ON PRINCIPAL SERIES REPRESENTATIONS OF QUASI-SPLIT REDUCTIVE p-ADIC GROUPS

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Abstract. Let G be a quasi-split reductive group over a non-archimedean local field. We establish a local Langlands correspondence for all irreducible smooth complex G-representations in the principal series. The parametrization map is injective, and its image is an explicitly described set of enhanced L-parameters. Our correspondence is determined by the choice of a Whittaker datum for G, and it is canonical given that choice.

We show that our parametrization satisfies many expected properties, among others with respect to the enhanced L-parameters of generic representations, temperedness, cuspidal supports and central characters. Our correspondence lifts to a categorical level, where it makes the appropriate Bernstein blocks of G-representations naturally equivalent to module categories of Hecke algebras coming from Langlands parameters. Along the way we characterize genericity of G-representations in terms of representations of an affine Hecke algebra.

Contents

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Introduction

Consider a quasi-split reductive group $G = \mathcal{G}(F)$ over a non-archimedean local field F. Let $\mathrm{Rep}(G)$ be the category of smooth G-representations on complex vector spaces and let $\mathrm{Irr}(G)$ be the set of equivalence classes of irreducible representations in $\mathrm{Rep}(G)$. The conjectural local Langlands correspondence (LLC) asserts that $\mathrm{Irr}(G)$ is canonically partitioned into finite L-packets $\Pi_\phi(G)$, indexed by L-parameters ϕ. Some time after the initial formulation in [Bor2], it was realized that $\Pi_\phi(G)$ should be parametrized by the set of irreducible representations of a finite component group R_ϕ. These conjectures have motivated a large part of the study of reductive groups over local fields in past decades, see the survey papers [ABPS2, Bor2, Kal, Kud, Vog].

This paper establishes a local Langlands correspondence for the most accessible class of G-representations, those in the principal series. To formulate the result precisely, we quickly recall some relevant notions.

Let $T \subset G$ be the centralizer of a maximal F-split torus in G, or equivalently a minimal Levi subgroup of G. Then T is itself a torus because G is quasi-split, and T is unique up to conjugation. Any representation of G that can be obtained from a smooth representation of T by parabolic induction and then taking a subquotient, is called a principal series G-representation. These representations form a product of Bernstein blocks in $\mathrm{Rep}(G)$. We denote the set of irreducible principal series G-representations by $\mathrm{Irr}(G,T)$. We warn that some L-packets contain elements of $\mathrm{Irr}(G,T)$ and also other elements of $\mathrm{Irr}(G)$.

It has turned out that the representation ρ_π of R_ϕ associated to a given $\pi \in \mathrm{Irr}(G)$ is not canonically determined. To specify it uniquely one needs additional input, namely a Whittaker datum for G. Such a Whittaker datum can be used to normalize relevant intertwining operators, which then determine exactly how $\rho_\pi \in \mathrm{Irr}(R_\phi)$ is related to π. For non-quasi-split groups G such a normalization should also be possible [Kal, Conjecture 2.5], but it is much more involved.

We fix a Borel subgroup $B = TU$ and a nondegenerate character ξ of the unipotent radical U of B. Then (U,ξ), or rather its G-conjugacy class, is a Whittaker datum for G. Recall that $\pi \in \mathrm{Irr}(G)$ is called (U,ξ)-generic if $\mathrm{Hom}_U(\pi,\xi)$ is nonzero.

Let $\mathbf{W}_F$ be the Weil group of F, let $G^\vee$ be the complex dual group of G and let ${}^L G = G^\vee \rtimes \mathbf{W}_F$ be the Langlands dual group. In this introduction (but not in the body of the paper) we realize L-parameters for G as Weil–Deligne representations

$$\phi : \mathbf{W}_F \ltimes \mathbb{C} \to {}^L G.$$

Here $\mathbf{W}_F$ acts on $\mathbb{C}$ by $w \cdot z = \|w\| z$, where $\|w\| \in p^{\mathbb{Z}}$ is determined by $w(x) = x^{\|w\|}$ for all x in an algebraic closure of the residue field of F. The appropriate component group of such an L-parameter is

$$R_\phi = \pi_0\big(Z_{G^\vee}(\phi(\mathbf{W}_F \ltimes \mathbb{C}))/Z(G^\vee)^{\mathbf{W}_F}\big),$$

where $Z(G^\vee)$ denotes the centre of $G^\vee$. An enhancement of ϕ is an irreducible R_ϕ-representation. Let $\Phi_e(G)$ be the set of enhanced L-parameters for G, considered up to $G^\vee$-conjugacy. An element $(\phi,\rho) \in \Phi_e(G)$ belongs to the principal series if its cuspidal support is an enhanced L-parameter for T. More explicitly, that means

- $\phi(\mathbf{W}_F) \subset T^\vee \rtimes \mathbf{W}_F$ (or for a $G^\vee$-conjugate of $T^\vee \rtimes \mathbf{W}_F$, because ϕ is only given up to $G^\vee$-conjugacy),
- ρ appears in the homology of a certain variety of Borel subgroups.

We denote the subset of $\Phi_e(G)$ associated to the principal series by $\Phi_e(G,T)$. For a given ϕ it may happen that some enhancements yield elements of $\Phi_e(G,T)$, while other enhancements bring us outside $\Phi_e(G,T)$.

Our main result is a canonical LLC for principal series representations:

Theorem A. [see Section 7]
The Whittaker datum (U,ξ) determines a canonical bijection

$$\begin{array}{ccc} \mathrm{Irr}(G,T) & \longleftrightarrow & \Phi_e(G,T) \\ \pi(\phi,\rho) & \leftarrow\!\shortmid & (\phi,\rho) \\ \pi & \mapsto & (\phi_\pi,\rho_\pi) \end{array}$$

with the following properties:

(*a*) *$\pi(\phi,\rho)$ is (U,ξ)-generic if and only if ρ is trivial and $u_\phi = \phi(1,1)$ lies in the dense $Z_{G^\vee}(\phi(\mathbf{W}_F))$-orbit in*

$$\{v \in G^\vee : v \text{ is unipotent and } \phi(w)v\phi(w)^{-1} = v^{\|w\|} \text{ for all } w \in \mathbf{W}_F\}.$$

(*b*) *$\pi(\phi,\rho)$ is tempered (resp. essentially square-integrable) if and only if ϕ is bounded (resp. discrete).*

(*c*) *The bijection is compatible with the cuspidal support maps on both sides.*

(*d*) *The bijection is equivariant for the canonical actions of $H^1(\mathbf{W}_F, Z(G^\vee))$.*

(*e*) *The bijection is compatible with the Langlands classification and (for tempered representations) with parabolic induction.*

All Borel's desiderata from [Bor2, §10] *are satisfied. When π is given, ϕ_π is uniquely determined by (a)–(e) and the local Langlands correspondence for tori.*

For non-split quasi-split groups, the vast majority of the groups under consideration here, very little in this direction was previously known. On the other hand, for split groups many instances of Theorem A have been established before:

- Kazhdan and Lusztig [KaLu] proved the bijection and (b,e) for Iwahori-spherical representations, assuming that G is F-split and that $Z(G)$ is connected as algebraic group. Their starting point is Borel's description [Bor1] of those representations, in terms of Hecke algebras.
- Reeder [Ree2] extended [KaLu] to $\mathrm{Irr}(G,T)$ when G is F-split, $Z(G)$ is connected and the residual characteristic p of F is not "too small". This is based on work of Roche [Roc1] and includes (a,b,e). We note that here the Whittaker datum is unique up to G-conjugacy because $Z(G)$ is connected.
- In [ABPS1] a (noncanonical) bijection satisfying (b,d,e) was established for $\mathrm{Irr}(G,T)$, when G is F-split and p is not too small.
- For quasi-split unitary groups with $p > 2$ a (noncanonical) bijection was constructed by the author's PhD student Badea [Bad].

In all cases, a study of affine Hecke algebras constitutes the largest part of the argument. Thanks to [ABPS2, Sol5], that technique is now available in complete generality (even outside the principal series). The main novelties of this paper are:

- The construction of the LLC is canonical and uniform, over all non-archimedean local fields F and all quasi-split reductive F-groups.
- We can handle generic representations, even when not all Whittaker data for G are equivalent.

- Our LLC lifts to a categorical level, as follows. For each involved Bernstein block of G-representations, the LLC comes from a canonical equivalence between that block and the module category of a certain Hecke algebra defined entirely in terms of Langlands parameters.

We will now discuss the content of the paper in more detail, at the same time explaining parts of the proof of the main theorem.

We start with a Bernstein block $\mathrm{Rep}(G)^{\mathfrak s}$ in the principal series, and a progenerator $\Pi_{\mathfrak s}$ thereof. Via [Sol5] $\mathrm{Rep}(G)^{\mathfrak s}$ is equivalent to the module category of some Hecke algebra $\mathrm{End}_G(\Pi_{\mathfrak s})^{op}$, which we analyse in Section 1. We show that $\mathrm{End}_G(\Pi_{\mathfrak s})$ is isomorphic to an affine Hecke algebra $\mathcal H(\mathfrak s)$ (extended with a twisted group algebra), and we determine its q-parameters.

In Section 2 we involve the Whittaker datum, and that enables us to make the aforementioned isomorphism canonical. In the same way we show that the twist in the extension part of $\mathcal H(\mathfrak s)$ is actually trivial, so that it is an extended affine Hecke algebra $\mathcal H(\mathfrak s)^\circ \rtimes \Gamma_{\mathfrak s}$. Here the Bernstein group $W_{\mathfrak s}$ associated to $\mathrm{Rep}(G)^{\mathfrak s}$ appears as $W(R_{\mathfrak s}^\vee) \rtimes \Gamma_{\mathfrak s}$ for a root system $R_{\mathfrak s}^\vee$.

A continuation of this analysis yields a useful criterion for genericity in terms of Hecke algebra modules. Let $\mathcal H(W(R_{\mathfrak s}^\vee), q_F^\lambda) \subset \mathcal H(\mathfrak s)^\circ$ be the finite dimensional Iwahori–Hecke algebra from the Bernstein presentation of $\mathcal H(\mathfrak s)^\circ$, where q_F is the cardinality of the residue field of F and $\lambda : R_{\mathfrak s}^\vee / W_{\mathfrak s} \to \mathbb Z_{\geq 0}$ is a label function. Recall that its Steinberg representation St is given by $T_{s_\alpha} \mapsto -1$ for every simple reflection $s_\alpha \in W(R_{\mathfrak s}^\vee)$. Let $\det : W_{\mathfrak s} \to \{\pm 1\}$ be the determinant of the action of $W_{\mathfrak s}$ on the lattice of F-rational cocharacters of T. We extend St to a onedimensional representation (still denoted St) of $\mathcal H(W(R_{\mathfrak s}^\vee), q_F^\lambda) \rtimes \Gamma_{\mathfrak s}$ by making it det on $\Gamma_{\mathfrak s}$.

Theorem B. [see Theorem 3.4]
Suppose that $\pi \in \mathrm{Rep}(G)^{\mathfrak s}$ has finite length. Then π is (U,ξ)-generic if and only if the associated $\mathcal H(\mathfrak s)^{op}$-module $\mathrm{Hom}_G(\Pi_{\mathfrak s}, \pi)$ contains the Steinberg representation of $(\mathcal H(W(R_{\mathfrak s}^\vee), q_F^\lambda) \rtimes \Gamma_{\mathfrak s})^{op}$.

The notion of principal series enhanced L-parameters is worked out in Section 4. There we also recall the Hecke algebras on the Galois side of the LLC, from [AMS3], and we compute their q-parameters. Via the LLC for tori we associate to $\mathrm{Rep}(G)^{\mathfrak s}$ a unique Bernstein component $\Phi_e(G)^{\mathfrak s^\vee}$ of $\Phi_e(G,T)$. That yields an extended affine Hecke algebra $\mathcal H(\mathfrak s^\vee, q_F^{1/2})$. The crucial step to pass from the p-adic side to the Galois side of the LLC is:

Theorem C. [see Theorem 5.4]
There exists a canonical algebra isomorphism $\mathcal H(\mathfrak s)^{op} \cong \mathcal H(\mathfrak s^\vee, q_F^{1/2})$.

The above steps make $\mathrm{Rep}(G)^{\mathfrak s}$ canonically equivalent to the module category of $\mathcal H(\mathfrak s^\vee, q_F^{1/2})$. In [AMS3], $\mathrm{Irr}\big(\mathcal H(\mathfrak s^\vee, q_F^{1/2})\big)$ is parametrized by $\Phi_e(G)^{\mathfrak s^\vee}$. We want to use that, but it does not quite suffice because we also need to keep track of genericity of representations. Therefore we revisit several constructions from [AMS3], in our setting of the principal series. The main point of Section 6 is to show that through all those steps the one-dimensional representation det of $(\mathcal H(W(R_{\mathfrak s}^\vee), q_F^\lambda) \rtimes \Gamma_{\mathfrak s})^{op}$ is transformed into an analogous representation det for an extended graded Hecke algebra. That enables us to normalize the parametrization of $\mathrm{Irr}\big(\mathcal H(\mathfrak s^\vee, q_F^{1/2})\big)$, so that it matches generic representations with the desired kind of enhanced L-parameters.

With that settled the preparations are complete, and the bijection in Theorem A is obtained as

$$\mathrm{Irr}(G)^{\mathfrak s} \leftrightarrow \mathrm{Irr}\big(\mathrm{End}_G(\Pi_{\mathfrak s})^{op}\big) \leftrightarrow \mathrm{Irr}\big(\mathcal H(\mathfrak s)^{op}\big) \leftrightarrow \mathrm{Irr}\big(\mathcal H(\mathfrak s^\vee, q_F^{1/2})\big) \leftrightarrow \Phi_e(G)^{\mathfrak s^\vee}.$$

The properties of the bijection $\mathrm{Irr}(G,T) \leftrightarrow \Phi_e(G,T)$, actually a few more than mentioned already, are checked in the remainder of Section 7.

Several further research topics are suggested by the above theorems.

- Like in [ABPS1, §17], one would like to show that the LLC is functorial with respect to those homomorphisms of reductive p-adic groups that have commutative kernel and commutative cokernel. That should be doable with the methods from [Sol3]. In particular that can be applied to automorphisms of G from conjugation with elements of $\mathcal G_{\mathrm{ad}}(F)$, then it will show how the LLC changes if one modifies the Whittaker datum.
- Suppose that ϕ is discrete and $Z(G)$ is compact. It is conjectured in [HII] that the formal degree of the square-integrable representation $\pi(\phi,\rho)$ equals $\dim(\rho)$ times the adjoint γ-factor of ϕ (with suitable normalizations on both sides). While this adjoint γ-factor can be computed as in [FOS2, Appendix A], it may be difficult to determine this formal degree. The reason is that one would like to use a type, but sometimes it is not known whether a type for the involved Bernstein block exists.
- Every L-packet conjecturally supports a stable distribution on G. For L-packets that are entirely contained in $\mathrm{Irr}(G,T)$, one could try to prove that the distribution $\sum_{\rho\in\mathrm{Irr}(R_\phi)} \dim(\rho)\,\mathrm{tr}\,\pi(\phi,\rho)$ is stable.
- A modern geometric approach to the Langlands correspondence [FaSc, Hel, Zhu] predicts that the derived category of $\mathrm{Rep}(G)$ embeds in a derived category of coherent sheaves on a stack of Langlands parameters. It would be interesting to transfer the obtained natural equivalence

  $$\mathrm{Rep}(G)^{\mathfrak s} \cong \mathrm{Mod}\big(\mathcal H(\mathfrak s^\vee, q_F^{1/2})\big)$$

  to a setting with such coherent sheaves, that would establish a part of the conjectures in [FaSc, Hel, Zhu]. It is reasonable to expect that can be done, because $\mathcal H(\mathfrak s^\vee, q_F^{1/2})$ is constructed from $\Phi_e(G)^{\mathfrak s^\vee}$ and because on the underlying cuspidal level the local Langlands correspondence for tori achieves it already.

1. Hecke algebras for principal series representations

Let F be a non-archimedean local field with ring of integers $\mathfrak o_F$. Let q_F be the cardinality of the residue field. Let $|\cdot|_F : F \to \mathbb R_{\geq 0}$ be the norm and fix an element $\varpi_F \in \mathfrak o_F$ with norm q_F^{-1}. We also fix a separable closure F_s of F and we let $\mathbf W_F \subset \mathrm{Gal}(F_s/F)$ be the Weil group. Field extensions of F will by default be contained in F_s.

Let $\mathcal G$ be a quasi-split reductive F-group, where we include connectedness in the definition of quasi-split. Let $\mathcal S$ be a maximal F-split torus in $\mathcal G$. We write $G = \mathcal G(F), S = \mathcal S(F)$ etcetera. Since $\mathcal G$ is F-quasi-split, the centralizer $\mathcal T$ of $\mathcal S$ in $\mathcal G$ is a maximal F-torus. It is also a minimal F-Levi subgroup, a Levi factor of a Borel subgroup $\mathcal{B}$ of $\mathcal{G}$. The Weyl group of $(\mathcal{G},\mathcal{S})$ and (G,S) is

$$W(\mathcal{G},\mathcal{S}) = N_{\mathcal{G}}(\mathcal{S})/Z_{\mathcal{G}}(\mathcal{S}) = N_{\mathcal{G}}(S)/\mathcal{T} \cong N_G(S)/T = N_G(T)/T.$$

This is also the Weyl group of the root system $R(\mathcal{G},\mathcal{S})$.

Let T_{cpt} be the unique maximal compact subgroup of T and let $X_{\mathrm{nr}}(T)$ be the group of unramified characters of T, that is, the characters that are trivial on T_{cpt}. Pick any smooth character $\chi_0 : T \to \mathbb{C}^\times$ and write $\chi_c = \chi_0|_{T_{\mathrm{cpt}}}$. Then χ_c determines $X_{\mathrm{nr}}(T)\chi_0$ and conversely.

We denote the category of smooth G-representations on complex vector spaces by $\mathrm{Rep}(G)$, and the set of equivalence classes of irreducible objects therein by $\mathrm{Irr}(G)$. The set $X_{\mathrm{nr}}(T)\chi_0 = \mathrm{Irr}(T)^{\mathfrak{s}_T}$ (with $\mathfrak{s}_T = [T,\chi_0]_T$) is also known as a Bernstein component of $\mathrm{Irr}(T)$. From $\mathrm{Irr}(T)^{\mathfrak{s}_T}$ one derives a Bernstein component $\mathrm{Irr}(G)^{\mathfrak{s}}$, where $\mathfrak{s} = [T,\chi_0]_G$. It consists of the irreducible subquotients of the normalized parabolic inductions

$$I_B^G(\chi) = \mathrm{ind}_B^G\big(\chi \otimes \delta_B^{1/2}\big) \text{ with } \chi \in X_{\mathrm{nr}}(T)\chi_0.$$

We recall that the Bernstein block $\mathrm{Rep}(G)^{\mathfrak{s}}$ is the full subcategory of $\mathrm{Rep}(G)$ made up by the representations π such that every irreducible subquotient of π belongs to $\mathrm{Irr}(G)^{\mathfrak{s}}$. The standard way to classify $\mathrm{Irr}(G)^{\mathfrak{s}}$ is by describing $\mathrm{Rep}(G)^{\mathfrak{s}}$ as the module category of a Hecke algebra, and then using the representation theory of Hecke algebras. We do so with the method that provides maximal generality, from [Hei, Sol5].

We denote smooth induction with compact supports by ind. The T-representation $\mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c) \cong \chi_c \otimes \mathbb{C}[T/T_{\mathrm{cpt}}]$ is a progenerator of $\mathrm{Rep}(T)^{\mathfrak{s}_T}$. By the first and second adjointness theorems,

$$\Pi_{\mathfrak{s}} = I_B^G\big(\mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c)\big)$$

is a progenerator of $\mathrm{Rep}(G)^{\mathfrak{s}}$. Let $\mathrm{End}_G(\Pi_{\mathfrak{s}})$ be the algebra of G-endomorphisms of $\Pi_{\mathfrak{s}}$, acting from the left on $\Pi_{\mathfrak{s}}$. Then the functors

$$\begin{array}{ccc} \mathrm{Rep}(G)^{\mathfrak{s}} & \longleftrightarrow & \mathrm{End}_G(\Pi_{\mathfrak{s}}) - \mathrm{Mod} \\ \rho & \mapsto & \mathrm{Hom}_G(\Pi_{\mathfrak{s}},\rho) \\ V \otimes_{\mathrm{End}_G(\Pi_{\mathfrak{s}})} \Pi_{\mathfrak{s}} & \leftarrow\!\shortmid & V \end{array} \tag{1.1}$$

are equivalences of categories [Roc2, Theorem I.8.2.1]. This is compatible with parabolic induction, in the following sense. Let $P = MR_u(P)$ be a parabolic subgroup of G, where $B \subset P$, M is a Levi factor of P and $T \subset M$. The diagram

$$\begin{array}{ccc} \mathrm{Rep}(G)^{\mathfrak{s}} & \to & \mathrm{End}_G(\Pi_{\mathfrak{s}}) - \mathrm{Mod} \\ \uparrow I_P^G & & \uparrow \mathrm{ind}_{\mathrm{End}_M(\Pi_{\mathfrak{s}_M})}^{\mathrm{End}_G(\Pi_{\mathfrak{s}})} \\ \mathrm{Rep}(M)^{\mathfrak{s}_M} & \to & \mathrm{End}_M(\Pi_{\mathfrak{s}_M}) - \mathrm{Mod} \end{array} \tag{1.2}$$

commutes, see [Sol2, Condition 4.1 and Lemma 5.1].

The algebra $\mathrm{End}_G(\Pi_{\mathfrak{s}})$ was investigated in [Sol5], in larger generality. We will make it more explicit in the current setting. Since $\dim \chi_0 = 1$, $\chi|_{T_{\mathrm{cpt}}}$ is irreducible and we may use [Sol5, §10] with $E = E_1 = \mathbb{C}$ and $\sigma_1 = \sigma|_{T_{\mathrm{cpt}}} = \chi_c$. For comparison with [Sol5] we also note that the group

$$X_{\mathrm{nr}}(T,\chi_0) = \{\chi \in X_{\mathrm{nr}}(T) : \chi \otimes \chi_0 \cong \chi_0\}$$

is trivial. We write

$$W_{\mathfrak{s}} = \mathrm{Stab}_{W(\mathcal{G},\mathcal{S})}(\mathfrak{s}_T) = \mathrm{Stab}_{W(\mathcal{G},\mathcal{S})}(X_{\mathrm{nr}}(T)\chi_0) = \mathrm{Stab}_{W(\mathcal{G},\mathcal{S})}(\chi_c).$$

This group acts naturally on the complex variety

$$T_{\mathfrak{s}} := \chi_0 X_{\mathrm{nr}}(T).$$

by $(w \cdot \chi)(t) = \chi(w^{-1} t w)$. The theory of the Bernstein centre [BeDe] says that

$$Z(\mathrm{Rep}(G)^{\mathfrak{s}}) \cong Z\big(\mathrm{End}_G(\Pi_{\mathfrak{s}})\big) \cong \mathcal{O}(X_{\mathrm{nr}}(T)\chi_0)^{W_{\mathfrak{s}}} = \mathcal{O}(X_{\mathrm{nr}}(T)\chi_0 / W_{\mathfrak{s}}).$$

The algebra $\mathrm{End}_G(\Pi_{\mathfrak{s}})$ contains $\mathcal{O}(X_{\mathrm{nr}}(T)\chi_0) = \mathcal{O}(T_{\mathfrak{s}})$ as maximal commutative subalgebra, and as module over that subalgebra it is free with a basis $\{N_w : w \in W_{\mathfrak{s}}\}$ [Sol5, Theorem 10.9].

We note that the inertial equivalence class $\mathfrak{s}$ for G can arise from different inertial equivalence classes for T. Namely, the possibilities are $w\mathfrak{s}_T = [T, w\chi_0]_T$ with $w \in W(\mathcal{G}, \mathcal{S})$. Thus $w\mathfrak{s} = \mathfrak{s}$ as inertial equivalence classes for G, but they are represented by different subsets of $\mathrm{Irr}(T)$. For any $w \in W(\mathcal{G}, \mathcal{S})$, the G-representations $\Pi_{\mathfrak{s}}$ and $\Pi_{w\mathfrak{s}} = I_B^G(\mathrm{ind}_{T_{\mathrm{cpt}}}^T(w\chi_c))$ are isomorphic, see [Ren, §VI.10.1]. That yields an algebra isomorphism

$$\mathrm{End}_G(\Pi_{\mathfrak{s}}) \cong \mathrm{End}_G(\Pi_{w\mathfrak{s}}), \tag{1.3}$$

unique up to inner automorphisms of $\mathrm{End}_G(\Pi_{\mathfrak{s}})$. In principle that suffices to compare the functors $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, ?)$ and $\mathrm{Hom}_G(\Pi_{w\mathfrak{s}}, ?)$ on the level of isomorphism classes of representations. Nevertheless, we will have to make (1.3) explicit later, and we prepare for that now.

Proposition 1.1. *Let $w \in W(\mathcal{G}, \mathcal{S})$ be of minimal length in $wW_{\mathfrak{s}}$. The isomorphism $\Pi_{\mathfrak{s}} \cong \Pi_{w\mathfrak{s}}$ can be chosen so that the induced algebra isomorphism* (1.3) *restricts to*

$$\mathcal{O}(T_{\mathfrak{s}}) \to \mathcal{O}(T_{w\mathfrak{s}}) : f \mapsto f \circ w^{-1}.$$

Proof. Let $w = s_r \cdots s_2 s_1$ be a reduced expression in the Weyl group $W(\mathcal{G}, \mathcal{S})$. Then each simple reflection s_j has minimal length in $s_j W_{s_{j-1} \cdots s_1 \mathfrak{s}}$. In this way we reduce the proposition to the case $w = s_\alpha$ for a simple root $\alpha \in W(\mathcal{G}, \mathcal{S})$, with $s_\alpha \mathfrak{s}_T \neq \mathfrak{s}_T$.

Let $G_\alpha \subset G$ be the subgroup generated by $T \cup U_\alpha \cup U_{-\alpha}$. As

$$\Pi_{\mathfrak{s}} = I_{BG_\alpha}^G I_{B \cap G_\alpha}^{G_\alpha} \mathrm{ind}_{T_{\mathrm{cpt}}}^T (\chi_c)$$

and similarly for $\Pi_{w\mathfrak{s}}$, it suffices to work with the reductive group G_α and its Borel subgroup $B \cap G_\alpha$. Equivalently, we may (and will) assume that $R(\mathcal{G}, \mathcal{S})$ has rank one. In this rank one setting, an isomorphism

$$\Pi_{\mathfrak{s}} \cong \Pi_{s_\alpha \mathfrak{s}} \tag{1.4}$$

is exhibited in [Ren, Lemme VI.10.1]. We analyse that construction.

