\[
\phi(F_{\text{Frob}}) \phi_b(F_{\text{Frob}})^{-1} \in \text{Lie}(X_{\text{nr}}(\mathcal{L}^\vee)_\text{un}).
\]

But by construction this lies in \( \text{Lie}(X_{\text{nr}}(\mathcal{L}^\vee)_\text{rs}) \), so the statement becomes \( \phi(F_{\text{Frob}}) = \phi_b(F_{\text{Frob}}) \). As \((\phi_b, v)\) is the only bounded \( L \)-parameter in \( X_{\text{nr}}(\mathcal{L}^\vee)_\text{rs}(\phi_b, v) \), this holds if and only if \( \phi \) is bounded. Since the map \( L^1 \) preserves \( \phi|_W F \), the statement \( \phi(F_{\text{Frob}}) = \phi_b(F_{\text{Frob}}) \) is also equivalent to

\[
L^1(\phi, \rho) = (\mathcal{L}^\vee \times W_F, \phi_b|_W F, v, q_\epsilon).
\]
Knowing these equivalences, the equality $M(\phi, \rho, \tilde{r}) = E(\phi, \rho, \tilde{r})$ is given in Proposition 1.6.b.

(c) Suppose that $\phi$ is discrete. Then

$$G^\circ_{\phi} = Z_{G^\circ_{\phi}}(\phi(W_F))^\circ = Z_{G^\circ_{\phi}}(\phi_b(W_F), \sigma)^\circ$$

is a reductive group in which $\phi(\text{SL}_2(C))$ has finite centralizer. This implies that $G^\circ_{\phi}$ is semisimple and that $u_\phi$ is distinguished in it. The first of these two properties implies that $G^\circ_{\phi}$ is a full rank subgroup of $G_{\phi_b}$, and that $G^\circ_{\phi_b}$ is also semisimple. In other words, $R(G^\circ_{\phi_b}, T)$ has rank equal to the dimension of $T$. Then $u_\phi$ is distinguished in $G^\circ_{\phi_b}$ as well, and [AMS2 (85)] says that $M(\phi, \rho, \tilde{r})$ is essentially discrete series.

Conversely, suppose that $M(\phi, \rho, \tilde{r})$ is essentially discrete series and that the rank of $R(G^\circ_{\phi_b}, T)$ equals $\dim_G(T)$. Then $G^\circ_{\phi_b}$ is semisimple and by [AMS2 (85)] $u_\phi \in G^\circ_{\phi_b}$ is distinguished in $G^\circ_{\phi_b}$. Hence $Z_{G^\circ_{\phi_b}}(u_\phi)^\circ$ is contained in the unipotent group $Z_{G^\circ_{\phi_b}}(u_\phi)^\circ$, and itself unipotent. It is known (see for example [Ree, §4.3]) that

$$Z_{G^\circ_{\phi_b}}(\phi)^\circ = Z_{G^\circ_{\phi}}(\phi(\text{SL}_2(C)))^\circ$$

is the maximal reductive quotient of $Z_{G^\circ_{\phi_b}}(u_\phi)^\circ$. Hence $Z_{G^\circ_{\phi_b}}(\phi)^\circ$ is trivial, which means that $\phi$ is discrete.

In this case Proposition 1.5.c says that $\sigma_0 \in Z(G^\circ_{\phi_b} \times X_m(L\mathcal{G}))$. Via the exponential map, that translates to the statement about $\phi(\text{Frob}_b \phi_b(\text{Frob}_F))^{-1}$. 

(d) This is a direct consequence of Proposition 1.6.d (and, for $E(\phi, \rho, \tilde{r})$, the proof thereof).

(e) Via (81), the condition becomes $\sigma_0 \in Z(G^\circ_{\phi_b} \times X_m(L\mathcal{G}))$. Apply Proposition 1.6.e.

(f) This follows from Lemma 1.8 with $\gamma = \phi|_{\text{SL}_2(C)}$. \qed

We conclude this paragraph with some remarks about parabolic induction. Suppose that $\mathcal{Q}(F) \subset \mathcal{G}(F)$ is a Levi subgroup such that $\phi$ has image in $L\mathcal{Q}$. Let $\mathcal{Q}^\vee_c$ be the inverse image of $\mathcal{Q}^\vee$ in $G^\vee_{\text{sc}}$, by [Bor, §3] it equals $Z_{G^\vee_{\text{sc}}}(Z(\mathcal{Q}_c^\vee \rtimes W_F)^\circ)$. Therefore

$$Z^1_{\mathcal{Q}^\vee_c}(\phi_b|W_F) = Z^1_{G^\vee_{\text{sc}}(\phi_b|W_F)} \cap Z_{G^\vee_{\text{sc}}}(Z(\mathcal{Q}_c^\vee \rtimes W_F)^\circ) = G^\circ_{\phi_b} \cap Z_{G^\vee_{\text{sc}}}(Z(\mathcal{Q}_c^\vee \rtimes W_F)^\circ).$$

(82)

This in turn shows that

$$G^\circ_{\phi_b} \cap Z^1_{\mathcal{Q}^\vee_c}(\phi_b|W_F) = Z_{\mathcal{Q}^\vee_c}(\phi_b(W_F))^\circ$$

is a Levi subgroup of $G^\circ_{\phi_b}$. Furthermore $Z^1_{\mathcal{Q}^\vee_c}(\phi_b|W_F)$ contains $M$, for the cuspidal quasi-support of $(\phi, \rho)$ with respect to $L\mathcal{G}$ is the same as the cuspidal quasi-support of $(\phi, \rho^Q)$ with respect to $L\mathcal{Q}$, for a suitable $\rho^Q \in \text{Irr}(S^Q_{\phi})$ [AMS1, Proposition 5.6.a].

Let $\zeta$ be the character of $Z(G^\vee_{\text{sc}})$ determined by $\rho^Q$, an extension of the character $\zeta_\mathcal{G} \in \text{Irr}(Z(G^\vee_{\text{sc}})^W_F)$ which was used to define $\mathcal{G}(F)$-relevance. Let $\zeta^Q \in \text{Irr}(Z(G^\vee_{\text{sc}})^W_F)$ be derived from $\zeta$ as in [AMS1 Lemma 7.4]. Let $p_{\zeta} \in \text{C}[S_{\phi}]$ and $p_{\zeta^Q} \in \text{C}[S^Q_{\phi}]$ be the central idempotents associated to these characters.

Let $S_{\phi, \mathcal{Q}}$ be the component group of the centralizer of $u_\phi$ in $Z^1_{\mathcal{Q}^\vee_c}(\phi|W_F)$, or equivalently the component group of the centralizer of $(\phi(\text{Frob}), u_\phi)$ in (82). By [AMS1 Lemma 7.4.c] there exist a canonical isomorphism and a canonical injection

$$p_{\zeta^Q} \text{C}[S^Q_{\phi}] \cong p_{\zeta} \text{C}[S_{\phi, \mathcal{Q}}] \rightarrow p_{\zeta} \text{C}[S_{\phi}].$$
This enables to restrict representations of $\mathcal{S}_\phi$ to $\mathcal{S}_\phi^G$, and it shows that enhancements for $\phi \in \Phi(Q(F))$ can just as well be constructed via \cite{AMS2} and $\mathcal{S}_\phi, Q$.

That is, $G_{\phi, v} \times X_{\text{nr}}(\ell \mathcal{G})$ and $Z_{\mathcal{G}, 1}(\phi|\mathcal{W}_F) \times X_{\text{nr}}(\ell \mathcal{G})$ fulfill the conditions of \cite{AMS2} Proposition 3.22] and Corollary 2.14. It follows that the families of representations

\[(\phi, \rho, \vec{r}) \mapsto E_{\log(u_0), \log(\phi(\text{Frob}_F)) - \phi_0(\text{Frob}_F)) + d\vec{r}(0_0 - \vec{r}), \vec{r}, \rho} \in \text{Mod}(\Phi(\phi, v, qe, \vec{r})) ,
\]

\[(\phi, \rho, \vec{r}) \mapsto M_{\log(u_0), \log(\phi(\text{Frob}_F)) - \phi_0(\text{Frob}_F)) + d\vec{r}(0_0 - \vec{r}), \vec{r}, \rho} \in \text{Irr}(\Phi(\phi, v, qe, \vec{r}))
\]

are compatible with parabolic induction in the same sense as \cite{AMS2} Proposition 3.22] and Corollary 2.14. In view of \cite{AMS2} (83) this does not change upon applying the Iwahori–Matsumoto involution, so it also goes for the representations $E(\phi, \rho, \vec{r})$ and $M(\phi, \rho, \vec{r})$ considered in Theorem 3.8.

3.2. Root systems.

Let us fix an inertial equivalence class $s_\triangleleft^\vee$ for $\Phi_\mathbb{R}(\mathcal{G}(F))$, and use the notations from Proposition 3.4. In particular $T = Z(M)^\circ = Z(\mathcal{L}_c^\vee)^{\mathcal{W}_F, \mathcal{O}}$.

For any $(\phi|\mathcal{W}_F, v, qe) \in s_\triangleleft^\vee$ we define

\[(83) \quad J := Z_{\mathcal{O}_\mathbb{R}}(\phi|1_F) ,
\]

a variation on $G_\phi$ from (75). These are possibly disconnected reductive groups with

\[(84) \quad G_\phi \subset J \quad \text{and} \quad G_\phi^0 = Z_J(\phi(\text{Frob}_F))^\circ .
\]

In this paragraph, we use the convention that a root system is a finite and integral root system.

**Proposition 3.9.** Define $R(J^\circ, T)$ as the set of $\alpha \in X^*(T) \setminus \{0\}$ which appear in the adjoint action of $T$ on $\text{Lie}(J^\circ)$.

(a) $R(J^\circ, T)$ is a root system.

(b) There exists a $(\phi_1|\mathcal{W}_F, v, qe)$ such that $R(G_{\phi_1}^0, T)_{\text{red}} = R(J^\circ, T)_{\text{red}}$.

(c) If $t \in T$ commutes with $\mathcal{G}^\vee$, then it lies in the kernel of every $\alpha \in R(J^\circ, T)$.

**Remark 3.10.** This result does not imply that $R(G_{\phi_1}^0, T) = R(J^\circ, T)$. For example if $\mathcal{G} = U_{2n+1}$ is an unramified unitary group and $\phi_1(1_F) = 1$, then $R(G_{\phi_1}^0, T)$ has type $B_n$ while $R(J^\circ, T)$ has type $BC_n$.

**Proof.** (a) From \cite{Lus2} Proposition 2.2] we know that every $R(G_{\phi_1}^0, T)$ is a root system. However, this result does not apply to our current $J^\circ$, as $(M, v, qe)$ need not be a cuspidal quasi-support for a group with neutral component $J^\circ$.

We will check the axioms of a root system for $R(J^\circ, T)$. Every single $\alpha \in R(J^\circ, T)$ appears in $R(G_{\phi_1}^0, T)$ for a suitable choice of $\phi$ (see the construction of $\phi_1$ below). This gives rise to a coroot $\alpha^\vee \in X_\alpha(T)$ with $\langle \alpha^\vee, \alpha \rangle = 2$. For arbitrary $\alpha, \beta \in R(J^\circ, T)$ we have to show that

1. $\langle \alpha^\vee, \beta \rangle \in \mathbb{Z}$;
2. $s_\alpha(\beta) \in R(J^\circ, T)$, where $s_\alpha : X^*(T) \to X^*(T)$ is the reflection associated to $\alpha$ and $\alpha^\vee$.

Assume first that $\alpha$ and $\beta$ are linearly independent in $X^*(T)$. The element $\phi(\text{Frob}_F) \in \mathcal{L}_c^\vee \times \mathcal{W}_F$ centralizes $T$ and normalizes $J^\circ$, so it stabilizes each of the root subspaces $g_\alpha \subset \text{Lie}(J^\circ)$. Let $\lambda_\alpha$ (respectively $\lambda_\beta$) be an eigenvalue of $\text{Ad}(\phi(\text{Frob}_F))|_{g_\alpha}$ (respectively $\text{Ad}(\phi(\text{Frob}_F))|_{g_\beta}$). Since $\alpha$ and $\beta$ are linearly independent, we can find a
\( t \in T \) with \( \alpha(t^{-1}) = \lambda_\alpha \) and \( \beta(t^{-1}) = \lambda_\beta \). Define \( (\phi_t|_{W_F}, v, q\varepsilon) \in \mathcal{S}_L^\varepsilon \) by \( \phi_t|_{I_F} = \phi|_{I_F} \) and
\[
(85) \quad \phi_t(\text{Frob}_F) = \phi(\text{Frob}_F)(\text{image of } t \in G_{\text{des}}^\varepsilon).
\]

Clearly \( \alpha, \beta \in R(G_{\phi_1}^\circ, T) \). Since this is a root system, (i) and (ii) hold for \( \alpha \) and \( \beta \) inside \( R(G_{\phi_1}^\circ, T) \). Then they are also valid in the larger set \( R(J^\circ, T) \).

Next we consider linearly dependent \( \alpha, \beta \). Then \( s_{\alpha}(\beta) = -\beta \), so (ii) is automatically fulfilled.

Suppose that there exists a \( \gamma \in R(J^\circ, T) \setminus \mathbb{Q} \alpha \) which is not orthogonal to \( \alpha \). As before, we can find \( \phi_2, \phi_3 \) such that \( \alpha, \gamma \in R(G_{\phi_2}^\circ, T) \) and \( \beta, \gamma \in R(G_{\phi_3}^\circ, T) \). Hence both \( \{\alpha, \gamma\} \) and \( \{\beta, \gamma\} \) generate rank two irreducible root systems in \( X^*(T) \), and these root systems have the same \( \mathbb{Q} \)-span. From the classification of rank two root systems we see that \( \mathbb{Q} \alpha \cap R(J^\circ, T) \) is either \( \{\pm \hat{\alpha}\} \) or \( \{\pm \hat{\alpha}, \pm 2\hat{\alpha}\} \) for a suitable \( \hat{\alpha} \). In particular (i) holds, because
\[
(\alpha^\vee, \beta) \in \{1, 2, 4\} \subset \mathbb{Z}.
\]

Finally we suppose that \( \mathbb{Q} \alpha \cap R(J^\circ, T) \) is orthogonal to \( R(J^\circ, T) \setminus \mathbb{Q} \alpha \). As above, we may pick \( \phi \) such that \( \alpha \in R(G_{\phi_1}^\circ, T) \). By assumption \( \beta = c\alpha \) for some \( c \in \mathbb{Q}^\times \). Pick \( \phi_t \) so that \( \beta \in R(G_{\phi_t}^\circ, T) \). As
\[
(\beta^\vee, \beta) = 2 = (\alpha^\vee, \alpha),
\]
we have \( c\alpha = \beta \in X^*(T) \) and \( -c^{-1}\alpha^\vee = \beta^\vee \in X_\ast(T) \). It follows that \( c \in \{1/2, 1, 2\} \) and \( (\alpha^\vee, \beta) \in \{1, 2, 4\} \).

(b) Let \( \Delta \) be a basis of the reduced root system \( R(J^\circ, T)_{\text{red}} \) — which is well-defined by part (a). Let \( \lambda_\alpha \in \mathbb{C} (\alpha \in \Delta) \) be an eigenvalue of \( \text{Ad}(\phi(\text{Frob}_F)) \) on \( g_\alpha \). Since \( \Delta \) is linearly independent, we can find \( t_1 \in T \) with \( \alpha(t_1^{-1}) = \lambda_\alpha \) for all \( \alpha \in \Delta \). We put \( \phi_1 := \phi_{t_1} \), where \( \phi_{t_1} \) is defined by [85]. Then \( \Delta \) is contained in the root system \( R(G_{\phi_1}^\circ, T) \). The Weyl group of \( (J^\circ, T) \) is generated by the reflections \( s_\alpha \) with \( \alpha \in \Delta \), so it equals the Weyl group of \( (G_{\phi_1}^\circ, T) \). In particular it stabilizes \( R(G_{\phi_1}^\circ, T) \). Every element of \( R(J^\circ, T)_{\text{red}} \) is in the Weyl group orbit of some \( \alpha \in \Delta \), so \( R(G_{\phi_1}^\circ, T) \) contains \( R(J^\circ, T)_{\text{red}} \).

(c) Such a \( t \) commutes with \( G_{\text{sc}}^\varepsilon \) and with \( J \), so its image under the adjoint representation of \( J^\circ \) is trivial. \( \square \)

We define
\[
(86) \quad W_{\text{sc}}^\varepsilon := W(R(J^\circ, T)) = N_{J^\circ}(T)/Z_{J^\circ}(T).
\]

Since \( \mathcal{L}_c^\varepsilon = Z_G^\varepsilon(T) \), it equals
\[
N_{Z_G^\varepsilon}((\phi(I_F)))(T)/Z_{\mathcal{L}_c^\varepsilon}(\phi(I_F))^\circ = N_{J^\circ}(\mathcal{L}_c^\varepsilon)/Z_{J^\circ}(\mathcal{L}_c^\varepsilon).
\]

By Proposition 3.9 [86] also equals
\[
W(R(G_{\phi_1}^\circ, T)) = N_{G_{\phi_1}^\circ}(T)/Z_{G_{\phi_1}^\circ}(T).
\]

Any element of \( G_{\phi_1}^\circ \) which normalizes \( T = T^\text{red} \) will also normalize \( \mathcal{L}^\varepsilon \ltimes W_F = Z_{G^\varepsilon \times W_F}(T) \) and \( M = Z_{G_{\phi_1}^\circ}(T) \), while by [AMS2, Lemma 2.1] it stabilizes \( C^\varepsilon_M \) and \( q\mathcal{E} \). The group
\[
(87) \quad W_{\text{sc}}^\varepsilon \subset N_{G^\varepsilon}(\mathcal{L}^\varepsilon \ltimes W_F, M, q\mathcal{E})/\mathcal{L}^\varepsilon
\]
From Lemma 3.7 we get a natural, finite covering of tori \( X \). By conjugation in \( G \) fixes 1.

Conjugating everything in \( \mathfrak{R}_\phi \) via this bijection we can retract the action of \( X \) but the group \( L \) naturally contains \( W_\phi \). Consequently the action of \( W_\phi \) on \( T \) is independent of the choice of \( \phi_1 \). On the other hand, the action of \( W_\phi \) on \( T_\phi \) may very well depend on the choice of the basepoint \( \phi_1 \).

Analogous to (88), we consider the finite group \( X_{\mathrm{nr}}(L_e)_{\phi_1} \) only depends on \( s_\phi^\vee \), not on \( \phi_1 \). Moreover it is finite, for it consists of elements coming from the finite group \( L_{\text{der}}^\vee \cap Z(\mathcal{L}^\vee) \). Writing

\[
T_\phi^\vee = X_{\mathrm{nr}}(L_e)/X_{\mathrm{nr}}(L_e)_{\phi_1},
\]

we obtain a bijection

\[
T_\phi^\vee \to s_\phi^\vee : z \mapsto [z \phi_1 | w_F, v, q e].
\]

From Lemma 3.7, we get a natural, finite covering of tori \( X_{\mathrm{nr}}(L) \times X_{\mathrm{nr}}(L) \to T_\phi \), which is injective on \( T/T_{\phi_1} \), and even if they do, we need not descend further to characters of \( T_\phi \). The latter problem already occurs for the Levi subgroup \( GL_2(F) \times GL_4(F) \).