By [Ren, Corollaire VII.1.3]

$$I_B^G : \mathrm{Rep}(T)^{\mathfrak{s}_T} \to \mathrm{Rep}(G)^{\mathfrak{s}} \quad \text{is an equivalence of categories.} \tag{1.5}$$

Let $J_{\overline{B}}^G : \mathrm{Rep}(G) \to \mathrm{Rep}(T)$ be the normalized Jacquet restriction functor with respect to the opposite Borel subgroup $\overline{B}$. As $\mathfrak{s}_T \neq s_\alpha \mathfrak{s}_T$, Bernstein's geometric lemma [Ren, Théorème VI.5.1] entails that

$$J_{\overline{B}}^G I_B^G \pi \cong \pi \oplus s_\alpha^{-1} \cdot \pi \qquad \text{for all } \pi \in \mathrm{Rep}(T)^{\mathfrak{s}_T}. \tag{1.6}$$

Let $\mathrm{pr}_{\mathfrak{s}_T} : \mathrm{Rep}(T) \to \mathrm{Rep}(T)^{\mathfrak{s}_T}$ be the projection provided by the Bernstein decomposition. From (1.6) we see that

$$\mathrm{pr}_{\mathfrak{s}_T} J_{\overline{B}}^G : \mathrm{Rep}(G)^{\mathfrak{s}} \to \mathrm{Rep}(T)^{\mathfrak{s}_T} \quad \text{is the inverse of (1.5).} \tag{1.7}$$

It follows (slightly varying on the proof of [Ren, Lemme VI.10.1] by using $\overline{B}$ instead of B) that (1.4) is determined by the choice of a T-isomorphism

$$\mathrm{pr}_{\mathfrak{s}_T} J_{\overline{B}}^G \Pi_{s_\alpha \mathfrak{s}} \cong \mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c). \tag{1.8}$$

Pick a representative for s_α in $N_G(T)$. From (1.6) with $\pi = \mathrm{ind}_{T_{\mathrm{cpt}}}^T(s_\alpha \chi_c)$ we see that evaluation at s_α in $I_B^G \mathrm{ind}_{T_{\mathrm{cpt}}}^T(s_\alpha \chi_c)$ provides an isomorphism of T-representations

$$\mathrm{ev}_{s_\alpha} : \mathrm{pr}_{\mathfrak{s}} J_{\overline{B}}^G \Pi_{s_\alpha \mathfrak{s}} \to s_\alpha^{-1} \cdot \mathrm{ind}_{T_{\mathrm{cpt}}}^T(s_\alpha \chi_c). \tag{1.9}$$

Further we have a canonical T-isomorphism

$$\begin{array}{ccc} s_\alpha^{-1} \cdot \mathrm{ind}_{T_{\mathrm{cpt}}}^T(s_\alpha \chi_c) & \to & \mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c) \\ f & \mapsto & [t \mapsto f(s_\alpha t s_\alpha^{-1})] \end{array} . \tag{1.10}$$

The composition of (1.9) and (1.10) gives us (1.8). Applying (1.5), we obtain (1.4).

The subalgebra $\mathcal{O}(T_{\mathfrak{s}})$ of $\mathrm{End}_G(\Pi_{\mathfrak{s}})$ arises as $I_B^G\big(\mathcal{O}(T_{\mathfrak{s}})\big)$, where $\mathcal{O}(T_{\mathfrak{s}})$ acts on $\mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c) \cong \mathcal{O}(T_{\mathfrak{s}})$ by multiplication, see [Sol5]. From (1.10) we see that $s_\alpha^{-1} \cdot \mathrm{ind}_{T_{\mathrm{cpt}}}^T(s_\alpha \chi_c)$ is naturally isomorphic to the regular representation of $\mathcal{O}(T_{\mathfrak{s}})$.

Similarly $\mathcal{O}(T_{w\mathfrak{s}})$ acts on $\mathrm{ind}_{T_{\mathrm{cpt}}}^T(s_\alpha \chi_c) \cong \mathcal{O}(T_{w\mathfrak{s}})$ by multiplication, and it becomes a subalgebra of $\mathrm{End}_G(\Pi_{s_\alpha \mathfrak{s}})$ via I_B^G. From (1.6) we see that its action on $s_\alpha^{-1} \cdot \mathrm{ind}_{T_{\mathrm{cpt}}}^T(s_\alpha \chi_c)$, obtained via $J_{\overline{B}}^G I_B^G$, is

$$s_\alpha^{-1} : \mathcal{O}(T_{s_\alpha \mathfrak{s}}) \to \mathcal{O}(T_{\mathfrak{s}})$$

followed by the regular representation. In other words, the action of $f \in \mathcal{O}(T_{\mathfrak{s}})$ on $s_\alpha^{-1} \cdot \mathrm{ind}_{T_{\mathrm{cpt}}}^T(s_\alpha \chi_c)$ via (1.8) coincides with the action of $s_\alpha(f) = f \circ s_\alpha^{-1} \in \mathcal{O}(T_{\mathfrak{s}})$ on $s_\alpha^{-1} \cdot \mathrm{ind}_{T_{\mathrm{cpt}}}^T(s_\alpha \chi_c)$ via (1.10). Applying the normalized parabolic induction functor I_B^G, we find that $I_B^G(f) \in \mathrm{End}_G(\Pi_{\mathfrak{s}})$ is transformed into $I_B^G(f \circ s_\alpha^{-1}) \in \mathrm{End}_G(\Pi_{s_\alpha \mathfrak{s}})$ by (1.4). $\Box$

We resume the analysis of $\mathrm{End}(\Pi_{\mathfrak{s}})$ with $\mathfrak{s} = [T, \chi_0]_G$. Let $R_{\mathfrak{s},\mu}$ be the set of roots $\alpha \in R(\mathcal{G}, \mathcal{S})$ for which Harish-Chandra's function μ_α is not constant on $X_{\mathrm{nr}}(T)\chi_0$. Then $R_{\mathfrak{s},\mu}$ is a root system and $W(R_{\mathfrak{s},\mu})$ is a normal subgroup of $W_{\mathfrak{s}}$ [Hei, Proposition 1.3]. As explained in [Sol5, §3], we can modify χ_0 inside $X_{\mathrm{nr}}(T)\chi_0$ so that $W(R_{\mathfrak{s},\mu})$ fixes χ_0. Let $R_{\mathfrak{s},\mu}^+$ be the positive system determined by the chosen Borel subgroup $\mathcal{B}$ of $\mathcal{G}$. Then

$$W_{\mathfrak{s}} = W(R_{\mathfrak{s},\mu}) \rtimes \Gamma_{\mathfrak{s}}$$

where $\Gamma_{\mathfrak{s}}$ denotes the stabilizer of $R_{\mathfrak{s},\mu}^+$ in $W_{\mathfrak{s}}$. Following [Sol5, §3], we use the lattice $T/T_{\mathrm{cpt}} \cong X^*(X_{\mathrm{nr}}(T))$, and the dual lattice $(T/T_{\mathrm{cpt}})^\vee \cong X_*(X_{\mathrm{nr}}(T))$. For $\alpha \in R_{\mathfrak{s},\mu}$ let $h_\alpha^\vee$ be the unique generator of $T/T_{\mathrm{cpt}} \cap \mathbb{Q}\alpha^\vee$ such that $|\alpha(h_\alpha^\vee)|_F > 1$. We put

$$R_{\mathfrak{s}}^\vee = \{h_\alpha^\vee : \alpha \in R_{\mathfrak{s},\mu}\} \qquad \subset T/T_{\mathrm{cpt}}$$

and we let $R_{\mathfrak{s}} \subset (T/T_{\mathrm{cpt}})^\vee$ be the dual root system. By [Sol5, Proposition 3.1]

$$\mathcal{R}_{\mathfrak{s}} = \big(R_{\mathfrak{s}}^\vee, T/T_{\mathrm{cpt}}, R_{\mathfrak{s}}, (T/T_{\mathrm{cpt}})^\vee\big)$$

is a root datum with Weyl group $W(R_{\mathfrak{s}}^\vee) = W(R_{\mathfrak{s},\mu})$. Moreover $W_{\mathfrak{s}}$ acts naturally on $\mathcal{R}_{\mathfrak{s}}$ and $\Gamma_{\mathfrak{s}}$ is the $W_{\mathfrak{s}}$-stabilizer of the basis of $\mathcal{R}_{\mathfrak{s}}$ determined by $\mathcal{B}$.

The complex variety $T_{\mathfrak{s}}$ is isomorphic to $X_{\mathrm{nr}}(T)$ via multiplication with χ_0. Let $\mathcal{H}(\mathfrak{s})^\circ$ be the vector space $\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathbb{C}[W(R_{\mathfrak{s}}^\vee)]$, identified with $\mathcal{O}(X_{\mathrm{nr}}(T)) \otimes \mathbb{C}[W(R_{\mathfrak{s}}^\vee)]$ via $X_{\mathrm{nr}}(T) \to T_{\mathfrak{s}}$. Given label functions λ, λ^* and $q \in \mathbb{C}^\times$, we build the affine Hecke algebra $\mathcal{H}(\mathcal{R}_{\mathfrak{s}}, \lambda, \lambda^*, q)$ (see for instance [AMS1, Proposition 2.2] with $\mathbf{z}_j$ specialized to q). Via the above isomorphism of vector spaces we make $\mathcal{H}(\mathfrak{s})^\circ$ into an algebra which is isomorphic to $\mathcal{H}(\mathcal{R}_\mathfrak{s}, \lambda, \lambda^*, q)$. The group $\Gamma_\mathfrak{s}$ acts on $\mathcal{H}(\mathfrak{s})^\circ$ by algebra isomorphisms:

$$\gamma(f \otimes w) = f \circ \gamma^{-1} \otimes \gamma w \gamma^{-1} \qquad f \in \mathcal{O}(T_\mathfrak{s}), w \in W(R_\mathfrak{s}^\vee). \tag{1.11}$$

That gives rise to a crossed product algebra

$$\mathcal{H}(\mathfrak{s}) := \mathcal{H}(\mathfrak{s})^\circ \rtimes \Gamma_\mathfrak{s},$$

which we would like to be isomorphic with $\mathrm{End}_G(\Pi_\mathfrak{s})$.

For s_α with $\alpha \in R_{\mathfrak{s},\mu}$ simple, and more generally for any $w \in W(R_\mathfrak{s}^\vee)$, an element $N_w \in \mathrm{End}_G(\Pi_\mathfrak{s})$ is constructed in [Sol5, Lemma 10.8 and remarks], it is called $q_F^{-\lambda(\alpha)/2} T'_w$ over there. It can be determined uniquely by the choice of a good maximal compact subgroup K of G, associated to a special vertex in the apartment for $\mathcal{T}$ in the Bruhat–Tits building of $(\mathcal{G}, F)$.

For $\gamma \in \Gamma_\mathfrak{s}$ we have to be more careful, mainly because it need not fix χ_0. (The group $W_\mathfrak{s}$ fixes χ_0 when $\mathcal{G}$ is F-split, but the argument in that case does not generalize to $\mathcal{G}$ that only split over a ramified extension of F.) Since $X_{\mathrm{nr}}(T, \chi_0) = 1$, there exists a unique $\chi_\gamma \in X_{\mathrm{nr}}(T)$ such that $\gamma \cdot \chi_0 = \chi_0 \otimes \chi_\gamma$. Then χ_γ is fixed by $W(R_\mathfrak{s})$ [Sol5, Lemma 3.5]. The element J_γ from [Sol5, Theorem 10.9] comes from A_γ in [Sol5, §5]. From [Sol5, start of §5.1] we see that A_γ depends on χ_γ (which is unique) and on some

$$\rho_\gamma \in \mathrm{Hom}_T(\gamma \chi_0, \chi_0 \otimes \chi_\gamma).$$

For the latter we have a canonical choice, namely the identity on $\mathbb{C}$. Apart from that A_γ depends only on the choice of K.

Theorem 1.2. *The above intertwining operators $N_w J_\gamma \in \mathrm{End}_G(\Pi_\mathfrak{s})$ give rise to an algebra isomorphism*

$$\mathrm{End}_G(\Pi_\mathfrak{s}) \cong \mathcal{H}(\mathfrak{s})^\circ \rtimes \mathbb{C}[\Gamma_\mathfrak{s}, \natural_\mathfrak{s}],$$

for a 2-cocycle $\natural_\mathfrak{s} : \Gamma_\mathfrak{s}^2 \to \mathbb{C}^\times$, suitable $W_\mathfrak{s}$-invariant label functions $\lambda : R_\mathfrak{s}^\vee \to \mathbb{Z}_{>0}, \lambda^* : R_\mathfrak{s}^\vee \to \mathbb{Z}_{\geq 0}$ and q-base $q_F^{1/2}$. This isomorphism is determined by the choice of a maximal compact subgroup K of G.

Proof. The isomorphism between $\mathcal{H}(\mathfrak{s})^\circ$ and the subalgebra of $\mathrm{End}_G(\Pi_\mathfrak{s})$ generated by $\mathcal{O}(T_\mathfrak{s})$ and the N_w with $w \in W(R_\mathfrak{s}^\vee)$ is given in [Sol5, Theorem 10.9]. The operators J_γ ($\gamma \in \Gamma_\mathfrak{s}$) in [Sol5, Theorem 10.9] coincide with the $A_\gamma \in \mathrm{End}_G(\Pi_\mathfrak{s})$ from [Sol5, §5.1]. The multiplication rules for the A_γ are given in [Sol5, Proposition 5.2.a]. As $X_{\mathrm{nr}}(T, \chi_0) = 1$, we get

$$A_\gamma A_{\gamma'} = \natural_\mathfrak{s}(\gamma, \gamma') A_{\gamma\gamma'} \qquad \gamma, \gamma' \in \Gamma_\mathfrak{s},$$

for some $\natural_\mathfrak{s}(\gamma, \gamma') \in \mathbb{C}^\times$. By the associativity of the multiplication, $\natural_\mathfrak{s}$ is a 2-cocycle. The other parts of [Sol5, Proposition 5.2] also simplify, because χ_γ is fixed by $W(R_\mathfrak{s}^\vee)$. They show that

$$A_\gamma A_w = A_{\gamma w} \quad \text{and} \quad A_w A_\gamma = A_{w\gamma} \quad \text{for } \gamma \in \Gamma_\mathfrak{s}, w \in W(R_\mathfrak{s}^\vee).$$

This implies

$$A_\gamma^{-1} A_w A_\gamma = A_{\gamma^{-1} w \gamma} = A_{\gamma^{-1}} A_w A_\gamma. \tag{1.12}$$

In view of how N_w is constructed from A_w [Sol5, §10], the relation (1.12) entails $A_\gamma^{-1} N_w A_\gamma = N_{\gamma^{-1} w \gamma}$. That and [Sol5, (5.2)] show that $\Gamma_\mathfrak{s}$ acts on the image of

$\mathcal{H}(\mathfrak{s})^\circ$ in $\mathrm{End}_G(\Pi_{\mathfrak{s}})$ as in (1.11). Combining that with [Sol5, Theorem 10.9] yields the required algebra isomorphism. $\square$

An important part of the structure of $\mathrm{End}_G(\Pi_{\mathfrak{s}})$ consists of the labels $\lambda(h_\alpha^\vee)$, $\lambda^*(h_\alpha^\vee)$ with $\alpha \in R_{\mathfrak{s},\mu}$. Here the eigenvalues of N_{s_α} are $q_F^{\lambda(h_\alpha^\vee)/2}$ and $-q_F^{-\lambda(h_\alpha^\vee)/2}$. When we recall the known formulas for these labels, it will be convenient to consider all $\alpha \in R(\mathcal{G}, \mathcal{S})$ such that $s_\alpha \in W_{\mathfrak{s}}$.

Suppose first that $\mathcal{G}$ is F-split. By [Sol7, Proposition 4.3], $\alpha \in R_{\mathfrak{s},\mu}$ if and only if $\chi \circ \alpha^\vee : F^\times \to \mathbb{C}^\times$ is unramified. Further, by [Sol7, Theorem 4.4]

$$\lambda\big(\alpha^\vee(\varpi_F^{-1})\big) = \lambda^*\big(\alpha^\vee(\varpi_F^{-1})\big) = 1. \tag{1.13}$$

Now we suppose that $\mathcal{G}$ quasi-split but not necessarily split. A special role is played by pairs of roots in type ${}^2A_{2n}$, such that the diagram automorphism permutes the pair. We settle the other cases before we turn to those exceptional roots.

Let $\alpha_{\mathcal{T}} \in R(\mathcal{G}, \mathcal{T})$ be a preimage of $\alpha \in R(\mathcal{G}, \mathcal{S})$ and let $\mathbf{W}_{F,\alpha_{\mathcal{T}}}$ be its stabilizer. The splitting field $F_\alpha = F_s^{\mathbf{W}_{F,\alpha_{\mathcal{T}}}}$ of α is unique up to isomorphism. Let $f(F_\alpha/F)$ be the residual degree of F_α/F.

Assuming that α is not exceptional, the issue can be reduced to (1.13). Indeed, by [Sol7, §4.2], $\alpha \in R_{\mathfrak{s},\mu}$ if and only if $\chi \circ \alpha^\vee : F_\alpha^\times \to \mathbb{C}^\times$ is unramified. Moreover, by [Sol7, Corollary 4.5]

$$\lambda\big(\alpha^\vee(\varpi_{F_\alpha}^{-1})\big) = \lambda^*\big(\alpha^\vee(\varpi_{F_\alpha}^{-1})\big) = f(F_\alpha/F). \tag{1.14}$$

In most cases $h_\alpha^\vee = \alpha^\vee(\varpi_{F_\alpha}^{-1})$ in T/T_{cpt}, and sometimes $\alpha^\vee(\varpi_{F_\alpha}^{-1}) = (h_\alpha^\vee)^2$ in T/T_{cpt}. In the latter cases, for instance $PGL_2(F)$,

$$\lambda(h_\alpha^\vee) = f(F_\alpha/F) \quad \text{and} \quad \lambda^*(h_\alpha^\vee) = 0. \tag{1.15}$$

The exceptional roots occur only when $R_{\mathfrak{s},\mu}$ has a component of type BC_n which comes from a component of type ${}^2A_{2n}$ in $R(\mathcal{G}, \mathcal{S})$. Consider an indivisible root $\alpha \in R_{\mathfrak{s},\mu}$ which comes from two adjacent roots in ${}^2A_{2n}$. As explained in [Sol7, §4.2], the computation of the parameters for this α can be reduced to a quasi-split group $SU_3(F_\alpha/F_{2\alpha})$ Moreover, since the groups of unramified characters of $SU_3(F_\alpha/F_{2\alpha})$, $U_3(F_\alpha/F_{2\alpha})$ and $PU_3(F_\alpha/F_{2\alpha})$ are naturally identified, the reductions from [Sol7, §2] apply to these groups in the strong sense that in these instances of [Sol7, Proposition 2.4] no doubling or halving of roots can occur. Consequently the labels for $\alpha \in R(\mathcal{G}, \mathcal{S})$ are precisely $f(F_{2\alpha}/F)$ times the labels for α as root for $U_3(F_\alpha/F_{2\alpha})$.

For $U_3(F_\alpha/F_{2\alpha})$ all q-parameters for principal series representations were computed via types by the author's PhD student Badea [Bad]. The outcome can be summarized as follows.

- If $F_\alpha/F_{2\alpha}$ is unramified and χ_c is trivial on $T_{\mathrm{cpt}} \cap SU_3(F_\alpha/F_{2\alpha})$, then $\alpha \in R_{\mathfrak{s},\mu}$ and $\lambda(h_\alpha^\vee) = 3, \lambda^*(h_\alpha^\vee) = 1$.
- If $F_\alpha/F_{2\alpha}$ is unramified and χ_c is nontrivial on $T_{\mathrm{cpt}} \cap SU_3(F_\alpha/F_{2\alpha})$, then $\alpha \in R_{\mathfrak{s},\mu}$ and $\lambda(h_\alpha^\vee) = \lambda^*(h_\alpha^\vee) = 1$.
- If $F_\alpha/F_{2\alpha}$ is ramified, then $\alpha \in R_{\mathfrak{s},\mu}$ if and only if $\chi \circ \alpha^\vee : F_\alpha^\times \to \mathbb{C}^\times$ is nontrivial on $\mathfrak{o}_{F_{2\alpha}}^\times$. (We note that $\chi^2 \circ \alpha^\vee|_{\mathfrak{o}_{F_{2\alpha}}^\times} = 1$ because $s_\alpha \chi_c = \chi_c$.)

  When this condition is fulfilled, we have $\lambda\big(\alpha^\vee(\varpi_{F_{2\alpha}}^{-1})\big) = \lambda^*\big(\alpha^\vee(\varpi_{F_{2\alpha}}^{-1})\big) = 1$ and $\lambda(h_\alpha^\vee) = 1, \lambda^*(h_\alpha^\vee) = 0$.

We warn that in [Bad] it is assumed throughout that the residual characteristic of F is not 2. For unramified characters χ this restriction is not necessary, because in those cases the Hecke algebras and the parameters were already known from [Bor1]. However, for other χ the tricky calculations in [Bad, §2.7 and §5.2.1] do not work in residual characteristic 2.

For F of arbitrary characteristic, the Hecke algebra parameters for $U_3(F_\alpha/F_{2\alpha})$ can also be determined via the endoscopic methods from [Mœ], see [Sol7, Theorem 4.9]. That shows that the above formulas also apply when the residual characteristic of F is 2.

2. Whittaker normalization

Unfortunately the isomorphism from Theorem 1.2 is not entirely canonical, because it depends on a good maximal compact subgroup K of G, and often G has more than one conjugacy class of such subgroups. Further, it may be expected that the 2-cocycle $\natural_{\mathfrak{s}}$ of $\Gamma_{\mathfrak{s}}$ is trivial, because G is quasi-split. We will fix both issues by using a Whittaker datum. Let U be the unipotent radical of B (since all Borel subgroups of G are conjugate, the choice of B is inessential.) Let $\xi : U \to \mathbb{C}^\times$ be a nondegenerate smooth character, which means that it is nontrivial on every root subgroup U_α with $\alpha \in R(\mathcal{G}, \mathcal{S})$ simple. Then the G-conjugacy class of (U, ξ) is a Whittaker datum for G.

Recall that a Whittaker functional for $\pi \in \mathrm{Rep}(G)$ is an element of

$$\mathrm{Hom}_U(\pi, \xi) \cong \mathrm{Hom}_G\big(\pi, \mathrm{Ind}_U^G(\xi)\big),$$

where Ind denotes smooth induction. We say that π is generic, or more precisely (U, ξ)-generic, if it admits a nonzero Whittaker functional. It is well-known [Rod, Shal] that every representation $I_B^G(\chi)$ with $\chi \in \mathrm{Irr}(T)$ is generic, and that its space of Whittaker functionals has dimension one. For the upcoming arguments we need a larger but modest supply of generic representations.

Proposition 2.1. *Suppose that $R(\mathcal{G}, \mathcal{S})$ and $R_{\mathfrak{s},\mu}$ have rank one. Then $|W_{\mathfrak{s}}| = 2$ and by Theorem 1.2 $\mathcal{H}(\mathfrak{s})$ is an affine Hecke algebra with a unique positive root $h_\alpha^\vee$. Let $\mathrm{St}_{\mathcal{H}(\mathfrak{s})}$ be the Steinberg representation of $\mathcal{H}(\mathfrak{s})$, the unique essentially discrete series representation with an $\mathcal{O}(T_{\mathfrak{s}})$-weight of the form $\chi_0|\alpha|_F^s$ with $s \in \mathbb{R}$.*

(*a*) *The G-representation $\mathrm{St}_{\mathfrak{s}} := \mathrm{St}_{\mathcal{H}(\mathfrak{s})} \otimes_{\mathrm{End}_G(\Pi_{\mathfrak{s}})} \Pi_{\mathfrak{s}}$ is generic.*

(*b*) *Suppose that $\lambda(h_\alpha^\vee) \neq \lambda^*(h_\alpha^\vee)$. In that case $\mathcal{H}(\mathfrak{s})$ has a unique essentially discrete series representation $\mathrm{St}_{\mathcal{H}(\mathfrak{s})-}$ with an $\mathcal{O}(T_{\mathfrak{s}})$-weight of the form $\chi_0|\alpha|_F^{ia+s}$ where $s, a \in \mathbb{R}$ and $|\alpha(h_\alpha^\vee)|_F^{ia} = -1$, see* [Sol4, §2.2]. *Then the G-representation $\mathrm{St}_{\mathfrak{s}-} := \mathrm{St}_{\mathcal{H}(\mathfrak{s})-} \otimes_{\mathrm{End}_G(\Pi_{\mathfrak{s}})} \Pi_{\mathfrak{s}}$ is generic.*

(*c*) *Suppose that the coroot of $h_\alpha^\vee$ lies in $2(T/T_{\mathrm{cpt}})^\vee$ and that $\lambda(h_\alpha^\vee) = \lambda^*(h_\alpha^\vee)$. Choose $a \in \mathbb{R}$ as in part (b). Then $I_B^G(\chi_0|\alpha|_F^{ia})$ is a direct sum of two irreducible subrepresentations. One of them, say $\pi_{\mathfrak{s}-}^g$, is (U, ξ)-generic and the other, say $\pi_{\mathfrak{s}-}^n$, is not.*

(*d*) *The irreducible G-representations in parts (a-c) are unitary.*

Proof. (a) As $R(\mathcal{G}, \mathcal{S})$ and $R_{\mathfrak{s},\mu}$ have the same rank, the equivalence of categories (1.1) translates "essentially square-integrable" into "essentially discrete series" [Sol5, Theorem 9.6.c]. In particular $\mathrm{St}_{\mathfrak{s}}$ is an essentially square-integrable G-representation. The assumptions of the proposition amount to the assumptions for [Shah, Theorem 8.1]. Part (b) of that result provides the desired conclusion, at least when

$\mathrm{char}(F) = 0$. The version of [Shah, Theorem 8.1] with $\mathrm{char}(F) > 0$ was established in [Lom, Theorem 5.5].
(b) This is analogous to part (a).
(c) It is well-known (see for instance [Sol4, §2.2]) that $\mathrm{ind}_{\mathcal{O}(T_\mathfrak{s})}^{\mathcal{H}(\mathfrak{s})}(\chi_0 |\alpha|_F^{ia})$ is a direct sum of two onedimensional representations, say $\pi^g_{\mathcal{H}(\mathfrak{s})-}$ and $\pi^n_{\mathcal{H}(\mathfrak{s})-}$. Writing $\pi^{g/n}_{\mathfrak{s}-} = \pi^{g/n}_{\mathcal{H}(\mathfrak{s})-} \otimes_{\mathrm{End}_G(\Pi_\mathfrak{s})} \Pi_\mathfrak{s}$, we obtain

$$I_B^G(\chi_0|\alpha|_F^{ia}) = \pi^g_{\mathfrak{s}-} \oplus \pi^n_{\mathfrak{s}-}.$$

Since $\dim \mathrm{Hom}_U\big(I_B^G(\chi_0|\alpha|_F^{ia}), \xi\big) = 1$, exactly one of these direct summands is generic (which one depends on ξ). By renaming if necessary, we can make $\pi^g_{\mathfrak{s}-}$ generic.
(d) This holds because these representations are tempered and irreducible [Ren, Corollaire VII.2.6]. $\Box$

2.1. **Modules of Whittaker functionals.**

For our purposes it is more convenient to analyse a perspective on generic representations which is dual to the traditional view. For $(\pi, V) \in \mathrm{Rep}(G)$ let $V^\dagger$ be the smooth Hermitian dual space, that is, the vector space of all conjugate-linear maps $\ell : V \to \mathbb{C}$ which factor through the projection $V \to V^K$ for some compact open subgroup K of G. The Hermitian dual representation $\pi^\dagger$ on $V^\dagger$ is defined by

$$(\pi^\dagger(g)\ell)(v) = \ell(\pi(g^{-1})v) \qquad \forall v \in V.$$

Equivalently, $\pi^\dagger$ is the smooth contragredient of the complex conjugate of π. If π is unitary and admissible, then $\pi^\dagger$ is isomorphic to π via the G-invariant inner product.