To set things up properly, we consider \( \phi(\text{Frob}_F) \) as a semisimple automorphism of \( J \). By [Ste Theorem 7.5] \( \phi(\text{Frob}_F) \) stabilizes a Borel subgroup \( B_J \) of \( J^0 \), and a maximal torus \( T_J \) thereof. Then \( B_J^\phi(\text{Frob}_F) \) is a Borel subgroup of \( G_\phi = J^0(\text{Frob}_F) \). By conjugation in \( G_\phi \), we may assume that \( u_\phi \in B_J^\phi(\text{Frob}_F) \). Then we can choose \( q \Psi_{G_\phi}(u_\phi, \rho) = [M, v, q e]_{G_\phi} \) so that

\[
M \supset T_J^\phi(\text{Frob}_F) \supset T = Z(M)^0.
\]

Conjugating everything in \( G^\vee \times W_F \), we can arrange further that \( W_F \) stabilizes a Borel subgroup \( B^\vee \) of \( G^\vee \) and a maximal torus \( T^\vee \) thereof, such that \( B_J \subset B_J^\vee \) and \( T_J \subset T_J^\vee \) (where the subscript \( e \) indicates the preimage in \( G^\vee \)). As in [AMS] Proposition 7.3 and Proposition 3.4, this implies the existence of a Levi subgroup \( \mathcal{L} \subset \mathcal{G} \) such that \( \mathcal{L}^\vee \supset T^\vee \).

\[
Z_{\mathcal{G}^\vee \times W_F}(Z(M)^0) = \mathcal{L}^\vee \times W_F
\]

and \( (\phi|w_F, v, q e) \) is a cuspidal L-parameter for \( \mathcal{L}(F) \). We define

\[
L_J := J \cap \mathcal{L}^\vee = Z_J(Z(\mathcal{L}_c^\vee \times I_F)^0),
\]

so that \( L_J^0 \) is a Levi subgroup of \( J^0 \).
Definition 3.11. For each \( \alpha \in R(J^o, T)_{\text{red}} \), we define \( m_\alpha \in \mathbb{Z}_{\geq 0} \) by the following requirements:

- Suppose that the preimage of \( \alpha \) in \( R(J^o, T_J) \) lies in a single irreducible component of that root system. Then \( m_\alpha \) is the smallest integer such that \( T_{\phi_1} \subset \ker(m_\alpha \alpha) \).
- Suppose that the preimage of \( \alpha \) in \( R(J^o, T_J) \) meets \( k > 1 \) irreducible components of that root system, permuted transitively by the action of \( \phi(\text{Frob}_F) \). Then \( m_\alpha = m_\alpha(\text{Frob}_F) \) equals \( k \) times the number \( m_\alpha(\text{Frob}_F) \) computed (as in the first bullet) with respect to the action of \( \phi(\text{Frob}_F) \). Equivalently, \( m_\alpha \) is \( k \) times the analogous number obtained by replacing \( W_F \) with the Weil group of the degree \( k \) unramified extension of \( F \).

These conditions guarantee that \( m_\alpha \alpha \) descends to a character of \( T/T_{\phi_1} \). Moreover \( m_\alpha \) is the minimal such integer, unless maybe when \( \alpha \in 2X^*(T) \) and \( k \) even. Extend \( m_\alpha \alpha \) to a character of \( T/T_{\phi_1} \times X_m(L^G) \), trivial on the second factor. In view of Proposition 3.9.c, \( m_\alpha \alpha \) is trivial on the kernel of (90), and hence descends naturally to a character of \( T_{s^\nu} \). We define

\[
R_{s^\nu} = \{ m_\alpha \alpha : \alpha \in R(J^o, T)_{\text{red}} \} \subset X^*(T_{s^\nu}).
\]

The construction of \( m_\alpha \) entails that \( (m_\alpha \alpha)^\vee := \alpha^\vee / m_\alpha \) is an element of \( X_*(T_{s^\nu}) \).

Lemma 3.12. (a) \( R_{s^\nu} \) is a reduced root system, and it is stable under the action of \( W_{s^\nu} \) on \( X^*(T_{s^\nu}) \).

(b) For \( \alpha \) and \( \beta \) in the same connected component of \( R(J^o, T)_{\text{red}} \), \( m_\alpha = m_\beta \), or \( m_\alpha = \| \alpha \|^2 \| \beta \|^2 m_\beta \).

Proof. (a) The choice of representatives for \( W_{s^\nu} \) after (87) makes that it normalizes \( T_{\phi_1} \). Conjugating these representatives further in \( L_J \), we can achieve that they normalize \( T_J \). Then \( W_{s^\nu} \) stabilizes all the data that go into the definition of \( m_\alpha \), so map \( \alpha \mapsto m_\alpha \) is constant on \( W_{s^\nu} \)-orbits. Consequently \( R_{s^\nu} \) is \( W_{s^\nu} \)-stable. In particular \( R_{s^\nu} \) is stable under all the reflections \( s_{m_\beta} = s_\beta \) with \( m_\beta \in R_{s^\nu} \).

As \( (m_\alpha \alpha)^\vee \in X_*(T_{s^\nu}) \) and \( m_\beta \in X^*(T_{s^\nu}) \), their natural pairing is an integer, which says that the root system \( R_{s^\nu} \) is integral.

(b) By the \( W_{s^\nu} \)-invariance of \( \alpha \mapsto m_\alpha \), it suffices to consider simple roots \( \alpha, \beta \in R(J^o, T)_{\text{red}} \). By definition, in \( R(J^o, T)_{\text{red}} \):

\[
s_\alpha(m_\beta \beta) = m_\beta \beta - m_\beta (\alpha^\vee, \beta) \alpha.
\]

On the other hand, in the integral root system \( R_{s^\nu} \):

\[
s_{m_\alpha}(m_\beta \beta) = m_\beta \beta - (m_\alpha \alpha)^\vee, m_\beta m_\alpha \alpha
\]

Comparing (91) and (92), we see that \( m_\beta (\alpha^\vee, \beta) \in m_\alpha \mathbb{Z} \). When \( (\alpha^\vee, \beta) = -1 \), this says that \( m_\beta \geq m_\alpha \). In that case

\[
(\beta^\vee, \alpha) = -\| \alpha \|^2 \| \beta \|^2 - 2 \in \{-1, -2, -3\}
\]

and \( m_\alpha \in m_\beta (\beta^\vee, \alpha)^{-1} \mathbb{Z} \), so \( m_\alpha = m_\beta \) or \( m_\alpha = \| \alpha \|^2 \| \beta \|^2 m_\beta \). Every pair of non-orthogonal simple roots is of this form, which allows to draw the same conclusion on an entire connected component of \( R(J^o, T) \).

Lemma 3.12 implies that

\[
R_{s^\nu} := (R_{s^\nu}, X^*(T_{s^\nu}), R_{s^\nu}, X_*(T_{s^\nu}))
\]
is a root datum with an action of \( W_{\nu} \).

**Lemma 3.13.** Let \( \alpha \in R(J^0, T)_{\text{red}} \) and \( t \in T \).

(a) If \( (m_\alpha \alpha)(t) = 1 \), then \( \alpha \in R(G_t^0, T) \).

(b) Suppose that \( R(G_t^0, T) \) contains \( \alpha \) or \( 2\alpha \). Then \( (m_\alpha \alpha)(t) = 1 \) or \( (m_\alpha \alpha)(t) = -1 \) and \( (m_\alpha \alpha)^\nu \in 2X_s(T_\nu) \).

**Proof.** (a) By the construction of \( \alpha \), \( t \phi_1 \) is \( \mathcal{L}^\nu \)-conjugate to a \( t' \phi_1 \) with \( \alpha(t') = 1 \). Hence \( \alpha \in R(G_t^0, T) = R(G_t^0, T) \).

(b) The reflection \( s_\alpha \) stabilizes \( \phi_1 \in s_\nu \equiv T_\nu^0 \). Thus \( s_\alpha \) fixes \( t \) considered as element of \( T_\nu \). As \( R_\nu \) is a root datum, we have reduced to the well-known setting of Weyl groups acting on complex tori associated to root data. In that setting we conclude with \( \text{Lus3} \) Lemma 3.15.

We endow \( R_\nu \) with the set of simple roots determined by the Borel subgroup \( B_J \subset J^0 \). We look for parameter functions \( \lambda \) and \( \lambda^* \) on \( R_\nu \) which are compatible with specialization to the graded Hecke algebras from Paragraph 3.1. Recall from [35] that \( \lambda^*(\alpha) \) is defined to be \( \lambda(\alpha) \) unless \( \alpha \) is a short root in a type B root subsystem of \( R_\nu \).

**Proposition 3.14.** (a) There exist unique \( W_\nu \)-invariant parameter functions

\[
\lambda : R_\nu \rightarrow \mathbb{Q}_{>0}, \quad \lambda^* : \{m_\alpha \alpha \in R_\nu : (m_\alpha \alpha)^\nu \in 2X_s(T_\nu)\} \rightarrow \mathbb{Q}
\]

such that, for every \( (\phi_1, v, qe) \in s_\nu \) with \( \phi_1 \) bounded, the reduction via Theorems 2.5 and 2.11 gives the graded Hecke algebra \( \mathbb{H}(\phi_1, v, qe, \tilde{v}) \) from (78).

(b) The basepoint \( \phi_1 \) of \( s_\nu \) can be chosen so that \( \lambda \) has image in \( \mathbb{Z}_{>0} \) and \( \lambda^* \) has image in \( \mathbb{Z}_{>0} \).

**Proof.** (a) The aforementioned reduction produces graded Hecke algebras with the roots \( m_\alpha \alpha \), whereas in (78) the root system is contained in \( R(J^0, T) \). We reconcile this by imposing \( c(m_\alpha \alpha) = m_\alpha c(\alpha) \), which is allowed because it preserves the braid relations in a graded Hecke algebra (Proposition 1.1).

For \( \phi_1 = \phi_1, \) imposes the conditions

\[
\lambda(m_\alpha \alpha) + \lambda^*(m_\alpha \alpha) = m_\alpha c(\alpha) \quad \alpha \in R(J^0, T)_{\text{red}},
\]

where \( c(\alpha) \in \mathbb{Z}_{>0} \) is computed as in Proposition 2.1 with respect to \( G_t^0 \).

For any \( z \in \mathbb{C} \), the variety

\[
\{m_\alpha \alpha = z\} = \{t \in T : (m_\alpha \alpha)(t) = z\}
\]

is a union of cosets of subtori of \( T \). If we fix an \( m_\alpha \)-th root \( z^{1/m_\alpha} \), then the definitions of \( m_\alpha \) and \( T_{\phi_1} \) entail that every element of \( \{m_\alpha \alpha = z\} \) is \( \mathcal{L}^\nu \)-associate to an element of \( \{\alpha = z^{1/m_\alpha}\} \), where associate means that the corresponding objects \( t_\phi \) are \( \mathcal{L}^\nu \)-conjugate. Given \( \phi_1 \in \mathbb{A}_{J^0, v, qe} \), the value of \( c(\alpha) \) depends only on the root subspaces for \( \alpha \) and \( 2\alpha \) in \( G_t^0 \). Thus in \( s_\nu \), \( c(\alpha) \) depends only on \( \alpha(t) \) and hence only on \( (m_\alpha \alpha)(t) \).

In view of Lemma 3.13 we need to consider at most two values of \( c(\alpha) \) for \( \phi_1 \in s_\nu \): one for \( \phi_1 \) and maybe another one, say \( c'(\alpha) \), for a \( t_\phi \) with \( (m_\alpha \alpha)(t) = -1 \). When \( R(G_t^0, T) \) contains \( 2\alpha \) but not \( \alpha \), we must rescale \( c'(\alpha) = c(2\alpha)/2 \) so that it really refers to \( \alpha \) like \( c(\alpha) \).
When \((m_\alpha \alpha)^\vee \notin 2X_*(T_\phi^\vee)\), Lemma 3.13 says that \(R(G_{t\phi_1}^\circ, T)\) contains \(\alpha\) or \(2\alpha\) if and only if \((m_\alpha \alpha)(t) = 1\). By convention \(\lambda^s(m_\alpha \alpha) = \lambda(m_\alpha \alpha)\), and the only way to solve (93) is setting
\[
\lambda(m_\alpha \alpha) = c(\alpha)m_\alpha/2 \in \mathbb{Q}_{>0}.
\]
Next consider an \(\alpha \in R(J^0, T)_{\text{rd}}\) with \((m_\alpha \alpha)^\vee \in 2X_*(T_\phi^\vee)\). Then \(s_{m_\alpha \alpha}\) fixes \(t\phi_1\) if \((m_\alpha \alpha)(t) = -1\), so we have to consider \(c^*(\alpha)\) in \(\mathbb{Z}_{\geq 0}\). If \(\alpha\) and \(2\alpha\) do not belong to \(R(G_{t\phi_1}^\circ, T)\) for one such \(t\), then the above argument shows that they do not lie in \(R(G_{t\phi_1}^\circ, T)\) for any such \(t\). In that case, in the twisted graded Hecke algebra \(\mathbb{H}(t\phi_1, v, q, \bar{r})\) the element \(N_{s_\alpha}\) satisfies a braid relation with trivial parameter \(c^*(\alpha) := 0\).

For any \(t \in T\) with \((m_\alpha \alpha)(t) = -1\), (51) imposes the new condition
\[
\lambda(m_\alpha \alpha) - \lambda^s(m_\alpha \alpha) = m_\alpha c^*(\alpha).
\]
Clearly (93) and (95) admit the unique solution
\[
\lambda(m_\alpha \alpha) = (c(\alpha) + c^*(\alpha))m_\alpha/2, \quad \lambda^s(m_\alpha \alpha) = (c(\alpha) - c^*(\alpha))m_\alpha/2.
\]
We address the \(W_\phi\)-invariance. Represent \(\gamma \in W_\phi\) in \(N_{G_{\phi_1}}(J)\) as in (87). Then it acts on the entire setting by conjugation, so \(\lambda \circ \gamma\) and \(\lambda^s \circ \gamma\) are parameter functions which also fulfill the requirements with respect to reduction to graded Hecke algebras. With the uniqueness of the solutions to the above equations, we find that \(\lambda \circ \gamma\) and \(\lambda^s \circ \gamma\) are unique.

(b) If \(c^*(\alpha) > c(\alpha)\), then we exchange them. This can be achieved with the method from the proof of Proposition 3.9b: take a new basepoint \(\phi_\nu\) such that \((m_\alpha \alpha)(t') = -1\) while \(t'\) lies in the kernel of every other simple roots of \(R(J^0, T)\). This assures that \(\lambda^s\) takes values in \(\mathbb{Q}_{\geq 0}\).

Case 1: \((m_\alpha \alpha)^\vee \notin 2X_*(T_\phi^\vee)\)
When \(2\alpha \notin R(J^0, T)\), Proposition 2.1a ensures that \(c(\alpha)\) is even. When \(2\alpha \in R(J^0, T)\) and still \((m_\alpha \alpha)^\vee \notin 2X_*(T_\phi^\vee)\), Lemma 3.12 shows that the relevant irreducible components of \(R(J^0, T)\) and \(R_{\phi_1}\) have type \(BC_n\) and \(C_n\), respectively. In particular \(m_\alpha = 2m_\beta\) for any other simple root in the same component of \(R(J^0, T)\), and \(m_\alpha\) is even. Hence (94) is always an integer.

Case 2: \((m_\alpha \alpha)^\vee \in 2X_*(T_\phi^\vee), 2\alpha \notin R(G_{\phi_1}^\circ, T)\)
In view of (96), we need to show that
\[
(c(\alpha) \pm c^*(\alpha))m_\alpha \quad \text{is even.}
\]
By Proposition 2.1a, \(c(\alpha)\) is even. Select \(t \in T\) with \((m_\alpha \alpha)(t) = -1\).

- If \(\alpha\) lies in \(R(G_{t\phi_1}^\circ, T)\) but \(2\alpha\) does not, then \(c^*(\alpha)\) is also even.
- If \(\alpha, \alpha \notin R(G_{t\phi_1}^\circ, T)\) then we argued in the proof of part (a) that \(c^*(\alpha) = 0\).
- Suppose \(\alpha \in R(G_{t\phi_1}^\circ, T)\). If \(m_\alpha\) would be odd, we could arrange that \(\alpha(t) = -1\). Then \((2\alpha)(t) = 1\), so \(2\alpha\) would lie in both \(R(G_{t\phi_1}^\circ, T)\) and \(R_{\phi_1}^\circ, T)\). That contradicts our assumptions, so \(m_\alpha\) is even. By Proposition 2.1 either \(c^*(\alpha) = c(\alpha)\) or \(c^*(\alpha) = c(\alpha)\) and it is always an integer.

In all these three instances (97) holds.

Case 3: \((m_\alpha \alpha)^\vee \in 2X_*(T_\phi^\vee), 2\alpha \in R(G_{\phi_1}^\circ, T)\)
Again we need to verify (97), and we pick a \(t \in T\) with \((m_\alpha \alpha)(t) = -1\). By Proposition 2.1b, \(c(\alpha)\) is odd.

- If \(\alpha, 2\alpha \in R(G_{t\phi_1}^\circ, T)\), then \(c^*(\alpha)\) is also odd.
• Suppose $m_\alpha$ is even and not $\alpha, 2\alpha \in R(G_{\tilde{t}_0}^0, T)$. If $\alpha, 2\alpha \notin R(G_{\tilde{t}_0}^0, T)$, then we argued in the proof of part (a) that $c^*(\alpha) = 0$. Otherwise, by Proposition 2.1, either $c^*(\alpha) = c(\alpha)$ or $c^*(\alpha) = c(2\alpha)/2$, and this is always an integer.

• Suppose $m_\alpha$ is odd and not $\alpha, 2\alpha \in R(G_{\tilde{t}_0}^0, T)$. Here we can arrange that $\alpha(t) = -1$, so that $(2\alpha)(t) = 1$. Then the root subspace $g_{2\alpha}$ is the same for $G_{\tilde{t}_0}^0$ as for $G_{\tilde{t}_0}^0$, so $2\alpha \in R(G_{\tilde{t}_0}^0, T) \neq \alpha$ and $c^*(\alpha)$ can be computed from $G_{\tilde{t}_0}^0$ alone. By Proposition 2.1b $c^*(\alpha) = c(2\alpha)/2 = 1$, which is odd.