Lemma 2.2. *If $\pi \in \mathrm{Rep}(G)^\mathfrak{s}$, then also $\pi^\dagger \in \mathrm{Rep}(G)^\mathfrak{s}$.*

Proof. Let $\mathfrak{s}' = [M, \sigma]_G$ be any inertial equivalence class different from $\mathfrak{s}$. We may assume that σ is unitary, so $\sigma^\dagger \cong \sigma$. Let $P \subset G$ be a parabolic subgroup with Levi factor M and let $M_1 \subset M$ be the subgroup generated by all compact subgroups of M. Then $I_P^G(\mathrm{ind}_{M_1}^M(\sigma))$ is a progenerator of $\mathrm{Rep}(G)^{\mathfrak{s}'}$, see [Ren, Théorème VI.10.1]. With Bernstein's second adjointness we compute

$$\begin{aligned}(2.1)\quad \mathrm{Hom}_G\big(I_P^G(\mathrm{ind}_{M_1}^M(\sigma)), \pi^\dagger\big) &\cong \mathrm{Hom}_M\big(\mathrm{ind}_{M_1}^M(\sigma), J_{\overline{P}}^G(\pi^\dagger)\big) \cong \\ \mathrm{Hom}_M\big(\mathrm{ind}_{M_1}^M(\sigma), (J_{\overline{P}}^G\pi)^\dagger\big) &\cong \mathrm{Hom}_{M_1}\big(\sigma, (J_{\overline{P}}^G\pi)^\dagger\big) \cong \mathrm{Hom}_{M_1}(J_{\overline{P}}^G(\pi), \sigma).\end{aligned}$$

Since $[M, \sigma]_G \neq \mathfrak{s}$, $J_{\overline{P}}^G(\pi)$ does not have any irreducible subquotient isomorphic with σ or an unramified twist of σ. Hence (2.1) is zero. This means that the component of $\pi^\dagger$ in $\mathrm{Rep}(G)^{\mathfrak{s}'}$ is zero for any $\mathfrak{s}' \neq \mathfrak{s}$. $\Box$

From [BuHe, (2.1.1)] one sees that the Hermitian dual of $\mathrm{ind}_U^G(\xi)$ is $\mathrm{Ind}_U^G(\xi)$, with respect to the pairing

$$\begin{array}{ccc} \mathrm{Ind}_U^G(\xi) \times \mathrm{ind}_U^G(\xi) & \to & \mathbb{C} \\ \langle f_1, f_2 \rangle & = & \int_{U \backslash G} f_1(g)\overline{f_2(g)}\mathrm{d}g \end{array}.$$

Hence there is a natural isomorphism

$$(2.2)\qquad \mathrm{Hom}_G\big(\pi, \mathrm{Ind}_U^G(\xi)\big) \cong \mathrm{Hom}_G(\mathrm{ind}_U^G(\xi), \pi^\dagger).$$

By Lemma 2.2 and (1.1), the right hand side is isomorphic with

$$(2.3)\qquad \mathrm{Hom}_{\mathrm{End}_G(\Pi_\mathfrak{s})}\big(\mathrm{Hom}_G(\Pi_\mathfrak{s}, \mathrm{ind}_U^G(\xi)), \mathrm{Hom}_G(\Pi_\mathfrak{s}, \pi^\dagger)\big).$$

Thus any nonzero Whittaker functional for π yields a nonzero element of (2.3). This prompts us to analyse $\mathrm{Hom}_G(\Pi_{\mathfrak s}, \mathrm{ind}_U^G(\xi))$ as $\mathrm{End}_G(\Pi_{\mathfrak s})^{op}$-module. By [BuHe, Theorem 2.2] there are canonical isomorphisms of T-representations

$$J_B^G \mathrm{ind}_U^G(\xi) \cong \mathrm{ind}_{U \cap T}^T(\xi) = \mathrm{ind}_{\{e\}}^T(\mathrm{triv}). \tag{2.4}$$

From that we compute

$$\begin{aligned} \mathrm{Hom}_G\big(\Pi_{\mathfrak s}, \mathrm{ind}_U^G(\xi)\big) &= \mathrm{Hom}_G\big(I_B^G(\mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c)), \mathrm{ind}_U^G(\xi)\big) \cong \\ &\mathrm{Hom}_T(\mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c), J_B^G \mathrm{ind}_U^G(\xi)\big) \cong \mathrm{Hom}_T(\mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c), \mathrm{ind}_{\{e\}}^T(\mathrm{triv})\big). \end{aligned} \tag{2.5}$$

The Bernstein decomposition of $\mathrm{Rep}(T)$ entails that only the part of $\mathrm{ind}_{\{e\}}^T(\mathrm{triv})$ on which T_{cpt} acts according to χ_c contributes to the right hand side. Hence (2.5) is naturally isomorphic with

$$\mathrm{Hom}_T\big(\mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c), \mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c)\big) \cong \mathrm{Hom}_{T_{\mathrm{cpt}}}(\chi_c, \mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c)\big) \cong \mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c). \tag{2.6}$$

This vector space contains a canonical unit vector, namely $\chi_c \in \mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c)$ or equivalently $\mathbf{1} \in \mathcal{O}(T_{\mathfrak s})$. We use the boldface to indicate that it is an element of (2.6), not of $\mathrm{End}_G(\Pi_{\mathfrak s})$.

We want to normalize our intertwining operators N_w so that they act on $\mathbf{1}$ in an easy way. Any $f \in \mathcal{O}(T_{\mathfrak s}) \cong \mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c)$ can be regarded as element of $\mathrm{End}_G(\Pi_{\mathfrak s})$, namely I_B^G applied to multiplication by f. The action of that on (2.6) is again multiplication by f. Thus (2.23) is free of rank one as $\mathcal{O}(T_{\mathfrak s})$-module, and $\mathbf{1}$ forms a canonical basis.

Let $\mathbb{C}(T_{\mathfrak s})$ be the field of rational functions on $T_{\mathfrak s}$, the quotient field of $\mathcal{O}(T_{\mathfrak s})$. It follows from Bernstein's geometric lemma [Ren, Théorème VI.5.1] that

$$\mathrm{End}_G\big(I_B^G \mathbb{C}(T_{\mathfrak s})\big) \cong \mathrm{End}_G\big(I_B^G \mathcal{O}(T_{\mathfrak s})\big) \otimes_{\mathcal{O}(T_{\mathfrak s})} \mathbb{C}(T_{\mathfrak s}), \tag{2.7}$$

see [Sol5, Lemma 5.3]. The natural isomorphisms (2.5) and (2.6) extend to

$$\mathrm{Hom}_G\big(I_B^G \mathcal{O}(T_{\mathfrak s}), \mathrm{ind}_U^G(\xi)\big) \otimes_{\mathcal{O}(T_{\mathfrak s})} \mathbb{C}(T_{\mathfrak s}) \cong \mathbb{C}(T_{\mathfrak s}), \tag{2.8}$$

and as module over (2.7) this is an extension of scalars of (2.6). The advantage of this setup is:

Proposition 2.3. *Theorem 1.2 extends to an algebra isomorphism*

$$\mathrm{End}_G\big(I_B^G \mathbb{C}(T_{\mathfrak s})\big) \cong \big(\mathbb{C}(T_{\mathfrak s}) \rtimes W(R_{\mathfrak s}^\vee)\big) \rtimes \mathbb{C}[\Gamma_{\mathfrak s}, \natural_{\mathfrak s}].$$

Proof. This is a direct consequence of Theorem 1.2 and §5.1 (in particular Corollary 5.8) of [Sol5]. □

In Proposition 2.3 the basis elements of $\mathbb{C}[\Gamma_{\mathfrak s}, \natural_{\mathfrak s}]$ are the same $J_\gamma = A_\gamma$ as in Theorem 1.2. The basis elements of

$$\mathbb{C}[W(R_{\mathfrak s}^\vee)] \subset \mathrm{End}_G\big(I_B^G \mathbb{C}(T_{\mathfrak s})\big)$$

are the $\mathcal{T}_w$ from [Sol5, Proposition 5.5], which are expressed in terms of the N_w in Lemma 10.8 and the preceding remarks of [Sol5]. Proposition 2.3 enables us to analyse the actions on (2.6) and on (2.8) more explicitly.

For $w \in W(R_{\mathfrak s}^\vee), \gamma \in \Gamma_{\mathfrak s}$ and $f \in \mathcal{O}(T_{\mathfrak s})$:

$$\begin{aligned} f \cdot \mathcal{T}_w J_\gamma = \mathbf{1} \cdot f \mathcal{T}_w J_\gamma &= (\mathbf{1} \cdot \mathcal{T}_w J_\gamma) \cdot (J_\gamma^{-1} \mathcal{T}_w^{-1} f \mathcal{T}_w J_\gamma) \\ &= (\mathbf{1} \cdot \mathcal{T}_w J_\gamma) \cdot (f \circ w\gamma) = (f \circ w\gamma)(\mathbf{1} \cdot \mathcal{T}_w J_\gamma) \end{aligned} \tag{2.9}$$

in (2.8). Notice that $\mathbf{1} \cdot J_\gamma$ must be invertible in $\mathcal{O}(T_{\mathfrak{s}})$, because J_γ is invertible in $\mathrm{End}_G(\Pi_{\mathfrak{s}})$.

We write $\theta_{n\alpha}$ for $\theta_{nh_\alpha^\vee}$, where $n \in \mathbb{Z}$ and $\alpha \in R_{\mathfrak{s}}^\vee$. We also abbreviate

$$q_\alpha = q_F^{(\lambda(h_\alpha^\vee)+\lambda^*(h_\alpha^\vee))/2} \qquad \text{and} \qquad q_{\alpha*} = q_F^{(\lambda(h_\alpha^\vee)-\lambda^*(h_\alpha^\vee))/2}.$$

Proposition 2.4. *For each simple root $h_\alpha^\vee \in R_{\mathfrak{s}}^\vee$ there exists $n_\alpha \in \mathbb{Z}$ such that $\mathbf{1} \cdot \mathcal{T}_{s_\alpha} = -\theta_{n_\alpha \alpha}$ in (2.8).*

Proof. The operators $N_{s_\alpha} \in \mathrm{End}_G(\Pi_{\mathfrak{s}})$ and $\mathcal{T}_{s_\alpha}$ arise by parabolic induction from the analogous elements for the Levi subgroup G_α of G generated by $T \cup U_\alpha \cup U_{-\alpha}$. Hence it suffices to work in G_α, which means that we may assume that $R(\mathcal{G}, \mathcal{S})$ and $R_{\mathfrak{s},\mu}$ have rank one.

First we consider the cases where $q_{\alpha*} \neq 1$, or equivalently $\lambda(h_\alpha^\vee) \neq \lambda^*(h_\alpha^\vee)$. From [Sol5, (5.19)] we know that $\mathcal{T}_{s_\alpha}(q_\alpha - \theta_{-\alpha})(q_{\alpha*} + \theta_{-\alpha}) \in \mathrm{End}_G(\Pi_{\mathfrak{s}})$. Hence we can write

$$\mathbf{1} \cdot \mathcal{T}_{s_\alpha} = f_1 (q_\alpha - \theta_{-\alpha})^{-1} (q_{\alpha*} + \theta_{-\alpha})^{-1} \quad \text{with } f_1 \in \mathcal{O}(T_{\mathfrak{s}}).$$

The relations $\mathcal{T}_{s_\alpha}^2 = 1$ and (2.9) imply that

$$1 = (\mathbf{1} \cdot \mathcal{T}_{s_\alpha})\, s_\alpha(\mathbf{1} \cdot \mathcal{T}_{s_\alpha}) = \frac{f_1 \; s_\alpha(f_1)}{(q_\alpha - \theta_\alpha)(q_{\alpha*} + \theta_\alpha)(q_\alpha - \theta_{-\alpha})(q_{\alpha*} + \theta_{-\alpha})}.$$

It follows that there exist $\epsilon \in \{\pm 1\}$ and $n_\alpha \in \mathbb{Z}$ such that

$$f_1 = \epsilon \theta_{n_\alpha \alpha} (q_\alpha - \theta_{\pm \alpha})(q_{\alpha*} + \theta_{\pm' \alpha}),$$

for suitable signs $\pm, \pm'$. Equivalently

$$\mathbf{1} \cdot \mathcal{T}_{s_\alpha} = \epsilon \theta_{n_\alpha \alpha} \Big(\frac{q_\alpha - \theta_\alpha}{q_\alpha - \theta_{-\alpha}} \Big)^\eta \Big(\frac{q_{\alpha*} + \theta_\alpha}{q_{\alpha*} + \theta_{-\alpha}} \Big)^{\eta'} =: \epsilon \theta_{n_\alpha \alpha} f_2, \tag{2.10}$$

where $\eta, \eta' \in \{0, 1\}$.

Under our assumption $\alpha^\sharp \in 2(T/T_{\mathrm{cpt}})^\vee$ and s_α fixes any $\chi \in X_{\mathrm{nr}}(T)$ with $\chi(h_\alpha^\vee) = -1$. Notice that $f_2(\chi) = 1$ whenever $\theta_\alpha(\chi) \in \{\pm 1\}$. As in [Sol5, 10.7.b] define

$$\epsilon_\alpha = \left\{ \begin{array}{ll} 1 & \text{if } I_B^G(\mathrm{ev}_\chi)\mathcal{T}_{s_\alpha} = -I_B^G(\mathrm{ev}_\chi), \\ 0 & \text{otherwise.} \end{array} \right. \tag{2.11}$$

By [Sol5, Lemma 10.8]

$$q_F^{\lambda(h_\alpha^\vee)/2} N_{s_\alpha} + 1 = (\mathcal{T}_{s_\alpha} \theta_{-\epsilon_\alpha \alpha} + 1)(\theta_\alpha q_\alpha - 1)(\theta_\alpha q_{\alpha*} + 1)(\theta_{2\alpha} - 1)^{-1} \tag{2.12}$$

belongs to $\mathrm{End}_G(\Pi_{\mathfrak{s}})$. In particular

$$\mathbf{1} \cdot (q_F^{\lambda(h_\alpha^\vee)/2} N_{s_\alpha} + 1) = \frac{(\epsilon \theta_{(n_\alpha - \epsilon_\alpha)\alpha} f_2 + 1)(\theta_\alpha q_\alpha - 1)(\theta_\alpha q_{\alpha*} + 1)}{\theta_{2\alpha} - 1} \tag{2.13}$$

lies in $\mathrm{Hom}_G\big(\Pi_{\mathfrak{s}}, \mathrm{ind}_U^G(\xi)\big) \cong \mathcal{O}(T_{\mathfrak{s}})$. Specializing the numerator of (2.13) at χ' with $\theta_\alpha(\chi') = 1$ gives $(\epsilon + 1)(q_\alpha - 1)(q_{\alpha*} + 1)$. Since $q_\alpha > 1$ and (2.13) has no poles, this implies $\epsilon = -1$.

Let $\mathcal{G}_{\mathrm{der}}$ be the derived group of $\mathcal{G}$ and write $\mathfrak{s}_{\mathrm{der}} = [\chi|_{T \cap G_{\mathrm{der}}}, T \cap G_{\mathrm{der}}]_{G_{\mathrm{der}}}$. By construction $\mathcal{H}(\mathfrak{s}_{\mathrm{der}})$ is the subalgebra of $\mathcal{H}(\mathfrak{s})$ generated by $\mathbb{C}[T \cap G_{\mathrm{der}} / T_{\mathrm{cpt}} \cap G_{\mathrm{der}}]$ and N_{s_α}. From [Sol4, §2.2] we recall that $\mathrm{St}_{\mathcal{H}(\mathfrak{s})} : \mathcal{H}(\mathfrak{s}) \to \mathbb{C}$ is given on $\mathcal{H}(\mathfrak{s}_{\mathrm{der}})$ by

$$\mathrm{St}_{\mathcal{H}(\mathfrak{s})}(N_{s_\alpha}) = -q_F^{-\lambda(h_\alpha^\vee)/2}, \qquad \mathrm{St}_{\mathcal{H}(\mathfrak{s})}(\theta_{n\alpha}) = q_\alpha^{-n}.$$

From Proposition 2.1.a,d we know that

$$\mathrm{Hom}_G(\mathrm{St}_{\mathfrak{s}}, \mathrm{Ind}_U^G(\xi)) \cong \mathrm{Hom}_G(\mathrm{ind}_U^G(\xi), \mathrm{St}_{\mathfrak{s}}) \quad \text{has dimension 1.}$$

As in (2.3), any nonzero Whittaker functional yields a surjection

$$\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \mathrm{ind}_U^G(\xi)) \cong \mathcal{O}(T_{\mathfrak{s}}) \to \mathrm{St}_{\mathcal{H}(\mathfrak{s})}.$$

This is an $\mathcal{O}(T_{\mathfrak{s}})$-module homomorphism, so up to rescaling it must be evaluation at χ_{St}, the unique $\mathcal{O}(T_{\mathfrak{s}})$-weight of $\mathrm{St}_{\mathcal{H}(\mathfrak{s})}$. Since $\mathcal{H}(W(R_{\mathfrak{s}}^\vee), q_F^\lambda)$ acts on $\mathrm{St}_{\mathcal{H}(\mathfrak{s})}$ via the sign representation,

$$\theta_x \cdot (q_F^{\lambda(h_\alpha^\vee)/2} N_{s_\alpha} + 1) \in \ker(\mathrm{ev}_{\chi_{\mathrm{St}}}) \quad \forall x \in T/T_{\mathrm{cpt}}. \tag{2.14}$$

For $x = 0$ we can make that more explicit with (2.12):

$$\begin{aligned} \mathbf{1} \cdot (q_F^{\lambda(h_\alpha^\vee)/2} N_{s_\alpha} + 1) &= (\theta_0 - \theta_{(n_\alpha - \epsilon_\alpha)\alpha} f_2)(\theta_\alpha q_\alpha - 1)(\theta_\alpha q_{\alpha *} + 1)(\theta_{2\alpha} - 1)^{-1} \\ &= \frac{(q_\alpha - \theta_{-\alpha})^\eta (q_{\alpha *} + \theta_{-\alpha})^{\eta'} - \theta_{(n_\alpha - \epsilon_\alpha)\alpha}(q_\alpha - \theta_\alpha)^\eta (q_{\alpha *} + \theta_\alpha)^{\eta'}}{(q_\alpha - \theta_{-\alpha})^\eta (q_{\alpha *} + \theta_{-\alpha})^{\eta'}} \frac{(\theta_\alpha q_\alpha - 1)(\theta_\alpha q_{\alpha *} + 1)}{(\theta_{2\alpha} - 1)} \end{aligned} \tag{2.15}$$

When $\eta = 1$, this reduces to

$$\frac{(q_\alpha - \theta_{-\alpha})(q_{\alpha *} + \theta_{-\alpha})^{\eta'} - \theta_{(n_\alpha - \epsilon_\alpha)\alpha}(q_\alpha - \theta_\alpha)(q_{\alpha *} + \theta_\alpha)^{\eta'}}{(q_{\alpha *} + \theta_{-\alpha})^{\eta'}} \frac{\theta_\alpha (\theta_\alpha q_{\alpha *} + 1)}{(\theta_{2\alpha} - 1)}.$$

Evaluation at χ_{St} sends this element to

$$\frac{-q_\alpha^{\epsilon_\alpha - n_\alpha}(q_\alpha - q_\alpha^{-1})(q_{\alpha *} + q_\alpha^{-1})^{\eta'} q_\alpha^{-1}(q_\alpha^{-1} q_{\alpha *} + 1)}{(q_{\alpha *} + q_\alpha)(q_\alpha^{-2} - 1)} \neq 0.$$

That contradicts (2.14), so that η must be 0.

We recall from [Sol4, proof of Theorem 2.4.c] that $\mathrm{St}_{\mathcal{H}(\mathfrak{s})-}$ is given on $\mathcal{H}(\mathfrak{s}_{\mathrm{der}})$ by

$$\mathrm{St}_{\mathcal{H}(\mathfrak{s})-}(N_{s_\alpha}) = -q_F^{-\lambda(h_\alpha^\vee)/2}, \qquad \mathrm{St}_{\mathcal{H}(\mathfrak{s})-}(\theta_{n\alpha}) = (-q_{\alpha *}^{-1})^n.$$

By Proposition 2.1.b,d,

$$\mathrm{Hom}_G(\mathrm{St}_{\mathfrak{s}-}, \mathrm{Ind}_U^G(\xi)) \cong \mathrm{Hom}_G(\mathrm{ind}_U^G(\xi), \mathrm{St}_{\mathfrak{s}-}) \quad \text{has dimension 1.}$$

As above, this gives a surjection

$$\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \mathrm{ind}_U^G(\xi)) \cong \mathcal{O}(T_{\mathfrak{s}}) \to \mathrm{St}_{\mathcal{H}(\mathfrak{s})-},$$

which (up to rescaling) is evaluation at the $\mathcal{O}(T_{\mathfrak{s}})$-weight $\chi_{\mathrm{St}-}$ of $\mathrm{St}_{\mathcal{H}(\mathfrak{s})-}$. Then $\ker(\mathrm{ev}_{\chi_{\mathrm{St}-}})$ contains

$$\begin{aligned} \mathbf{1} \cdot (q_F^{\lambda(h_\alpha^\vee)/2} N_{s_\alpha} + 1) &= \mathbf{1} \cdot (\mathcal{T}_{s_\alpha} \theta_{-\epsilon_\alpha \alpha} + 1)(\theta_\alpha q_\alpha - 1)(\theta_\alpha q_{\alpha *} + 1)(\theta_{2\alpha} - 1)^{-1} \\ &= \big(\theta_0 - \theta_{(n_\alpha - \epsilon_\alpha)\alpha}(q_{\alpha *} + \theta_\alpha / q_{\alpha *} + \theta_{-\alpha})^{\eta'}\big)(\theta_\alpha q_\alpha - 1)(\theta_\alpha q_{\alpha *} + 1)(\theta_{2\alpha} - 1)^{-1} \\ &= \frac{(q_{\alpha *} + \theta_{-\alpha})^{\eta'} - \theta_{(n_\alpha - \epsilon_\alpha)\alpha}(q_{\alpha *} + \theta_\alpha)^{\eta'}}{(q_{\alpha *} + \theta_{-\alpha})^{\eta'}} \frac{(\theta_\alpha q_\alpha - 1)(\theta_\alpha q_{\alpha *} + 1)}{(\theta_{2\alpha} - 1)}. \end{aligned} \tag{2.16}$$

When $\eta' = 1$, (2.16) simplifies to

$$(q_{\alpha *} + \theta_{-\alpha} - \theta_{(n_\alpha - \epsilon_\alpha)\alpha}(q_{\alpha *} + \theta_\alpha)) \frac{(\theta_\alpha q_\alpha - 1)\theta_\alpha}{(\theta_{2\alpha} - 1)}.$$

Evaluation at $\chi_{\mathrm{St}-}$ results in

$$\frac{(-q_{\alpha*}^{-1})^{n_\alpha - \epsilon_\alpha}(q_\alpha - q_{\alpha*}^{-1})(-q_{\alpha*}^{-1} q_\alpha - 1) q_{\alpha*}^{-1}}{q_{\alpha*}^{-2} - 1}.$$

This is nonzero because $q_\alpha \geq q_{\alpha*} > 1$. But then (2.16) does not lie in the kernel of $\mathrm{ev}_{\chi_{\mathrm{St}-}}$, a contradiction. Therefore η' must be 0.

Now we consider the cases with $q_{\alpha*} = 1$, or equivalently $\lambda(h_\alpha^\vee) = \lambda^*(h_\alpha^\vee)$. Then we can omit all factors $q_{\alpha*} + \theta_{\pm\alpha}$, and we can replace $\theta_{2\alpha} - 1$ by $\theta_\alpha - 1$. The above argument with $\mathrm{St}_{\mathcal{H}(\mathfrak{s})}$ still applies, and shows that $\eta = 0$. $\square$

For the moment we continue to work in G_α. Assume that $\alpha^\sharp \in 2(T/T_{\mathrm{cpt}})^\vee$ and $\lambda(h_\alpha^\vee) = \lambda^*(h_\alpha^\vee)$. The onedimensional $\mathcal{H}(\mathfrak{s})$-representation $\pi^g_{\mathcal{H}(\mathfrak{s})-}$ from Proposition 2.1.c extends canonically to a representation of $\mathcal{H}(\mathfrak{s}) + \mathcal{T}_{s_\alpha}\mathcal{H}(\mathfrak{s})$, because $\mathcal{T}_{s_\alpha}$ does not have a pole at $|\alpha|_F^{ia}$. In particular $\pi^g_{\mathcal{H}(\mathfrak{s})-}$ determines a character of the order two group $\langle \mathcal{T}_{s_\alpha} \rangle$. We define

$$\epsilon_\alpha = \begin{cases} 1 & \pi^g_{\mathcal{H}(\mathfrak{s})-}|_{\langle \mathcal{T}_{s_\alpha}\rangle} = \text{triv}, \\ 0 & \pi^g_{\mathcal{H}(\mathfrak{s})-}|_{\langle \mathcal{T}_{s_\alpha}\rangle} = \text{sign}. \end{cases} \tag{2.17}$$

This complements the definition of ϵ_α when $\alpha^\sharp \in 2(T/T_{\mathrm{cpt}})^\vee$ and $\lambda(h_\alpha^\vee) \neq \lambda^*(h_\alpha^\vee)$, see (2.11). Together these provide a function

$$\epsilon_? : \{h_\alpha^\vee \in R_{\mathfrak{s}}^\vee \text{ simple}, \alpha^\sharp \in 2(T/T_{\mathrm{cpt}})^\vee\} \to \{0, 1\}. \tag{2.18}$$

Lemma 2.5. (*a*) *The function* (2.18) *is* $\Gamma_{\mathfrak{s}}$*-invariant.*
(*b*) *Take* α *in the domain of* $\epsilon_?$ *and let* n_α *be as in Proposition 2.4. Then* $n_\alpha - \epsilon_\alpha$ *is even.*

Proof. (a) For $\gamma \in \Gamma_{\mathfrak{s}}$, represented in $N_G(T)$, we have $\gamma G_\alpha \gamma^{-1} = G_{\gamma(\alpha)}$ and $J_\gamma \mathcal{T}_{s_\alpha} J_\gamma^{-1} = \mathcal{T}_{s_{\gamma(\alpha)}}$. Further

$$\mathrm{Ad}(\gamma) I_{B\cap G_\alpha}^{G_\alpha}(\chi_0 |\alpha|_F^z) \cong I_{B\cap G_{\gamma(\alpha)}}^{G_{\gamma(\alpha)}}(\chi_0|\alpha|_F^z) \text{ for any } z \in \mathbb{C}.$$

When $\lambda(h_\alpha^\vee) = \lambda^*(h_\alpha^\vee)$, we apply this with $z = ia$. We note that $I_{G_\alpha B}^G(\pi^g_{\mathfrak{s}-})$ is generic while $I_{G_\alpha B}^G(\pi^n_{\mathfrak{s}-})$ is not. As $I_{G_\alpha B}^G(\pi^g_{\mathfrak{s}-}) = I_{G_{\gamma(\alpha)} B}^G \mathrm{Ad}(\gamma)(\pi^g_{\mathfrak{s}-})$, we conclude that $\pi^g_{\mathfrak{s}-}$ for $G_{\gamma(\alpha)}$ is obtained from $\pi^g_{\mathfrak{s}-}$ for G_α by $\mathrm{Ad}(\gamma)$. Hence $\epsilon_{\gamma(\alpha)} = \epsilon_\alpha$.

When $\lambda(h_\alpha^\vee) \neq \lambda^*(h_\alpha^\vee)$, the same argument works with the irreducible representation $I_{B\cap G_\alpha}^{G_\alpha}(\chi_0|\alpha|_F^{ia})$.
(b) Suppose that $\lambda(h_\alpha^\vee) \neq \lambda^*(h_\alpha^\vee)$. Recall from the proof of Proposition 2.4 that $\epsilon f_2 = -1$. Specializing the numerator of (2.13) at χ with $\theta_\alpha(\chi) = -1$ gives

$$(-(-1)^{n_\alpha - \epsilon_\alpha} + 1)(-q_\alpha - 1)(-q_{\alpha*} + 1) = ((-1)^{n_\alpha - \epsilon_\alpha} - 1)(q_\alpha + 1)(1 - q_{\alpha*}).$$

Again this must be 0 by (2.13). Using $q_{\alpha*} \neq 1$ we find that $n_\alpha - \epsilon_\alpha$ is even.

Suppose that $\lambda(h_\alpha^\vee) = \lambda^*(h_\alpha^\vee)$ and $\pi^g_{\mathcal{H}(\mathfrak{s})-}|_{\langle\mathcal{T}_{s_\alpha}\rangle} = \text{triv}$. By Proposition 2.1.d, any Whittaker functional for $\pi^g_{\mathcal{H}(\mathfrak{s})-}$ gives a surjection

$$\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \mathrm{ind}_U^G(\xi)) \cong \mathcal{O}(T_{\mathfrak{s}}) \to \pi^g_{\mathcal{H}(\mathfrak{s})-}.$$

As $\mathcal{O}(T_{\mathfrak{s}})$-module homomorphism it is (up to scaling) evaluation at $\chi_- := \chi_0|\alpha|_F^{ia}$, a character such that $\theta_\alpha(\chi_-) = -1$. Then $\ker(\mathrm{ev}_{\chi_-})$ contains

$$\mathbf{1} \cdot (\mathcal{T}_{s_\alpha} - 1) = -\theta_{n_\alpha \alpha} - \theta_0,$$

so n_α is odd. Recall that $\epsilon_\alpha = 1$ in this case.

Suppose that $\lambda(h_\alpha^\vee) = \lambda^*(h_\alpha^\vee)$ and $\pi^g_{\mathcal{H}(\mathfrak{s})-}|_{\langle \mathcal{T}_{s_\alpha}\rangle} = \text{sign}$. Then $\ker(\text{ev}_{\chi_-})$ contains

$$\mathbf{1} \cdot (\mathcal{T}_{s_\alpha} + 1) = -\theta_{n_\alpha \alpha} + \theta_0,$$

so n_α is even. Here $\epsilon_\alpha = 0$, so again $n_\alpha - \epsilon_\alpha$ is even. $\square$

2.2. **Normalization of intertwining operators.**

With Lemma 2.5.a, we can extend $\epsilon_?$ to a $W_\mathfrak{s}$-invariant function on $\{h_\alpha^\vee \in R_\mathfrak{s}^\vee : \alpha^\sharp \in 2(T/T_{\text{cpt}})^\vee\}$. In [Sol5], ϵ_α was only defined when $\lambda(h_\alpha^\vee) \neq \lambda^*(h_\alpha^\vee)$, implicitly saying that it is 0 otherwise. We can just as well use ϵ_α for any simple $h_\alpha^\vee$ with $\alpha^\sharp \in 2(T/T_{\text{cpt}})^\vee$, Lemma 2.5.a ensures that all the computations from [Sol5] remain valid. In particular we can now (re)define $N_{s_\alpha} \in \text{End}_G(\Pi_\mathfrak{s})$ by

$$q_F^{\lambda(h_\alpha^\vee)/2} N_{s_\alpha} + 1 = (\mathcal{T}_{s_\alpha} \theta_{-\epsilon_\alpha \alpha} + 1)\ (\theta_\alpha q_\alpha - 1)(\theta_\alpha q_{\alpha *} + 1)(\theta_{2\alpha} - 1)^{-1} \tag{2.19}$$

for any simple $h_\alpha^\vee$ with $\alpha^\sharp \in 2(T/T_{\text{cpt}})^\vee$. The analogous formula when $\alpha^\sharp \notin 2(T/T_{\text{cpt}})^\vee$ is slightly simpler:

$$q_F^{\lambda(h_\alpha^\vee)/2} N_{s_\alpha} + 1 = (\mathcal{T}_{s_\alpha} + 1)\ (\theta_\alpha q_\alpha - 1)(\theta_\alpha - 1)^{-1}. \tag{2.20}$$

Recall that the isomorphism in Theorem 1.2 was determined by the choice of a good maximal compact subgroup K of G, associated to a special vertex in the apartment for $\mathcal{T}$ in the Bruhat–Tits building of $(\mathcal{G}, F)$.

Lemma 2.6. *The good maximal compact subgroup K can be replaced by a G-conjugate, such that the isomorphism in Theorem 1.2 satisfies, for all simple roots $h_\alpha^\vee \in R_\mathfrak{s}^\vee$:*

$$\mathbf{1} \cdot \mathcal{T}_{s_\alpha} \theta_{-\epsilon_\alpha \alpha} = -\mathbf{1} \quad \textit{and} \quad \mathbf{1} \cdot N_{s_\alpha} = -q_F^{-\lambda(h_\alpha^\vee)/2} \mathbf{1}.$$

Proof. Recall the integers n_α from Proposition 2.4. We will tacitly put $\epsilon_\alpha = 0$ when $\alpha^\sharp \notin 2(T/T_{\text{cpt}})^\vee$. Select y in $\text{Hom}_\mathbb{Z}(\mathbb{Z}R_\mathfrak{s}, \mathbb{Z})$ so that $\langle y, \alpha^\sharp \rangle = n_\alpha - \epsilon_\alpha$ for every simple root $\alpha^\sharp \in R_\mathfrak{s}$. By Lemma 2.5.b y can be extended to an element of $\text{Hom}_\mathbb{Z}((T/T_{\text{cpt}})^\vee, \mathbb{Z}) = T/T_{\text{cpt}}$, which we still denote by y. The automorphism $\text{Ad}(\theta_y)$ of $\mathcal{H}(\mathfrak{s})^\circ$ extends uniquely to an automorphism of $\mathcal{H}(\mathfrak{s})^\circ \otimes_{\mathcal{O}(T_\mathfrak{s})} \mathbb{C}(T_\mathfrak{s})$, which satisfies

$$\text{Ad}(\theta_y)(\mathcal{T}_{s_\alpha} \theta_{-\epsilon_\alpha}) = \mathcal{T}_{s_\alpha} \theta_{s_\alpha(y) - y} \theta_{-\epsilon_\alpha} = \mathcal{T}_{s_\alpha} \theta_{(-n_\alpha)\alpha}. \tag{2.21}$$

By Proposition 2.4

$$\mathbf{1} \cdot \text{Ad}(\theta_y)(\mathcal{T}_{s_\alpha} \theta_{-\epsilon_\alpha \alpha}) = -\mathbf{1}. \tag{2.22}$$

For any representative y_G of y in T, $K' = \text{Ad}(y_G)K$ is another good maximal compact subgroup of G. If we replace K by K', then we must replace the representatives $\tilde{w} \in K$ for $w \in W(\mathcal{G}, \mathcal{S})$, which are used in the constructions behind Theorem 1.2, by representatives in K'. Which choice in K' does not matter, we take

$$\text{Ad}(y_G^{-1})\tilde{w} = \tilde{w} w^{-1}(y_G^{-1}) y_G \in K'.$$

For a simple root, that means

$$\text{Ad}(y_G) \tilde{s_\alpha} T_{\text{cpt}} = \tilde{s_\alpha}\ (h_\alpha^\vee)^{\langle y, \alpha^\sharp \rangle} \in N_G(T)/T_{\text{cpt}}.$$

According to [Hei, Proposition 3.1], the effect of this replacement on $\mathcal{T}_{s_\alpha}$ is left composition with $s_\alpha(\theta_\alpha^{-\langle y, \alpha^\sharp \rangle}) = \theta_\alpha^{\langle y, \alpha^\sharp \rangle}$ or equivalently right multiplication with $\theta_\alpha^{-\langle y, \alpha^\sharp \rangle}$. In view of (2.21), the effect of $\text{Ad}(y_G^{-1})$ on $\mathcal{H}(\mathfrak{s})^\circ$ is precisely $\text{Ad}(\theta_y)$.

By (2.22), the new element $\mathcal{T}'_{s_\alpha} = \mathcal{T}_{s_\alpha} \theta_{\langle y, \alpha^\sharp \rangle \alpha}$ has the same ϵ_α as before, and n_α has become ϵ_α. Now (2.22) says that $\mathbf{1} \cdot \mathcal{T}'_{s_\alpha} \theta_{-\epsilon_\alpha \alpha} = -\mathbf{1}$. The equations (2.19) and (2.20) for the new elements N'_{s_α} become

$$q_F^{\lambda(h_\alpha^\vee)/2} N'_{s_\alpha} + 1 = (\mathcal{T}'_{s_\alpha} \theta_{-\epsilon_\alpha \alpha} + 1)(\theta_\alpha q_\alpha - 1)(\theta_\alpha q_{\alpha *} + 1)(\theta_{2\alpha} - 1)^{-1}$$

In view of (2.22), this implies

$$\mathbf{1} \cdot (q_F^{\lambda(h_\alpha^\vee)/2} N'_{s_\alpha} + 1) = 0 \in \mathbb{C}(T_\mathfrak{s}).$$

Equivalently, we obtain $\mathbf{1} \cdot N_{s_\alpha} = -q_F^{-\lambda(h_\alpha^\vee)/2} \mathbf{1}$. □

From now on we choose K as in the statement of Lemma 2.6. For $w \in W_\mathfrak{s}$ let $\det(w)$ be the determinant of the action of w on $(T/T_{\mathrm{cpt}}) \otimes_\mathbb{Z} \mathbb{R}$. Then $\det : W_\mathfrak{s} \to \mathbb{R}^\times$ is a quadratic character extending the sign character of $W(R_\mathfrak{s}^\vee)$.

For $\gamma \in \Gamma_\mathfrak{s}$ we write $\mathbf{1} \cdot J_\gamma = z_\gamma \theta_{x_\gamma}$ with $z_\gamma \in \mathbb{C}^\times$ and $x_\gamma \in T/T_{\mathrm{cpt}}$. Consider the operators

$$N_\gamma = \det(\gamma) z_\gamma^{-1} \theta_{-x_\gamma} J_\gamma \in \mathrm{End}_G(\Pi_\mathfrak{s}).$$

From (2.9) we see that

$$\mathbf{1} \cdot N_\gamma = \det(\gamma) \mathbf{1} \qquad \gamma \in \Gamma_\mathfrak{s}. \tag{2.23}$$

Theorem 2.7. *The operators $N_w N_\gamma$, for $w \in W(R_\mathfrak{s}^\vee)$ with K as in Lemma 2.6 and N_γ with $\gamma \in \Gamma_\mathfrak{s}$ as above, provide an algebra isomorphism*

$$\mathrm{End}_G(\Pi_\mathfrak{s}) \cong \mathcal{H}(\mathfrak{s})^\circ \rtimes \Gamma_\mathfrak{s} = \mathcal{H}(\mathfrak{s}).$$

Given the Whittaker datum (U, ξ), this isomorphism is canonical.

Proof. By direct computation, using Lemma 2.6:

$$\mathbf{1} \cdot \mathcal{T}_{s_\alpha} \theta_{-\epsilon_\alpha \alpha} = -\det(\gamma) z_\gamma \theta_{s_\alpha(x_\gamma)}. \tag{2.24}$$

A similar computation shows that

$$\mathbf{1} \cdot \mathcal{T}_{s_{\gamma(\alpha)}} \theta_{\epsilon_\alpha \gamma(\alpha)} J_\gamma = -\mathbf{1} \cdot J_\gamma = -\det(\gamma) z_\gamma \theta_{x_\gamma}. \tag{2.25}$$

As $J_\gamma \mathcal{T}_{s_\alpha} \theta_{-\epsilon_\alpha \alpha}$ equals $\mathcal{T}_{s_{\gamma(\alpha)}} \theta_{\epsilon_\alpha \gamma(\alpha)} J_\gamma$, (2.24) and (2.25) are equal, and we deduce that $s_\alpha(x_\gamma) = x_\gamma$. Hence x_γ is fixed by each such s_α, and by the entire group $W(R_\mathfrak{s}^\vee)$. Now we can easily check that the N_γ satisfy the desired relations:

$$\begin{aligned} &\mathbf{1} \cdot N_{\gamma \tilde{\gamma}} = \det(\gamma \tilde{\gamma}) \mathbf{1} = \det(\gamma) \det(\tilde{\gamma}) \mathbf{1} = \mathbf{1} \cdot N_\gamma N_{\tilde{\gamma}}, \\ &N_\gamma f N_\gamma^{-1} = J_\gamma f J_\gamma^{-1} = f \circ \gamma^{-1} \qquad f \in \mathcal{O}(T_\mathfrak{s}), \\ &N_\gamma \mathcal{T}_{s_\alpha} N_\gamma^{-1} = z_\gamma^{-1} \theta_{-x_\gamma} J_\gamma \mathcal{T}_{s_\alpha} J_\gamma^{-1} \theta_{x_\gamma} z_\gamma = \theta_{-x_\gamma} \mathcal{T}_{s_{\gamma\alpha}} \theta_{x_\gamma} = \mathcal{T}_{s_{\gamma\alpha}}. \end{aligned}$$

The first two of these relations imply that

$$N_{\gamma \tilde{\gamma}} = N_\gamma N_{\tilde{\gamma}} \quad \text{for all } \gamma, \tilde{\gamma} \in \Gamma_\mathfrak{s}.$$

We deduce that, with respect to the given $\mathcal{O}(T_\mathfrak{s})$-basis, $\mathrm{End}_G(\Pi_\mathfrak{s})$ becomes $\mathcal{H}(\mathfrak{s})^\circ \rtimes \Gamma$.

Any two isomorphisms of this kind differ by an automorphism ψ of $\mathcal{H}(\mathfrak{s})^\circ \rtimes \Gamma_\mathfrak{s}$. Since the subalgebra $\mathcal{O}(T_\mathfrak{s})$ is mapped naturally to $\mathrm{End}_G(\Pi_\mathfrak{s})$, ψ is the identity on that subalgebra. Hence ψ extends to an automorphism of the version of $\mathcal{H}(\mathfrak{s})^\circ \rtimes \Gamma_\mathfrak{s}$ with $\mathbb{C}(T_\mathfrak{s})$. Then (2.9) entails that ψ multiplies each basis element $\mathcal{T}_w N_\gamma$ by an element of $\mathcal{O}(T_\mathfrak{s})$. Combining that with Lemma 2.6 and (2.23), we find that ψ is the identity. □

Theorem 2.7 shows in particular that the 2-cocycle $\natural_{\mathfrak s}$ from Theorem 1.2 becomes trivial in $H^2(\Gamma_{\mathfrak s}, \mathbb C^\times)$.

Recall from page 7 that $\mathfrak s$ can also arise from $w \mathfrak s_T = [T, w\chi_0]_T$ for any $w \in W(\mathcal G, \mathcal S)$. To compare all these cases, it suffices to consider one w from every left coset of $W_{\mathfrak s} = \mathrm{Stab}_{W(\mathcal G,\mathcal S)}(\mathfrak s_T)$.

Proposition 2.8. *Let $w \in W(\mathcal G, \mathcal S)$ be of minimal length in $w W_{\mathfrak s}$.*

(a) *The isomorphism $\Pi_{\mathfrak s} \cong \Pi_{w\mathfrak s}$ from* [Ren, §VI.10.1] *can be normalized so that it sends $\mathbf 1 \in \mathrm{Hom}_G(\Pi_{\mathfrak s}, \mathrm{ind}_U^G(\xi))$ to $\mathbf 1 \in \mathrm{Hom}_G(\Pi_{w\mathfrak s}, \mathrm{ind}_U^G(\xi))$.*

(b) *In that situation the induced algebra isomorphism $\mathrm{End}_G(\Pi_{\mathfrak s}) \cong \mathrm{End}_G(\Pi_{w\mathfrak s})$ is given by $f \mapsto f \circ w^{-1}$ for $f \in \mathcal O(T_{\mathfrak s})$ and $N_v \mapsto N_{wvw^{-1}}$ for $v \in W_{\mathfrak s}$.*

Proof. (a) The isomorphism of G-representations

$$\phi_w : \Pi_{w\mathfrak s} \xrightarrow{\sim} \Pi_{\mathfrak s}$$

from Proposition 1.1 induces a map

$$\mathcal O(T_{\mathfrak s}) \cong \mathrm{Hom}_G(\Pi_{\mathfrak s}, \mathrm{ind}_U^G(\xi)) \longrightarrow \mathrm{Hom}_G(\Pi_{w\mathfrak s}, \mathrm{ind}_U^G(\xi)) \cong \mathcal O(T_{w\mathfrak s}) \tag{2.26}$$

and a compatible algebra isomorphism

$$\mathrm{Ad}(\phi_w^{-1}) : \mathrm{End}_G(\Pi_{\mathfrak s}) \to \mathrm{End}_G(\Pi_{w\mathfrak s}).$$

In view of (2.4)–(2.6) and Proposition 1.1, (2.26) must be $f \mapsto f \circ w^{-1}$ followed by multiplication with some element of $\mathcal O(T_{w\mathfrak s})$.

Like in the proof of Proposition 1.1, we reduce to the case where $R(\mathcal G, \mathcal S)$ has rank one, $w = s_\alpha$ is a simple reflection and $s_\alpha \mathfrak s_T \neq \mathfrak s_T$. We represent s_α in the maximal compact subgroup K from Lemma 2.6. Consider $\chi_c \in \mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c)$ and

$$\mathbf 1 \in \mathrm{Hom}_G(\Pi_{\mathfrak s}, \mathrm{ind}_U^G(\xi)) \cong \mathrm{Hom}_T\big(\mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c), \mathrm{ind}_{\{e\}}^T(\mathrm{triv})\big).$$

Recall from (2.5) that here the isomorphism is given by J_B^G and the natural transformation $\mathrm{id} \to J_B^G I_B^G$. By definition $J_B^G(\mathbf 1)\chi_c = \chi_c$. We want to determine

$$J_B^G(\mathbf 1) J_B^G(\phi_{s_\alpha})(s_\alpha \chi_c) \in J_B^G \mathrm{ind}_U^G(\xi), \tag{2.27}$$

where $s_\alpha \chi$ is considered as element of $\mathrm{ind}_{T_{\mathrm{cpt}}}^T(s_\alpha \chi_c) \subset J_B^G \Pi_{s_\alpha \mathfrak s}$. From (1.6) we see that $J_B^G(\phi_{s_\alpha})$ on $\mathrm{ind}_{T_{\mathrm{cpt}}}^T(s_\alpha\chi_c)$ equals

$$s_\alpha \circ \big[J_B^G(\phi_{s_\alpha}) \text{ on } s_\alpha^{-1} \cdot \mathrm{ind}_{T_{\mathrm{cpt}}}^T(s_\alpha \chi_c)\big] \circ s_\alpha^{-1}.$$

Similarly $J_B^G(\mathbf 1)$ on $s_\alpha \cdot \mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c)$ equals

$$s_\alpha \circ \big[J_B^G(\mathbf 1) \text{ on } \mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c)\big] \circ s_\alpha^{-1}.$$

Now we can compute (2.27). First $s_\alpha \chi$ is mapped to χ_c by s_α^{-1}, then (1.8)–(1.10) show that $J_B^G(\phi_{s_\alpha})$ sends that to $\chi_c \in \mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c)$. Applying $J_B^G(\mathbf 1)$ returns $\chi_c \in \mathrm{ind}_{\{e\}}^T(\mathrm{triv})$ and finally the action of s_α yields $s_\alpha \chi_c$. Hence

$$J_B^G(\mathbf 1) J_B^G(\phi_{s_\alpha})(s_\alpha \chi_c) = s_\alpha \chi_c = J_B^G(\mathbf 1)(s_\alpha \chi_c),$$

where the second $\mathbf 1$ comes from $\Pi_{s_\alpha \mathfrak s}$. In view of (2.5) and (2.6), that implies

$$J_B^G(\mathbf 1) J_B^G(\phi_{s_\alpha}) = J_B^G(\mathbf 1) \quad \text{and} \quad \mathbf 1 \circ \phi_{s_\alpha} = \mathbf 1.$$

(b) Since w has minimal length in $wW_{\mathfrak s}$, it also has minimal length in $wW(R_{\mathfrak s,\mu})$. According to [AMS3, Lemma 2.4.a], $w(R_{\mathfrak s,\mu}^+) \subset R(\mathcal B, \mathcal S)$. By the $W(\mathcal G,\mathcal S)$-equivariance of Harish-Chandra μ-functions also $w(R_{\mathfrak s,\mu}^+) \subset R_{w\mathfrak s,\mu}$, and therefore $w(R_{\mathfrak s,\mu}^+) = R_{w\mathfrak s,\mu}^+$.

The proof of part (a) shows that (2.26) equals $f \mapsto f \circ w^{-1}$. With the rough description of the $\mathcal H(\mathfrak s)$-action on $\mathrm{Hom}_G(\Pi_{\mathfrak s}, \mathrm{ind}_U^G(\xi))$ from (2.9) we deduce that $\mathrm{Ad}(\phi_w^{-1})$ sends each $N_v \in \mathcal H(\mathfrak s)$ to $N_{wvw^{-1}}$ times an element of $\mathcal O(T_{\mathfrak s^\vee})$. The more precise descriptions from Lemma 2.6 and (2.23) show that in fact $\mathrm{Ad}(\phi_w^{-1}) N_v$ equals $N_{wvw^{-1}}$ for any $v \in W_{\mathfrak s}$. □

3. Characterization of generic representations

We want to parametrize $\mathrm{Irr}(G,T)$ so that the generic representations correspond to the expected kind of enhanced L-parameters. To that end a simple characterization of genericity in terms of Hecke algebras will be indispensable.

We start with a complete description of $\mathrm{Hom}_G\big(\Pi_{\mathfrak s}, \mathrm{ind}_U^G(\xi)\big)$ as right $\mathcal H(\mathfrak s)$-module. Let $\mathcal H(W(R_{\mathfrak s}^\vee, q_F^\lambda) \subset \mathcal H(\mathfrak s)^\circ$ be the finite dimensional Iwahori–Hecke algebra spanned by the N_w with $w \in W(R_{\mathfrak s}^\vee)$. The Steinberg representation of $\mathcal H(W(R_{\mathfrak s}^\vee, q_F^\lambda))$ is defined by

$$\mathrm{St}(N_{s_\alpha}) = -q_F^{\lambda(h_\alpha^\vee)/2} \quad \text{for simple} \quad h_\alpha^\vee \in R_{\mathfrak s}^\vee. \tag{3.1}$$

We extend this to a representation St of

$$\mathcal H(W_{\mathfrak s}, q_F^\lambda) := \mathcal H(W(R_{\mathfrak s}^\vee), q_F^\lambda) \rtimes \Gamma_{\mathfrak s}$$

by $\mathrm{St}(N_w N_\gamma) = \mathrm{St}(N_w)\det(\gamma)$. Notice that this formula also defines a representation of the opposite algebra $\mathcal H(W_{\mathfrak s}, q_F^\lambda)^{op}$.

Special cases of the next result were established in [ChSa] (for the Iwahori-spherical Bernstein component of a split group) and in [MiPa] (for principal series representations of split reductive p-adic groups, with some extra conditions).

Lemma 3.1. *There is an isomorphism of $\mathcal H(\mathfrak s)^{op}$-representations*

$$\mathrm{Hom}_G\big(\Pi_{\mathfrak s}, \mathrm{ind}_U^G(\xi)\big) \cong \mathrm{ind}_{\mathcal H(W_{\mathfrak s}, q_F^\lambda)^{op}}^{\mathcal H(\mathfrak s)^{op}}(\mathrm{St}).$$

Proof. Let $w \in W(R_{\mathfrak s}^\vee)$ and $\gamma \in \Gamma_{\mathfrak s}$. By Lemma 2.6 and (2.23)

$$\mathbf 1 \cdot N_w N_\gamma = \mathbf 1 \cdot \mathrm{St}(N_w N_\gamma) \in \mathrm{Hom}_G\big(\Pi_{\mathfrak s}, \mathrm{ind}_U^G(\xi)\big).$$

As vector spaces

$$\mathcal H(\mathfrak s) = \mathcal O(T_{\mathfrak s}) \otimes \mathcal H(W_{\mathfrak s}, q_F^\lambda).$$

Further $\mathrm{Hom}_G\big(\Pi_{\mathfrak s}, \mathrm{ind}_U^G(\xi)\big)$ is isomorphic to $\mathcal O(T_{\mathfrak s})$ as $\mathcal O(T_{\mathfrak s})$-module, with basis vector $\mathbf 1$. Hence

$$\begin{array}{ccc} \mathrm{ind}_{\mathcal H(W_{\mathfrak s}, q_F^\lambda)^{op}}^{\mathcal H(\mathfrak s)^{op}}(\mathrm{St}) & \to & \mathrm{Hom}_G\big(\Pi_{\mathfrak s}, \mathrm{ind}_U^G(\xi)\big) \\ h \otimes 1 & \mapsto & h \cdot \mathbf 1 \end{array}$$

is an isomorphism of $\mathcal H(\mathfrak s)^{op}$-modules. □

The criterium for genericity, namely that (2.2) is nonzero, can be put in a more manageable form with Lemma 3.1. For any $\pi \in \mathrm{Rep}(G)^{\mathfrak s}$:

$$\begin{aligned} \mathrm{Hom}_G(\pi, \mathrm{Ind}_U^G(\xi)) &\cong \mathrm{Hom}_G(\mathrm{ind}_U^G(\xi), \pi^\dagger) \\ &\cong \mathrm{Hom}_{\mathrm{End}_G(\Pi_{\mathfrak s})^{op}}\big(\mathrm{Hom}_G(\Pi_{\mathfrak s}, \mathrm{ind}_U^G(\xi)), \mathrm{Hom}_G(\Pi_{\mathfrak s}, \pi^\dagger)\big) \\ &\cong \mathrm{Hom}_{\mathcal H(\mathfrak s)^{op}}\big(\big(\mathrm{ind}_{\mathcal H(W_{\mathfrak s}, q_F^\lambda)^{op}}^{\mathcal H(\mathfrak s)^{op}}(\mathrm{St})\big), \mathrm{Hom}_G(\Pi_{\mathfrak s}, \pi^\dagger)\big) \\ &\cong \mathrm{Hom}_{\mathcal H(W_{\mathfrak s}, q_F^\lambda)^{op}}\big(\mathrm{St}, \mathrm{Hom}_G(\Pi_{\mathfrak s}, \pi^\dagger)\big). \end{aligned} \tag{3.2}$$

Corollary 3.2. *A representation $\pi \in \mathrm{Rep}(G)^{\mathfrak s}$ is (U,ξ)-generic if and only if the $\mathcal H(W_{\mathfrak s}, q_F^\lambda)^{op}$-module $\mathrm{Hom}_G(\Pi_{\mathfrak s}, \pi^\dagger)$ contains St.*

In this corollary the effect of $\pi \mapsto \pi^\dagger$ on $\mathcal H(\mathfrak s)^{op}$-modules is not obvious, we analyse that in several steps. With the *-structure and the trace from [Sol1, §3.1],

$$\mathcal H(W_{\mathfrak s}, q_F^\lambda) = \mathcal H(W(R_{\mathfrak s}^\vee), q_F^\lambda) \rtimes \Gamma_{\mathfrak s} \tag{3.3}$$

is a finite dimensional Hilbert algebra, so in particular semisimple.

Recall that a standard G-representation is of the form $I_P^G(\tau \otimes \chi)$, where $P = MN$ is a parabolic subgroup of G, τ is an irreducible tempered M-representation and $\chi : M \to \mathbb R_{>0}$ is an unramified character in positive position with respect to P. By conjugating P and M, we may assume that $T \subset M$.

Lemma 3.3. *Suppose that $\tau \in \mathrm{Rep}(M)^{[T,\chi_0]_M}$. Then $I_P^G(\tau \otimes \chi) \in \mathrm{Rep}(G)^{\mathfrak s}$ and*

$$\mathrm{Hom}_G\big(\Pi_{\mathfrak s}, I_P^G(\tau\otimes\chi)^\dagger\big) \cong \mathrm{Hom}_G\big(\Pi_{\mathfrak s}, I_P^G(\tau\otimes\chi)\big) \text{ as } \mathcal H(W_{\mathfrak s}, q_F^\lambda)^{op}\text{-modules.}$$

Proof. The representation $I_P^G(\tau\otimes\chi)$ has cuspidal support $\mathrm{Sc}(\tau)\otimes\chi|_T \in [T,\chi_0]_T$, so it belongs to $\mathrm{Rep}(G)^{[T,\chi_0]_G}$. By [Ren, IV.2.1.2]

$$I_P^G(\tau\otimes\chi)^\dagger \cong I_P^G((\tau\otimes\chi)^\dagger) \cong I_P^G(\tau^\dagger \otimes \chi^\dagger).$$

Since χ is real-valued and τ is unitary [Ren, Corollaire VII.2.6], the right hand side is isomorphic with $I_P^G(\tau\otimes\chi^{-1})$. Consider the continuous path

$$[-1,1] \to \mathrm{Rep}(G)^{\mathfrak s} : t \mapsto I_P^G(\tau\otimes\chi^t).$$

Via the equivalence of categories (1.1), we obtain a continuous path in $\mathrm{Mod}(\mathcal H(\mathfrak s)^{op})$. Modules of a finite dimensional semisimple algebra are stable under continuous deformations, so

$$\mathrm{Hom}_G\big(\Pi_{\mathfrak s}, I_P^G(\tau\otimes\chi)^\dagger\big) \cong \mathrm{Hom}_G\big(\Pi_{\mathfrak s}, I_P^G(\tau\otimes\chi^{-1})\big) \cong \mathrm{Hom}_G\big(\Pi_{\mathfrak s}, I_P^G(\tau\otimes\chi)\big)$$

as $\mathcal H(W_{\mathfrak s}, q_F^\lambda)^{op}$-modules. □

We are ready to establish a useful characterization of genericity, without Hermitian duals. The next result is formulated for finite length representations, but we believe it is also valid without that condition. To study it for arbitrary representations in $\mathrm{Rep}(G)^{\mathfrak s}$ one probably needs Hermitian duals of modules over affine Hecke algebras.