In these three instances, (97) is indeed valid. \qed

3.3. **Affine Hecke algebras.**

Recall that $W_s^\gamma$ acts naturally on the root system $R(J^0, T)$. Let $R^+(J^0, T)$ be the positive system defined by the $\phi(Frob)$-stable Borel subgroup $B_J$ of $J^0$. By Proposition 3.9a any two such $B_J$ are $J^0$-conjugate, so the choice is inessential. Since $W_s^\gamma$ acts simply transitively on the collection of positive systems for $R(J^0, T)$, we obtain a semi-direct factorization

$$W_s^\gamma = W_s^\gamma \rtimes R_s^\gamma,$$

$$R_s^\gamma = \{ w \in W_s^\gamma : wR^+(J^0, T) = R^+(J^0, T) \}.$$

To $s^\gamma$ we can associate the affine Hecke algebra $H(R_s^\gamma, \lambda, \lambda^*, \tilde{z})$, where $\phi_1$ is as in Proposition 3.14 and $\lambda$ and $\lambda^*$ satisfy (93) and (95). However, this algebra takes only the subgroup $W_s^\gamma$ of $W_s^\gamma$ into account. To see $W_s^\gamma, \phi_1, v, q, t$, we can enlarge it to

$$H(R_{s^\gamma}, \lambda, \lambda^*, \tilde{z}) = \mathbb{C}[R_{s^\gamma}, \phi_1, v, q, t; s^\gamma, \phi_1, \tilde{z}] =$$

(99) $$H(R_s^\gamma, \lambda, \lambda^*, \tilde{z}) \rtimes \text{End}_{\text{Lie}(J_{\tilde{t}_0})}^+(q\pi_s(\tilde{q}E)).$$

But $W_s^\gamma$ can also contain elements that do not fix $\phi_1$. In fact, in some cases $W_s^\gamma$ even acts freely on $T_{s^\gamma}$.

**Proposition 3.15.** Assume that the almost direct factorization (7) of $J^0$ induces a decomposition of $R(J^0, T)$ which is $W_s^\gamma$-stable.

(a) The group $R_s^\gamma$ acts naturally on $H(R_s^\gamma, \lambda, \lambda^*, \tilde{z})$, by algebra automorphisms.

(b) This can be realized in a twisted affine Hecke algebra

$$H(R_s^\gamma, \lambda, \lambda^*, \tilde{z}) = \mathbb{C}[R_{s^\gamma}, \phi_1, v, q, t; s^\gamma, \phi_s^\gamma, \tilde{z}].$$

which (98) is canonically embedded.

**Proof.** (a) The action of $R_s^\gamma$ on $T_{s^\gamma}$ comes from (89). This determines an action on $O(T_{s^\gamma}) \cong \mathbb{C}[X^*(T_{s^\gamma})]$. Any $\gamma \in R_s^\gamma$ maps $\theta_x$ to an invertible element of $\mathbb{C}[X^*(T_{s^\gamma})]$. That is,

$$\gamma \cdot \theta_x = \theta_{\gamma x} \lambda_{\gamma, x} \in \mathbb{C}.$$

The linear part $x \mapsto \gamma x$ is an automorphism of $X^*(T_{s^\gamma})$, and the translation part of $\gamma : T_{s^\gamma} \to T_{s^\gamma}$ is given by $\lambda_{\gamma, x}^{-1} = x(\gamma(1))$. Since $W_s^\gamma$ is normal in $W_{s^\gamma}$,

$$(W_{s^\gamma})_{\gamma(1)} = (\gamma W_{s^\gamma} \gamma^{-1})_{\gamma(1)} = (W_{s^\gamma})_{\gamma(1)} = W_{s^\gamma}.$$

In other words, the translation part of $\gamma$ commutes with all the reflections $s_\alpha (\alpha \in R_{s^\gamma})$. According to [AMS] Lemma 9.2 there exists a canonical algebra isomorphism

$$\psi_{\gamma, \phi_1, v, q} : \mathbb{C}[W_{s^\gamma}, \phi_1, v, q; \kappa_{\phi_1, v, q}] \to \mathbb{C}[W_{s^\gamma}, \gamma(\phi_1), v, q, \kappa_{\gamma \phi_1, v, q}].$$
Let us recall its construction. There is a $G_{\phi_1}$-equivariant local system $q\pi_*(\widetilde{qE})$ on $(G_{\phi_1})_{RS}$, an analogue of $K$ and $K^*$. It satisfies
\begin{equation}
\mathbb{C}[W^0_{g'v},\phi_1,v,q_e,\kappa_{\phi_1,v,q_e}] \cong \text{End}_{D(G_{\phi_1})_{RS}}(q\pi_*(\widetilde{qE})).
\end{equation}
Choosing a lift $n_\gamma \in N_{G_{\phi_1}}(M)$ of $\gamma$ and following the proof of [AMS1] Lemma 5.4, we find an isomorphism
\begin{equation}
q\pi_{\gamma}(\widetilde{qE}) \rightarrow q\pi_*(\text{Ad}(n_\gamma)^*qE).
\end{equation}
Then $\psi_{\gamma,\phi_1,v,q_e}$ is conjugation with $qW_{g'}$.

In this context [AMS1] Lemma 5.4 says that there are canonical elements $qW_{g'} \in \text{End}_{D(G_{\phi_1})_{RS}}(q\pi_*(qE))$ ($w \in W^0_{g'v}$) which via \ref{100} become a basis of $C[W^0_{g'v}]$. Since $W^0_{g'v}$ is normal in $W^0_{g'}$, $\psi_{\gamma,\phi_1,v,q_e}$ stabilizes the set $\{qW_{g'} : W^0_{g'v}\}$. Moreover $\gamma \in R_{g'}$, so $\psi_{\gamma,\phi_1,v,q_e}$ permutes the set of simple reflections in $W^0_{g'}$.

By Proposition 3.14 the parameter functions $\lambda$ and $\lambda^*$ are $W^0_{g'}$-invariant. Hence the map $N_s \mapsto N_{\gamma_s}s_{\gamma-1}$ extends uniquely to an automorphism of the Iwahori–Hecke algebra $H(W^0_{g'},\mathbb{Z}^{2\lambda})$ which fixes $\mathbb{Z}$.

Now we have canonical group actions of $R_{g'}$ on the algebras
\begin{equation}
O(X_{M}((L)\times (\mathcal{C}^x)^d) = \mathbb{C}[X^* (X_{M}((L)\times (\mathcal{C}^x)^d) \otimes \mathbb{C}[z,\mathbb{Z}^{-1}]
\end{equation}
and $H(W^0_{g'},\mathbb{Z}^{2\lambda})$, and as vector spaces
\begin{equation}
H(R_{g'},\lambda,\lambda^*,\mathbb{Z}) = O(X_{M}((L)\times (\mathcal{C}^x)^d) \otimes H(W^0_{g'},\mathbb{Z}^{2\lambda}).
\end{equation}
The relation involving $\theta_s N_{s_a} - N_{s_s} \theta_s N_{s_a}$ in Proposition 2.2 is also satisfied by $\gamma$, because $x(\gamma(1)) = s_{a}(x)(\gamma(1))$. So $R_{g'}$ acts canonically on $H(R_{g'},\lambda,\lambda^*,\mathbb{Z})$ by algebra automorphisms.

(b) The same construction as in the proof of Proposition 2.2 yields an algebra
\begin{equation}
H(R_{g'\gamma},\lambda,\lambda^*,\mathbb{Z}) \cong \mathbb{C}[\kappa_{\lambda^*},\kappa_{g'}],
\end{equation}
in which the action of $R_{g'}$ on $H(R_{g'\gamma},\lambda,\lambda^*,\mathbb{Z})$ has become an inner automorphism. This works for any 2-cocycle $\kappa_{g'}$. It only remains to pick it in a good way, such that $\kappa_{g'}(v_{g'\gamma},v_{g'\gamma}) = \kappa_{g'\gamma},v_{g'\gamma},v_{g'\gamma}$. For this we, again, use the maps $qW_{g'}$ from \ref{101}. The cuspidal local system $\text{Ad}(n_\gamma)^*qE$ does not depend on the choice of $n_\gamma$, because $qE$ is $M$-equivariant. Furthermore $qW_{g'}$ is unique up to scalars, so
\begin{equation}
qW_{g'} \cdot qW_{g'} = \lambda_{\gamma,\gamma'}qW_{g'\gamma} \text{ for a unique } \lambda_{\gamma,\gamma'} \in \mathbb{C}^x.
\end{equation}
We define $\kappa_{g'\gamma}(\gamma,\gamma') = \lambda_{\gamma,\gamma'}$. This is a slight generalization of the construction in Section 4 and in [AMS1] Lemma 5.4. As over there,
\begin{align*}
End_{P_{g}^{\text{Lie}(J)_{RS}}}(q\pi_*(\widetilde{qE})) & \cong \mathbb{C}[W_{g'},\kappa_{g'}], \\
End_{P_{g}^{\text{Lie}(J)_{RS}}}(q\pi_*(qE)) & \cong \mathbb{C}[R_{g'},\kappa_{g'}].
\end{align*}
As the $J$-equivariant sheaf $q\pi_*(\widetilde{qE})$ on $\text{Lie}(J)_{RS}$ contains the $G_{\phi}$-equivariant sheaf $q\pi_*(qE)$ on $\text{Lie}(G_{\phi})_{RS}$,
\begin{equation}
\kappa_{g'} : (W^0_{g'})^2 \rightarrow (W^0_{g'}/W^0_{g'})^2 = R_{g'}^{2} \rightarrow \mathbb{C}^x
\end{equation}
extends $\kappa_{g'\gamma},v_{g'\gamma} : (W^0_{g'\gamma},v_{g'\gamma})^2 \rightarrow \mathbb{C}^x$, for every $(\phi|w,v,q_e) \in s_{g'\gamma}$. For $\phi = \phi_1$ this means that
\begin{equation}
H(R_{g'},\lambda,\lambda^*,\mathbb{Z}) \cong \mathbb{C}[W^0_{g'},\phi_1,v,q_e,\kappa_{\phi_1,v,q_e}].
\end{equation}
is canonically embedded in \ref{102}. □
The algebra from Proposition 3.15 is attached to \( s^\vee \) and the basepoint \( \phi_1 \) of \( s^\vee \). To remove the dependence on the basepoint, we reinterpret \( H \langle R_{s^\vee}, \lambda, d^* \rangle \). Recall that \( W_{s^\vee} \) acts naturally on \( s^\vee \) (which is diffeomorphic to \( T_{s^\vee} \)). Every \( \alpha \in R_{s^\vee} \) is by definition a character of \( T_{s^\vee} \) and by Proposition 3.9 it does not depend on the choice of \( \phi_1 \), so it canonically determines a function on \( s^\vee \). In the same way as in Proposition 2.2, we can define an algebra structure on

\[
O(s^\vee) \otimes \mathbb{C}[\bar{Z}, \bar{Z}^{-1}] \otimes \mathbb{C}[W_{s^\vee}].
\]

It becomes an algebra \( H(s^\vee, W_{s^\vee}, \lambda, d^*, \bar{Z}) \) which is isomorphic to \( H(R_{s^\vee}, \lambda, d^*, \bar{Z}) \), but only via the choice of a basepoint of \( s^\vee \). In Proposition 3.15 we showed that \( R_{s^\vee} \) acts naturally on \( H(s^\vee, W_{s^\vee}, \lambda, d^*, \bar{Z}) \). Applying Proposition 3.15, we obtain an algebra

\[
H(s^\vee, W_{s^\vee}, \lambda, d^*, \bar{Z}) \times \text{End}_{\mathcal{P}, \text{Lie}(R)}^+(q\pi_*(q\bar{E})), \quad J = Z_{\mathcal{G}^{s^\vee}}(\phi_{1^R}).
\]

Now we suppose that the almost direct factorization of \( J^0 \) induces a \( W_{s^\vee} \)-stable decomposition of \( R(J^0, T) \) (and, equivalently, of \( R_{s^\vee} \)). We focus on two algebras obtained in this way:

- \( H(s^\vee, z) \), the algebra \((103)\) when \( J = J^0_{\text{der}} \), with only one variable \( z \);
- \( H(s^\vee, \bar{z}) \), the algebra \((103)\) when \((7)\) induces the finest possible \( W_{s^\vee} \)-stable decomposition of \( R(J^0, T) \).

**Lemma 3.16.** The algebras \( H(s^\vee, z) \) and \( H(s^\vee, \bar{z}) \) depend only on \( s^\vee \), up to canonical isomorphisms.

**Proof.** The above construction shows that \( H(s^\vee, z) \) and \( H(s^\vee, \bar{z}) \) are uniquely determined by \( (s^\vee, M, P) \). Up to \( G^\vee \)-conjugation, this triple is completely determined by \( s^\vee \). The normalizer of \( s^\vee \) is contained in \( J \), and the pointwise stabilizer of \( s^\vee \) in \( J \) is just \( M \). Given \( s^\vee \) and \( M \), [AMS2, Lemma 1.1] shows that all possible choices for \( P \) are conjugate by unique elements of \( N_{J^0}(M^0)/M^0 \). Thus all possible \( (s^\vee, M', P') \) underlying \( s^\vee \) are conjugate to \( (s^\vee, M, P) \) in a canonical way. Any element of \( G^\vee \) which realizes such a conjugation provides a canonical isomorphism between \( H(s^\vee, z) \) (respectively \( H(s^\vee, \bar{z}) \)) and its version based on \((s^\vee, M', P')\).

**Example 3.17.** Suppose that \( \phi \) is itself cuspidal, so \( L^\vee = G^\vee \) and \( q \phi = \rho \). Then \( J^0 = M^0 \), \( v \) is distinguished in that group, \( T = 1 \) and \( R(J^0, T) \) is empty. Furthermore \( W_{s^\vee} = 1 \) because \( N_{G^\vee}(L^\vee \times \mathbb{W}_F)/L^\vee = 1 \). Consequently

\[
H(s^\vee, z) = O(T_{s^\vee}) \otimes \mathbb{C}[z, z^{-1}] \quad \text{and} \quad H(s^\vee, \bar{z}) = O(T_{s^\vee}) \otimes \mathbb{C}[z_1, z_1^{-1}, \ldots, z_d, z_d^{-1}],
\]

where \( d \) is the number of simple factors of \( J^0_{\text{der}} \).

For \( (\phi, \rho) \) as in \((7)\), let \( \bar{M}(\phi, \rho, \bar{Z}) \) be the irreducible \( H(s^\vee, \bar{Z}) \)-module obtained from \( M(\phi, \rho, \log \bar{Z}) \in \text{Irr}(\mathbb{H}(\phi, v, q \epsilon, \bar{F})) \) via Theorems 2.5 and 2.11. Up to \( G^\vee \)-conjugation, every element of \( \Phi_\epsilon(G(F))^{s^\vee} \) is of the form described in \((7)\), so this definition extends naturally to all possible \( (\phi, \rho) \). Similarly we define \( E(\phi, \rho, \bar{Z}) \) as the "standard" \( H(s^\vee, \bar{Z}) \)-module obtained from \( E(\phi, \rho, \log \bar{Z}) \in \text{Mod}(\mathbb{H}(\phi, v, q \epsilon, \bar{F})) \) via Theorems 2.5 and 2.11.

We formulate the next result only for \( H(s^\vee, \bar{Z}) \), but there is also a version for \( H(s^\vee, z) \). In view of \((36)\), the latter can be obtained by assuming that all \( z_j \) are equal.
**Theorem 3.18.** (a) For every \( \vec{z} \in \mathbb{R}^d_{\geq 0} \) there exists a canonical bijection

\[
\Phi_e(G(F))^s \to \text{Irr}_\mathbb{F}(s^\vee, \vec{z}) : (\phi, \rho) \mapsto M(\phi, \rho, \vec{z}).
\]

(b) Both \( \tilde{M}(\phi, \rho, \vec{z}) \) and \( \tilde{E}(\phi, \rho, \vec{z}) \) admit the central character \( W_{\vec{z}}(\tilde{\phi}|_{W_F}, v, q_\mathbb{F}) \in \Phi_e(\mathcal{L}(F))^{s_\mathbb{F}}/W_{s_\mathbb{F}}, \) where \( \tilde{\phi}|_{I_F} = \phi|_{I_F} \) and \( \tilde{\phi}(\text{Frob}_F) = \phi(\text{Frob}_F)\chi_{\phi,v}(\vec{z}) \) with \( \chi_{\phi,v} \) as in (80). We may also take \( \chi_{\phi,v}^{-1} \) instead of \( \chi_{\phi,v} \).

(c) Suppose that \( \vec{z} \in \mathbb{R}^d_{\leq 1} \). Equivalent are:

- \( \phi \) is bounded;
- \( \tilde{E}(\phi, \rho, \vec{z}) \) is tempered;
- \( \tilde{M}(\phi, \rho, \vec{z}) \) is tempered.

(d) Suppose that \( \vec{z} \in \mathbb{R}^d_{\leq 1} \). Then \( \phi \) is discrete if and only if \( \tilde{M}(\phi, \rho, \vec{z}) \) is essentially discrete and the rank of \( R_{s_\mathbb{F}} \) equals \( \dim_{\mathbb{C}}(T_{s_\mathbb{F}}/X_{nr}(L_G)) \).

In this case \( \phi(\text{Frob}_F)^{-1} \) comes from an element of \( Z(J^\circ) \times X_{nr}(L_G) \) via Lemma 3.7 and (89).

(e) Suppose that \( \zeta \in Z(\mathcal{G}^\vee) \) stabilizes \( \Phi_e(G(F))^s \). Via (89) \( \zeta \) determines a unique element \( t_\zeta \in T_{s_\mathbb{F}} \). (For instance \( \zeta \in X_{nr}(L_G) \), in which case \( t_\zeta = \zeta X_{nr}(L_G)^{s_\mathbb{F}} \)). Then

\[
\tilde{M}(\zeta, \rho, \vec{z}) = t_\zeta \otimes \tilde{M}(\phi, \rho, \vec{z}) \quad \text{and} \quad \tilde{E}(\zeta, \rho, \vec{z}) = t_\zeta \otimes \tilde{E}(\phi, \rho, \vec{z}).
\]

(f) Suppose that \( \vec{z} \in \mathbb{R}^d_{\leq 1} \) and that \( \phi(\text{Frob}_F)^{-1} \) comes from an element of \( Z(J^\circ) \times X_{nr}(L_G) \) via Lemma 3.7 and (89). Then \( \tilde{E}(\phi, \rho, \vec{z}) = \tilde{M}(\phi, \rho, \vec{z}) \).

**Proof.** (a) Let us fix the bounded part \( \phi_b \) and consider only \( \phi \) in \( X_{nr}(L_G)_{rs}\phi_b \). We need to construct a bijection between such \( (\rho, \vec{z}) \) and the set of irreducible \( \mathcal{H}(s^\vee, \vec{z}) \)-modules on which \( \vec{z} \) acts as \( \vec{z} \) and with \( \mathcal{O}(s^\vee_L) \)-weights in

\[
W_{s^\vee}(X_{nr}(L_G)_{rs}\phi_b, v, q_\mathbb{F}) \subset s^\vee_L.
\]

We want to apply Theorem 2.5 a here, although \( \mathcal{H}(s^\vee, \vec{z}) \) and \( \mathcal{H}(R_{s^\vee}, \lambda, \lambda^*, \vec{z}) \) need not be of the form \( \mathcal{H}(G, M, q_\mathbb{C}) \). To see that this is allowed, pick a basepoint \( \phi_1 \) as in Proposition 3.9. Then \( \mathcal{H}(s^\vee, \vec{z}) \) becomes a twisted affine Hecke algebra associated to a root datum, parameters, a finite group and a 2-cocycle. For such an algebra the proof of Theorem 2.5 works, it does not matter that the parameters can be different and that \( R_{s^\vee} \) need not fix the basepoint of \( T_{s^\vee} \).

Consider the twisted affine Hecke algebra \( \mathcal{H}(s^\vee, \phi_b) \) with as data the torus \( s^\vee_L \), roots \( \{ \alpha \in R_{s^\vee} : s_\alpha(\phi_b) = \phi_b \} \), the finite group \( W_{s^\vee, \phi_b, v, q_\mathbb{F}} \), parameters \( \lambda, \lambda^* \) as in (93) and (95) and 2-cocycle \( z_{q_\mathbb{F}} \). The upshot of Theorem 2.5 a is a canonical bijection between the above irreducible \( \mathcal{H}(s^\vee, \vec{z}) \)-modules and the irreducible modules of \( \mathcal{H}(s^\vee, \phi_b) \) with central character in \( (X_{nr}(L_G)_{rs}\phi_b, v, q_\mathbb{F}) \times \{ \vec{z} \} \).

With respect to the new basepoint \( \phi_b \), \( \mathcal{H}(s^\vee, \phi_b) \) becomes isomorphic to a twisted affine Hecke algebra of the form described in Proposition 2.2. Then we can apply Theorem 2.11 to it, which relates its modules to those over a twisted graded Hecke algebra. Again it does not matter that the parameters of the affine Hecke algebra can differ from those in Theorem 2.11, this result applies to all possible parameters. The parameters of the resulting graded Hecke algebra are given by (51) and (50). Comparing that with (93), (95) and (78), we see that that graded Hecke algebra is none other than \( \mathcal{H}(\phi_b, v, q_\mathbb{F}) \).

Thus Theorem 2.5 a yields a bijection between the above set of irreducible modules and the irreducible \( \mathcal{H}(\phi_b, v, q_\mathbb{F})\)-modules with central character in \( \text{Lie}(X_{nr}(L_G)_{rs}) \times \)
The resulting bijection between (104) and the subset of Irr($\mathcal{H}(s^\vee, \mathbf{z})$) with the appropriate central character could depend on the choice of an element in the $W_{s^\vee}$-orbit of $\phi_b$. Fortunately, the proof of Lemma 2.9 applies also in this setting, and it entails that the bijection does not depend on such choices. Now we combine all these bijections, for the various $\phi_b$. This gives a canonical bijection between $\Phi_e(G^\vee)^s$ and $\text{Irr}_E(\mathcal{H}(s^\vee, \mathbf{z}))$.

(b) By Theorem 3.8, if $E(\phi, \rho, \log \mathbf{z})$ admits the central character

$$L^\Psi^{-1}(\mathcal{L}^\vee \times W_F, X_{\text{nr}}(L^\mathcal{L}) \phi_b | W_F, v, q\epsilon),$$

where $\sigma_0$ is given by (81). Applying Theorems 2.11 and 2.5 to the central character that sends $\text{Frob}_F$ to $\phi(\text{Frob}_F) \chi_{\phi, v}(\mathbf{z})^{\pm 1}$, we obtain $W_{s^\vee}(\phi) \chi_{w, v}(\mathbf{z})$. The same holds for the quotient $M(\phi, \rho, \mathbf{z})$ of $E(\phi, \rho, \mathbf{z})$.

(c) This follows from Theorem 3.8(b), Theorem 2.11(d) and Proposition 2.7(d).

(d) Notice that by the very definition of $R_{s^\vee}$, it has the same rank as $R(J^\vee, T)$.

Suppose that $\phi$ is discrete. By Theorem 3.8(c), $M(\phi, \rho, \log \mathbf{z})$ is essentially discrete and a module for $\mathbb{H}(\phi_b, v, q\epsilon, \log \mathbf{z})$, and the rank of $R(G_{\phi_b}, T)$ equals dim$_C(T)$. Now Theorems 2.11 and Proposition 2.7(c) say that $M(\phi, \rho, \mathbf{z})$ is essentially discrete and that the root system $R(J^\vee, T)$ contains $R(G_{\phi_b}, T)$, so its rank is at least dim$_C(T)$ — and hence precisely that, for it obviously cannot be strictly larger. By Lemma 3.7, $T$ is a finite cover of $T/\text{nr}(L^G)$, so both these tori have the same dimension.

Conversely, suppose that $M(\phi, \rho, \mathbf{z})$ is essentially discrete and that the rank of $R(J^\vee, T)$ equals dim$_C(T_{s^\vee}/\text{nr}(L^G))$. By Proposition 2.7(c), the root system $R(G_{\phi_b}, T)$ has the same rank, which we already saw equals dim$_C(T)$. In combination with Theorem 2.11 we also obtain that the $\mathbb{H}(\phi_b, v, q\epsilon, \log \mathbf{z})$-module $M(\phi, \rho, \log \mathbf{z})$ is essentially discrete and that $\text{dim}_C(T_{s^\vee}/\text{nr}(L^G)) = \text{dim}_C(T)$.

By Theorem 3.8(d), $\phi(\text{Frob}_F)\phi_b(\text{Frob}_F)^{-1}$ lies in $Z(G^\circ)$ for a complex reductive group $G^\circ$ with maximal torus $T_{s^\vee}$ and Weyl group $W_{s^\vee}$. That is the Weyl group of $(J^\vee, T)$, so via Lemma 3.7, $\phi(\text{Frob}_F)\phi_b(\text{Frob}_F)^{-1}$ must come from an element of $G_{\phi_1}^\circ \times X_{\text{nr}}(L^G)$ which is central by $J^\vee$.

As $L^\Psi(\zeta, \phi, \rho) = \zeta^L \Psi(\zeta, \rho) \in s^\vee_\mathcal{E}$, $\zeta$ determines a unique element of $T_{s^\vee}$. It is invariant under $G_{\phi_1}$ and $G_{\phi_b}$, because $\zeta$ comes from $Z(G^\vee)$. Now the claim follows from Theorem 3.8(d) in the same way as Theorem 3.13 was derived from Proposition 1.6.

(f) Reasoning as in the last lines of the proof of part (d), we see that $\phi(\text{Frob}_F)\phi_b(\text{Frob}_F)^{-1} \in Z(G^\circ)$. Apply Theorem 2.13(d).

Comparing Theorem 3.18(b) with [AMS1, Definition 7.7] we see that, when $\mathbf{z} = q^{\mathbf{z}/2}$, the central character of $M(\phi, \rho, q^{\mathbf{z}/2})$ equals the cuspidal support of $(\phi, \rho)$.

Part (e) says that Theorem 3.18 is equivariant with respect to twists by $X_{\text{nr}}(L^G)$, that is, equivariant with respect to twisting by unramified characters of $G(F)$.

The bijection obtained in part (a) is compatible with parabolic induction in the same sense as Corollary 2.14. For reference, we formulate this precisely. We use the notations as in (82) and after that. Recall from pages 31 and 12 that $\epsilon_{u, \mathbf{z}}(\phi(\text{Frob}_F)\phi_b(\text{Frob}_F)^{-1}, \mathbf{z}) = \epsilon_{\log(u, \mathbf{z})}(d\mathbf{z}, (F_0 \psi_0, F_0 \psi)) + \log(\phi(\text{Frob}_F)^{-1}\phi_b(\text{Frob}_F), F)$.


Lemma 3.19. Let \( Q = \mathcal{Q}(F) \) be a Levi subgroup of \( \mathcal{G}(F) \) and assume that, for each \( j = 1, \ldots, d \), \( \epsilon_{\phi,j}(\phi(\text{Frob}_F)\phi(\text{Frob}_F)^{-1}, z_j) \neq 0 \) or \( z_j = 1 \).

(a) There is a natural isomorphism of \( \mathcal{H}(s^\vee, \bar{z}) \)-modules

\[
\mathcal{H}(s^\vee, \bar{z}) \otimes_{\mathcal{H}(s_Q^\vee, \bar{z})} \bar{E}(\phi, \rho, z, \bar{z}) \cong \bigoplus_{\rho} \text{Hom}_{S^Q}(\rho^Q, \rho) \otimes \bar{E}(\phi, \rho, z, \bar{z}),
\]

where the sum runs over all \( \rho \in \text{Irr}(S^Q) \) with \( L\Psi^Q(\phi, \rho^Q) = L\Psi(\phi, \rho) \).

(b) The multiplicity of \( \bar{M}(\phi, \rho, z) \) in \( \mathcal{H}(s^\vee, \bar{z}) \otimes_{\mathcal{H}(s_Q^\vee, \bar{z})} \bar{E}(\phi, \rho, z, \bar{z}) \) is \( [\rho^Q : \rho]_{S^Q} \). It already appears that many times as a quotient of \( \mathcal{H}(s^\vee, \bar{z}) \otimes_{\mathcal{H}(s_Q^\vee, \bar{z})} \bar{M}(\phi, \rho^Q, z, \bar{z}) \).

Proof. As observed after (82), the bijection in Theorem 3.8.a is compatible with parabolic induction in the sense of Corollary 2.14. The bijection in Theorem 3.18.a is obtained from Theorem 3.8 by means of the reduction Theorems 2.5 and 2.11. Since these reduction theorems respect parabolic induction, Corollary 2.14 remains valid in the setting of Theorem 3.8 and it gives the desired results. \( \square \)

4. The relation with the stable Bernstein center

Let \( \Phi(L\mathcal{G}) \) be the collection of \( \mathcal{G}^\vee \)-orbits of L-parameters for \( L\mathcal{G} \). Recently, inspired by [Vog], Haines has considered the stable Bernstein center in [Hai]. We will explore below the relation of the latter with the Bernstein components \( \Phi_e(L\mathcal{G})^s \).

The notion of stable Bernstein center which we employ here naturally lives on the Galois side. In principle it should be related to stable distributions on \( \mathcal{G}(F) \) [Hai §5.5], but that connection is currently highly conjectural. Because of that, we will consider it for all inner twists of a given reductive connected \( p \)-adic group \( \mathcal{G}(F) \) simultaneously. Let \( \mathcal{G}^*(F) \) be a quasi-split \( F \)-group which is an inner twist of \( \mathcal{G}(F) \). The equivalence classes of inner twists of \( \mathcal{G}^* \) are parametrized by the Galois cohomology group \( H^1(F, \mathcal{G}_{\text{ad}}^*) \). For every \( \alpha \in H^1(F, \mathcal{G}_{\text{ad}}^*) \), we will denote by \( \mathcal{G}_\alpha(F) \) an inner twist of \( \mathcal{G}^*(F) \) which is parametrized by \( \alpha \). By construction

\[
\Phi_e(L\mathcal{G}) = \bigsqcup_{\alpha \in H^1(F, \mathcal{G}_{\text{ad}}^*)} \Phi_e(\mathcal{G}_\alpha(F)).
\]

Definition 4.1. The infinitesimal character of an L-parameter \( \phi \in \Phi(L\mathcal{G}) \) (or an enhanced L-parameter \( (\phi, \rho) \in \Phi_e(L\mathcal{G}) \)) is the \( \mathcal{G}^\vee \)-conjugacy class of the admissible morphism \( \lambda_\phi: W_F \to \mathcal{G}^\vee \times W_F \) (trivial on \( SL_2(\mathbb{C}) \)) defined by

\[
\lambda_\phi(w) := \phi \left( w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix} \right) \quad w \in W_F.
\]

With this notion we can reinterpret Theorem 3.18.b as: the infinitesimal character of \( (\phi, \rho) \) equals the infinitesimal character of the central character of \( \bar{M}(\phi, \rho, \pm^{1/2}) \).

Remark 4.2. As noticed in [Hai §5], if \( \phi \) is relevant for \( \mathcal{G}(F) \), it may happen that \( \lambda_\phi \) is not relevant for \( \mathcal{G}(F) \) anymore. This is why \( \lambda_\phi \) is called an admissible morphism, i.e. an L-parameter without the relevance condition. In contrast, for every \( \phi \in \Phi(L\mathcal{G}) \), we have \( \lambda_\phi \in \Phi_e(L\mathcal{G}) \), for \( \lambda_\phi \) is relevant for \( \mathcal{G}^*(F) \).
**Definition 4.3.** An inertial infinitesimal datum $i$ for $\Phi(LG)$ is a pair $(LM, i_L M)$, where $LM$ is a Levi $L$-subgroup of $LG$, i.e. $LM = M^\lor \ltimes W_L$ with $M^\lor$ a $W_L$-stable Levi subgroup of $G^\lor$ and $i_L M$ is the $M^\lor$-conjugacy class of the $X_{nr}(LM)$-orbit of a discrete admissible morphism $\lambda : W_L \to M^\lor \ltimes W_L$ (trivial on $SL_2(\mathbb{C})$). Another such object is regarded as equivalent if the two are conjugate by an element of $G^\lor$. The equivalence class is denoted

$$i = (M^\lor \ltimes W_L, i_L M)_{G^\lor} = [M^\lor \ltimes W_L, \lambda]_{G^\lor}.$$

We will write $\mathcal{B}_s^\lor(LG)$ for the set of inertial infinitesimal equivalence classes.

For every inertial infinitesimal datum $i = (M^\lor \ltimes W_L, i_L M)_{G^\lor}$, $i_L M$ has the structure of an affine variety over $\mathbb{C}$ (see [Hal] § 5.3). The stable Bernstein center for $LG$ is the ring of regular functions on the disjoint union $\bigsqcup_{i = (LM, i_L M) \in \mathcal{B}_s^\lor(LG)} i_L M$.

We will attach to each inertial equivalence class for $\Phi_s(G(F))$ an inertial infinitesimal datum, as follows:

**Definition 4.4.** For every cuspidal inertial equivalence class $s^\lor = (\mathcal{L} \ltimes W_F, X_{nr}(LM) \cdot (\phi, \rho)) \in \mathcal{B}^\lor(G(F))$, we set

$$\inf(s^\lor) := (M^\lor \ltimes W_F, (X_{nr}(LM) \cdot \lambda_\phi)_{LM})_{G^\lor},$$

where $M^\lor \ltimes W_F$ is a Levi $L$-subgroup of $LG$ which minimally contains $\lambda_\phi(W_F)$.

We remark that if $\phi$ has nontrivial restriction to $SL_2(\mathbb{C})$, then we may have $M^\lor \ltimes W_F \subseteq \mathcal{L}^\lor \ltimes W_F$ and $X_{nr}(LM) \subseteq X_{nr}(LM)$.

For every $i = (M^\lor \ltimes W_F, \lambda)_{G^\lor} \in \mathcal{B}_s^\lor(LG)$ we set:

$$\Phi_s(LG)_i := \left\{ (\phi, \rho) \in \Phi_s(LG) : \lambda_\phi is minimally contained in M^\lor \ltimes W_F \right\}.$$

In this way, we obtain a partition of the set $\Phi_s(LG)$ (a "stable Bernstein decomposition"):

$$\Phi_s(LG) = \bigsqcup_{i \in \mathcal{B}_s^\lor(LG)} \Phi_s(LG)_i. \tag{105}$$

It is worth to observe that, in contrast with Section 3, the above definitions involve only the Langlands parameter $\phi \in \Phi(LG)$ and not the enhancement of $\phi$. In particular $(\phi, \rho)$ and $(\phi, \rho')$ are always contained in the same $\Phi_s(LG)_i$. Consequently the decomposition (105) is coarser than the Bernstein decomposition of $\Phi_s(LG)$ from (113). However, under the local Langlands conjecture, it is a union of L-packets. Indeed, let $i = [M^\lor \ltimes W_F, \lambda]_{G^\lor} \in \mathcal{B}_s^\lor(LG)$. From the definition of $i$ we see that

$$\Phi_s(LG)_i = \bigsqcup_{\alpha \in H^1(F, G^\lorad) (\lambda_\chi)_{LM} \in i_L M} \left[ (\lambda_\phi)_{G^\lor} = (\lambda_\chi)_{G^\lor} \right] \Pi_\phi(G_\alpha(F)) \Pi_\chi(G_\alpha(F)).$$

Define

$$\mathcal{B}^\lor(LG) := \bigsqcup_{\alpha \in H^1(F, G^\lorad)} \mathcal{B}^\lor(G_\alpha(F)).$$

**Theorem 4.5.** For $i \in \mathcal{B}_s^\lor(LG)$, we write $\mathcal{B}^\lor(LG)_i := \{ s^\lor \in \mathcal{B}^\lor(LG) : \inf(s^\lor) = i \}$. Then

$$\Phi_s(LG)_i = \bigsqcup_{s^\lor \in \mathcal{B}^\lor(LG)_i} \Phi_s(LG)_{s^\lor}.$$
Proof. Use that for any enhanced Langlands parameter \((\phi, \rho) \in \Phi_\epsilon(LG)\), the infinitesimal character \(\lambda_\phi\) of \(\phi\) coincides with the infinitesimal character \(\lambda_\varphi\) of its cuspidal support \((\varphi, q)\) \[\text{AMS}, (108)\].

This theorem implies that \([105]\) is a partition of \(\Phi_\epsilon(LG)\) in subsets which are at the same time unions of Bernstein components and unions of L-packets.

Combining Theorems 4.5 and 3.18, we obtain:

**Corollary 4.6.** For every \(i \in \mathfrak{B}_{sk}(LG)\) and every \(\tilde{z} \in \mathbb{R}_d^+\), there is a canonical bijection

\[
\Phi_\epsilon(LG)^i \longleftrightarrow \bigcup_{\mathfrak{s}^\vee \in \mathfrak{B}^\vee(LG)_i} \text{Irr}_{\tilde{z}}(\mathcal{H}(\mathfrak{s}^\vee, \tilde{z})).
\]

**Remark 4.7.** It is natural to expect that a certain compatibility should exist between the algebras \(\mathcal{H}(\mathfrak{s}^\vee, \tilde{z})\) when \(\mathfrak{s}^\vee\) runs over the set \(\mathfrak{B}^\vee(LG)_i\), for a fixed \(i = [\mathcal{M}^\vee \times W_F, \lambda]_{G^\vee}\). A naive guess would be that there exist "spectral transfer morphisms" (as introduced for affine Hecke algebras by Opdam \[\text{Opd2}\]) between the algebras \(\mathcal{H}(\mathfrak{s}^\vee, \tilde{z})\) for \(\mathfrak{s}^\vee \in \mathfrak{B}^\vee(LG)_i\), the role of the lowest algebra being played by an algebra \(\mathcal{H}(\mathfrak{s}^\vee_1, \tilde{z})\), with \(\mathfrak{s}^\vee_1 = [\mathcal{M}^\vee \times W_F, \lambda, 1]_{G^\vee}\).