Theorem 3.4. *Suppose that $\pi \in \mathrm{Rep}(G)^{\mathfrak s}$ has finite length. Then π is (U,ξ)-generic if and only if $\mathrm{Hom}_{\mathcal H(W_{\mathfrak s}, q_F^\lambda)^{op}}\big(\mathrm{Hom}_G(\Pi_{\mathfrak s}, \pi), \mathrm{St}\big)$ is nonzero.*

Proof. Since π has finite length, we can form it semisimplification π_{ss}. Then $\pi^\dagger_{ss}$ is the semisimplification of $\pi^\dagger$. By (3.3) the module category of $\mathcal{H}(W_{\mathfrak{s}}, q_F^\lambda)^{op}$ is semisimple. In particular

$$\mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}}, q_F^\lambda)^{op}} \big(\mathrm{St}, \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi^\dagger) \big) \tag{3.4}$$

does not change if we replace $\pi^\dagger$ by $\pi^\dagger_{ss}$. Since we only need semisimplifications of modules here, we may pass to the Grothendieck group of finite length representations in $\mathrm{Rep}(G)^{\mathfrak{s}}$. The standard modules in $\mathrm{Rep}(G)^{\mathfrak{s}}$ form a $\mathbb{Z}$-basis of that Grothendieck group. Indeed, that is a consequence of the Langlands classification [Ren, Théorème VII.4.2] and the property that the irreducible quotient of a standard module is the unique maximal constituent in a certain sense [BoWa, §XI.2].

For each such standard module we have Lemma 3.3, and hence the conclusion of Lemma 3.3 extends to the entire Grothendieck group of the finite length part of $\mathrm{Rep}(G)^{\mathfrak{s}}$. In particular

$$\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi^\dagger) \cong \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi^\dagger_{ss}) \cong \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi_{ss}) \cong \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi)$$

as $\mathcal{H}(W_{\mathfrak{s}}, q_F^\lambda)^{op}$-modules. Hence the vector space (3.4) is isomorphic with

$$\mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}}, q_F^\lambda)^{op}} \big(\mathrm{St}, \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi) \big).$$

By the semisimplicity of the involved algebra, this has the same dimension as

$$\mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}}, q_F^\lambda)^{op}} \big(\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi), \mathrm{St} \big). \tag{3.5}$$

We conclude by applying Corollary 3.2 to (3.4) and (3.5). $\square$

From Theorem 3.4 it is easy to prove an analogue of the uniqueness (up to scalars) of Whittaker functionals [Rod, Shal] in the context of Hecke algebras. Let M be a standard Levi subgroup of G and write $\mathfrak{s}_M = [T, \chi_0]_M$. Via parabolic induction $\mathcal{H}(\mathfrak{s}_M) \cong \mathrm{End}_M(\Pi_{\mathfrak{s}_M})$ becomes a subalgebra of $\mathcal{H}(\mathfrak{s}) \cong \mathrm{End}_G(\Pi_{\mathfrak{s}})$. In fact the constructions in [Sol5, §10.2] show that $\mathcal{H}(\mathfrak{s}_M)$ is a parabolic subalgebra of $\mathcal{H}(\mathfrak{s})$, in the sense of [Sol2, p. 216]. The functor $\mathrm{ind}_{\mathcal{H}(\mathfrak{s}_M)^{op}}^{\mathcal{H}(\mathfrak{s})^{op}}$ corresponds to parabolic induction from M to G, see [Roc2, Proposition 1.8.5.1].

Lemma 3.5. *Let V be an irreducible $\mathcal{H}(\mathfrak{s}_M)^{op}$-module. Then*

$$\dim \mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}}, q_F^\lambda)^{op}} \big(\mathrm{ind}_{\mathcal{H}(\mathfrak{s}_M)^{op}}^{\mathcal{H}(\mathfrak{s})^{op}} V, \mathrm{St} \big) \leq 1.$$

Proof. By the Bernstein presentation of $\mathcal{H}(\mathfrak{s})^{op}$ we can simplify the module:

$$\mathrm{Res}^{\mathcal{H}(\mathfrak{s})^{op}}_{\mathcal{H}(W_{\mathfrak{s}}, q_F^\lambda)^{op}} \big(\mathrm{ind}_{\mathcal{H}(\mathfrak{s}_M)^{op}}^{\mathcal{H}(\mathfrak{s})^{op}} V \big) = \mathrm{ind}^{\mathcal{H}(W_{\mathfrak{s}}, q_F^\lambda)^{op}}_{\mathcal{H}(W_{\mathfrak{s}_M}, q_F^\lambda)^{op}} \big(\mathrm{Res}^{\mathcal{H}(\mathfrak{s}_M)^{op}}_{\mathcal{H}(W_{\mathfrak{s}_M}, q_F^\lambda)^{op}} V \big).$$

With Frobenius reciprocity it follows that

$$\mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}}, q_F^\lambda)^{op}} \big(\mathrm{ind}_{\mathcal{H}(\mathfrak{s}_M)^{op}}^{\mathcal{H}(\mathfrak{s})^{op}} V, \mathrm{St} \big) \cong \mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}_M}, q_F^\lambda)^{op}} (V, \mathrm{St}). \tag{3.6}$$

This reduces the lemma to the case $M = G$, which investigate next.

As $\mathcal{H}(\mathfrak{s})$ has finite rank as module over its centre, V has finite dimension. Hence V contains an eigenvector for $\mathcal{O}(T_{\mathfrak{s}})$, say with character t. Then

$$0 \neq \mathrm{Hom}_{\mathcal{O}(T_{\mathfrak{s}})}(t, V) \cong \mathrm{Hom}_{\mathcal{H}(\mathfrak{s})^{op}} \big(\mathrm{ind}_{\mathcal{O}(T_{\mathfrak{s}})}^{\mathcal{H}(\mathfrak{s})^{op}} (t), V \big),$$

so V is a quotient of $\mathrm{ind}_{\mathcal{O}(T_\mathfrak{s})}^{\mathcal{H}(\mathfrak{s})^{op}}(t)$. For multiplicities upon restriction to the finite dimensional semisimple subalgebra $\mathcal{H}(W_\mathfrak{s}, q_F^\lambda)^{op}$, that means

$$\dim \mathrm{Hom}_{\mathcal{H}(W_\mathfrak{s},q_F^\lambda)^{op}}(V, \mathrm{St}) \leq \dim \mathrm{Hom}_{\mathcal{H}(W_\mathfrak{s},q_F^\lambda)^{op}}\big(\mathrm{ind}_{\mathcal{O}(T_\mathfrak{s})}^{\mathcal{H}(\mathfrak{s})^{op}}(t), \mathrm{St}\big). \tag{3.7}$$

By the presentation of $\mathcal{H}(\mathfrak{s})$, $\mathrm{ind}_{\mathcal{O}(T_\mathfrak{s})}^{\mathcal{H}(\mathfrak{s})^{op}}(t) \cong \mathcal{H}(W_\mathfrak{s}, q_F^\lambda)^{op}$ as $\mathcal{H}(W_\mathfrak{s}, q_F^\lambda)^{op}$-modules. Hence the right hand side of (3.7) is 1. $\square$

4. Hecke algebras for principal series L-parameters

Recall that we fixed a separable closure F_s of F. Let $\mathbf{I}_F \subset \mathbf{W}_F \subset \mathrm{Gal}(F_s/F)$ be the inertia subgroup of the Weil group and pick a geometric Frobenius element Frob_F of $\mathbf{W}_F$. Let $G^\vee$ be the complex dual group of G and let ${}^LG = G^\vee \rtimes \mathbf{W}_F$ be the Langlands dual group. Let $\Phi(G)$ be the set of L-parameters $\phi : \mathbf{W}_F \times SL_2(\mathbb{C}) \to {}^LG$, considered modulo $G^\vee$-conjugacy.

For an L-parameter ϕ we have the component group

$$R_\phi = \pi_0(Z_{G^\vee}(\phi)/Z(G^\vee)^{\mathbf{W}_F}),$$

this is the appropriate version because G is quasi-split. We define a (G-relevant) enhancement of ϕ to be an irreducible representation of the finite group R_ϕ. Compared to [AMS1], the quasi-splitness of G allows us to focus on the enhancements whose $Z({G^\vee}_{\mathrm{sc}})$-character is trivial, and that eliminates the need to consider the centralizer of ϕ in ${G^\vee}_{\mathrm{sc}}$. We denote the set of $G^\vee$-conjugacy classes of enhanced L-parameters for G by $\Phi_e(G)$.

Recall [AMS1] that there exists a notion of cuspidality and a cuspidal support map Sc for enhanced L-parameters. The map Sc associates to each $(\phi, \rho) \in \Phi_e(G)$ a F-Levi subgroup L of G and a cuspidal enhanced L-parameter for L (unique up to $G^\vee$-conjugation). We say that (ϕ, ρ) is a principal series L-parameter if $\mathrm{Sc}(\phi, \rho)$ is an enhanced L-parameter for T (or a G-conjugate of T). In that case $\mathrm{Sc}(\phi, \rho)$ is unique up to $N_{G^\vee}(T^\vee \rtimes \mathbf{W}_F)$-conjugacy. In other words, $\mathrm{Sc}(\phi, \rho)$ as element of $\Phi_e(T)$ is unique up to conjugacy by $N_{G^\vee}(T^\vee \rtimes \mathbf{W}_F)/T^\vee$.

For the maximal torus T, the dual group $T^\vee$ is a complex torus. In particular any L-parameter for T is trivial on $SL_2(\mathbb{C})$ and has trivial component group. Hence an element of $\Phi_e(T)$ is just the $T^\vee$-conjugacy class of a homomorphism $\hat{\chi} : \mathbf{W}_F \to {}^LT$. Every element of $\Phi_e(T)$ is cuspidal, because T has no proper Levi subgroups.

To describe principal series (enhanced) L-parameters more explicitly, we consider an arbitrary $(\phi, \rho) \in \Phi_e(G)$. We want to determine $\mathrm{Sc}(\phi, \rho) = (L, \psi, \epsilon)$. By construction

$$\psi|_{\mathbf{I}_F} = \phi|_{\mathbf{I}_F} \text{ and } \psi\Big(\mathrm{Frob}_F, \begin{pmatrix} q_F^{-1/2} & 0 \\ 0 & q_F^{1/2} \end{pmatrix}\Big) = \phi\Big(\mathrm{Frob}_F, \begin{pmatrix} q_F^{-1/2} & 0 \\ 0 & q_F^{1/2} \end{pmatrix}\Big). \tag{4.1}$$

In order that $L = T$, it is necessary that $\phi\Big(\mathrm{Frob}_F, \begin{pmatrix} q_F^{-1/2} & 0 \\ 0 & q_F^{1/2} \end{pmatrix}\Big) \in T^\vee \mathrm{Frob}_F$ and $\phi(i) \in T^\vee i$ for all $i \in \mathbf{I}_F$. The group

$$H^\vee := Z_{G^\vee}(\phi(\mathbf{W}_F))$$

is reductive [Vog, (4.4.f)] and

$$R_\phi = \pi_0\big(Z_{G^\vee}(\phi)/Z(G^\vee)^{\mathbf{W}_F}\big) \quad \text{equals} \quad \pi_0\big(Z_{H^\vee}\big(\phi(SL_2(\mathbb{C}))\big)/Z(G^\vee)^{\mathbf{W}_F}\big).$$

This group is a quotient of

$$\pi_0\big(Z_{H^\vee}\big(\phi(SL_2(\mathbb{C}))\big)\big) \cong \pi_0\big(Z_{H^\vee}(u_\phi)\big),$$

where $u_\phi = \phi\big(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\big)$. Thus we can regard ρ as an irreducible representation of $\pi_0\big(Z_{H^\vee}(u_\phi)\big)$. Let $(M^\vee, u_\psi, \epsilon)$ be the cuspidal quasi-support of (u_ϕ, ρ) for $H^\vee$, as in [AMS1, §5]. Then ψ is the L-parameter determined (up to conjugacy) by (4.1) and u_ψ, while ϵ is as above and $L^\vee = Z_{H^\vee}(Z(M^\vee)^\circ)$.

For $L^\vee = T^\vee$ we need $M^\vee = T^\vee$, which implies that $u_\psi = 1$. There is an explicit criterium for $\mathrm{Sc}(u_\phi, \rho) = (T^\vee, 1, \epsilon)$ with arbitrary ϵ, as follows. Let $\mathcal{B}_{H^\vee}^{u_\phi}$ be the variety of Borel subgroups of $H^{\vee,\circ}$ that contain u_ϕ, it carries a natural action of $Z_{H^\vee}(u_\phi)$. As $Z_{H^{\vee,\circ}}(u_\phi)$ is a union of connected components of $Z_{H^\vee}(u_\phi)$, its component group $\pi_0(Z_{H^{\vee,\circ}}(u_\phi))$ is a subgroup of $\pi_0(Z_{H^\vee}(u_\phi))$. Let ρ° be any irreducible constituent of $\rho|_{\pi_0(Z_{H^{\vee,\circ}}(u_\phi))}$. Then the criterium says: ρ° appears in the action of $\pi_0(Z_{H^{\vee,\circ}}(u_\phi))$ on the (top degree) homology of $\mathcal{B}_{H^\vee}^{u_\phi}$.

Summarising, we found the following necessary and sufficient conditions for $(\phi, \rho) \in \Phi_e(G)$ to be a principal series enhanced L-parameter:

(i) $\phi\Big(\mathrm{Frob}_F, \begin{pmatrix} q_F^{-1/2} & 0 \\ 0 & q_F^{1/2} \end{pmatrix}\Big), \phi(i) \in T^\vee \rtimes \mathbf{W}_F$ for any $i \in \mathbf{I}_F$;

(ii) ρ° appears in $H_*\big(\mathcal{B}_{H^\vee}^{u_\phi}\big)$ where $H^\vee = Z_{G^\vee}(\phi(\mathbf{W}_F))$.

We note that under these conditions $\mathrm{Sc}(\phi, \rho)$ does not depend on u_ϕ or ρ. Moreover it equals $\mathrm{Sc}(\phi, \mathrm{triv})$, because $H^{top}\big(\mathcal{B}_{H^\vee}^{u_\phi}\big)$ is a permutation representation of R_ϕ (with as permuted objects the irreducible components of $\mathcal{B}_{H^\vee}^{u_\phi}$), and that always contains the trivial representation. With this in mind, we call $\phi \in \Phi(G)$ a principal series L-parameter if (i) holds.

Recall from [Hai, §3.3.1] that there is a natural isomorphism

$$X_{\mathrm{nr}}(G) \cong \big(Z(G^\vee)^{\mathbf{I}_F}\big)^\circ_{\mathrm{Frob}}, \tag{4.2}$$

that the group of unramified characters $X_{\mathrm{nr}}(T)$ is naturally isomorphic to $(T^{\vee,\mathbf{I}_F})^\circ_{\mathbf{W}_F}$. We will sometimes identify these groups and write simply $X_{\mathrm{nr}}(T)$. We note that $(T^{\vee,\mathbf{I}_F})_{\mathbf{W}_F}$ acts on $\Phi(T)$ by

$$(z\hat\chi)|_{\mathbf{I}_F} = \hat\chi|_{\mathbf{I}_F},\ (z\hat\chi)(\mathrm{Frob}_F) = z(\hat\chi(\mathrm{Frob}_F))$$

for $z \in T^{\vee,\mathbf{I}_F}$ and $\hat\chi \in \Phi(T)$. A Bernstein component of $\Phi_e(T) = \Phi(T)$ is by definition one $X_{\mathrm{nr}}(T)$-orbit in $\Phi(T)$. We will usually write this as $\mathfrak{s}_T^\vee = X_{\mathrm{nr}}(T)\hat\chi$ for one $\hat\chi \in \Phi(T)$. It gives rise to a Bernstein component $\Phi_e(G)^{\mathfrak{s}^\vee} := \mathrm{Sc}^{-1}(T, \mathfrak{s}_T^\vee)$ in the principal series part of $\Phi_e(G)$.

Next we make the extended affine Hecke algebra $\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z})$ from [AMS3] explicit. The maximal commutative subalgebra of $\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z})$ is $\mathcal{O}(\mathfrak{s}^\vee) \otimes \mathbb{C}[\mathbf{z}, \mathbf{z}^{-1}]$, where $\mathbf{z}$ is a formal variable. In this context we prefer to write $T_{\mathfrak{s}^\vee}$ for $\mathfrak{s}^\vee$, to emphasize that it is a complex torus (as a variety, in general it does not have a canonical multiplication).

The group $N_{G^\vee}(T^\vee \rtimes \mathbf{W}_F)/T^\vee$ acts naturally on $\Phi(T)$, by conjugation. Let $W_{\mathfrak{s}^\vee}$ denote the stabilizer of $\mathfrak{s}^\vee$ in $N_{G^\vee}(T^\vee \rtimes \mathbf{W}_F)/T^\vee$. Although $W_{\mathfrak{s}^\vee}$ need not be a Weyl group, it always contains the Weyl group of a root system. Namely, consider the group $J = Z_{G^\vee}(\hat\chi(\mathbf{I}_F))$, with the torus $(T^{\vee,\mathbf{W}_F})^\circ$ and the maximal torus $T^\vee$. According to [AMS3, Proposition 3.9.a and (3.22)], $R\big(J^\circ, (T^{\vee,\mathbf{W}_F})^\circ\big)$ is a root system and $W_{\mathfrak{s}^\vee}$ acts naturally on it. Moreover [AMS3, Proposition 3.9.b] says that for a suitable choice of $\hat\chi$ in $T_{\mathfrak{s}^\vee}$ the set of indivisible roots

$$R\big(J^\circ, (T^{\vee,\mathbf{W}_F})^\circ\big)_{\mathrm{red}} \quad \text{equals} \quad R\big(Z_{G^\vee}(\hat\chi(\mathbf{W}_F))^\circ, (T^{\vee,\mathbf{W}_F})^\circ\big)_{\mathrm{red}}.$$

For such a choice of a basepoint $\hat\chi$ of $T_{\mathfrak{s}^\vee}$,

$$W^\circ_{\mathfrak{s}^\vee} := W\big(R(J^\circ, (T^{\vee,\mathbf{W}_F})^\circ)\big)$$

is a normal subgroup of $W_{\mathfrak{s}}$. Let $R^+\big(J^\circ, (T^{\vee,\mathbf{W}_F})^\circ\big)$ be the positive root system determined by the Borel subgroup $B^\vee$ of $G^\vee$. Then $W_{\mathfrak{s}^\vee} = W^\circ_{\mathfrak{s}^\vee} \rtimes \Gamma_{\mathfrak{s}^\vee}$, where $\Gamma_{\mathfrak{s}^\vee}$ denotes the stabilizer of $R^+\big(J^\circ, (T^{\vee,\mathbf{W}_F})^\circ\big)$ in $W_{\mathfrak{s}^\vee}$.

The root system for $\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z})$ will essentially be $R\big(J^\circ, (T^{\vee,\mathbf{W}_F})^\circ\big)$, but we still need to rescale the elements [AMS3, §3.2]. We note that the inclusion $(T^{\vee,\mathbf{W}_F})^\circ \to T^\vee$ induces a surjection

$$\mathrm{pr} : R(J^\circ, T^\vee) \cup \{0\} \to R\big(J^\circ, (T^{\vee,\mathbf{W}_F})^\circ\big) \cup \{0\}.$$

In [AMS3, Definition 3.11], positive integers m_α for $\alpha^\vee \in R\big(J^\circ, (T^{\vee,\mathbf{W}_F})^\circ\big)_{\mathrm{red}}$ are defined, as follows.

- Suppose that $\mathrm{pr}^{-1}(\{\alpha^\vee\})$ meets $k > 1$ connected components of $R(J^\circ, T^\vee)$. These k components are permuted transitively by Frob_F. Then m_α equals k times the analogous number m'_α obtained by replacing F by its degree k unramified extension (or equivalently replacing Frob_F by Frob_F^k).
- Suppose that $\mathrm{pr}^{-1}(\{\alpha^\vee\})$ lies in a single connected component of $R(J^\circ, T^\vee)$. Then m_α is the smallest natural number such that $\ker(m_\alpha \alpha^\vee)$ contains the kernel of the canonical surjection

$$(T^{\vee,\mathbf{W}_F})^\circ \to (T^{\vee,\mathbf{I}_F})_{\mathbf{W}_F}^\circ \cong X_{\mathrm{nr}}(T). \tag{4.3}$$

In fact it is easy to identify the kernel of (4.3) as

$$(T^{\vee,\mathbf{W}_F})^\circ_{\mathrm{Frob}_F} := (T^{\vee,\mathbf{W}_F})^\circ \cap (1 - \mathrm{Frob}_F) T^{\vee,\mathbf{I}_F}.$$

Lemma 4.1. *The number m_α equals $f(F_\alpha/F)$, where $\mathbf{W}_{F_\alpha}$ is the $\mathbf{W}_F$-stabilizer of a lift of $\alpha^\vee$ to $R(G^\vee, T^\vee)$.*

Proof. The number m_α can be related to the structure of the F-group $\mathcal{G}$. Let $\mathcal{G}_\alpha$ be the F-simple almost direct factor of $\mathcal{G}$ such that $\mathrm{pr}^{-1}(\{\alpha^\vee\})$ consists of roots coming from $G_\alpha^\vee$. Write $\mathcal{G}_\alpha = \mathrm{Res}_{E_\alpha/F} \mathcal{H}_\alpha$, where $\mathcal{H}_\alpha$ is absolutely simple. The injection $\mathcal{G}_\alpha \to \mathcal{G}$ induces a surjection ${}^L G \to {}^L G_\alpha$ which does not change $\alpha^\vee\big((T^{\vee,\mathbf{W}_F})^\circ_{\mathrm{Frob}_F}\big)$. Knowing that, the first bullet above says that m_α equals $f(E_\alpha/F)$ times the number m'_α for $\mathcal{H}_\alpha(E_\alpha)$.

Let $\mathcal{T}_\alpha$ be the maximal torus of $\mathcal{H}_\alpha$ with $\mathrm{Res}_{E_\alpha/F} \mathcal{T}_\alpha = \mathcal{T} \cap \mathcal{G}_\alpha$. The Weil group $\mathbf{W}_{E_\alpha}$ acts on the irreducible root system $R(\mathcal{H}_\alpha^\vee, \mathcal{T}_\alpha^\vee)$, and the set of orbits is in bijection with the irreducible component of $R(J^\circ, (T^{\vee,\mathbf{W}_F})^\circ)$ containing $\alpha^\vee$. Let $\mathbf{W}_{F_\alpha}$ be the $\mathbf{W}_{E_\alpha}$-stabilizer of an element $\alpha'^\vee \in R(\mathcal{H}_\alpha^\vee, \mathcal{T}_\alpha^\vee)$ that corresponds to $\alpha^\vee$. Then $\alpha^\vee = \alpha'^\vee|_{T_\alpha^{\mathbf{W}_{F_\alpha}}}$.

Suppose that the elements of $\mathbf{W}_{E_\alpha} \alpha'^\vee$ are mutually orthogonal, which happens in most cases. From the definitions we see that

$$\big|\alpha^\vee\big((T^{\vee,\mathbf{W}_F})^\circ_{\mathrm{Frob}_F}\big)\big| = f(F_\alpha/E_\alpha).$$

Here the relevant elements of $(T^{\vee,\mathbf{W}_F})^\circ_{\mathrm{Frob}_F}$ are the powers of

$$(1 - \mathrm{Frob}_F)t \quad \text{where} \quad (\mathrm{Frob}^n \alpha'^\vee)(t) = \exp(2\pi i n / f(F_\alpha/E_\alpha)).$$

We find $m'_\alpha = f(F_\alpha/E_\alpha)$ and $m_\alpha = f(F_\alpha/F)$.

Next we consider the cases where the elements of $\mathbf{W}_{E_\alpha}\alpha'^\vee$ are not mutually orthogonal. Classification shows that $R(\mathcal{H}^\vee, \mathcal{T}_\alpha^\vee)$ has type ${}^2A_{2n}$ and

$$|\mathbf{W}_{E_\alpha}\alpha'^\vee| = [F_\alpha : E_\alpha] = 2,$$

so that $H_{\alpha,\mathrm{ad}} \cong PU_{2n+1}(F_\alpha/E_\alpha)$. Direct computations show that, when F_α/E_α is ramified, $m'_\alpha = 1$ and $m_\alpha = f(E_\alpha/F) = f(F_\alpha/F)$. Similarly, when F_α/E_α is unramified, $m'_\alpha = 2$ and $m_\alpha = 2f(E_\alpha/F) = f(F_\alpha/F)$. □

Lemma 4.1 and [AMS3, Lemma 3.12] yield the precise definition of the root system for $\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z})$:

$$R_{\mathfrak{s}^\vee} = \big\{ m_\alpha \alpha^\vee : \alpha^\vee \in R\big(J^\circ, (T^{\vee,\mathbf{W}_F})^\circ\big)_{\mathrm{red}} \big\}.$$

This root system is endowed with an action of $W_{\mathfrak{s}^\vee}$. Hence $W_{\mathfrak{s}^\vee}$ also acts on the resulting root datum from [AMS3, §3.2]:

$$\begin{aligned}\mathcal{R}_{\mathfrak{s}^\vee} &= \big(R_{\mathfrak{s}^\vee}, X^*\big((T^{\vee,\mathbf{I}_F})^\circ_{\mathbf{W}_F}\big), R_{\mathfrak{s}^\vee}^\vee, X_*\big((T^{\vee,\mathbf{I}_F})^\circ_{\mathbf{W}_F}\big)\big)\\ &= \big(R_{\mathfrak{s}^\vee}, T/T_{\mathrm{cpt}}, R_{\mathfrak{s}^\vee}^\vee, (T/T_{\mathrm{cpt}})^\vee\big).\end{aligned}$$

The label functions λ, λ^* for $\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z})$ are determined in [AMS3, Proposition 3.14]. Suppose first that the elements of $\mathbf{W}_{E_\alpha}\alpha'^\vee$ are mutually orthogonal (in the notation from the proof of Lemma 4.1), and that the same holds for $\alpha^\vee/2$ whenever $\alpha^\vee/2$ can be lifted to $R(G^\vee, (T^{\vee,\mathbf{W}_F})^\circ)$. In these non-exceptional cases

$$\lambda(m_\alpha \alpha^\vee) = \lambda^*(m_\alpha \alpha^\vee) = m_\alpha = f(F_\alpha/F). \tag{4.4}$$

If in addition $m_\alpha \alpha^\vee \in 2X^*\big((T^{\vee,\mathbf{I}_F})^\circ_{\mathbf{W}_F}\big) = 2(T/T_{\mathrm{cpt}})$, then we can get the same Hecke algebras with $m_\alpha \alpha^\vee/2$ instead of $m_\alpha \alpha^\vee$, and

$$\lambda(m_\alpha \alpha^\vee/2) = m_\alpha = f(F_\alpha/F), \ \lambda^*(m_\alpha \alpha^\vee/2) = 0. \tag{4.5}$$

We call the remaining cases exceptional, these occur only when $R(G^\vee, T^\vee)$ has a component of type ${}^2A_{2n}$ and $\alpha^\vee$ or $\alpha^\vee/2$ comes from two non-orthogonal roots that are exchanged by the diagram automorphism. As noted in the proof of Lemma 4.1, $H_{\alpha,\mathrm{ad}} \cong PU_{2n+1}(F_\alpha/E_\alpha)$. The groups $PU_{2n+1}(F_\alpha/E_\alpha)$, $SU_{2n+1}(F_\alpha/E_\alpha)$ and $U_{2n+1}(F_\alpha/E_\alpha)$ give the same root system, the same unramified characters and the same groups $(T^{\vee,\mathbf{W}_F})^\circ$. Hence the relevant data for H_α can be reduced (via its derived group) to those for $U_{2n+1}(F_\alpha/E_\alpha)$, and it suffices to continue the analysis in the latter group.