5. Examples

In this section we will work out some affine Hecke algebras attached to Bernstein components of Langlands parameters. In the examples that we consider the local Langlands correspondence is known, and it matches Bernstein components on the Galois side with Bernstein components on the \(p\)-adic side. We will compare the Hecke algebras associated to Bernstein components that correspond under the LLC.

All our examples are inner forms of split groups, so \(X_{\text{int}}(LG) = Z(L^\vee)^0\) and we may replace \(LG\) by \(G^\vee\).

5.1. Inner twists of \(GL_n(F)\).

Recall that \(F\) is a local non-archimedean field, and let \(q_F\) be the cardinality of its residue field. Let \(D\) be a division algebra with centre \(F\) and \(\text{dim}_F(D) = d^2\). Take \(m \in \mathbb{N}\) and consider \(G(F) = GL_n(D)\). It is an inner form of \(GL_n(F)\) with \(n = md\). In fact \(G(F)\) becomes an inner twist if we regard \(D\), the Hasse invariant \(h(D) \in \{z \in \mathbb{C}^\times : z^d = 1\}\) or the associated character \(\chi_D\) of \(Z(SL_n(\mathbb{C}))\) as part of the data. Up to conjugacy every Levi subgroup of \(G(F)\) is of the form

\[
\mathcal{L}(F) = \prod_j GL_{m_j}(D) \quad \text{with} \quad \sum_j m_j = m.
\]

Let \((\phi = \bigoplus_j \phi_j, \rho = \bigotimes_j \rho_j) \in \Phi_{\text{cusp}}(\mathcal{L}(F))\). In \[\text{AMS}, \text{Example 6.11}\] we worked out the shape of cuspidal Langlands parameters \((\phi_j, \rho_j)\) for \(GL_{m_j}(D)\). Namely

* \(\phi_j = \phi_j|W_F \otimes S_{d_j}\) where \(S_{d_j}\) is the irreducible \(d_j\)-dimensional representation of \(SL_2(\mathbb{C})\) and \(\phi_j|W_F\) is an irreducible representation of dimension \(m_jd/d_j\). (This says that \(\phi_j\) is discrete.)

* \(S_{\rho_j} = Z(SL_{m_jd}(\mathbb{C}))\) and \(\rho_j\) is the character associated to \(GL_{m_j}(D)\), that is, \(\rho_j(\exp(2\pi ik/(m_jd)))I_{m_jd} = h(D)^k\). (So \((\phi_j, \rho_j)\) is relevant for \(GL_{m_j}(D)\).)

* \(\text{lcm}(d, m_jd/d_j) = m_jd\), or equivalently \(\text{gcd}(d, m_jd/d_j) = d/d_j\). (This guarantees cuspidality.)

It is known that two irreducible representation \(\phi_j\) and \(\phi_k\) of \(W_F\) are isomorphic up to an unramified character, if and only if their restrictions to \(1_F\) are isomorphic. Hence
we can adjust the indexing so that \( \phi|_{I_F} = \bigoplus_i \phi_i^{\otimes e_i}|_{I_F} \). Because the restriction of each \( \phi_i \) to \( I_F \) decomposes as sum of irreducible representations of \( I_F \) with multiplicity one, we find that \( R(F,T) \cong \prod_i A_{e_i-1} \). To determine the Hecke algebra of the associated Bernstein component \( \Phi^\vee \) of \( \Phi_{e}(G(F)) \), we make a simplifying assumption: if \( m_1 = m_j \) and \( \phi_i \) differs from \( \phi_j \) by an unramified twist, then \( \phi_i = \phi_j \).

We adjust the indexing so that

\[
\mathcal{L}(F) = \prod_i \text{GL}_{m_i}(D)^{e_i}, \quad \phi = \bigoplus_i \phi_i^{\otimes e_i}, \quad \rho = \bigotimes_i \rho_i^{\otimes e_i},
\]

where \( \phi_i \) and \( \phi_j \) are not inertially equivalent if \( i \neq j \). Let \( s^i \) be the Bernstein component of \( \Phi_{e}(\text{GL}_{m_i}(D)) \) determined by \( (\phi_i^{\otimes e_i}, \rho_i^{\otimes e_i}) \). Choose an isomorphism \( M_{d,e_i}(\mathbb{C}) \cong M_{m_i/d_i}(\mathbb{C}) \otimes M_{d,e_i}(\mathbb{C}) \) and let \( 1_m \) be the multiplicative unit of the matrix algebra \( M_m(\mathbb{C}) \). Then

\[
G_\phi = Z_{\text{SL}_n(\mathbb{C})}(\phi(W_F)) = \text{SL}_n(\mathbb{C}) \cap \prod_i (1_{m_i/d_i} \otimes \text{GL}_{d_i,e_i}(\mathbb{C})) = \text{SL}_n(\mathbb{C}) \cap \prod_i G_{\phi,i},
\]

\[
M \cong \text{SL}_n(\mathbb{C}) \cap \prod_i (1_{m_i/d_i} \otimes \text{GL}_{d_i}(\mathbb{C})^{e_i}),
\]

\[
T \cong \text{SL}_n(\mathbb{C}) \cap \prod_i (1_{m_i/d_i} \otimes \mathbb{Z}(\text{GL}_{d_i}(\mathbb{C})^{e_i})),
\]

\[
R(G_\phi,T) \cong \prod_i A_{e_i-1},
\]

\[
T_i = \{ \phi_i \otimes \chi_i \in \Phi(\text{GL}_{m_i}(D)) : \chi_i \in X_{nr}(\mathbb{C}) \}/\mu_{t_{\phi,i}},
\]

\[
T_{s^i} = \prod_i T_{s^i} = \prod_i T_i^{e_i}, \quad W_{s^i} = W_{s^i,\phi} \cong \prod_i S_i.
\]

Here \( \mu_k \) denotes the functor of taking \( k \)-th roots of unity and \( t_{\phi,i} \) denotes the numbers of unramified twists \( z_i \in X_{nr}(\mathbb{C}) \) such that \( z_i \phi_i \cong \phi_i \) in \( \Phi_{cusp}(\text{GL}_{m_i}(D)) \). The cyclic group \( \mu_{t_{\phi,i}}(\mathbb{C}) \) is naturally embedded in the one-dimensional complex torus \( X_{nr}(\mathbb{C}) \). Furthermore we can decompose \( u_{\phi} = \prod_i (u_{\phi,i}) \), where \( u_{\phi,i} \) belongs to the unique distinguished unipotent class of \( 1_{m_i/d_i} \otimes \text{GL}_{d_i}(\mathbb{C})^{e_i} \). By \([\text{Lus}2, 2.13]\) this implies \( c(\alpha) = 2d \) for all \( \alpha \in R(G_\phi,T) \). Then \( \lambda(\alpha) = t_{\phi,i} \) on \( R(G_\phi,T) \), whereas \( \lambda^* \) does not occur. We conclude that

\[
H(s^i, \bar{z}) = H(R_{s^i}, \lambda, \bar{z}) \cong \bigotimes_i H(\text{GL}_{d_i}(\mathbb{C}), \text{GL}_{d_i}(\mathbb{C})^{e_i}, \nu_i, \rho_i^{\otimes e_i}, z_i),
\]

a tensor product of affine Hecke algebras of type \( \text{GL}_{e_i} \) with parameters \( z_i^{t_{\phi,i}d_i} \). The most appropriate specialization of \((106)\) is at \( z_i = q_i^{1/2} \). Indeed this recovers the exact parameters found by Sécherre in \([\text{Sec}1, \text{Théorème 4.6}]\), see \((108)\).

Now we consider Hecke algebras on the \( p \)-adic side. By the local Langlands correspondence for \( \text{GL}_{m_i}(D) \) (see \([\text{HiS}2, \S 11]\) and \([\text{ABPS}2, \S 2]\)), \((\phi_i, \rho_i)\) is associated to a unique essentially square-integrable representation \( \sigma_i \in \text{Irr}(\text{GL}_{m_i}(D)) \). Moreover the condition \( \text{lcm}(d, m_i/d_i) = m_i d \) guarantees that \( \sigma_i \) is supercuspidal, by \([\text{DKV, Théorème B.2.b}]\). (This is a formal consequence of the Jacquet–Langlands correspondence, so in view of \([\text{Bad}]\) it also holds in positive characteristic.) Hence

\( (\phi_i^{\otimes e_i}, \rho_i^{\otimes e_i}) \in \Phi_{cusp}(\text{GL}_{m_i}(D)^{e_i}) \) corresponds to \( \sigma_i^{\otimes e_i} \in \text{Irr}_{cusp}(\text{GL}_{m_i}(D)^{e_i}) \).

Let \( s_i \) denote the inertial equivalence class for \( \text{GL}_{m_i}(D) \) determined by \( (\text{GL}_{m_i}(D)^{e_i}, \sigma_i^{\otimes e_i}) \). In \([\text{Sec}1, \text{Théorème 5.23}]\) a \( s_i \)-type \((J_i, \tau_i)\) was constructed. The Hecke algebra for \((J_i, \tau_i)\) was analysed in \([\text{Sec}1, \text{Théorème 4.6}]\), Sécherre found an isomorphism

\[
H(\text{GL}_{m_i}(D), J_i, \tau_i) \cong H(\text{GL}_{e_i}, q_i^{t_i}),
\]
where the right hand side denotes an affine Hecke algebra of type $GL_e$, with parameter $q^f_i$ (for a suitable $f_i \in \mathbb{N}$ depending only on $\sigma_i$ or $\phi_i$, see below). From the explicit description in \textbf{Sec}1 §4 one sees readily that the isomorphism \textbf{[107]} respects the natural Hilbert algebra structures on both sides.

\textbf{Remark 5.1.} Let $t_{\sigma_i}$ denote the torsion number of $\sigma_i$, \textit{i.e.}, the number of unramified characters $\chi_i$ of $GL_{m_i}(D)$ such that $\chi_i \otimes \sigma_i \cong \sigma_i$. It equals $t_{\phi_i}$.

If $D = F$, then $f_i = t_{\sigma_i}$. In general, $f_i = s_{\sigma_i} t_{\sigma_i}$, where $s_{\sigma_i}$ is the reducibility number of $\sigma_i$, as defined in \textbf{SeSt2} Introduction (see also \textbf{SeSt2} Theorem 4.6). The number $s_{\sigma_i}$ coincides with the invariant introduced in \textbf{DKV} Théorème B.2.b (as it follows for instance from \textbf{BHLs} Eqn. (1.1) and Definition 2.2), itself equal to the integer $d_i$. Hence $f_i$ admits the following description in terms of Langlands parameters:

\begin{equation}
(108) \quad f_i = s_{\sigma_i} t_{\sigma_i} = d_i t_{\phi_i}.
\end{equation}

Write $\mathcal{M}(F) = \prod_i GL_{m_i}(D)^{\epsilon_i}$, $\sigma = \bigotimes_i \sigma_i^{s_{\sigma_i}}$ and let $s$ be the inertial equivalence class of $(\mathcal{M}(F), \sigma)$ for $GL_m(D)$. In \textbf{SeSt2} Theorem C a $s$-type $(J, \tau)$ was constructed, as a cover of the product of the types $(J_i, \tau_i)$ for $s_i$. Moreover it was shown that

\begin{equation}
(109) \quad \mathcal{H}(GL_m(D), J, \tau) \cong \bigotimes_i \mathcal{H}(GL_{e_i}, q^{f_i}_e).
\end{equation}

Since \textbf{[107]} was an isomorphism of Hilbert algebras, so is \textbf{[109]}. Notice that the right hand side is also the specialization of $\mathcal{H}(s^\vee, \bar{z})$ at $z_i = q^{1/2}_i$. Thus there are equivalences of categories

\begin{equation}
(110) \quad \text{Rep}(GL_m(D))^s \cong \text{Mod}\left(\bigotimes_i \mathcal{H}(GL_{e_i}, q^{f_i}_e)\right) \cong \text{Mod}\left(\mathcal{H}(s^\vee, \bar{z})/\left\{z_i - q^{1/2}_i\right\}\right).
\end{equation}

It was shown in \textbf{BaCi} §5.4 that, since these equivalences come from isomorphisms of Hilbert algebras, they preserve temperedness of representations. Then \textbf{ABPS3} Lemma 16.5 proves that \textbf{[110]} maps essentially square-integrable representations to essentially discrete series representations and conversely.

The torus underlying $\bigotimes_i \mathcal{H}(GL_{e_i}, q^{f_i}_e)$ is $T_s = [\mathcal{M}(F), \sigma]_{\mathcal{M}(F)}$, which by the LLC for $GL_m(D)$ is naturally isomorphic to the torus $T_{s^\vee}$ underlying $\mathcal{H}(s^\vee, \bar{z})$. Then \textbf{ABPS3} Theorem 4.1 shows that, with the interpretation as in Lemma 3.16 (which highlights the tori in these affine Hecke algebras), the equivalences \textbf{[110]} become canonical. This means in essence that we use the local Langlands correspondence for supercuspidal representations as input. With Theorem 3.18 we obtain canonical bijections

\begin{equation}
(111) \quad \text{Irr}(GL_m(D))^s \leftrightarrow \text{Irr}\left(\mathcal{H}(s^\vee, \bar{z})/\left\{z_i - q^{1/2}_i\right\}\right) \leftrightarrow \Phi_e(\text{GL}_m(D))^s^\vee.
\end{equation}

\textbf{Proposition 5.2.} The union of the bijections \textbf{[111]} over all Bernstein components for $GL_m(D)$ equals the local Langlands correspondence for $GL_m(D)$.

\textbf{Proof.} In \textbf{ABPS2} §2 the LLC for $GL_m(D)$ was constructed by starting with irreducible essentially square-integrable representations of Levi subgroups, then applying parabolic induction and finally taking Langlands quotients. In the context of types and covers thereof, \textbf{BuKu1} Corollary 8.4 shows that the maps \textbf{[110]} commute
with parabolic induction. They also commute with taking Langlands quotients, because for these groups every Langlands quotient is the unique irreducible quotient of a suitable representation.

Thus we have reduced the claim to the case of irreducible essentially square-integrable representations. From [DKV] §B.2 we see that Rep(GL\(_m(D)\))\(^2\) only contains such representations if \(m_1e_1 = m\). We may just as well consider the group GL\(_{m,e_1}(D)\), which we prefer because then we can stick to the above notation. All its irreducible essentially square-integrable representations are generalized Steinberg representations built from \(T_{s_i}\). By construction the bijection \(\psi\) for GL\(_m(D)\)\(^{\epsilon_i}\) sends \(T_{s_i}\) to \(T_{s_i'}\).

Let \(\chi_i \in X_m(\text{GL}_{m_i}(D))\), with Langlands parameter \(t_i \in X_{m_i}(\text{GL}_{m_i}(D))\). The generalized Steinberg representation St(\(\sigma'\)) based on \(\sigma' = (\chi_i \sigma_i)\otimes e_i\) is the irreducible essentially square-integrable subrepresentation of the parabolic induction of

\[
\nu_i^{(1-e_i)/2} \chi_i \sigma_i \otimes \cdots \otimes \nu_i^{(e_i-1)/2} \chi_i \sigma_i
\]

to \(\prod_i \text{GL}_{m_i,e_i}(D)\), where \(\nu_i\) denotes the absolute value of reduced norm map for \(\text{GL}_{m_i}(D)\). There is a unique such subrepresentation by [DKV] Théorème B.2.b. By definition [ABPS2] (12) St(\(\sigma'\)) has Langlands parameter \(t_i \phi_i \otimes S_{s_i}\).

Now we plug St(\(\sigma'\)) in \(\psi\) and use the property discussed under \(\psi\). Thus we end up with an essentially discrete series representation of \(\mathcal{H}(s^\vee, z)/(\{1 - q_{s_i}^{\frac{1}{2}}\})\). By Theorem 3.18 it corresponds to a discrete element of \(\Phi_\iota(\text{GL}_{m,e_i}(D))\). Its enhancement \(\rho_i\) is uniquely determined by the requirement that it is relevant for \(\text{GL}_{m,e_i}(D)\), so we can ignore that and focus on the L-parameter. The image of \(W_F\) under this L-parameter is contained in \(\text{GL}_{m_i}(D)\)^\(e_i\)^\(\vee\) = \(\text{GL}_{m_i,d}(\mathbb{C})\)^\(e_i\), so it can only be discrete if it is of the form \(\psi_i \otimes \pi_{e_i,\text{SL}_2(\mathbb{C})}\) for some irreducible \(m_id\)-dimensional representation of \(W_F\). Since the cuspidal support of the enhanced L-parameter lies in \(T_{s_i'}\), \(\psi_i\) must be an unramified twist of \(\phi_i\). From (112) and the expression for the central character of \(M(\psi_i \otimes \pi_{e_i,\text{SL}_2(\mathbb{C})}, \rho_i, z_i)\) given in Theorem 3.18 we deduce that \(\psi_i = t_i \phi_i\). Thus (111) agrees with the local Langlands correspondence for essentially square-integrable representations.

\(\square\)

5.2. **Inner twists of SL\(_n(F)\).**

This paragraph is largely based on [ABPS2, ABPS3]. We keep the notations from the previous paragraph. For any subgroup of \(\text{GL}_m(D)\), we indicate the subgroup of elements of reduced norm 1 by a \(\sharp\). Thus

\[
\mathcal{G}^\sharp(F) = \text{GL}_m(D)^\sharp = \{g \in \text{GL}_m(D) : \text{Nrd}(g) = 1\} = \text{SL}_m(D).
\]

The inner twists of \(\text{GL}_n(F)\) are in bijection with the inner twists of \(\text{SL}_n(F)\), via

\[
\text{GL}_m(D) \leftrightarrow \text{GL}_m(D)^\sharp = \text{SL}_m(D).
\]

The L-parameters for \(\text{GL}_m(D)^\sharp\) are the same as for \(\text{GL}_m(D)\), only their image is considered in \(\text{PGL}_n(\mathbb{C})\). In particular every discrete L-parameter

\[
\phi^\sharp : W_F \times \text{SL}_2(\mathbb{C}) \to \text{PGL}_n(\mathbb{C})
\]

lifts to an irreducible \(n\)-dimensional representation of \(W_F \times \text{SL}_2(\mathbb{C})\). The local Langlands correspondence for these groups was worked out in [HiSa, ABPS2]. It provides a bijection between the Bernstein components on both sides of the LLC, which will use implicitly as \(s^\sharp \leftrightarrow s^{\vee}\).
Let $\phi = \otimes_i \phi_i^{e_i}$ be as before, and let $\phi^\sharp \in \Phi(L^\sharp(F))$ be the obtained by composition with the projection $GL_n(\mathbb{C}) \to PGL_n(\mathbb{C})$. Every Bernstein component contains L-parameters of this form. There is a central extension

$$1 \to Z_{\phi^\sharp} \to S_{\phi^\sharp} \to \mathcal{R}_{\phi^\sharp} \to 1$$

where $\mathcal{R}_{\phi^\sharp} = \pi_0(Z_{PGL_n(\mathbb{C})(\text{im}\phi^\sharp)})$ and

$$Z_{\phi^\sharp} = Z(SL_n(\mathbb{C}))/Z(SL_n(\mathbb{C})) \cap ZSL_n(\mathbb{C})(\phi^\sharp)^\circ.$$ 

Let $\rho^\sharp$ be an enhancement of $\phi^\sharp$. The restriction $\rho = \rho^\sharp|_{Z_{\phi^\sharp}}$ is an enhancement of $\phi$, so as before we may assume that it has the form $\rho = \otimes_i \rho_i^{e_i}$. Cuspidality of $(\phi^\sharp, \rho^\sharp)$ depends only $(\phi, \rho)$, it holds whenever $\rho_i$ is associated to the inner twist $GL_{m_i}(D)$ of $GL_n(F)$ via the Kottwitz isomorphism. We assume that this is the case, and that $(\phi^\sharp, \rho^\sharp) \in \Phi_{\text{cusp}}(L^\sharp(F))$. We note that $G_m \gives$ is the same for $GL_m(D)$ and $SL_m(D)$, and that $\phi$ and $\phi^\sharp$ have the same connected centralizer. Consequently

$$G^\circ_{\phi^\sharp} = G^\circ_\phi, \ G_{\phi^\sharp}/G^\circ_{\phi^\sharp} \cong \mathcal{R}_{\phi^\sharp}, \ M^\circ_{\phi^\sharp} = M^\circ_\phi,$$

$$R(G^\circ_{\phi^\sharp}, T) = \prod_i A_{e_i - 1}, \ \lambda(\alpha) = t_\phi, d_i \forall \alpha \in R(G_{\phi^\sharp}, T) \subset R(G^\circ_{\phi^\sharp}, T).$$

Let $s^{\sharp \vee}$ be the inertial equivalence class for $\Phi_{e}(GL_m(D)^\sharp)$ determined by $(\phi^\sharp, \rho^\sharp)$. (In spite of the notation $s^\vee$ does not determine it uniquely.) Then

$$T_{s^{\sharp \vee}} = \left( \prod_i T_{\phi_i}^{e_i} \right)/Z(GL_n(\mathbb{C})), \ W^\circ_{s^{\sharp \vee}} \cong \prod_i S_{e_i}.$$ 

The cuspidal local system $q\mathcal{E}$ associated to $(\phi^\sharp, \rho^\sharp)$ satisfies

$$\mathcal{R}_{q\mathcal{E}} \cong W_{s^{\sharp \vee}}/W^\circ_{s^{\sharp \vee}} = \mathcal{R}_{s^{\sharp \vee}} \cong \mathcal{R}_{\phi^\sharp}.$$

The algebra

$$(113) \quad \mathcal{H}(\mathcal{R}_{s^{\sharp \vee}}, \lambda, \bar{z}) = \mathcal{H}(G^\circ_{\phi^\sharp}, M^\circ_{\phi^\sharp}, v, \rho, \bar{z})$$

is a subalgebra of $\mathcal{H}(\mathcal{R}_{s^{\vee}}, \lambda, \bar{z})$, corresponding to the projection $T_{s^{\vee}} \to T_{s^{\sharp \vee}}$. It is contained in

$$\mathcal{H}(s^{\vee}, \bar{z}) = \mathcal{H}(\mathcal{R}_{s^{\vee}}, \lambda, \bar{z}) \rtimes \mathbb{C}[\mathcal{R}_{\phi^\sharp}, \bar{z}_{\phi^\sharp}].$$

Here the twisted group algebra and the 2-cocycle $\bar{z}_{\phi^\sharp} = \bar{z}_{s^{\sharp \vee}}$ are given by

$$\mathbb{C}[\mathcal{R}_{\phi^\sharp}, \bar{z}_{\phi^\sharp}] = p_{\rho}\mathbb{C}[S_{\phi^\sharp}],$$

while the action of $\mathcal{R}_{\phi^\sharp}$ on $(113)$ comes from its natural action on $\mathcal{R}_{s^{\vee}}$.

For better comparison with the $p$-adic side we also determine the graded Hecke algebras attached to $s^{\vee}$. Let $(\phi^\flat, \rho^\flat) \in \Phi_{\text{cusp}}(L^\vee(F))$ be an unramified twist of $(\phi^\sharp, \rho^\sharp)$ which is bounded. Let $W_{\phi^\flat}$ be the stabilizer of $\phi^\flat$ in $W_{s^{\vee}}$. Then $W^\circ_{\phi^\flat} = W(G^\circ_{\phi^\flat}, T)$ is the subgroup of $W_{\phi^\flat} \cap W^\circ_{s^{\vee}}$ generated by the reflections it contains. The parabolic subgroup of $G^\circ_{\phi^\flat}$ generated by $M^\circ_{\phi^\flat}$ and upper triangular matrices determines a group $\mathcal{R}_{\phi^\flat}$ such that

$$W_{\phi^\flat} = W^\circ_{\phi^\flat} \rtimes \mathcal{R}_{\phi^\flat}.$$
The 2-cocycle $\zeta_{\phi_b^s}$ on $W_{\phi_b}$ is the restriction of $\zeta_{\phi_b^s} : W_{\phi_b^s}^2 \to \mathbb{C}^\times$. The root system $R_{\phi_b^s}$ is again a product of systems of type $A$, namely $\prod_j A_{e_j-1}$ if $\phi_b^s = \otimes_j \phi_j^s$. Then $W_{\phi_b^s} \cong \prod_j S_{e_j}$ and

$$t_{s^TV} = \text{Lie}(T_{s^TV}) = \left( \sum_i \text{Lie}(T_{s^iV}) \right)/Z(\mathfrak{gl}_n(\mathbb{C})).$$

It follows that

$$\mathbb{H}(\phi_b, v, q, \vec{r}) \cong \mathbb{H}(t_{s^TV}, W_{\phi_b^s}, \vec{r}, \zeta_{\phi_b^s}) \cong \mathbb{H}(t_{s^TV}, W_{\phi_b}, \vec{r}) \rtimes \mathbb{C}[\zeta_{\phi_b^s}, \zeta_{\phi_b^s}].$$

The Hecke algebras for Bernstein components of $\text{SL}_m(D)$ were computed in [ABPS3]. They are substantially more complicated than their counterparts for $\text{GL}_m(D)$, and in particular do not match entirely with the above affine Hecke algebras for Langlands parameters. To describe them, we need some notations. Let $\mathcal{P}$ be a parabolic subgroup of $\text{GL}_m(D)$, with Levi factor $\mathcal{M}$. Consider the inertial equivalence classes $s_{\mathcal{M}} = [\mathcal{M}, \sigma]_{\mathcal{M}}$ and $s = [\mathcal{M}, \sigma]_{\text{GL}_m(D)}$. Recall from (109) that $\mathcal{H}(\text{GL}_m(D))^s$ is Morita equivalent with

$$\mathcal{H}(\mathcal{R}_s, \lambda, q_s) = \bigotimes_i \mathcal{H}(\text{GL}_{e_i}, q_i f_i).$$

We need the groups

$$X^{\mathcal{M}}(s) = \{ \gamma \in \text{Irr}(\mathcal{M}/\mathcal{M}^s Z(\text{GL}_m(D))) : \gamma \otimes \sigma \in s_{\mathcal{M}} \},$$

$$X^{\text{GL}_m(D)}(s) = \{ \gamma \in \text{Irr}(\text{GL}_m(D)/\text{GL}_m(D)^s Z(\text{GL}_m(D))) : \gamma \otimes \mathcal{I}^{\text{GL}_m(D)}(\sigma) \in s \},$$

$$W_s^s = \{ w \in N_{\text{GL}_m(D)}(\mathcal{M})/(\mathcal{M}^s) \otimes \mathcal{I}^{\text{GL}_m(D)}(\sigma) : w(\gamma \otimes \sigma) \in s_{\mathcal{M}} \}.$$  

By [ABPS3] Lemma 2.3 $W_s^s = W_s \rtimes \mathcal{R}_s^s$ for a suitable subgroup $\mathcal{R}_s^s$, and by [ABPS3] Lemma 2.4 $X^{\text{GL}_m(D)}(s)/X^{\mathcal{M}}(s) \cong \mathcal{R}_s^s$. The group $X^{\text{GL}_m(D)}(s)$ acts naturally on $T_s \rtimes W_s$.

Let $\sigma^s$ be an irreducible constituent of $\sigma|_{\mathcal{M}^s}$. Every inertial equivalence class for $\text{SL}_m(D) = \text{GL}_m(D)^s$ is of the form $s^s = [\mathcal{M}^s, \sigma^s]_{\text{GL}_m(D)}$. By [ABPS3] Theorem 1 there exists a finite dimensional projective representation $V_{\mu}$ of $X^{\text{GL}_m(s)}$ such that $\mathcal{H}(\text{GL}_m(D)^s)^s_{\sigma^s}$ is Morita equivalent with one direct summand of

$$\mathcal{H}(\mathcal{R}_s, \lambda, q_s) \otimes \text{End}_{\mathbb{C}}(V_{\mu})^{X^{\mathcal{M}}(s)} \times X^{\mathcal{M}}(s)_{\mathcal{M}/\mathcal{M}^s} \rtimes \mathcal{R}_s^s.$$  

The other direct summands correspond to different constituents of $\sigma|_{\mathcal{M}^s}$. In (115) the group

$$X_{nr}(\mathcal{M}/\mathcal{M}^s) = \{ \chi \in X_{nr}(\mathcal{M}) : \mathcal{M}^s \subset \ker \chi \}$$

acts only via translations of $T_s$. We denote the quotient torus $T_s/X_{nr}(\mathcal{M}/\mathcal{M}^s)$ by $T_s^s$ and its Lie algebra by $\mathfrak{t}_s^s$.

From now on we will be more sketchy. The below can be made precise, but for that one would have to delve into some of the technicalities of [ABPS3], which are not so relevant for this paper. Although it is not so easy to write down all direct summands of (115) explicitly, we can say that they look like

$$\mathcal{H}(X^{s}(T_s^s), R_s, X_s(T_s^s), R_s^s, \lambda, q_s) \otimes \text{End}_{\mathbb{C}}(V_{\mu})^{X^{\mathcal{M}}(s, \sigma^s)} \rtimes \mathcal{R}_s^s.$$  

for suitable $X^{\mathcal{M}}(s, \sigma^s) \subset X^{\mathcal{M}}(s)$ and $V_{\mu^s} \subset V_{\mu}$. (From the above argument for graded Hecke algebras one sees approximately how (116) arises from (115).) This algebra need not be Morita equivalent to a twisted affine Hecke algebra as studied.
in this paper. The problem comes from the simultaneous action of \(X^M(\mathfrak{s}, \sigma^2)\) on \(T_s^\mathfrak{s}\) and \(V_{\mu^s}\): if that is complicated, it prevents \((116)\) from being Morita equivalent to a similar algebra without \(\text{End}_C(V_{\mu^s})\). If we consider \((116)\) as a kind of algebra bundle over \(T_s^\mathfrak{s}\), then these remarks mean that \(V_{\mu^s}\) could introduce some extra twists in this bundle, which take the algebra outside the scope of this paper. Examples can be constructed by combining the ideas in \[ABPS3\] Examples 5.2 and 5.5.

That being said, the other data involved in \((116)\) are as desired. It was checked in \[ABPS3\] Lemma 5.5 that:

(i) The underlying torus \(T^\mathfrak{s}_s = T^\mathfrak{s}_s/\Gamma^M(\mathfrak{s}, \sigma^2)\) is naturally isomorphic to \(T^\mathfrak{s}_sV = \Phi^s_\mathfrak{s}(\mathfrak{m}^s)[\mathfrak{m}^s, \sigma^2]\).

(ii) \(W_s \rtimes \mathcal{R}_s^\mathfrak{s} = W_s^\mathfrak{s}\) is isomorphic to \(W_s^\mathfrak{s}V = W_s^\mathfrak{s} \rtimes \mathcal{R}_s^\mathfrak{s}V\).

(iii) The space of irreducible representations of \((116)\) is isomorphic to a twisted extended quotient

\[
(T^\mathfrak{s}_s//W^\mathfrak{s}_s)_{\kappa^s} \cong (T^\mathfrak{s}_sV//W^\mathfrak{s}_sV)_{\kappa^s},
\]

and the 2-cocycle \(\kappa^s\) of \(W^\mathfrak{s}_s\) is equivalent to the 2-cocycle \(\kappa^sV\) of \(W^\mathfrak{s}_sV\).

Let us also discuss the graded Hecke algebras which can be derived from \((115)\) and \((116)\). The algebra \(\mathcal{O}(T^\mathfrak{s}_s)^{-X^M(\mathfrak{s})W^\mathfrak{s}_s}\) is naturally contained in the centre of \((115)\). This entails that we can localize at suitable subsets of \(T^\mathfrak{s}_s\). By localization at a small neighborhood of \(U\) of \(W^\mathfrak{s}_sX^M(\mathfrak{s})t(T^\mathfrak{s}_s)\), we can effectively replace \(X^M(\mathfrak{s})\) by the stabilizer of \(X^M(\mathfrak{s})t\), and \(\mathcal{R}^\mathfrak{s}_t\) by the stabilizer \(\mathcal{R}^\mathfrak{s}_t\) of \(W^\mathfrak{s}_sX^M(\mathfrak{s})t\). Then \((115)\) is transformed into the algebra

\[
(117) \quad C_{an}(U)^{-X^M(\mathfrak{s})W^\mathfrak{s}_s} \otimes \mathcal{O}(T^\mathfrak{s}_s)^{-X^M(\mathfrak{s})W^\mathfrak{s}_s} (\mathcal{H}(\mathcal{R}^\mathfrak{s}_t, \lambda, q_s) \otimes \text{End}_C(V_{\mu^s}))^{-X^M(\mathfrak{s})t} \times \mathcal{R}^\mathfrak{s}_t(t)
\]

where \(\mathcal{R}^\mathfrak{s}_s = (\mathcal{X}^s(T^\mathfrak{s}_s), \mathcal{R}_s, \mathcal{X}_s(T^\mathfrak{s}_s), \mathcal{R}_s^\mathfrak{s})\). But \(X^M(\mathfrak{s})t\) acts by translations on \(T^\mathfrak{s}_s\), so \(X^M(\mathfrak{s})t\) consists of all the elements that fix \(T^\mathfrak{s}_s\) entirely. From the description of the actions on \((115)\) in \[ABPS3\] Lemma 4.11 we see that \(X^M(\mathfrak{s})t\) acts only on \(\text{End}_C(V_{\mu^s})\). Then

\[
(118) \quad \text{End}_C(V_{\mu^s})^{-X^M(\mathfrak{s})t} = \text{End}_{X^M(\mathfrak{s})t}(V_{\mu^s}) \cong \bigoplus_{\mu^s} \text{End}_C(V_{\mu^s})
\]

is a finite dimensional semisimple algebra. The direct summands of \((115)\) and of \((117)\) are in bijection with the \(\mathcal{R}^\mathfrak{s}_t(t)\)-orbits on the set of direct summands of \((118)\).

That holds for any \(t \in (T^\mathfrak{s}_s)\), in particular for some \(t\) with \(\mathcal{R}^\mathfrak{s}_t(t) = 1\), so in fact the direct summands \(\text{End}_C(V_{\mu^s})\) of \((118)\) parametrize the direct summands of \((115)\) and of \((117)\). Thus \((117)\) is a direct sum of algebras

\[
(119) \quad C_{an}(U)^{-X^M(\mathfrak{s})W^\mathfrak{s}_s} \otimes \mathcal{O}(T^\mathfrak{s}_s)^{-X^M(\mathfrak{s})W^\mathfrak{s}_s} (\mathcal{H}(\mathcal{R}^\mathfrak{s}_t, \lambda, q_s) \otimes \text{End}_C(V_{\mu^s})) \times \mathcal{R}^\mathfrak{s}_t(t).
\]

Here \((\mu^s, V_{\mu^s})\) is a projective representation of \(\mathcal{R}^\mathfrak{s}_t(t)\). In such situations there is a Morita equivalent algebra embedding\[
\mathbb{C}[\mathcal{R}^\mathfrak{s}_t(t), z] \rightarrow \text{End}_C(V_{\mu^s}) \times \mathcal{R}^\mathfrak{s}_t(t)
\]
\[
r \mapsto \mu^s(r)^{-1}r,
\]
for a suitable 2-cocycle \( \zeta \). Via this method (119) is Morita equivalent with
\[
C_{an}(U)^{X^M(s)}W^2_t \otimes \mathcal{O}(T^2)^{X^M(s)}W^2_t^{\ast} \overset{\mathcal{H}(R^2_{\ast}, \lambda, q_{\ast})}{\times} \mathbb{C}[\mathcal{R}_{\ast}(t), \zeta].
\]

From the property (iii) of the algebra (116) we see that \( \zeta \) has to be the restriction of \( \zeta_{S^t} \) to \( \mathcal{R}_{\ast}(t)^{2} \). By Theorems 2.5\(a\) and 2.11\(a\) the algebra (120) is Morita equivalent with
\[
C_{an}(U)^{X^M(s)}W^2_t \otimes \mathbb{H}(t_{\ast}, W(R_{\ast}), q_{\ast}) \times \mathbb{C}[\mathcal{R}_{\ast}(t), \zeta_{S^t\ast}].
\]

Hence the equivalence between \( \text{Rep}(\text{SL}_m(D))^{s^d} \cong \text{Mod}(\mathcal{H}(\text{GL}_m(D)^{s^d})) \) and the module category of (116) restricts to an equivalence between
\[
\text{Mod}_{f, W^2_t \otimes X^M(s) (T^2)^{s^d}} (\mathcal{H}(\text{GL}_m(D)^{s^d})) \quad \text{and} \quad \\
\text{Mod}_{f, (t_{\ast}) (\mathbb{H}(t_{\ast}, W(R_{\ast}), q_{\ast}) \times \mathbb{C}[\mathcal{R}_{\ast}(t), \zeta_{S^t\ast}]).}
\]

Every finite length representation in \( \text{Rep}(\text{SL}_m(D))^{s^d} \) decomposes canonically as a direct sum of generalized weight spaces for \( \mathcal{O}(T^2)^{X^M(s)}W^2_t \), so by varying \( t \) in \( (T^2)^{s^d} \) we can describe all such representations in terms of these equivalences of categories. In this sense
\[
\mathbb{H}(t_{\ast}, W(R_{\ast}), q_{\ast}) \times \mathbb{C}[\mathcal{R}_{\ast}(t), \zeta_{S^t\ast}]
\]
is the graded Hecke algebra attached to \( (s^d, t) \). Suppose that \( t \) corresponds to \( (\phi^d_{\ast}, \rho^d) \in \Phi_{\text{cusp}}(\mathcal{L}^d(F)) \), where \( M = \mathcal{L}(F) \). Then we can compare (122) with (114). Using the earlier comparison results (i), (ii) and (iii), we see that (122) is the specialization of (114) and \( \mathcal{L} = \log(q_{\ast}) \).