For $U_{2n+1}(F_\alpha/E_\alpha)$ all the labels were computed in [AMS4, §5]. For convenience we provide an overview, where we remark that the labels from [AMS4] still have to be multiplied by $f(E_\alpha/F)$ to account for the restriction of scalars $\mathcal{G}_\alpha(F) = \mathcal{H}_\alpha(E_\alpha)$, as in the proof of Lemma 4.1. We write $\alpha^\vee = \alpha'^\vee + \alpha''^\vee$, where $\alpha'^\vee$ and $\alpha''^\vee$ are non-orthogonal roots in A_{2n} exchanged by the diagram automorphism.

- F_α/E_α unramified and $\hat{\chi}(\mathbf{W}_{E_\alpha}) \subset Z(GL_{2n+1}(\mathbb{C})) \rtimes \mathbf{W}_{E_\alpha}$. Then $\lambda(\alpha^\vee) = 3$ and $\lambda^*(\alpha^\vee) = 1$.
- F_α/E_α unramified and $\hat{\chi}(\mathbf{W}_{E_\alpha}) \not\subset Z(GL_{2n+1}(\mathbb{C})) \rtimes \mathbf{W}_{E_\alpha}$. Here we need $\hat{\chi}(\mathbf{W}_{E_\alpha})$ to fix $U_{\alpha^\vee}$ pointwise for $\alpha^\vee \in R_{\mathfrak{s}^\vee}$. Under that condition $\lambda(\alpha^\vee) = \lambda^*(\alpha^\vee) = 1$.

- F_α / E_α is ramified. When $\hat\chi \circ \alpha^\vee : F_\alpha \to \mathbb{C}^\times$ is conjugate-orthogonal, $\alpha^\vee \notin R_{\mathfrak{s}^\vee}$. Otherwise $\hat\chi \circ \alpha^\vee$ is conjugate-symplectic, then $\alpha^\vee \in R_{\mathfrak{s}^\vee}$ and $\lambda(\alpha^\vee) = \lambda^*(\alpha^\vee) = 1$. Equivalently, using $\alpha^\vee/2$ as root:

$$\lambda(\alpha^\vee/2) = 1, \quad \lambda^*(\alpha^\vee/2) = 0. \tag{4.6}$$

The algebra $\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z})$ has a subalgebra $\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z})^\circ$, whose underlying vector space is

$$\mathcal{O}(T_{\mathfrak{s}^\vee}) \otimes \mathbb{C}[\mathbf{z}, \mathbf{z}^{-1}] \otimes \mathbb{C}[W^\circ_{\mathfrak{s}^\vee}].$$

It is isomorphic to the affine Hecke algebra $\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \mathbf{z})$, for suitable label functions λ, λ^*. The identification of the vector spaces comes from the elements $N_w \in \mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \mathbf{z})$ and the bijection

$$(T^{\mathbf{I}_F})^\circ_{\mathbf{W}_F} \to T_{\mathfrak{s}^\vee} : t \mapsto t\hat\chi. \tag{4.7}$$

Theorem 4.2. *There is a canonical algebra isomorphism*

$$\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z}) \cong \mathcal{H}(\mathfrak{s}^\vee, \mathbf{z})^\circ \rtimes \Gamma_{\mathfrak{s}^\vee}.$$

Proof. By design $\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z})$ is free as $\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z})^\circ$-module, with a basis indexed by $\Gamma_{\mathfrak{s}^\vee}$. More precisely, by [AMS3, Proposition 3.15.a] the actions of $\Gamma_{\mathfrak{s}^\vee}$ on $R_{\mathfrak{s}^\vee}, T_{\mathfrak{s}^\vee}$ and $\mathcal{O}(T_{\mathfrak{s}^\vee})$ naturally induce an action of $\Gamma_{\mathfrak{s}^\vee}$ on $\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z})^\circ$. For every $\gamma \in \Gamma_{\mathfrak{s}^\vee}$, that yields an element of $\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z})$, unique up to scaling.

For the Langlands parameters under consideration, the sheaf $q\mathcal{E}$ from [AMS1, AMS3] is just the constant sheaf with stalk $\mathbb{C}$ on the point $1 \in T^\vee$. It follows that there is a canonical choice for the map qb_γ from [AMS3, (3.39)], namely the identity. Then $\gamma \mapsto qb_\gamma$ is multiplicative, the scalars $\lambda_{\gamma,\gamma'}$ in the proof of [AMS3, Proposition 3.15.b] reduce to 1 and $\mathbb{C}[\Gamma_{\mathfrak{s}^\vee}]$ embeds in $\mathcal{H}(\mathfrak{s}^\vee, \mathbf{z})$ as the span of these qb_γ. With this in place, [AMS3, Proposition 3.15.a] provides the desired statement. $\square$

Next we specialize $\mathbf{z}$ to $q_F^{1/2}$, that yields the algebra

$$\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2}) = \mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})^\circ \rtimes \Gamma_{\mathfrak{s}^\vee} \cong \mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, q_F^{1/2}) \rtimes \Gamma_{\mathfrak{s}^\vee}. \tag{4.8}$$

We note that here the isomorphism depends on the choice of the basepoint $\hat\chi$ of $T_{\mathfrak{s}^\vee}$. From (4.8) we see that the centre of $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$ is

$$Z\big(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})\big) = \mathcal{O}(T_{\mathfrak{s}^\vee})^{W_{\mathfrak{s}}} = \mathcal{O}(T_{\mathfrak{s}^\vee}/W_{\mathfrak{s}^\vee})$$

The main use of the algebras (4.8) lies in the following result.

Theorem 4.3. [AMS3, Theorem 3.18] *There exists a canonical bijection*

$$\begin{array}{ccc} \Phi_e(G)^{\mathfrak{s}^\vee} & \to & \mathrm{Irr}\big(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})\big) \\ (\phi, \rho) & \mapsto & \bar M(\phi, \rho, q_F^{1/2}) \end{array}$$

such that:

(*a*) $\bar M(\phi, \rho, q_F^{1/2})$ *admits the central character* $W_{\mathfrak{s}^\vee}\tilde\phi \in T_{\mathfrak{s}^\vee}/W_{\mathfrak{s}^\vee}$*, where* $\tilde\phi|_{\mathbf{I}_F} = \phi|_{\mathbf{I}_F}$ *and* $\tilde\phi(\mathrm{Frob}_F) = \phi\Big(\mathrm{Frob}_F, \begin{pmatrix} q_F^{-1/2} & 0 \\ 0 & q_F^{1/2} \end{pmatrix}\Big)$.

(*b*) ϕ *is bounded if and only if* $\bar M(\phi, \rho, q_F^{1/2})$ *is tempered.*

(*c*) ϕ *is discrete if and only if* $\bar M(\phi, \rho, q_F^{1/2})$ *is essentially discrete series and the rank of* $R_{\mathfrak{s}^\vee}$ *equals the* F*-split rank of* $\mathcal{T}/Z(\mathcal{G})$.

(*d*) *The bijection is equivariant for the canonical actions of* $Z(G^\vee)^{\mathbf{I}_F} \cap (T^{\vee,\mathbf{I}_F})^\circ$.

We note that in [AMS3] the canonicity is obtained in a slightly weaker sense, by interpreting the subalgebra of $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$ spanned by the N_γ with $\gamma \in \Gamma_{\mathfrak{s}^\vee}$ as the endomorphism algebra of a certain perverse sheaf [AMS3, (2.5)]. We got rid of that subtlety in the proof of Theorem 4.2.

For part (d) we recall that any element $t \in Z(G^\vee)^{\mathbf{I}_F}$ determines a weakly unramified character of G [Hai, §3.3.1], and that character is trivial on T_{cpt} if and only if $t \in (T^{\vee,\mathbf{I}_F})^\circ$. To $t \in Z(G^\vee)^{\mathbf{I}_F} \cap (T^{\vee,\mathbf{I}_F})^\circ$ we associate the automorphism

$$xN_w \mapsto x(t)xN_w \qquad x \in T/T_{\mathrm{cpt}}, w \in W_{\mathfrak{s}^\vee}$$

of $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$, where x is regarded as function on $T_{\mathfrak{s}^\vee}$ via (4.7). The action of t on $\mathrm{Irr}\big(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})\big)$ is composition with the above automorphism.

5. Comparison of Hecke algebras

We start with a Bernstein component $\mathfrak{s}_T$ for T. Recall that this is just a $X_{\mathrm{nr}}(T)$-coset in $\mathrm{Irr}(T)$. The local Langlands correspondence for tori [Lan2, Yu] associates to $\mathfrak{s}_T$ a $X_{\mathrm{nr}}(T)$-orbit in $\Phi(T)$, that is, one Bernstein component $\mathfrak{s}_T^\vee$ in $\Phi_e(T)$. Let $W_{\mathfrak{s}}$ be the stabilizer of $\mathfrak{s}_T$ in $N_G(T)/T$ and let $W_{\mathfrak{s}^\vee}$ be the stabilizer of $\mathfrak{s}_T^\vee$ in $N_{G^\vee}(T^\vee \rtimes \mathbf{W}_F)/T^\vee$.

Lemma 5.1. *There is a natural isomorphism $W_{\mathfrak{s}} \cong W_{\mathfrak{s}^\vee}$.*

Proof. The root datum

$$(R(\mathcal{G},\mathcal{T}), X^*(\mathcal{T}), R(G^\vee, T^\vee), X^*(T^\vee))$$

comes with a natural group isomorphism

$$W(\mathcal{G},\mathcal{T}) \cong W(G^\vee, T^\vee). \tag{5.1}$$

From [ABPS2, Proposition 3.1 and its proof] we know that (5.1) restricts to an isomorphism

$$N_G(T)/T \cong N_{G^\vee}(T^\vee \rtimes \mathbf{W}_F)/T^\vee = W(G^\vee, T^\vee)^{\mathbf{W}_F}. \tag{5.2}$$

The LLC for tori is natural, so compatible with isomorphisms of

$$(X^*(T), X_*(T)) = (X_*(T^\vee), X^*(T^\vee)).$$

In particular it is equivariant for the action of (5.1), and hence for the action of (5.2). Thus the action of $N_G(T)/T$ on $\mathrm{Irr}(T)$ is turned into the conjugation action of $W(G^\vee, T^\vee)^{\mathbf{W}_F}$ by the LLC. In particular (5.2) matches $\mathrm{Stab}_{N_G(T)/T}(\mathfrak{s}_T)$ with the $W(G^\vee, T^\vee)^{\mathbf{W}_F}$-stabilizer of the image $\mathfrak{s}_T^\vee$ of $\mathfrak{s}_T$ in $\Phi_e(T)$. $\square$

Similarly we can compare root systems on both sides of the LLC.

Lemma 5.2. *There exists a natural bijection between $R_{\mathfrak{s}}^\vee$ and $R_{\mathfrak{s}^\vee}$, which preserves positivity.*

Proof. Pick any $\chi \in \mathfrak{s}_T$.

By construction $R_{\mathfrak{s}}^\vee$ consists of positive multiples of the $\alpha^\vee \in R(\mathcal{G},\mathcal{S})^\vee$ for which $\alpha \in R_{\mathfrak{s},\mu}$. Similarly $R_{\mathfrak{s}^\vee}$ consists of positive multiples of the $\alpha^\vee$ in

$$R\big(Z_{G^\vee}(\hat\chi(\mathbf{I}_F)), T^{\vee,\mathbf{W}_F,\circ}\big)_{\mathrm{red}} \subset R\big(G^\vee, T^{\vee,\mathbf{W}_F,\circ}\big)_{\mathrm{red}} \cong R(G^\vee, S^\vee)_{\mathrm{red}} \cong R(\mathcal{G},\mathcal{S})^\vee_{\mathrm{red}}$$

for which $\hat\chi(\mathbf{I}_F)$ fixes $U_{\alpha^\vee}$ or $U_{2\alpha^\vee}$ in $Z_{G^\vee}(\hat\chi(\mathbf{I}_F))$. Since both $R_{\mathfrak{s}}^\vee$ and $R_{\mathfrak{s}^\vee}$ are reduced root systems, this means that there exists at most one bijection $R_{\mathfrak{s}}^\vee \to R_{\mathfrak{s}^\vee}$

which scales each root by a positive real number. Positivity in $R_{\mathfrak{s}}^\vee$ is determined by $\mathcal{B}$ and positivity in $R_{\mathfrak{s}^\vee}$ is determined by $B^\vee$, so such a bijection would automatically preserve positivity of roots.

It remains to check that for $R_{\mathfrak{s}}^\vee$ and $R_{\mathfrak{s}^\vee}$ the same elements of $R(\mathcal{G},\mathcal{S})^\vee_{\mathrm{red}}$ are relevant. For the non-exceptional roots we know from (1.13)–(1.14) that $\alpha \in \Sigma_{\mathfrak{s},\mu}$ if and only if $\chi \circ \alpha^\vee : F_\alpha^\times \to \mathbb{C}^\times$ is unramified. Via the LLC for tori that becomes:

$$\alpha^\vee \circ \hat\chi : \mathbf{W}_{F_\alpha} \to \mathbb{C} \rtimes \mathbf{W}_{F_\alpha} \text{ restricts to the identity on } \mathbf{I}_{F_\alpha}.$$

In this setting the roots in the associated $\mathbf{W}_F$-orbit in $R(G^\vee, T^\vee)$ are mutually orthogonal, permuted by $\mathbf{W}_F$ and fixed by $\mathbf{W}_{F_\alpha}$. Hence $\alpha^\vee \circ \hat\chi(\mathbf{I}_{F_\alpha})$ fixes $U_{\alpha^\vee}$ pointwise, which means that α belongs to $R\big(Z_{G^\vee}(\hat\chi(\mathbf{I}_F)), T^{\vee,\mathbf{W}_F,\circ}\big)_{\mathrm{red}}$. This argument also works in the opposite direction, so $\alpha^\vee \in R_{\mathfrak{s}^\vee}$ if and only if $\alpha \in R_{\mathfrak{s},\mu}$.

For the exceptional roots $\alpha^\vee$ with $s_\alpha \in W_{\mathfrak{s}} \cong W_{\mathfrak{s}^\vee}$, we saw on page 10 and after (4.5) that on both sides the issue can be reduced to a unitary group U_{2n+1}. From the list of cases at the end of Section 1 it is clear that if U_{2n+1} is unramified, $\alpha^\vee$ is relevant for $R_{\mathfrak{s}}^\vee$ if and only if it is relevant for $R_{\mathfrak{s}^\vee}$.

When the involved group U_{2n+1} only splits over a ramified extension, we need to check one more detail to arrive at the same conclusion. Namely, if $\alpha^\vee \circ \hat\chi : \mathbf{W}_{E_\alpha} \to \mathbb{C}^\times$ is conjugate-orthogonal (respectively conjugate-symplectic) then $\chi \circ \alpha^\vee : \mathfrak{o}_{E_\alpha}^\vee \to \mathbb{C}^\times$ must be trivial (respectively of order two). This is exactly [GGP, Lemma 3.4]. $\square$

Lemma 5.2 implies that the isomorphism (5.2) restricts to $W_{\mathfrak{s}}^\circ \cong W_{\mathfrak{s}^\vee}^\circ$. We choose a $W_{\mathfrak{s}^\vee}^\circ$-invariant base point $\hat\chi_0$ of $\mathfrak{s}_T^\vee$ as in Section 4. We use the image χ_0 of $\hat\chi_0$ under the LLC as basepoint of $\mathfrak{s}_T$. By the aforementioned equivariance of the LLC for tori, χ_0 is invariant under $W_{\mathfrak{s}}^\circ$.

Recall that $h_\alpha^\vee \in R_{\mathfrak{s}}^\vee$ generates $\mathbb{Q}\alpha^\vee \cap T/T_{\mathrm{cpt}}$. The element $m_\alpha \alpha^\vee$ does not necessarily generate $\mathbb{Q}\alpha^\vee \cap T/T_{\mathrm{cpt}}$. However, since $R_{\mathfrak{s}^\vee}$ is part of the root datum $\mathcal{R}_{\mathfrak{s}^\vee}$, $m_\alpha\alpha^\vee$ is at most divisible by 2 in T/T_{cpt} (namely when it is a long root in a type C root system). For a better comparison, we replace $m_\alpha\alpha^\vee$ by $m_\alpha\alpha^\vee/2$ whenever that is possible. That option was already taken into account in Section 4. We denote the new multiple of $\alpha^\vee$ by $\tilde m_\alpha \alpha^\vee$ and we write

$$\tilde R_{\mathfrak{s}^\vee} = \big\{ \tilde m_\alpha \alpha^\vee : \alpha^\vee \in R(J^\circ, T^{\vee,\mathbf{W}_F,\circ}) \big\}.$$

Now Lemma 5.2 entails that the isomorphism

$$X^*(T_{\mathfrak{s}}) \cong X^*(X_{\mathrm{nr}}(T)) \cong T/T_{\mathrm{cpt}} \cong X^*\big((T^{\vee,\mathbf{I}_F})^\circ_{\mathbf{W}_F} \big), \tag{5.3}$$

induced by the LLC for tori, sends $R_{\mathfrak{s}}^\vee$ bijectively to $R_{\mathfrak{s}^\vee}$.

Lemma 5.3. *For any $\alpha \in R_{\mathfrak{s},\mu}$: $\lambda(h_\alpha^\vee) = \lambda(\tilde m_\alpha \alpha^\vee)$ and $\lambda^*(h_\alpha^\vee) = \lambda^*(\tilde m_\alpha\alpha^\vee)$.*

Proof. For the non-exceptional roots, this was checked in (1.14), (1.15), Lemma 4.1 and (4.4). For exceptional roots (i.e. those for which the issue can be reduced to a unitary group U_3), it is verified case-by-case in the lists at the end of Section 1 and just before (4.6). $\square$

We are ready to prove that the desired isomorphism between Hecke algebras on the two sides of the LLC.

Theorem 5.4. *There is a canonical algebra isomorphism $\psi_{\mathfrak{s}} : \mathcal{H}(\mathfrak{s})^{op} \to \mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$, given by*

- *on* $\mathcal{O}(T_{\mathfrak{s}})$*,* $\psi_{\mathfrak{s}}$ *is induced by the bijection* $T_{\mathfrak{s}} \cong T_{\mathfrak{s}^\vee}$ *from the LLC for tori,*
- $\psi_{\mathfrak{s}}(N_w) = N_{w^{-1}}$ *for all* $w \in W_{\mathfrak{s}} \cong W_{\mathfrak{s}^\vee}$*.*

Proof. By Theorem 2.7 there is a unique isomorphism of $\mathcal{O}(T_{\mathfrak{s}^\vee})$-modules with these properties. By the $W_{\mathfrak{s}}$-equivariance of the LLC for tori via (5.2), $\mathcal{O}(T_{\mathfrak{s}}) \xrightarrow{\sim} \mathcal{O}(T_{\mathfrak{s}^\vee})$ is $W_{\mathfrak{s}}$-equivariant. Combine that with Lemma 5.3 and the multiplication rules in extended affine Hecke algebras [AMS3, Proposition 2.2]. $\square$

We note that Theorem 5.4 is compatible with parabolic induction from standard parabolic and standard Levi subgroups of G. Indeed, for a standard Levi subgroup M of G one obtains the same isomorphism as in Theorem 5.4, on the subalgebra generated by $\mathcal{O}(T_{\mathfrak{s}})$ and the N_w with $w \in N_M(T)/T$.

Remark 5.5. It is also possible to construct a canonical algebra isomorphism $\mathcal{H}(\mathfrak{s}) \cong \mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$. To that end, we only have to change the condition $\psi_{\mathfrak{s}}(N_w) = N_{w^{-1}}$ in Theorem 5.4 to $\psi_{\mathfrak{s}}(N_w) = N_w$. The two options are related by the canonical isomorphism

$$\begin{array}{ccc} \mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2}) & \xrightarrow{\sim} & \mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})^{op} \\ f N_w & \to & N_{w^{-1}} f \qquad\qquad f \in \mathcal{O}(T_{\mathfrak{s}^\vee}), w \in W_{\mathfrak{s}^\vee}. \end{array}$$

6. Parameters of generic representations

With Theorem 5.4 and (5.3) we can reformulate Theorem 3.4 in terms of $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$-modules. Then it says: π is (U,ξ)-generic if and only if

$$\mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}^\vee}, q_F^{\lambda})}\big(\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi), \mathrm{St}\big) \quad \text{is nonzero.} \tag{6.1}$$

We want to investigate which Langlands parameters should correspond to generic representations in Theorem 4.3. With the reduction theorems from [Lus3, §8–9] we translate the study of (irreducible) representations of $\mathcal{H}(\mathfrak{s})^{op} \cong \mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$ to representations of graded Hecke algebras. Subsequently we take a closer look at the geometric construction of the representations of such algebras. We need to revisit the methods from [Lus3] and [AMS2, AMS3], because the aspects we are interested in were not considered previously and require quite some details.

6.1. Reduction to graded Hecke algebras.

To ease the notation, from now on the elements of $R_{\mathfrak{s}^\vee}$ will be called just $\alpha^\vee$, instead of $m_\alpha \alpha^\vee$ as previously. For a $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$-module V and $t \in T_{\mathfrak{s}^\vee}$ we write

$$V_t = \{ v \in V : \text{ there exists } n \in \mathbb{N} \text{ such that } (\theta_x - x(t))^n v = 0 \text{ for all } x \in X \}.$$

If V_t is nonzero, then we call t a weight of V. For a $W_{\mathfrak{s}^\vee}$-stable subset $U \subset T_{\mathfrak{s}^\vee}$, let $\mathrm{Mod}(\mathcal{H}_{\mathfrak{s}^\vee})_U$ be the category of finite length $\mathcal{H}_{\mathfrak{s}^\vee}$-modules all whose $\mathcal{O}(T_{\mathfrak{s}^\vee})$-weights belong to U. There is a natural equivalence of categories

$$\begin{array}{ccc} \mathrm{Mod}\big(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})\big)_U & \to & \bigoplus_{t \in U / W_{\mathfrak{s}^\vee}} \mathrm{Mod}\big(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})\big)_{W_{\mathfrak{s}^\vee} t} \\ V & \mapsto & \bigoplus_{t \in U / W_{\mathfrak{s}^\vee}} \Big(\sum_{w \in W_{\mathfrak{s}^\vee}} V_{wt} \Big) \end{array}.$$

Let $T_{\mathfrak{s}^\vee,\mathrm{un}} \subset T_{\mathfrak{s}^\vee}$ be the maximal compact real subtorus. It is homeomorphic to the set of unitary characters in $T_{\mathfrak{s}} = X_{\mathrm{nr}}(T)\chi_0$. For $u \in T_{\mathfrak{s}^\vee,\mathrm{un}}$ we put

$$R_{\mathfrak{s}^\vee,u} = \{ \alpha^\vee \in R_{\mathfrak{s}^\vee} : s_\alpha(u) = u \}.$$

This is a root system and its Weyl group is contained in $W_{\mathfrak{s}^\vee,u}$. Recall that we fixed a Borel subgroup $B^\vee \subset G^\vee$, which provides $R_{\mathfrak{s}^\vee,u}$ with a notion of positive roots. Let $\Gamma_{\mathfrak{s}^\vee,u}$ be the stabilizer of $R^+_{\mathfrak{s}^\vee,u} = R^+_{\mathfrak{s}^\vee} \cap R_{\mathfrak{s}^\vee,u}$ in $W_{\mathfrak{s}^\vee,u}$, then

$$W_{\mathfrak{s}^\vee,u} = W(R_{\mathfrak{s}^\vee,u}) \rtimes \Gamma_{\mathfrak{s}^\vee,u}.$$

From these objects we build a new root datum

$$\mathcal{R}_{\mathfrak{s}^\vee,u} = \big(R_{\mathfrak{s}^\vee,u}, X^*(T_{\mathfrak{s}^\vee}), R^\vee_{\mathfrak{s}^\vee,u}, X_*(T_{\mathfrak{s}^\vee})\big),$$

which is endowed with an action of $\Gamma_{\mathfrak{s}^\vee,u}$. That gives rise to an extended affine Hecke algebra

$$\mathcal{H}_{\mathfrak{s}^\vee,u} = \mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee,u}, \lambda, \lambda^*, q_F^{1/2}) \rtimes \Gamma_{\mathfrak{s}^\vee,u}.$$

We denote the standard generators of this algebra (as $\mathcal{O}(T_{\mathfrak{s}^\vee})$-module) by $N_{w,u}$, where $w \in W_{\mathfrak{s}^\vee,u}$.

The positive part of $X_{\mathrm{nr}}(T)$ is $X^+_{\mathrm{nr}}(T) = \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{R}_{>0})$. Via the isomorphism (4.2), $X^+_{\mathrm{nr}}(T)$ can be regarded as a subgroup of $(T^{\vee,\mathbf{I}_F})^\circ_{\mathbf{W}_F}$, and as such it acts on $T_{\mathfrak{s}^\vee}$. In particular that yields a subset $W_{\mathfrak{s}^\vee,u} X^+_{\mathrm{nr}}(T) u$ of $T_{\mathfrak{s}^\vee}$. Notice that every element of $T_{\mathfrak{s}^\vee}$ lies in a subset of the form $X^+_{\mathrm{nr}}(T)u$ with $u \in T_{\mathfrak{s}^\vee,\mathrm{un}}$.

Theorem 6.1. *There exists a canonical equivalence of categories*

$$\begin{array}{cccc} \mathrm{ind}_u : & \mathrm{Mod}(\mathcal{H}_{\mathfrak{s}^\vee,u})_{X^+_{\mathrm{nr}}(T)u} & \to & \mathrm{Mod}\big(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})\big)_{W_{\mathfrak{s}^\vee} X^+_{\mathrm{nr}}(T)u} \\ & V_{X^+_{\mathrm{nr}}(T)u} := \sum_{t \in X^+_{\mathrm{nr}}(T)} V_{tu} & \leftarrow\!\shortmid & V \end{array}$$

such that:

(*a*) ind_u *is given by localization of the centres on both sides, followed by induction.*
(*b*) ind_u *and* ind_u^{-1} *preserve central characters.*
(*c*) *For* $V \in \mathrm{Mod}(\mathcal{H}_{\mathfrak{s}^\vee,u})_{X^+_{\mathrm{nr}}(T)u}$ *there is an isomorphism*

$$\mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}^\vee}, q_F^\lambda)}(\mathrm{ind}_u V, \mathrm{St}) \cong \mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}^\vee,u}, q_F^\lambda)}(V, \mathrm{St}).$$

Proof. The original version of this equivalence is [Lus3, Theorem 8.6], but the setup is slightly different there. The version we need, including the canonicity and the group $\Gamma_{\mathfrak{s}^\vee}$, is shown in [Sol1, Theorem 2.1.2]. Strictly speaking $\Gamma_{\mathfrak{s}_\vee}$ must fix a point of $T_{\mathfrak{s}^\vee}$ in [Sol1]. Fortunately, that does not play a role in the proof, it works in the generality of our setting because we consider u that need not be fixed by $W(R_{\mathfrak{s}^\vee})$. The properties (a) and (b) are checked in [AMS3, Theorem 2.5].