We conclude that, for a Bernstein component \( s^d \) of \( \text{SL}_m(D) \), corresponding to a Bernstein component \( s^{d^d} \) of enhanced L-parameters:
- The twisted graded Hecke algebras attached to \( s^d \) and to \( s^{d^d} \) are isomorphic.
- The twisted affine Hecke algebras attached to \( s^d \) and to \( s^{d^d} \) need not be isomorphic, but they are sufficiently close, so that their categories of finite length modules are equivalent.

5.3. Pure inner twists of classical groups.

Take \( n \in \mathbb{N} \) and let \( \mathcal{G}^s_n \) be a \( F \)-split connected classical group of rank \( n \). That is, \( \mathcal{G}^s_n \) is one of the following groups:
- (i) \( \text{Sp}_{2n} \), the symplectic group in \( 2n \) variables defined over \( F \),
- (ii) \( \text{SO}_{2n+1} \), the split special orthogonal group in \( 2n+1 \) variables defined over \( F \),
- (iii) \( \text{SO}_{2n} \), the split special orthogonal group in \( 2n \) variables defined over \( F \).

Let \( V^* \) be a finite dimensional \( F \)-vector space equipped with a non-degenerate symplectic or orthogonal form such that \( \mathcal{G}^s_n(F) \) equals \( \text{Sp}(V^*) \) or \( \text{SO}(V^*) \). The pure inner twists \( \mathcal{G}_n \) of \( \mathcal{G}^s_n \) correspond bijectively to forms \( V \) of the space \( V^* \) with its bilinear form \( \langle \cdot, \cdot \rangle \) [KMRT] §29D–E. If \( \mathcal{G}^s_n(F) = \text{Sp}(V^*) \), then the pointed set \( H_1(F, \mathcal{G}^s_n) \) has only one element and there are no nontrivial pure inner twists of \( \mathcal{G}^s_n \). If \( \mathcal{G}^s_n(F) = \text{SO}(V^*) \), then elements of \( H_1(F, \mathcal{G}^s_n) \) correspond bijectively to the isomorphism classes of orthogonal spaces \( V \) over \( F \) with \( \dim(V) = \dim(V^*) \) and \( \text{disc}(V) = \text{disc}(V^*) \). The corresponding pure inner twist of \( \mathcal{G}^s_n(F) \) is the special orthogonal group \( \text{SO}(V) \).
Let \( \mathcal{G}_n(F) \) be a pure inner twist of \( \mathcal{G}_n^+(F) \) (we allow \( \mathcal{G}_n(F) = \mathcal{G}_n^+(F) \)). It is known (see for instance \([\text{ChGo}]\)), that up to conjugacy every Levi subgroup of \( \mathcal{G}_n(F) \) is of the form

\[
(123) \quad \mathcal{L}(F) = \mathcal{G}_{n-}(F) \times \prod_j \GL_{m_j}(F),
\]

where \( \sum_j m_j + n^- = n \) and \( \mathcal{G}_{n-}(F) \) is an inner twist of the split connected classical group \( \mathcal{G}_{n-}^* \) defined over \( F \), of rank \( n^- \), which has the same type as \( \mathcal{G}_n^+(F) \). There is a natural embedding \( \text{Std}_{L'} \) of \( L' \) into \( \GL_{N'}(\mathbb{C}) \times W_F \), where \( N' = 2n + 1 \) if \( \mathcal{G}_n = \text{Sp}_{2n} \), and \( N' = 2n \) otherwise.

Let \((\phi, \rho) \in \Phi_{\text{cusp}}(\mathcal{L}(F))\). The factorization \((123)\) leads to

\[
(124) \quad \text{Std}_{L'} \circ \phi = \varphi \oplus \bigoplus_j (\phi_j \oplus \phi_j^\vee).
\]

Because we consider only pure inner twists in this section, it would be superfluous to replace \( \mathcal{G}^\vee \) by \( \mathcal{G}^\vee_{\text{sc}} \). We refrain from doing so in this section, and we use the objects, which before where defined in terms of \( \mathcal{G}^\vee_{\text{sc}} \), now with the same definition involving just \( \mathcal{G}^\vee \). For instance, instead of the group \( S_\phi \) defined in Definition \([3.2]\) we will take the component group \( \pi_0(Z_{\mathcal{G}^\vee}(\phi)) \) and we use a variation on \( \Phi_{\text{cusp}}(L') \) with that component group. The restriction of an enhancement \( \rho \) to the center of \( \mathcal{L}^\vee \) still determines the relevance. For instance, if the restriction to \( Z(\mathcal{L}^\vee) \) is trivial, then it corresponds to the split form, otherwise it corresponds to a non-split form. Hence, we can decompose \( \rho = \varphi \otimes \bigotimes_j j_{\rho_j} \), where \((\varphi, \varrho)\in \Phi_{\text{cusp}}(\mathcal{G}_{n-}(F))\) and \((\phi_j, \rho_j) \in \Phi_{\text{cusp}}(\GL_{m_j}(F))\) for each \( j \).

Let \( I^+_\phi \) (resp. \( I^-_\phi \)) be the set of (classes of) self-dual irreducible representations of \( W_F \) which occur in \( \text{Std}_{L'} \circ \phi \) and which factor through a group of the type of \( \mathcal{G}^\vee \) (resp. of opposite type of \( \mathcal{G}^\vee \)). Let \( I^+_\phi \) be a set of (classes of) non self-dual irreducible representations of \( W_F \) which occur in \( \text{Std}_{L'} \circ \phi \), such that if \( \tau \in I^+_\phi \) then \( \tau^\vee \not\in I^-_\phi \), and maximal for this property. We denote the irreducible \( a \)-dimensional representation of \( SL_n(\mathbb{C}) \) by \( S_a \).

On the one hand \((\phi_j, \rho_j)\) satisfy the conditions stated in Paragraph \([5.1]\), i.e. \( \phi_j \) is an irreducible representation of \( W_F \) and \( \rho_j \) is the trivial representation of \( \pi_0(Z_{\GL_{m_j}(\mathbb{C})} \langle \phi_j \rangle) \). On the other hand, by [\text{Mou}, Proposition 3.6] we have

\[
(125) \quad \text{Std}_{L'_{\mathcal{G}_{n-}}} \circ \varphi = \bigoplus_{\tau \in I^+_\varphi} \left( \bigoplus_{a \text{ odd}, a = 1} a_{\tau} \right) (\tau \otimes S_a) + \bigoplus_{\tau \in I^-_\varphi} \left( \bigoplus_{a \text{ even}, a = 2} a_{\tau} \right) (\tau \otimes S_a),
\]

where \( a_{\tau} \in \mathbb{Z}_{\geq 0} \). As introduced by Mœglin, let \( \text{Jord}(\varphi) \) be the set of pairs \((\tau, a)\) with \( \tau \in \text{Irr}(W_F) \), \( a \in \mathbb{Z}_{\geq 0} \) such that \( \tau \otimes S_a \) is a reducible subrepresentation of \( \text{Std}_{L'_{\mathcal{G}_{n-}}} \circ \varphi \).

The group \( S_\phi \) is isomorphic to \( (\mathbb{Z}/2\mathbb{Z})^p \) for some integer \( p \). It is generated by elements of order two, by \( \tau_{\tau,a} \) for \((\tau, a) \in \text{Jord}(\varphi)\) without hypothesis on the parity of \( a \) and by \( z_{\tau,a} \) when \( a \) is even. The character \( \rho \) satisfies \( \rho(z_{\tau, 2i-1} z_{\tau, 2i+1}) = -1 \) for all \( \tau \in I^+_\varphi \) and \( i \in [1, \frac{a_{\tau} - 1}{2}] \) and \( \rho(z_{\tau, 2i}) = (-1)^i \) for all \( \tau \in I^-_\varphi \) and \( i \in [1, \frac{a_{\tau}}{2}] \).

If \( \tau \) is an irreducible representation of \( W_F \) and of dimension \( m \) such that \( \tau|_{L'} \cong \tau^\vee|_{L'} \), then \( \tau \cong \tau^\vee z \) with \( z \in X_{\text{nr}}(L' \GL_m(F)) \). Replacing \( \tau \) by \( \tau z^{1/2} \) (where \( z^{1/2} \) is any square root of \( z \)), we can assume that \( \tau \cong \tau^\vee \). In the following, for all \( j \)
we assume that, if \( \phi_i^\lor \) is inertially equivalent to \( \phi_j \), then \( \phi_i^\lor \cong \phi_j \). Note that a self-dual irreducible representation of \( W_F \) is necessarily of symplectic-type or of orthogonal-type.

We choose a basepoint \( \phi \) (inside its inertial equivalence class) as follows:

- if \( m_i = m_j \) and \( \phi_i \) differs from \( \phi_j \) by an unramified twist, then \( \phi_i = \phi_j \);
- if \( \phi_i^\lor \) is an unramified twist of \( \phi_i \), then we can assume that \( \phi_i^\lor \cong \phi_i \);
- if \( \phi_i^\lor \cong \phi_j \), then \( i = j \).

For an irreducible representation \( \tau \) of \( W_F \), we will denote by \( e_\tau \) the number of times that \( \tau \) appears in a GL factor of \( \mathcal{L}^\lor \), by \( \ell_\tau \) the multiplicity of \( \tau \) in \( \varphi|_{W_F} \). With the above choice of \( \phi \), for all \( \tau \in I_\phi^+ \cup I_\phi^- \cup I_\phi^0 \),

\[
\phi = \bigoplus_{\tau \in I_\phi^+ \cup I_\phi^-} 2e_\tau \tau \oplus \bigoplus_{\tau \in I_\phi^0} e_\tau (\tau \oplus \tau^\lor) \oplus \varphi;
\]

\[
\phi|_{W_F} = \bigoplus_{\tau \in I_\phi^+ \cup I_\phi^-} (2e_\tau + \ell_\tau) \tau \oplus \bigoplus_{\tau \in I_\phi^0} e_\tau (\tau \oplus \tau^\lor),
\]

\[
G_\phi \cong \prod_{\tau \in I_\phi^+} \text{Sp}_{2e_\tau + \ell_\tau} (\mathbb{C}) \times \prod_{\tau \in I_\phi^- \text{dim } \tau \text{ even}} \text{O}_{2e_\tau + \ell_\tau} (\mathbb{C}) \times S \left( \prod_{\tau \in I_\phi^+ \text{dim } \tau \text{ odd}} \text{O}_{2e_\tau + \ell_\tau} (\mathbb{C}) \right) \times \prod_{\tau \in I_\phi^0} \text{GL}_{e_\tau} (\mathbb{C}),
\]

\[
M \cong \prod_{\tau \in I_\phi^+ \text{dim } \tau \text{ even}} (\mathbb{C}^\times)^{e_\tau} \times \text{Sp}_{\ell_\tau} (\mathbb{C}) \times \prod_{\tau \in I_\phi^+ \text{dim } \tau \text{ odd}} (\mathbb{C}^\times)^{e_\tau} \times
\]

\[
\prod_{\tau \in I_\phi^- \text{dim } \tau \text{ even}} \text{O}_{\ell_\tau} (\mathbb{C}) \times S \left( \prod_{\tau \in I_\phi^- \text{dim } \tau \text{ odd}} \text{O}_{\ell_\tau} (\mathbb{C}) \right) \times \prod_{\tau \in I_\phi^0} (\mathbb{C}^\times)^{e_\tau}.
\]

Here \( S(H) \), for a matrix group \( H \), means the elements of determinant 1 in \( H \). The above expression for \( G_\phi \) naturally factors as \( \prod_{\tau \in I_\phi^+ \cup I_\phi^- \cup I_\phi^0} G_\tau^\circ \), and similarly for \( M^\circ \).

This is an almost direct factorization of \( G_\phi^\circ \) in the sense of \( \text{[7]} \). With that we can write

\[
(126) \quad T \cong \prod_{\tau \in I_\phi^+ \cup I_\phi^- \cup I_\phi^0} (\mathbb{C}^\times)^{e_\tau}, \quad R(G_\phi^\circ, T) \cong \prod_{\tau \in I_\phi^+ \cup I_\phi^0 \cup I_\phi^0} R(G_\tau^\circ T, T).
\]

Let us record the root systems \( R_\tau = R(G_\tau^\circ T, T) \):

<table>
<thead>
<tr>
<th>( \tau )</th>
<th>( e_\tau )</th>
<th>( R_\tau )</th>
<th>( R_{\tau, \text{red}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau \in I_\phi^- )</td>
<td>( e_\tau = 0 )</td>
<td>( \varnothing )</td>
<td>( \varnothing )</td>
</tr>
<tr>
<td>( e_\tau \neq 0, \ell_\tau = 0 )</td>
<td>( C_{e_\tau} )</td>
<td>( C_{e_\tau} )</td>
<td></td>
</tr>
<tr>
<td>( e_\tau \neq 0, \ell_\tau \neq 0 )</td>
<td>( BC_{e_\tau} )</td>
<td>( B_{e_\tau} )</td>
<td></td>
</tr>
<tr>
<td>( \tau \in I_\phi^+ )</td>
<td>( e_\tau = 0 )</td>
<td>( \varnothing )</td>
<td>( \varnothing )</td>
</tr>
<tr>
<td>( e_\tau \neq 0, \ell_\tau = 0 )</td>
<td>( D_{e_\tau} )</td>
<td>( D_{e_\tau} )</td>
<td></td>
</tr>
<tr>
<td>( e_\tau \neq 0, \ell_\tau \neq 0 )</td>
<td>( B_{e_\tau} )</td>
<td>( B_{e_\tau} )</td>
<td></td>
</tr>
<tr>
<td>( \tau \in I_\phi^0 )</td>
<td>( e_\tau \leq 1 )</td>
<td>( \varnothing )</td>
<td>( \varnothing )</td>
</tr>
<tr>
<td>( e_\tau \geq 2 )</td>
<td>( A_{e_\tau - 1} )</td>
<td>( A_{e_\tau - 1} )</td>
<td></td>
</tr>
</tbody>
</table>
To justify the above choice of a basepoint $\phi$, we need to check that $G_{\phi}^0$ detects as many roots as possible. Let us consider the restriction $\phi|_{I_F}$:
\[
\operatorname{Std}_{\tau} \circ \phi|_{I_F} = \varphi|_{I_F} \oplus \bigoplus_j (\phi_j|_{I_F} \oplus \phi_j^\vee|_{I_F})
\]
\[
= \bigoplus_{\tau \in I_0^r \cup I_0^-} (2c_{\tau} + \ell_{\tau}) \tau|_{I_F} \oplus \bigoplus_{\tau \in I_0^r} c_{\tau}(\tau|_{I_F} \oplus \tau^\vee|_{I_F}).
\]
We have assumed that for $\tau \in I_0^r$, $\tau|_{I_F} \not\cong \tau^\vee|_{I_F}$ and we know that an irreducible representation $\tau$ of $W_F$ decomposes upon restriction to $I_F$ as
\[
(127) 
\tau|_{I_F} = \theta \oplus \theta^{\text{Frob}}_F \oplus \cdots \oplus \theta^{\text{Frob}}_{F}^{r-1},
\]
for some irreducible representation $\theta$ of $I_F$. Here for all $w \in I_F$, $\theta^{\text{Frob}}_F(w) = \theta(\text{Frob}_F^{k} w \text{Frob}_F^{k})$. If we assume $\tau|_{I_F} \cong \tau^\vee|_{I_F}$, then $\theta^\vee \cong \theta^{\text{Frob}}_F$ for some integer $i$ between 0 and $t_\tau - 1$. Then we have $\theta \cong \theta^{\text{Frob}}_F$ for $\theta^\vee \cong \theta^{\text{Frob}}_F$. This implies that $i = 0$ or $t_\tau$ is even and $i = t_\tau/2$. In the first case, $\theta^\vee \cong \theta$ and in the second case $\theta^\vee \cong \theta^{\text{Frob}}_F^{r/2}$. We denote by $I_{\phi}^{++}$ (resp. $I_{\phi}^{-}$) the subset of $I_{\phi}^+$ (resp. $I_{\phi}^-$) corresponding to the first case, and define $I_{\phi}^{+,-}$ as the remaining subset of $I_{\phi}^+ \cup I_{\phi}^-$. For any $\tau$, let $\tau'$ be a twist of $\tau$ by an unramified character of $W_F$, such that $\tau'$ is self-dual but not isomorphic to $\tau$. For $\tau \in I_{\phi}^{+,-}$ the type of $\tau'$ is opposite to that of $\tau$, which motivates the superscript $+-$. The three sets $I_{\phi}^{++}, I_{\phi}^{-,-}, I_{\phi}^{-+}$ are considered modulo the relation $\tau \sim \tau'$. We find that
\[
(128) 
J^0 = Z_{G^\vee}(\phi|_{I_F})^0 \cong \prod_{\tau \in I_{\phi}^{+,-}} \text{Sp}_{2e_\tau + \ell_\tau + \ell_{\tau'}}(\mathbb{C})^{t_\tau} \times \prod_{\tau \in I_{\phi}^{++}} \text{SO}_{2e_\tau + \ell_\tau + \ell_{\tau'}}(\mathbb{C})^{t_\tau} \times \prod_{\tau \in I_{\phi}^{-,-}} \text{GL}_{2e_\tau + \ell_\tau + \ell_{\tau'}}(\mathbb{C})^{t_\tau/2} \times \prod_{\tau \in I_{\phi}^{+-}} \text{GL}_{e_\tau}(\mathbb{C})^{t_\tau}.
\]
For all $\tau \in I_{\phi}^{++}$, we have an embedding of $(\mathbb{C}^r)^{e_\tau}$ into $(\mathbb{C}^r)^{e_\tau} \times \text{SO}_{2e_\tau + \ell_\tau + \ell_{\tau'}}(\mathbb{C})^{t_\tau}$ and the latter is embedded diagonally as Levi subgroup in $\text{SO}_{2e_\tau + \ell_\tau + \ell_{\tau'}}(\mathbb{C})^{t_\tau}$. We have the same kind of embedding for $\tau \in I_{\phi}^{-,-}$ and $\tau \in I_{\phi}^0$. For $\tau \in I_{\phi}^{+-}$, the embedding of $(\mathbb{C}^r)^{e_\tau}$ in $\text{GL}_{2e_\tau + \ell_\tau + \ell_{\tau'}}(\mathbb{C})$ is given by
\[
(z_1, \ldots, z_{e_\tau}) \mapsto \text{diag}(z_1, \ldots, z_{e_\tau}, 1, \ldots, 1, z_{e_\tau}^{-1}, \ldots, z_1^{-1}),
\]
with $\ell_\tau + \ell_{\tau'}$ times 1 in the middle.

From (128) we see that $R(J^0, T)$ is a union of irreducible components $R(J^0, T)_{\tau}$. Comparing these data with the earlier description from (126) and the subsequent table, we deduce that $R(J^0, T)_{\tau,\text{red}} = R(G_{\phi}^0, T)_{\tau,\text{red}}$ for all $\tau$. Hence $R(J^0, T)_{\text{red}} = R(G_{\phi}^0, T)_{\text{red}}$, as required for a good basepoint $\phi$. In particular $W_{G^\vee}^0 \cong W(G_{\phi}^0, T)$.

We note that $Z(L^\vee)^0 = T$, see (126). Since $\phi_j : W_F \times \text{SL}_2(\mathbb{C}) \to \text{GL}_{m_j}(\mathbb{C})$ is cuspidal, it is irreducible and trivial on $\text{SL}_2(\mathbb{C})$. Thus we can write
\[
L^\vee = G_{\phi}^v(\mathbb{C}) \times \prod_j \text{GL}_{m_j}(\mathbb{C}) = G_{\phi}^v(\mathbb{C}) \times \prod_{\tau \in I_{\phi}^{+,-} \cup I_{\phi}^{++}} \text{GL}_{\dim(\tau)}(\mathbb{C})^{e_\tau}.
\]
It follows from (127) that $\dim(\tau) = t_\tau \dim(\theta)$ with $\theta \in \text{Irr}(I_F)$, and that
\[
X_{nr}(L^\vee)_{\phi} \cong \prod_{\tau \in I_{\phi}^{+,-} \cup I_{\phi}^{++} \cup I_{\phi}^{+-}} \mu_{t_\tau}(\mathbb{C})^{e_\tau}.
\]
Here μ_k denotes the functor of taking the k-th roots of unity in ring. In particular \( t_\tau = |X_{\text{un}}(\mathbb{L}\text{GL}_m)_\tau| \), the number of unramified characters \( \chi \) such that \( \chi \tau = \tau \).

In the following table, which stems largely from [Mou §4.1], we describe the root systems and the Weyl groups. We may omit the cases \( e_\tau = 0 \), because there all the root systems and Weyl groups are trivial.

<table>
<thead>
<tr>
<th>( \tau \in I_{\phi}^- )</th>
<th>( (\mathbb{C}^X)^e \times \text{Sp}_k(\mathbb{C}) )</th>
<th>( \ell_\tau = 0 )</th>
<th>( C_{e_\tau} )</th>
<th>( S_{e_\tau} \times (\mathbb{Z}/2\mathbb{Z})^{e_\tau} )</th>
<th>( S_{e_\tau} \times (\mathbb{Z}/2\mathbb{Z})^{e_\tau} )</th>
<th>( S_{e_\tau} \times (\mathbb{Z}/2\mathbb{Z})^{e_\tau} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I_{\phi}^+ )</td>
<td>( (\mathbb{C}^X)^e \times \text{Sp}_k(\mathbb{C}), (\mathbb{C}^X)^e_0 \times \text{O}_k(\mathbb{C}) )</td>
<td>( \ell_\tau = 0 )</td>
<td>( C_{e_\tau} )</td>
<td>( S_{e_\tau} \times (\mathbb{Z}/2\mathbb{Z})^{e_\tau} )</td>
<td>( S_{e_\tau} \times (\mathbb{Z}/2\mathbb{Z})^{e_\tau} )</td>
<td>( S_{e_\tau} \times (\mathbb{Z}/2\mathbb{Z})^{e_\tau} )</td>
</tr>
<tr>
<td>( I_{\phi}^+ )</td>
<td>( (\mathbb{C}^X)^e \times \text{O}_k(\mathbb{C}) )</td>
<td>( \ell_\tau = 0 )</td>
<td>( B_{e_\tau} )</td>
<td>( S_{e_\tau} \times (\mathbb{Z}/2\mathbb{Z})^{e_\tau} )</td>
<td>( S_{e_\tau} \times (\mathbb{Z}/2\mathbb{Z})^{e_\tau} )</td>
<td>( S_{e_\tau} \times (\mathbb{Z}/2\mathbb{Z})^{e_\tau} )</td>
</tr>
<tr>
<td>( I_{\phi}^0 )</td>
<td>( (\mathbb{C}^X)^e )</td>
<td>( e_\tau \leq 1 )</td>
<td>( \emptyset )</td>
<td>( {1} )</td>
<td>( {1} )</td>
<td></td>
</tr>
<tr>
<td>( I_{\phi}^0 )</td>
<td>( (\mathbb{C}^X)^e )</td>
<td>( e_\tau \geq 2 )</td>
<td>( A_{e_\tau-1} )</td>
<td>( S_{e_\tau} )</td>
<td>( S_{e_\tau} )</td>
<td></td>
</tr>
</tbody>
</table>

For all \( \tau \in I_{\phi}^0 \) such that \( \ell_\tau = 0 \neq e_\tau \), take

\[
r_\tau = \text{diag}(1, \ldots, 1, (0 \ 1 \ 0), 1, \ldots, 1) \in O_{2e_\tau}(\mathbb{C}) \setminus SO_{2e_\tau}(\mathbb{C}).
\]

It normalizes \( M, T, \phi \) and generates \( W_{M_\tau}^{G_\tau}/W_{M_\tau}^{G_\tau} \). The finite group \( \mathfrak{R}_{e_\tau} \) is generated by such elements \( r_\tau \). More precisely, let

\[
C = \{ \tau \in I_{\phi}^0 \mid \ell_\tau = 0 \},
\]

\[
C_{\text{even}} = \{ \tau \in C \mid \text{dim } \tau \text{ even} \},
\]

\[
C_{\text{odd}} = \{ \tau \in C \mid \text{dim } \tau \text{ odd} \}.
\]

It was shown in [Mou §4.1] that:

- if \( \mathcal{G} = \text{Sp}_N \) or \( \mathcal{G} = \text{SO}_N \) with \( N \) odd, then

\[
\mathfrak{R}_{e_\tau} \cong \prod_{\tau \in C} \langle r_\tau \rangle;
\]

- if \( \mathcal{G} = \text{SO}_N \) and \( \mathcal{L} = \text{GL}_{d_1} \times \ldots \times \text{GL}_{d_r} \times \text{SO}_{N'} \) with \( N \) even and \( N' \geq 4 \), then

\[
\mathfrak{R}_{e_\tau} \cong \prod_{\tau \in C} \langle r_\tau \rangle;
\]

- if \( \mathcal{G} = \text{SO}_N \) and \( \mathcal{L} = \text{GL}_{d_1} \times \ldots \times \text{GL}_{d_r} \) with \( N \) even, then

\[
\mathfrak{R}_{e_\tau} \cong \prod_{\tau \in C_{\text{even}}} \langle r_\tau \rangle \times \langle r_\tau r' \rangle \mid \tau, \tau' \in C_{\text{odd}}.
\]

From the shape of \( M_\tau^0 \) we can describe the unipotent element \( v_\tau \):

<table>
<thead>
<tr>
<th>( M_\tau^0 )</th>
<th>( v_\tau )</th>
<th>( \ell )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (\mathbb{C}^X)^e_0 \times \text{Sp}_k(\mathbb{C}) )</td>
<td>( (1^e) \times (2, 4, \ldots, 2d - 2, 2d) )</td>
<td>( \ell_\tau = d(d + 1) )</td>
</tr>
<tr>
<td>( (\mathbb{C}^X)^e \times \text{SO}_{k}(\mathbb{C}) )</td>
<td>( (1^e) \times (1, 3, \ldots, 2d - 3, 2d - 1) )</td>
<td>( \ell_\tau = d^2 )</td>
</tr>
<tr>
<td>( (\mathbb{C}^X)^e_0 )</td>
<td>( (1^e) )</td>
<td></td>
</tr>
</tbody>
</table>

To be complete, let us describe the cuspidal representations of \( A_{M_\tau^0} (v_\tau) \). We have

\[
A_{M_\tau^0} (v_\tau) \cong \begin{cases} 
(\mathbb{Z}/2\mathbb{Z})^d = \langle z_{\tau, 2a}, a \in [1, d] \rangle & \text{if } \tau \in I_{\phi}^- \\
(\mathbb{Z}/2\mathbb{Z})^{d-1} = \langle z_{\tau, 2a-1} z_{\tau, 2a+1}, a \in [1, d - 1] \rangle & \text{if } \tau \in I_{\phi}^+
\end{cases}
\]
Moreover, the cuspidal irreducible representation $\epsilon_\tau$ of $A_{M_\psi}(v_\tau)$ satisfies

\[ \epsilon_\tau(z_{\tau,2a}) = (-1)^a \text{ if } \tau \in I_\phi^- \quad \text{and} \quad \epsilon_\tau(z_{\tau,2a-1}z_{\tau,2a+1}) = -1 \text{ if } \tau \in I_\phi^+. \]

For all $\tau \in I_\phi^+ \sqcup I_\phi^-$, denote by $a_\tau$ the biggest part of the partition of $v_\tau$ and by $a'_\tau$ the biggest part of the partition of $v_{\tau'}$. In case $v_{\tau'} = 1$, we will assume that $a'_\tau = 0$ if $\tau \in I_\phi^-$ and $a'_\tau = -1$ if $\tau \in I_\phi^+$ (this is compatible with Proposition 3.14).

Finally, we consider the parameter functions. The number $m_\alpha$ from Definition 3.11 equals $t_\tau$ unless $\tau \in \text{Irr}(W_F)^{\tau^-}$, $\ell_\tau = 0$ and $\alpha$ is a long root in a type $C$ root system, then $m_\alpha = t_\tau/2$. Recall that $R_\psi^\tau$ consists of the roots $m_\alpha \alpha$ with $\alpha \in R(J^0,T)_{\text{red}}$. Multiplication by $m_\alpha$ does not change the type of $R(J^0,T)_{\tau'}$, only in the exceptional case, there $C_{\epsilon_\tau}$ is turned into $B_{\epsilon_\tau}$.

If $\alpha \in R_{\tau,\text{red}}$ is not a short root in a type $B$ root system, then by [Luc2, 2.13] $c(\alpha) = 2$, so $\lambda(\alpha) = m_\alpha$. For the simple short root $\alpha_\tau \in R_{\tau,\text{red}}$ we have $c(\alpha_\tau) = a_\tau + 1$, $c^*(\alpha_\tau) = a'_\tau + 1$ and $m_\alpha = t_\tau$. Hence

\[ \lambda(\alpha_\tau) = (a_\tau + a'_\tau + 2)t_\tau/2 \quad \text{and} \quad \lambda^*(\alpha_\tau) = |a_\tau - a'_\tau|t_\tau/2. \]

We conclude that

\[ (129) \quad \mathcal{H}(s^\vee, \bar{z}) = \mathcal{H}(R_{\psi^\tau}, \lambda, \lambda^*, \bar{z}) \cong \mathbb{C}[R_{\psi^\tau}]. \]

Via the specialization of $z_\tau$ at $q_F^{1/2}$, (129) becomes the extended affine Hecke algebra given in [Hei2]. Moreover, it was shown in [Hei2] that there is an equivalence of categories between $\text{Rep}(G(F))^\psi$ and the right modules over $\mathcal{H}(s^\vee, \bar{z})/\langle \{z_\tau - q_F^{1/2} \} \rangle$. Together with the LLC for $G(F)$ we get bijections

\[ (130) \quad \text{Irr} \left( \mathcal{H}(s^\vee, \bar{z})/\langle \{z_\tau - q_F^{1/2} \} \rangle \right) \leftrightarrow \text{Irr}(G(F))^\psi \leftrightarrow \Phi_\psi(G(F))^{s^\vee}. \]

It does not seem unlikely that this works out to the same bijection as in Theorem 3.18. But at present that is hard to check, because the LLC is not really explicit.

**Example 5.3.** We consider an example that illustrates many of the above aspects. Let $\tau : W_F \to GL_4(\mathbb{C})$ be an irreducible representation of $W_F$, self-dual of sympletic type and let $\varphi : W_F \times SL_2(\mathbb{C}) \to SO_{37}(\mathbb{C})$ be defined by

\[ \text{Std}_{GL_{37}} \circ \varphi = 1 \boxtimes (S_5 \oplus S_3 \oplus S_1) \oplus \xi \boxtimes (S_3 \oplus S_1) \oplus \tau \boxtimes (S_1 \oplus S_2), \]

with $\xi : W_F \to \mathbb{C}^\times$ an unramified quadratic character. We have

\[ Z_{SO_{37}(\mathbb{C})}(\varphi|W_F)^0 \cong SO_9(\mathbb{C}) \times SO_4(\mathbb{C}) \times Sp_6(\mathbb{C}), \]

and $\varphi$ defines a $L$-packet $\Pi_\varphi(\text{Sp}_{36}(F))$ with $2^5$ elements, of which two are supercuspidal. Let $\sigma \in \Pi_\varphi(\text{Sp}_{36}(F))$ be supercuspidal, corresponding to an enhanced Langlands parameter $(\varphi, \varepsilon)$ with $\varepsilon$ cuspidal. Consider $G(F) = \text{Sp}_{58}(F)$, the Levi subgroup

\[ L(F) = GL_4(F)^2 \times GL_1(F)^3 \times \text{Sp}_{30}(F) \]

and an irreducible supercuspidal representation $\pi_\varphi \boxtimes 1 \boxtimes 3 \boxtimes \sigma$ of $L(F)$. The cuspidal pair $s = [L(F), \pi_\varphi \boxtimes 1 \boxtimes 3 \boxtimes \sigma]$ of $G(F)$ admits $s^\vee = [L^\vee, \phi, \varepsilon]$ as dual inertial equivalence class, where $\phi : W_F \times SL_2(\mathbb{C}) \to L^\vee$,

\[ L^\vee = GL_4(\mathbb{C})^2 \times GL_1(\mathbb{C})^3 \times SO_{37}(\mathbb{C}) \quad \text{and} \quad \text{Std}_{L^\vee} \circ \phi = (\tau \oplus \tau^\vee) \boxtimes (1 \oplus 1^\vee) \boxtimes 3 \boxtimes \varphi. \]
We assume that $\tau|_{I_F} = \theta \oplus \theta^{\text{Frob}_F}$ with $\theta^\vee \cong \theta$, so $t_\tau = 2$. We first compute $W_\zeta^\circ$:

$$\phi|_{I_F} = \tau|_{I_F} = \theta \oplus \theta^{\text{Frob}_F} = \theta \oplus \theta^{\text{Frob}_F} \oplus 1 \oplus 1.$$

$$J^0 = Z G^\circ(\phi|_{I_F})^\circ \cong \text{Sp}_{10}(\mathbb{C}) \times \text{SO}_{10}(\mathbb{C}).$$

The torus $T$ is decomposed as $T = (\mathbb{C}^\times)^2 \times (\mathbb{C}^\times)^3$. The first part $(\mathbb{C}^\times)^2$ is embedded in an obvious way in $(\mathbb{C}^\times)^2 \times \text{Sp}_6(\mathbb{C})$ and then in $\text{Sp}_{10}(\mathbb{C})$ diagonally as Levi subgroup. The second part $(\mathbb{C}^\times)^3$ is embedded in $(\mathbb{C}^\times)^3 \times \text{SO}_{13}(\mathbb{C})$ and then in $\text{SO}_{19}(\mathbb{C})$ as Levi subgroup as well. The root system $R(J^0, T)$ (resp. $R(J^0, T)_\text{red}$) is $BC_2 \times B_3$ (resp. $B_2 \times B_3$), so $W_\zeta^\circ = W_{B_2} \times W_{B_3}$.

From the above discussion, we can see that $\phi$ is already a basepoint. If we denote by $\phi'$ the parameter defined by $\phi' = (\tau' \circ \tau^\vee) \oplus (\xi \circ \xi^\vee) \oplus \phi$, then $\phi'$ is another basepoint. Indeed, we have:

$$\phi|_{W_F} = \tau^\oplus 1 \oplus 1 \oplus \xi^\oplus$$

$$G^\circ = G^\circ(\phi|_{W_F})^\circ \cong \text{Sp}_{10}(\mathbb{C}) \times \text{Sp}_{15}(\mathbb{C}) \times \text{SO}_4(\mathbb{C})$$

$$M^\circ = M^\circ(\phi|_{W_F})^\circ \cong ((\mathbb{C}^\times)^2 \times \text{Sp}_6(\mathbb{C})) \times ((\mathbb{C}^\times)^3 \times \text{SO}_9(\mathbb{C})) \times \text{SO}_4(\mathbb{C})$$

$$\phi'|_{W_F} = \tau^\oplus \oplus 1^\oplus \oplus \xi^\oplus$$

$$C^\circ = C^\circ(\phi|_{W_F})^\circ \cong \text{Sp}_{10}(\mathbb{C}) \times \text{Sp}_{10}(\mathbb{C}) \times \text{SO}_{10}(\mathbb{C})$$

$$M^\circ_\phi = M^\circ(\phi|_{W_F})^\circ \cong (\mathbb{C}^\times)^2 \times \text{Sp}_6(\mathbb{C}) \times \text{SO}_9(\mathbb{C}) \times ((\mathbb{C}^\times)^3 \times \text{SO}_4(\mathbb{C})).$$

Here $R_\zeta^\circ$ is trivial, so $W_\zeta^\circ = W_\zeta^\circ$. Denote by $\alpha_1, \alpha_2$ (resp. $\beta_1, \beta_2, \beta_3$) the simple roots of $B_2$ (resp. $B_3$) with $\alpha_2$ (resp. $\beta_3$) the short root. Then $a_1 = 5, a_2 = 3, a_3 = 4$ and $a'_3 = 0$. The parameters are given by $\lambda(\alpha_1) = t_\tau = 2,$ $\lambda(\beta_1) = 1$ and

$$\lambda(\alpha_2) = t_\tau \frac{4 + 2}{2} = 6, \lambda(\beta_3) = \frac{5 + 3 + 2}{2} = 5, \lambda^*(\alpha_2) = t_\tau \frac{4}{2} = 4, \lambda^*(\beta_3) = \frac{5 - 3}{2} = 1.$$

Specializing $\tilde{z}$ to $q_F^{1/2}$, the quadratic relations in the Hecke algebra become

$$(N_{s_{\alpha_1}} - q_F^2)(N_{s_{\alpha_1}} + q_F^{-2}) = 0, (N_{s_{\alpha_2}} - q_F^{-3})(N_{s_{\alpha_2}} + q_F^{-3}) = 0,$$

$$(N_{s_{\beta_3}} - q_F^{5/2})(N_{s_{\beta_3}} + q_F^{-5/2}) = 0, (N_{s_{\beta_1}} - q_F^{1/2})(N_{s_{\beta_1}} + q_F^{-1/2}) = 0 \quad (i = 1, 2).$$

REFERENCES


AFFINE HECKE ALGEBRAS FOR LANGLANDS PARAMETERS


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