By [AMS3, Theorem 2.5] the effect of the thus obtained functor ind_u on $\mathcal{H}(W_{\mathfrak{s}^\vee}, q_F^\lambda)$-modules is

$$V \;\mapsto\; \mathrm{ind}_{\mathcal{H}(W_{\mathfrak{s}^\vee,u}, q_F^\lambda)}^{\mathcal{H}(W_{\mathfrak{s}^\vee}, q_F^\lambda)} V. \tag{6.2}$$

In this expression $\mathcal{H}(W(R_{\mathfrak{s}^\vee}), q_F^\lambda)$ and $\mathbb{C}[\Gamma_{\mathfrak{s}^\vee} \cap \Gamma_{\mathfrak{s}^\vee,u}]$ are naturally subalgebras of $\mathcal{H}(W_{\mathfrak{s}^\vee}, q_F^\lambda)$, but we have to be careful with the $\tilde{N}_{w,u}$ for which $w \in \Gamma_{\mathfrak{s}^\vee,u}$ but $w \notin \Gamma_{\mathfrak{s}^\vee}$. From [Lus3, §8] and [Sol1, §2.1] one sees that $\tilde{N}_{w,u}$ is sent to

$$\tilde{\mathcal{T}}_w 1_{X^+_{\mathrm{nr}}(T)u} = \mathcal{T}_w 1_{X^+_{\mathrm{nr}}(T)u}$$

in a suitable completion of $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$. Here $\mathcal{T}_w$ is as in Section 2, transferred to completions of $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$ via Theorem 5.4. From (6.2) and Frobenius reciprocity (in a suitably completed algebra) we obtain (c). $\square$

With Theorem 6.1 we can reduce the study of $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$-modules that admit a central character to modules of another affine Hecke algebra, $\mathcal{H}_{\mathfrak{s}^\vee,u}$, such that for the new modules the compact part of the central character is fixed by the new extended Weyl group. In this process all relevant properties of modules are preserved.

Let $\mathfrak{T}_u(T_{\mathfrak{s}^\vee})$ be the tangent space of $T_{\mathfrak{s}^\vee}$ at u. It can be identified with $\mathbb{C} \otimes_{\mathbb{Z}} X_*(T_{\mathfrak{s}^\vee})$, so $R_{\mathfrak{s}^\vee,u}$ can be regarded as a subset of the cotangent space $\mathfrak{T}_u^*(T_{\mathfrak{s}^\vee})$. For $\alpha^\vee \in R_{\mathfrak{s}^\vee}$ we define a parameter

$$k^u_{\alpha^\vee} = (\lambda(h_\alpha^\vee) + \alpha(u)\lambda^*(h_\alpha^\vee)) \log(q_F)/2 \in \mathbb{R}_{\geq 0}.$$

The graded Hecke algebra $\mathbb{H}(W(R_{\mathfrak{s}^\vee,u}), \mathfrak{T}_u(T_{\mathfrak{s}^\vee}), k^u)$ is the vector space $\mathcal{O}(\mathfrak{T}_u(T_{\mathfrak{s}^\vee})) \otimes \mathbb{C}[W(R_{\mathfrak{s}^\vee,u})]$ with multiplication defined by

- $\mathcal{O}(\mathfrak{T}_u(T_{\mathfrak{s}^\vee}))$ and $\mathbb{C}[W(R_{\mathfrak{s}^\vee,u})]$ are embedded as unital subalgebras,
- for $\alpha^\vee \in R_{\mathfrak{s}^\vee,u}$ simple and $f \in \mathcal{O}(\mathfrak{T}_u(T_{\mathfrak{s}^\vee}))$:

$$s_\alpha f - s_\alpha(f) s_\alpha = k^u_{\alpha^\vee}(f - s_\alpha(f))/\alpha^\vee.$$

The group $\Gamma_{\mathfrak{s}^\vee,u}$ acts naturally on this algebra, by

$$\gamma(wf) = (\gamma w \gamma^{-1})\, f \circ \gamma^{-1} \qquad w \in W(R_{\mathfrak{s}^\vee,u}), f \in \mathcal{O}(\mathfrak{T}_u(T_{\mathfrak{s}^\vee})).$$

This action gives rise to the extended graded Hecke algebra

$$\mathbb{H}_{\mathfrak{s},u} = \mathbb{H}(W(R_{\mathfrak{s}^\vee,u}), \mathfrak{T}_u(T_{\mathfrak{s}^\vee}), k^u) \rtimes \Gamma_{\mathfrak{s}^\vee,u}.$$

Its centre is $\mathcal{O}(\mathfrak{T}_u(T_{\mathfrak{s}^\vee}))^{W_{\mathfrak{s}^\vee,u}}$ and weights of $\mathbb{H}_{\mathfrak{s}^\vee,u}$-modules are by default considered with respect to the maximal commutative subalgebra $\mathcal{O}(\mathfrak{T}_u(T_{\mathfrak{s}^\vee}))$. Like for affine Hecke algebras, for a $W_{\mathfrak{s}^\vee,u}$-stable subset $U \subset \mathfrak{T}_u(T_{\mathfrak{s}^\vee})$ we have the category $\mathrm{Mod}(\mathbb{H}_{\mathfrak{s}^\vee,u})_U$ of finite length modules all whose $\mathcal{O}(\mathfrak{T}_u(T_{\mathfrak{s}^\vee}))$-weights belong to U.

Recall the exponential map for $T_{\mathfrak{s}^\vee}$ based at u:

$$\begin{array}{cccc} \exp_u : & \mathfrak{T}_u(T_{\mathfrak{s}^\vee}) & \to & T_{\mathfrak{s}^\vee} \\ & y & \mapsto & u \exp(y) \end{array}.$$

Theorem 6.2. *The map* $\exp_u$ *induces a canonical equivalence of categories*

$$\exp_{u*} : \mathrm{Mod}(\mathbb{H}_{\mathfrak{s}^\vee,u})_{\mathbb{R}\otimes X_*(T_{\mathfrak{s}^\vee})} \to \mathrm{Mod}(\mathcal{H}_{\mathfrak{s}^\vee,u})_{X^+_{\mathrm{nr}}(T)u},$$

such that:

(*a*) $\exp_{u*}$ *comes from an isomorphism (induced by* $\exp_u$*) between localized versions of* $\mathbb{H}_{\mathfrak{s}^\vee,u}$ *and of* $\mathcal{H}_{\mathfrak{s}^\vee,u}$.
(*b*) $\exp_{u*}$ *does not change the vector spaces underlying the modules.*
(*c*) *The effect of* $\exp_{u*}$ *on* $\mathcal{O}(\mathfrak{T}_u(T_{\mathfrak{s}^\vee}))$*-weights is* $\exp_u$.
(*d*) *For any* $V \in \mathrm{Mod}(\mathbb{H}_{\mathfrak{s}^\vee,u})_{\mathbb{R}\otimes X_*(T_{\mathfrak{s}^\vee})}$ *there is an isomorphism*

$$\mathrm{Hom}_{W_{\mathfrak{s}^\vee,u}}(V, \det) \cong \mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}^\vee,u}, q_F^\lambda)}(\exp_{u*} V, \mathrm{St}).$$

Proof. The original version of this equivalence is [Lus3, Theorem 9.3]. We use the version from [Sol1, Theorem 2.1.4 and Corollary 2.1.5]. This includes the canonicity and the properties (a),(b) and (c).

One way to see (d) is via deformations of the parameters. We can scale the parameters k^u linearly to 0. That gives a family of extended graded Hecke algebras

$$\mathbb{H}_{\mathfrak{s}^\vee,u,\epsilon} = \mathbb{H}(W(R_{\mathfrak{s}^\vee,u}), \mathfrak{T}_u(T_{\mathfrak{s}^\vee}), \epsilon k^u) \rtimes \Gamma_{\mathfrak{s}^\vee,u} \qquad \epsilon \in \mathbb{R}_{\geq 0}.$$

A module V can be "scaled" to modules V_ϵ, via the scaling homomorphisms $\mathbb{H}_{\mathfrak{s}^\vee,u,\epsilon} \to \mathbb{H}_{\mathfrak{s}^\vee}$ for $\epsilon \geq 0$ [Sol1, (1.11)]. For $\epsilon = 0$ we obtain a module V_0 of

$$\mathbb{H}_{\mathfrak{s}^\vee,u,0} = \mathcal{O}(\mathfrak{T}_u(T_{\mathfrak{s}^\vee})) \rtimes W_{\mathfrak{s}^\vee,u},$$

which equals V as $\mathbb{C}[W_{\mathfrak{s}^\vee,u}]$-module and on which $\mathcal{O}(\mathfrak{T}_u(T_{\mathfrak{s}^\vee}))$ acts by evaluation at $0 \in \mathfrak{T}_u(T_{\mathfrak{s}^\vee})$.

For the affine Hecke algebra $\mathcal{H}_{\mathfrak{s}^\vee,u}$, the parameters can be scaled via $q_F \mapsto q_F^\epsilon$ with $\epsilon \in [0,1]$. That yields a family of algebras

$$\mathcal{H}_{\mathfrak{s}^\vee,u,\epsilon} = \mathcal{H}(\mathcal{R}_u, \lambda, \lambda^*, q_F^\epsilon) \rtimes \Gamma_{\mathfrak{s}^\vee,u} \qquad \epsilon \in \mathbb{R}_{\geq 0}.$$

The module $\exp_{u*} V$ can be "scaled" accordingly via a functor

$$\tilde{\sigma}_\epsilon : \mathrm{Mod}(\mathcal{H}_{\mathfrak{s}^\vee,u})_{\chi_+ u} \to \mathrm{Mod}(\mathcal{H}_{\mathfrak{s}^\vee,u,\epsilon})_{\chi_+^\epsilon u} \qquad \epsilon \in [0,1],$$

see [Sol1, Corollary 4.2.2]. In this process $\mathcal{H}(W_{\mathfrak{s}^\vee,u}, q_F^\lambda)$ is replaced by the isomorphic semisimple algebra $\mathcal{H}(W_{\mathfrak{s}^\vee,u}, q_F^{\epsilon\lambda})$. The multiplicities

$$\dim \mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}^\vee,u}, q_F^{\epsilon\lambda})}\big(\tilde{\sigma}_\epsilon(\exp_{u*} V), \mathrm{St}\big)$$

depend continuously on $\epsilon \in [0,1]$ and they are integers, so in fact they are constant as functions of ϵ. It is known from [Sol1, (4.6)–(4.7)] that

$$\exp_{u*}(V_\epsilon) = \tilde{\sigma}_\epsilon(\exp_{u*} V) \qquad \text{for all } \epsilon \in [0,1].$$

We conclude that

$$\begin{aligned} \mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}^\vee,u}, q_F^\lambda)}(\exp_{u*} V), \mathrm{St}) &\cong \mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}^\vee,u}, q_F^0)}\big(\tilde{\sigma}_0(\exp_{u*} V), \mathrm{St}\big) \\ &\cong \mathrm{Hom}_{W_{\mathfrak{s}^\vee,u}}(V_0, \det) = \mathrm{Hom}_{W_{\mathfrak{s}^\vee,u}}(V, \det). \quad \Box \end{aligned}$$

In view of Theorems 3.4, 6.1 and 6.2, the role of genericity for $\mathbb{H}_{\mathfrak{s}^\vee,u}$ is played by modules that the contain the character det of $\mathbb{C}[W_{\mathfrak{s}^\vee,u}]$. To analyse those, we bring the algebra in an easier form.

Let $R_{u>0}$ be the subset of $R_{\mathfrak{s}^\vee,u}$ consisting of the roots $\alpha^\vee$ for which $k^u_{\alpha^\vee} > 0$. Let $\Gamma_{u>0}$ be the stabilizer of $R^+_{u>0} = R_{u>0} \cap R^+_{\mathfrak{s}^\vee}$ in $W_{\mathfrak{s}^\vee,u}$.

Lemma 6.3. *The set $R_{u>0}$ is a root system and*

$$\mathbb{H}_{\mathfrak{s}^\vee,u} = \mathbb{H}(W(R_{u>0}), \mathfrak{T}_u(T_{\mathfrak{s}^\vee}), k^u) \rtimes \Gamma_{u>0}.$$

Proof. The set $R_{u>0}$ is $W_{\mathfrak{s}^\vee,u}$-stable by the invariance properties of the labels. In particular it is stable under the reflections with respect to its roots, so it is a root system. In every irreducible component of $R_{\mathfrak{s}^\vee,u}$, $R_{u>0}$ is either everything or empty or the roots of one given length. By the simple transitivity of the action of $W(R_{u>0})$ on the collection of positive systems in $R_{u>0}$:

$$W_{\mathfrak{s}^\vee,u} = W(R_{u>0}) \rtimes \Gamma_{u>0}.$$

We note that

$$R_{\mathfrak{s}^\vee,u} \setminus R_{u>0} = \{\alpha^\vee \in R_{\mathfrak{s}^\vee,u} : \lambda(h_\alpha^\vee) = \lambda^*(h_\alpha^\vee), \alpha^\vee(u) = -1\}. \tag{6.3}$$

By reduction to irreducible root systems, and the classification thereof, one checks that $W(R_{\mathfrak{s}^\vee,u})$ is the semidirect product of $W(R_{u>0})$ and the subgroup generated by the reflections with respect to the simple roots in $R_{\mathfrak{s}^\vee,u} \setminus R_{u>0}$. For such reflections the multiplication relations in $\mathbb{H}_{\mathfrak{s}^\vee,u}$ simplify to $s_\alpha f = s_\alpha(f) s_\alpha$. That implies

$$\mathbb{H}(W(R_{\mathfrak{s}^\vee,u}), \mathfrak{T}_u(T_{\mathfrak{s}^\vee}), k^u) = \mathbb{H}(W(R_{u>0}), \mathfrak{T}_u(T_{\mathfrak{s}^\vee}), k^u) \rtimes (\Gamma_{u>0} \cap W(R_{\mathfrak{s}^\vee,u})),$$

which in turn implies the lemma. $\square$

The advantage of Lemma 6.3 is that via the new presentation the algebra becomes isomorphic to a graded Hecke algebra with equal parameters.

Lemma 6.4. $\mathbb{H}_{\mathfrak{s}^\vee,u}$ *is isomorphic to a graded Hecke algebra (extended by* $\Gamma_{u>0}$*) such that every root from* $R_{u>0}$ *has the parameter* $\log(q_F)$.

Proof. By [AMS3, Proposition 3.14], $\mathbb{H}_{\mathfrak{s}^\vee,u}$ is isomorphic to the graded Hecke algebra associated to a certain complex reductive group $\tilde{G}$ and a cuspidal L-parameter with values in a quasi-Levi subgroup $\tilde{M}$ of $\tilde{G}$. In our specific setting $\tilde{M}^\circ$ is a torus, because we only work with principal series L-parameters. In particular the cuspidal L-parameter is trivial on $SL_2(\mathbb{C})$. Thus $\mathbb{H}_{\mathfrak{s}^\vee,u}$ is a graded Hecke algebra associated to $\tilde{G}$ and a cuspidal support whose unipotent (or nilpotent) element is trivial. By construction [AMS3, §1] all the nonzero parameters are of the form $k^u_{\alpha^\vee} = c(\alpha^\vee) r_i$, where $r_i \in \mathbb{C}$ depends only on the connected component of the root system that contains $\alpha^\vee$. Further $c(\alpha^\vee) = 2$ by [Lus2, §2] and our earlier specialization of $\mathbf{z}$ to $q_F^{1/2}$ entails $k_i = \log(q_F^{1/2}) = \log(q_F)/2$. Combine that with Lemma 6.3. $\square$

6.2. Geometric representations of graded Hecke algebras.

Recall that u corresponds to a unitary character of T, so it is a bounded L-parameter for T. By [AMS3, Proposition 3.14], the algebra $\mathbb{H}_{\mathfrak{s}^\vee,u}$ is of the form $\mathbb{H}(u, 0, \mathrm{triv}, \log(q_F)/2)$, where triv means the trivial local system on the trivial nilpotent orbit 0. The meaning of this statement is explained somewhat further in [AMS3, (3.9) and below]. It can be formulated as

$$\mathbb{H}(u, 0, \mathrm{triv}, \log(q_F)/2) \cong \mathbb{H}(G_u^\vee, M^\vee, \mathrm{triv}, \log(q_F)/2).$$

In [AMS3] the group $G_u^\vee$ is defined as $Z^1_{G^\vee_{sc}}(u) \times X_{\mathrm{nr}}(G)$, but in our current setting we have just $G_u^\vee = Z_{G^\vee}(u)$. The reason is that at the start of Section 4 we refrained from involving the simply connected cover of $G^\vee_{\mathrm{der}}$, that would be superfluous for quasi-split groups. Similarly the group $M^\vee$, which is a quasi-Levi subgroup of $Z^1_{G^\vee_{sc}}(u) \times X_{\mathrm{nr}}(G)$ in [AMS3], becomes simply $T^\vee$ in our setup.

Notice that $G_u^\vee$ need not be connected. In fact the isomorphism

$$\mathbb{H}(G_u^\vee, T^\vee, \mathrm{triv}, \log(q_F)/2) \cong \mathbb{H}_{\mathfrak{s}^\vee,u} = \mathbb{H}(W(R_{u>0}), \mathfrak{T}_u(T_{\mathfrak{s}^\vee}), k^u) \rtimes \Gamma_{u>0} \tag{6.4}$$

and Lemma 6.3 imply that $\pi_0(G_u^\vee) \cong \Gamma_{u>0}$. When we replace $G_u^\vee$ by its identity component, we obtain the subalgebra

$$\mathbb{H}^\circ_{\mathfrak{s}^\vee,u} := \mathbb{H}(G_u^{\vee,\circ}, T^\vee, \mathrm{triv}, \log(q_F)/2) \cong \mathbb{H}(W(R_{u>0}), \mathfrak{T}_u(T_{\mathfrak{s}^\vee}), k^u).$$

The irreducible representations of such graded Hecke algebras were parametrized and constructed geometrically in [Lus2, Lus4]. The parameters are triples (σ, y, ρ°) where:

(i) $\sigma \in \mathrm{Lie}(G_u^{\vee,\circ})$ is semisimple,
(ii) $y \in \mathrm{Lie}(G_u^{\vee,\circ})$ is nilpotent,
(iii) $[\sigma, y] = \log(q_F) y$,
(iv) ρ° is an irreducible representation of $\pi_0\big(Z_{G_u^{\vee,\circ}}/Z(G_u^{\vee,\circ})\big)$ satisfying the analogue of (ii) on page 24.

By [Lus4] $G_u^{\vee,\circ}$-conjugacy classes of such triples are naturally in bijection with $\mathrm{Irr}\big(\mathbb{H}(G_u^{\vee,\circ}, T^\vee, \mathrm{triv}, \log(q_F)/2)\big)$. Let us write that as

$$(\sigma, y, \rho^\circ) \mapsto M^\circ_{y,\sigma,\rho^\circ}.$$

In [Lus2, Lus4] there is an extra parameter $r \in \mathbb{C}$, but we suppress that because in this paper it will always be equal to $\log(q_F)/2$. From these parameters σ can always be chosen in $\mathrm{Lie}(T^\vee)$, and then $W(R_{u>0})\sigma$ is the central character of $M^\circ_{y,\sigma,\rho^\circ}$.

Lusztig's parametrization was slightly modified in [AMS2, §3.5], essentially by composing it with the Iwahori–Matsumoto involution IM of $\mathbb{H}^\circ_{\mathfrak{s}^\vee,u}$. To make that consistent, the above condition (iii) must be replaced by

(iii') $[\sigma, y] = -\log(q_F) y$.

We denote the resulting parametrization of $\mathrm{Irr}\big(\mathbb{H}(G_u^{\vee,\circ}, T^\vee, \mathrm{triv}, \log(q_F)/2)\big)$, which is the one used in [AMS3], by

$$(\sigma, y, \rho^\circ) \mapsto \bar{M}^\circ_{y,\sigma,\rho^\circ} := \mathrm{IM}^* M^\circ_{y,-\sigma,\rho^\circ}. \tag{6.5}$$

Proposition 6.5. *The irreducible $\mathbb{H}(G_u^{\vee,\circ}, T^\vee, \mathrm{triv}, \log(q_F)/2)$-representation $\bar{M}^\circ_{y,\sigma,\rho^\circ}$ contains the sign representation of $\mathbb{C}[W(R_{u>0})]$ if and only if ρ° is the trivial representation and the $Z_{G_u^{\vee,\circ}}(\sigma)$-orbit of y is dense in*

$$\{Y \in \mathrm{Lie}(G_u^{\vee,\circ}) : [\sigma, Y] = -\log(q_F)Y\}.$$

Proof. We may replace $G_u^{\vee,\circ}$ by any finite covering group, that does not change the associated graded Hecke algebra. In particular we may assume that the derived group of $G_u^{\vee,\circ}$ is simply connected.

Via [AMS3, Theorems 2.5 and 2.11], analogous to Theorems 6.1 and 6.2, $\bar{M}^\circ_{y,\sigma,\rho^\circ}$ becomes an irreducible representation of the affine Hecke algebra associated to $(G_u^{\vee,\circ}, T^\vee, \mathrm{triv})$, with parameter q_F. By [AMS3, Proposition 2.18], $\bar{M}^\circ_{y,\sigma,\rho^\circ}$ is turned into the module $\tilde{M}_{\exp(\sigma),\exp(y),\rho^\circ}$ associated by Kazhdan–Lusztig [KaLu] to $(\exp(\sigma), \exp(y), \rho^\circ)$ and q_F. The paper [KaLu] assumed that the derived group of the involved complex reductive group is simply connected, but that condition was lifted in [Ree2]. Furthermore, it was shown in [Ree2, §7.2–7.3] that $\tilde{M}_{\exp(\sigma),\exp(y),\rho^\circ}$ contains the Steinberg representation of $\mathcal{H}(W(R_{u>0}), q_F)$ if and only if ρ° is trivial and the $Z^\circ_{G_u^{\vee,\circ}}$-orbit of y is dense in

$$\{Y \in \mathrm{Lie}(G_u^{\vee,\circ}) : \mathrm{Ad}(\exp\sigma)Y = q_F^{-1}Y\}.$$

Now we go back to $\mathbb{H}(G_u^{\vee,\circ}, T^\vee, \mathrm{triv}, \log(q_F)/2)$-modules, and we conclude with a version of Theorem 6.2.d. $\square$

The parametrization of $\mathrm{Irr}(\mathbb{H}^\circ_{\mathfrak{s}^\vee,u})$ from (6.5) has been generalized to $\mathbb{H}_{\mathfrak{s}^\vee,u}$ in [AMS2, Theorem 3.20] and [AMS3, Theorem 3.8]. The parameters are $G_u^\vee$-conjugacy classes of triples (σ, y, ρ) as above, with the only difference that ρ is now an irreducible representation of $\pi_0\big(Z_{G_u^\vee}(\sigma, y)/Z(G_u^{\vee,\circ})\big)$.

The two constructions are related as follows. To (σ, y) one associates [Lus2] a $\mathbb{H}^\circ_{\mathfrak{s}^\vee,u} \times \pi_0(Z_{G_u^{\vee,\circ}})$-representation $E^\circ_{y,-\sigma}$. Then

$$E^\circ_{y,-\sigma,\rho^\circ} = \mathrm{Hom}_{\pi_0(Z_{G_u^{\vee,\circ}}(\sigma,y))}(\rho^\circ, E^\circ_{y,-\sigma})$$

and $M^\circ_{y,-\sigma,\rho^\circ}$ is the unique irreducible quotient of that module.

Similarly a $\mathbb{H}_{\mathfrak{s}^\vee,u} \times \pi_0(Z_{G_u^\vee})$-representation $E_{y,-\sigma}$ can be constructed [AMS2], and by [AMS2, Lemma 3.3] there is a canonical isomorphism

$$E_{y,-\sigma} \cong \mathrm{ind}_{\mathbb{H}^\circ_{\mathfrak{s}^\vee,u}}^{\mathbb{H}_{\mathfrak{s}^\vee,u}} E^\circ_{y,\sigma}. \tag{6.6}$$

One defines

$$E_{y,-\sigma,\rho} = \mathrm{Hom}_{\pi_0(Z_{G_u^\vee}(\sigma,y))}(\rho, E_{y,-\sigma}), \tag{6.7}$$

and then $M_{y,-\sigma,\rho}$ is the unique irreducible quotient of $E_{y,-\sigma,\rho}$.

Lemma 6.6. *Every semisimple $\sigma \in G_u^{\vee,\circ}$ can be extended to a triple as used in (6.7), such that $M_{y,-\sigma,\rho}$ contains the trivial representation of $\mathbb{C}[W(R_{u>0}) \rtimes \Gamma_{u>0}]$. Moreover (y,ρ) is unique up to $Z_{G_u^\vee}(\sigma)$-conjugacy, y lies in the dense $Z_{G_u^{\vee,\circ}}(\sigma)$-orbit in*

$$\{Y \in \mathrm{Lie}(G_u^{\vee,\circ}) : [\sigma, Y] = -\log(q_F)Y\}$$

and the restriction of ρ to $\pi_0(Z_{G_u^{\vee,\circ}}(\sigma,y))$ is a multiple of the trivial representation.

Proof. Let $M_{y,-\sigma}$ be the maximal semisimple quotient $\mathbb{H}_{\mathfrak{s}^\vee,u}$-module of $E_{y,-\sigma}$. Then

$$M_{y,-\sigma,\rho} = \mathrm{Hom}_{\pi_0(Z_{G_u^\vee}(y,\sigma))}(\rho, M_{y,-\sigma}), \tag{6.8}$$

for any eligible ρ. The same can be done for the analogous $\mathbb{H}^\circ_{\mathfrak{s}^\vee,u}$-modules. It follows from (6.6), (6.7) and (6.8) that

$$M_{y,-\sigma} \cong \mathrm{ind}_{\mathbb{H}^\circ_{\mathfrak{s}^\vee,u}}^{\mathbb{H}_{\mathfrak{s}^\vee,u}} M^\circ_{y,-\sigma}. \tag{6.9}$$

Recall that $\mathbb{H}_{\mathfrak{s}^\vee,u} = \mathbb{H}^\circ_{\mathfrak{s}^\vee,u} \rtimes \Gamma_{u>0}$. By Frobenius reciprocity and (6.9) the multiplicity of $\mathrm{triv}_{W(R_{u>0}) \rtimes \Gamma_{u>0}}$ in $M_{y,-\sigma}$ equals the multiplicity of $\mathrm{triv}_{W(R_{u>0})}$ in $M^\circ_{y,-\sigma}$.

For any given σ, Proposition 6.5 and (6.8) for $\mathbb{H}^\circ_{\mathfrak{s}^\vee,u}$ entail that $\mathrm{triv}_{W(R_{u>0})}$ appears with multiplicity one in $M^\circ_{y,-\sigma}$ if y satisfies the density condition, and otherwise that multiplicity is zero. Hence $M_{y,-\sigma}$ contains $\mathrm{triv}_{W(R_{u>0}) \rtimes \Gamma_{u>0}}$ if and only if y satisfies the condition from the statement, and then the multiplicity is one.

For such (σ, y), multiplicity one ensures that there exists a unique ρ such that $M_{y,\sigma,\rho}$ contains $\mathrm{triv}_{W(R_{u>0}) \rtimes \Gamma_{u>0}}$. Let ρ° be an irreducible subrepresentation of ρ restricted to the normal subgroup $\pi_0(Z_{G_u^{\vee,\circ}}(\sigma,y))$. By Clifford theory the restriction of ρ to $\pi_0(Z_{G_u^{\vee,\circ}}(\sigma,y))$ is a multiple of $\bigoplus_g g \cdot \rho^\circ$, where g runs over $\pi_0(Z_{G_u^\vee}(\sigma,y))$ modulo the stabilizer of ρ°.

Suppose that ρ° is nontrivial. Then $g \cdot \rho^\circ$ is nontrivial for any $g \in \pi_0(Z_{G_u^\vee}(\sigma,y))$, and $M_{y,-\sigma,\rho}$ cannot contain any $\mathbb{H}^\circ_{\mathfrak{s}^\vee,u}$-submodule of the form $M^\circ_{y',\sigma',\mathrm{triv}}$. In this case $M_{y,-\sigma,\rho}$ does not contain $\mathrm{triv}_{W(R_{u>0})}$. □

To see that the parametrization of $\mathrm{Irr}(\mathbb{H}^{\mathfrak{s}^\vee,u})$ from [AMS3, Theorem 3.8] has a property like Proposition 6.5, it remains to analyse the ρ determined by Lemma 6.6. To that end we have to delve more deeply into the underlying constructions.

By the naturality of the parametrization (6.5), the $\Gamma_{u>0}$-stabilizer of $\bar{M}^\circ_{y,\sigma,\rho^\circ}$ (or equivalently of $M^\circ_{y,-\sigma,\rho^\circ}$) equals the $\Gamma_{u>0}$-stabilizer of the $G_u^{\vee,\circ}$-orbit of (σ, y, ρ°). When $\rho^\circ = \mathrm{triv}$, that group depends only on (σ, y). When furthermore y satisfies the density condition from Proposition 6.5, the $\Gamma_{u>0}$-stabilizer of $\bar{M}^\circ_{y,\sigma,\mathrm{triv}}$ equals the $\Gamma_{u>0}$-stabilizer of the $G_u^{\vee,\circ}$-orbit of σ, which we denote by $\Gamma_{[\sigma]}$.

Lemma 6.7. *Let $(\sigma, y, \mathrm{triv})$ be as in Proposition 6.5. The action of $\mathbb{H}^\circ_{\mathfrak{s}^\vee,u}$ on $M^\circ_{y,-\sigma,\mathrm{triv}}$ extends in a unique way to an action of $\mathbb{H}^\circ_{\mathfrak{s}^\vee,u} \rtimes \Gamma_{[\sigma]}$ that contains the trivial representation of $\mathbb{C}[W(R_{u>0}) \rtimes \Gamma_{[\sigma]}]$.*

Proof. By Proposition 6.5 $M^\circ_{y,-\sigma,\rho^\circ}$ contains the trivial representation of $\mathbb{C}[W(R_{u>0})]$, and by Lemma 3.5 it does so with multiplicity one. Any $\gamma \in \Gamma_{[\sigma]}$ stabilizes $M^\circ_{y,-\sigma,\mathrm{triv}}$, so there exists a linear bijection I_γ such that

$$I_\gamma \circ h = \gamma(h) \circ I_\gamma : M^\circ_{y,-\sigma,\rho^\circ} \to M^\circ_{y,-\sigma,\rho^\circ} \quad \text{for all } h \in \mathbb{H}^\circ_{\mathfrak{s}^\vee,u}.$$

Schur's lemma says that I_γ is unique up to scalars. Since $\mathrm{triv}_{W(R_{u>0})}$ is $\Gamma_{[\sigma]}$-stable and appears with multiplicity on in $M^\circ_{y,-\sigma,\rho^\circ}$, I_γ stabilizes the one-dimensional subspace which affords $\mathrm{triv}_{W(R_{u>0})}$. We normalize I_γ by requiring that it fixes $\mathrm{triv}_{W(R_{u>0})} \subset M^\circ_{y,-\sigma,\rho^\circ}$ pointwise, that is the only possibility if we want to end up with the trivial representation of $W(R_{u>0}) \rtimes \Gamma_{[\sigma]}$.

For any $\gamma, \gamma' \in \Gamma_{[\sigma]}$, $I_\gamma \circ I_{\gamma'}$ satisfies the same condition as $I_{\gamma\gamma'}$, so equals $I_{\gamma\gamma'}$. These I_γ provide the desired extension. $\square$

The module $M^\circ_{y,\sigma,\rho^\circ}$ comes as the unique irreducible quotient of a standard module $E^\circ_{y,\sigma,\rho^\circ}$ [Lus5, Theorem 1.15.a]. The latter is a subspace of the homology of the variety $\mathcal{B}^y$ of Borel subgroups of $G_u^{\vee,\circ}$ that contain $\exp(y)$, with coefficients in a certain local system $\dot{\mathcal{L}}$. In our setting $\dot{\mathcal{L}}$ is trivial because it comes from the trivial local system on $\{0\}$. In terms of the $\Gamma_{u>0}$-stable Borel subgroup $B^\vee \cap G_u^{\vee,\circ}$ we have

$$E^\circ_{y,\sigma,\rho^\circ} \subset H_*(\mathcal{B}^y) = H_*\big(\{g \in G_u^{\vee,\circ}/B^\vee \cap G_u^{\vee,\circ} : \mathrm{Ad}(g^{-1})y \in \mathrm{Lie}(B^\vee \cap G_u^{\vee,\circ})\}\big).$$

From that and (6.7) we see that

$$E^\circ_{y,-\sigma,\mathrm{triv}} = H_*(\mathcal{B}^y)^{Z_{G_u^{\vee,\circ}}(y,\sigma)}. \tag{6.10}$$

Lemma 6.8. *Let $(\sigma, y, \mathrm{triv})$ be as in Proposition 6.5. Then*

$$M^\circ_{y,-\sigma,\mathrm{triv}} = E^\circ_{y,-\sigma,\mathrm{triv}} = H_*(\mathcal{B}^y)^{Z_{G_u^{\vee,\circ}}(y,\sigma)}$$

as vector spaces. The subspace $H_0(\mathcal{B}^y)^{Z_{G_u^{\vee,\circ}}(y,\sigma)}$ has dimension one and $\mathbb{C}[W(R_{u>0})]$ acts on it as the trivial representation.

Proof. By [Lus4, §10.4–10.8], every irreducible subquotient of $E^\circ_{y,-\sigma,\mathrm{triv}}$ different from $M^\circ_{y,-\sigma,\mathrm{triv}}$ is of the form $M^\circ_{y',-\sigma,\rho^\circ}$ with

$$\mathrm{Ad}(Z_{G_u^{\vee,\circ}})y \;\subset\; \overline{\mathrm{Ad}(Z_{G_u^{\vee,\circ}})y'}.$$

By the density condition on y, such a y' does not exist. Therefore $E^\circ_{y,-\sigma,\mathrm{triv}}$ is irreducible and equal to $M^\circ_{y,-\sigma,\mathrm{triv}}$.

Again by [Lus4, §10.4–10.8], $M^\circ_{y,-\sigma,\mathrm{triv}}$ is a subquotient of $E^\circ_{0,-\sigma,\mathrm{triv}}$. As

$$\overline{\mathrm{Ad}(Z_{G_u^{\vee,\circ}})y} = \{Y \in \mathrm{Lie}(G_u^{\vee,\circ}) : [\sigma, Y] = -\log(q_F)Y\} \tag{6.11}$$

is a vector space, the intersection cohomology complex from the constant sheaf on $\mathrm{Ad}(Z_{G_u^{\vee,\circ}})y$ is simply the constant sheaf on (6.11). In view of [Lus4, §10], restricting that sheaf to $\{0\}$ provides a natural nonzero $\mathbb{H}^\circ_{\mathfrak{s}^\vee,u}$-module homomorphism

$$E^\circ_{y,-\sigma,\mathrm{triv}} \to E^\circ_{0,-\sigma,\mathrm{triv}}.$$

This realizes $M^\circ_{y,-\sigma,\mathrm{triv}}$ as subrepresentation of $E^\circ_{0,-\sigma,\mathrm{triv}}$.

Consider the algebra $A = \mathcal{O}(\mathrm{Lie}(G_u^{\vee,\circ})/\mathrm{Ad}(G) \times \mathbb{C})$ of conjugation invariant functions on the Lie algebra of $G_u^{\vee,\circ} \times \mathbb{C}^\times$. We recall from [Lus2] that

$$E^\circ_{0,-\sigma,\rho^\circ} = \mathrm{Hom}\big(\rho^\circ, E^\circ_{y,\sigma}\big) = \mathrm{Hom}\big(\rho^\circ, \mathbb{C}_{-\sigma,\log(q_F)/2} \otimes_A H_*^A(\mathcal{B}^0)\big).$$

If we replace $\log(q_F)/2$ by an arbitrary $r \in \mathbb{C}$, we still obtain a module for a graded Hecke algebra, namely $\mathbb{H}(G_u^{\vee,\circ}, T^\vee, \mathrm{triv}, r)$. It is known from [Lus2, Proposition 7.2] that $H_*^A(\mathcal{B}^0)$ is a free A-module. That implies that the modules $\mathbb{C}_{-\sigma,r} \otimes_A H_*^A(\mathcal{B}^0)$ form an algebraic family parametrized by $r \in \mathbb{C}$ and a semisimple $\sigma \in \mathrm{Lie}(G_u^{\vee,\circ})$. In particular, as modules for the finite dimensional semisimple subalgebra $\mathbb{C}[W(R_{u>0})]$ they do not depend on (σ, r).

For $r = 0, \sigma = 0$ the group $Z_{G_u^{\vee,\circ}}(\sigma, 0) = G_u^{\vee,\circ}$ is connected, and we obtain $E^\circ_{0,0} = H_*(\mathcal{B}^0)$. This is a $\mathbb{C}[W(R_{u>0})]$-representation with which the classical Springer correspondence can be constructed. Here we must use the version of the Springer correspondence from [Lus1], which by [Lus1, Theorem 9.2] means that the trivial $W(R_{u>0})$-representation appears as

$$H_0(\mathrm{pt}) = H_0(\mathcal{B}^x) \cong H_0(\mathcal{B}^0)$$

for a regular unipotent element $x \in G_u^{\vee,\circ}$. As $\dim H_0(\mathcal{B}^0) = 1$, the parts of

$$M^\circ_{y,-\sigma,\mathrm{triv}} \ \subset \ E^\circ_{0,-\sigma,\mathrm{triv}} \ \subset \ \mathbb{C}_{-\sigma,\log(q_F)/2} \otimes_A H_*^A(\mathcal{B}^0)$$

in homological degree zero also have dimension one and carry the trivial representation of $W(R_{u>0})$. $\square$

We are ready to prove the desired generalization of Proposition 6.5.

Theorem 6.9. *There exists a canonical bijection between* $\mathrm{Irr}(\mathbb{H}_{\mathfrak{s}^\vee,u})$ *and the set of* $G_u^\vee$*-conjugacy classes of triples* (σ, y, ρ)*, where*

- $\sigma, y \in \mathrm{Lie}(G_u^\vee)$ *with* σ *semisimple,* y *nilpotent and* $[\sigma, y] = -\log(q_F)y$*,*
- ρ *is an irreducible representation of* $\pi_0\big(Z_{G_u^\vee}(\sigma, y)/Z(G_u^{\vee,\circ})\big)$*, such that any irreducible* $\pi_0\big(Z_{G_u^{\vee,\circ}}(\sigma, y)/Z(G_u^{\vee,\circ})\big)$*-subrepresentation appears in the homology of the variety of Borel subgroups of* $G_u^{\vee,\circ}$ *that contain* $\exp(\sigma)$ *and* $\exp(y)$*.*

The module $\bar{M}_{y,\sigma,\rho}$ *associated to* (σ, y, ρ) *contains the character* $\det$ *of* $\mathbb{C}[W_{\mathfrak{s}^\vee,u}]$ *if and only if* ρ *is trivial and the* $Ad(Z_{G_u^{\vee,\circ}}(\sigma))$*-orbit of* y *is dense in*

$$\{Y \in \mathrm{Lie}(G_u^{\vee,\circ}) : [\sigma, Y] = -\log(q_F)Y\}.$$

Proof. We start with the parametrization of $\mathrm{Irr}\big(\mathbb{H}(G_u^\vee, T^\vee, \mathrm{triv}, \log(q_F)/2)\big)$ provided by [AMS2, §3.5] and [AMS3, Theorem 3.8]. This has almost all the required properties, only the action of $\mathbb{C}[\Gamma_{u>0}]$ on the thus constructed modules can still be normalized in several ways.

Fix a nilpotent $y \in \mathrm{Lie}(G_u^{\vee,\circ})$ and consider the variety

$$\mathcal{P}_y := \big\{g \in G_u^\vee/B^\vee \cap G_u^{\vee,\circ} : \mathrm{Ad}(g^{-1})y \in \mathrm{Lie}(B^\vee \cap G_u^{\vee,\circ})\big\}.$$

The $\mathbb{H}_{\mathfrak{s}^\vee,u} \times \pi_0(Z_{G_u^\vee}(\sigma, y))$-representation $E_{y,-\sigma}$ equals $H_*(\mathcal{P}^y)$ as vector space. The action of $\pi_0(Z_{G_u^\vee}(\sigma, y))$ on $H_*(\mathcal{P}^y)$ is induced by the natural left action of $Z_{G_u^\vee}(\sigma, y)$ on $\mathcal{P}_y$. An element $\gamma \in \Gamma_{u>0}$ acts on $\mathcal{P}^y$ by

$$r_\gamma^{-1} : g(B^\vee \cap G_u^{\vee,\circ}) \mapsto g\gamma^{-1}B^\vee \cap G_u^{\vee,\circ}, \tag{6.12}$$

which in fact makes $\mathcal{P}_y$ isomorphic to $\mathcal{B}^y \times \Gamma_{u>0}$. We normalize the action of $\mathbb{C}[\Gamma_{u>0}]$ on $E_{y,-\sigma}$, by defining it as $H_*(r_\gamma^{-1})$. (This normalization was not possible in [AMS2], because there the homology of $\mathcal{P}_y$ had coefficients in a local system that could be nontrivial.)

From now on we assume that y satisfies the density condition from the statement. In view of Lemma 6.6, it remains to analyse the $\pi_0(Z_{G_u^\vee}(\sigma, y))$-invariants in $E_{y,-\sigma}$. We recall from [AMS2, Lemma 3.12] that

$$\Gamma_{[\sigma]} \cong \pi_0(Z_{G_u^\vee}(\sigma, y))/\pi_0(Z_{G_u^{\vee,\circ}}(\sigma, y)).$$

Let $\gamma_{\sigma,y} \in Z_{G_u^\vee}(\sigma, y)$ be a representative of $\gamma \in \Gamma_{[\sigma]}$. Then $H_*(\mathcal{P}_y)^{\pi_0(Z_{G_u^\vee}(\sigma,y))}$ consists of the invariants for $\{\gamma_{\sigma,y} : \gamma \in \Gamma_{[\sigma]}\}$ in

$$H_*(\mathcal{P}_y)^{\pi_0(Z_{G_u^{\vee,\circ}}(\sigma,y))} = \bigoplus\nolimits_{w \in \Gamma_{u>0}} H_*(w \cdot \mathcal{B}^y)^{\pi_0(Z_{G_u^{\vee,\circ}}(\sigma,y))}. \tag{6.13}$$

Fix $\gamma \in \Gamma_{[\sigma]}$ and consider the map

$$\begin{array}{cccc} f_\gamma : & \mathcal{B}^y & \to & \mathcal{B}^y \\ & g(B^\vee \cap G_u^{\vee,\circ}) & \mapsto & \gamma_{\sigma,y} g \gamma^{-1}(B^\vee \cap G_u^{\vee,\circ}) \end{array}. \tag{6.14}$$

It can be decomposed as

$$f_\gamma = l_{\gamma_{y,\sigma}} \circ \rho_\gamma^{-1} = r_\gamma^{-1} \circ l_{\gamma_{y,\sigma}}$$

The induced map on $H_*(\mathcal{B}^y)^{\pi_0(Z_{G_u^{\vee,\circ}}(\sigma,y))}$ is the composition of the action of an $\mathbb{H}_{\mathfrak{s}^\vee,u}$-intertwiner $H_*(l_{\gamma_{y,\sigma}})$ from $\pi_0(Z_{G_u^{\vee,\circ}}(\sigma, y))$ and the action $H_*(\rho_\gamma^{-1})$ of $\gamma \in \mathbb{H}_{\mathfrak{s}^\vee,u}$, so it is an $\mathbb{H}^\circ_{\mathfrak{s}^\vee,u}$-intertwiner

$$H_*(f_\gamma) : E^\circ_{y,-\sigma,\mathrm{triv}} \to \gamma \cdot E^\circ_{y,-\sigma,\mathrm{triv}}.$$

Let $\pi_{y,-\sigma}$ be the extension of $E^\circ_{y,-\sigma,\mathrm{triv}} = M^\circ_{y,-\sigma,\mathrm{triv}}$ to an $\mathbb{H}_{\mathfrak{s}^\vee,u} \rtimes \Gamma_{[\sigma]}$-representation from Lemma 6.8. Consider the composition

$$\pi_{y,-\sigma}(\gamma^{-1}) \circ H_*(f_\gamma) \in \mathrm{End}_{\mathbb{H}^\circ_{\mathfrak{s}^\vee,u}}(E^\circ_{y,-\sigma,\mathrm{triv}}).$$

By Schur's lemma this is a scalar, say $\lambda \in \mathbb{C}$. We know from Lemma 6.8 that $H_0(\mathcal{B}^y)^{\pi_0(Z_{G_u^{\vee,\circ}}(\sigma,y))}$ has dimension one, so in terms of simplicial homology it is spanned by an element v that is the sum of one point from every connected component of $\mathcal{B}^y$. That v is fixed by $H_0(f_\gamma)$ because (6.14) is a homeomorphism. By Lemmas 6.7 and 6.8 also $\pi_{y,-\sigma}(\gamma^{-1})v = v$. Hence $\lambda = 1$ and $\pi_{y,-\sigma}(\gamma^{-1}) \circ H_*(f_\gamma)$ is the identity. Equivalently,

$$H_*(l_{\gamma_{y,\sigma}}) = H_*(r_\gamma) \circ \pi_{y,-\sigma}(\gamma) : H_*(\mathcal{B}^y)^{\pi_0(Z_{G_u^{\vee,\circ}}(\sigma,y))} \to H_*(\gamma \cdot \mathcal{B}^y)^{\pi_0(Z_{G_u^{\vee,\circ}}(\sigma,y))}.$$

Specializing to $H_0(\mathcal{B}^y)^{\pi_0(Z_{G_u^{\vee,\circ}}(\sigma,y))} = \mathbb{C}v$ we obtain

$$H_0(l_{\gamma_{y,\sigma}}) = H_0(r_\gamma) : H_0(\mathcal{B}^y)^{\pi_0(Z_{G_u^{\vee,\circ}}(\sigma,y))} \to H_0(\gamma \cdot \mathcal{B}^y)^{\pi_0(Z_{G_u^{\vee,\circ}}(\sigma,y))}.$$

It follows that $H_0(\mathcal{P}^y)^{\pi_0(Z_{G_u^\vee}(\sigma,y))}$ contains the nonzero vector

$$\sum_{\gamma \in \Gamma_{u>0}} H_0(r_\gamma^{-1})v = \sum_{w \in \Gamma_{u>0}/\Gamma_{[\sigma]}} H_0(r_w^{-1}) \sum_{\gamma \in \Gamma_{[\sigma]}} H_0(l_{\gamma_{y,\sigma}})v_0.$$

Lemma 6.8 shows that this an element of $E_{y,-\sigma,\mathrm{triv}}$ fixed by $W(R_{u>0}) \rtimes \Gamma_{u>0}$. In other words,

(6.15) $\mathrm{IM}^* M_{y,-\sigma,\mathrm{triv}}$ contains the $\mathbb{C}[W(R_{u>0}) \rtimes \Gamma_{u>0}]$-representation sign $\rtimes$ triv.

Lemma 6.6 says that only the triples (y, σ, ρ) of the kind indicated in the statement have that property.

Finally, we slightly modify the construction from [AMS2, §3.5]. Instead of extending the Iwahori–Matsumoto involution from $\mathbb{H}^\circ_{\mathfrak{s}^\vee,u}$ to $\mathbb{H}_{\mathfrak{s}^\vee,u}$ by making it the identity on $\mathbb{C}[\Gamma_{u>0}]$,

$$\text{we extend IM to } \mathbb{H}_{\mathfrak{s}^\vee,u} \text{ as multiplication by } \det \text{ on } \mathbb{C}[\Gamma_{u>0}]. \tag{6.16}$$

Then (6.15) becomes: $\mathrm{IM}^* M_{y,-\sigma,\rho}$ contains $\det_{W_{\mathfrak{s}^\vee,u}}$ if and only if (σ, y, ρ) is as stated in the theorem. $\square$

We note that (6.16) only differs from the usual Iwahori–Matsumoto involution on the extended graded Hecke algebra $\mathbb{H}_{\mathfrak{s}^\vee,u}$ by the automorphism

$$\det{}_{\Gamma_{u>0}} : \gamma w f \mapsto \det(\gamma)\gamma w f \qquad \gamma \in \Gamma_{u>0}, w \in W(R_{u>0}), f \in \mathcal{O}(\mathfrak{T}_u(T_{\mathfrak{s}^\vee})). \tag{6.17}$$

Since $\det_{\Gamma_{u>0}}$ is the identity on $\mathcal{O}(\mathfrak{T}_u(T_{\mathfrak{s}^\vee}))$, it preserves all the properties (e.g. temperedness) that we need later on. The advantage of (6.16) over IM from [AMS2] is that all reflections in $W(R_{u>0}) \rtimes \Gamma_{u>0}$ are treated in the same way, irrespective of their q-parameters in some Hecke algebra. This is needed to express Theorem 6.9 with $\det|_{W_{\mathfrak{s}^\vee,u}}$.

We wrap up this section by combining the main results.

Lemma 6.10. *We modify* [AMS3, Theorem 3.18] *(see Theorem 4.3) by using* (6.16) *instead of the involution* IM *from* [AMS2, §3.5]. *That yields a canonical bijection*

$$\begin{array}{ccc} \Phi_e(G)^{\mathfrak{s}^\vee} & \to & \mathrm{Irr}\big(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})\big) \\ (\phi, \rho) & \mapsto & \bar{M}(\phi, \rho, q_F^{1/2}) \end{array}$$

such that:

- *It has all the properties listed in* [AMS3, Theorem 3.18].
- $\bar{M}(\phi, \rho, q_F^{1/2})$ *contains the Steinberg representation of* $\mathcal{H}(W_{\mathfrak{s}^\vee}, q_F^\lambda)$ *if and only if* ρ *is trivial and* $\log \phi(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right))$ *lies in the dense* $Z_{G^\vee}\big(\tilde{\phi}(\mathbf{W}_F)\big)$*-orbit in*
$$\big\{Y \in \mathrm{Lie}\big(Z_{G^\vee}(\phi(\mathbf{I}_F))\big) : \mathrm{Ad}(\tilde{\phi}(\mathrm{Frob}_F))Y = q_F^{-1} Y\big\}.$$
Here $\tilde{\phi}|_{\mathbf{I}_F} = \phi|_{\mathbf{I}_F}$ *and* $\tilde{\phi}(\mathrm{Frob}_F) = \phi\big(\mathrm{Frob}_F, \begin{pmatrix} q_F^{-1/2} & 0 \\ 0 & q_F^{1/2} \end{pmatrix}\big)$.

Proof. By design [AMS3, Theorem 3.18] for $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$ is the composition of Theorems 6.1, 6.2 and 6.9, with (6.16) as only difference. Since each of the three involved bijections is canonical, so is our version of [AMS3, Theorem 3.18]. As explained after (6.17), the automorphism $\det_{\Gamma_{u>0}}$ does not destroy any of the properties from [AMS3, Theorem 3.18], so our bijection still satisfies all those properties.

In Theorem 6.9 we found a necessary and sufficient condition so that $\bar{M}_{y,\sigma,\rho} \in \mathrm{Irr}(\mathbb{H}_{\mathfrak{s}^\vee})$ contains $\det_{W_{\mathfrak{s}^\vee,u}}$. With Theorem 6.2 we can translate that to $\exp_{u*} \bar{M}_{y,\sigma,\rho}$. Thus the latter module contains the representation St of $\mathcal{H}(W_{\mathfrak{s}^\vee,u}, q_F^\lambda)$ if and only if ρ is trivial and the orbit of y is dense in

$$\{Y \in \mathrm{Lie}(Z_{G^\vee}(\phi(\mathbf{I}_F)) : \mathrm{Ad}(u \exp(\sigma))Y = q_F^{-1} Y\}.$$

With Theorem 6.1 we transfer that to a property of

$$\mathrm{ind}_u^{-1} \exp_{u*} \bar{M}_{y,\sigma,\rho} \in \mathrm{Irr}\big(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})\big).$$

The translation to L-parameters from [AMS3] is such that $\mathrm{Sc}(\phi,\rho) = u \exp(\sigma)$ and $y = \log \phi(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right))$. Thus we recover the characterization of genericity stated in the lemma. $\square$

7. A canonical local Langlands correspondence

Recall that we fixed a quasi-split group $G = \mathcal{G}(F)$, a maximal split torus S of G, a Borel subgroup $B \subset G$ containing $T = Z_G(S)$ and a Whittaker datum (U,ξ). Given G, only ξ is really a choice, the other objects are unique up to G-conjugacy.

We denote the space of irreducible G-representations in the principal series by $\mathrm{Irr}(G,T)$, and we write $\Phi_e(G,T)$ for the set of principal series enhanced L-parameters in $\Phi_e(G)$.

Theorem 7.1. *The Whittaker datum (U,ξ) determines a canonical bijection*

$$\begin{array}{ccc} \mathrm{Irr}(G,T) & \longleftrightarrow & \Phi_e(G,T) \\ \pi & \mapsto & (\phi_\pi,\rho_\pi) \\ \pi(\phi,\rho) & \leftarrow\!\shortmid & (\phi,\rho) \end{array} .$$

Proof. Recall from (5.2) that the LLC for tori provides a $N_G(T)/T$-equivariant bijection between the Bernstein components of $\mathrm{Irr}(T)$ and the Bernstein components of $\Phi_e(T)$, say $\mathfrak{s}_T \mapsto \mathfrak{s}_T^\vee$.

Every principal series Bernstein component $\mathrm{Irr}(G)^{\mathfrak{s}}$ of $\mathrm{Irr}(G)$ determines a unique $N_G(T)/T$-orbit of Bernstein components $\mathrm{Irr}(T)^{\mathfrak{s}_T}$. Similarly every principal series Bernstein component $\Phi_e(G)^{\mathfrak{s}^\vee}$ determines a unique $N_G(T)/T$-orbit of Bernstein components $\Phi_e(G)^{\mathfrak{s}_T^\vee}$. Thus the LLC for tori induces a natural bijection between the Bernstein components of $\mathrm{Irr}(G,T)$ and those of $\Phi_e(G,T)$. We denote it by $\mathrm{Irr}(G)^{\mathfrak{s}} \mapsto \Phi_e(G)^{\mathfrak{s}^\vee}$, where typically $\mathfrak{s} = [T,\chi_0]_G$ and $\mathfrak{s}^\vee = (T, \hat\chi_0 X_{\mathrm{nr}}(T))$. From respectively (1.1) and Theorem 2.7, Theorem 5.4 and Theorem 4.3 in the form of Lemma 6.10, we obtain canonical bijections

$$\mathrm{Irr}(G)^{\mathfrak{s}} \leftrightarrow \mathrm{Irr}(\mathrm{End}_G(\Pi_{\mathfrak{s}})^{op}) \leftrightarrow \mathrm{Irr}(\mathcal{H}(\mathfrak{s})^{op}) \leftrightarrow \mathrm{Irr}\big(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})\big) \leftrightarrow \Phi_e(G)^{\mathfrak{s}^\vee}. \tag{7.1}$$

Suppose we represent $\mathfrak{s}$ instead by $w\mathfrak{s}_T = [T, w\chi_0]_T$ with $w \in W(\mathcal{G},\mathcal{S}) = N_G(T)/T$. Clearly we may assume that w has minimal length in $wW_{\mathfrak{s}}$. Start with any $\pi \in \mathrm{Irr}(G)^{\mathfrak{s}}$ and follow (7.1) to obtain

$$\pi_{\mathfrak{s}} \in \mathrm{Irr}(\mathcal{H}(\mathfrak{s})^{op}),\ \pi_{\mathfrak{s}^\vee} \in \mathrm{Irr}\big(\mathcal{H}(\mathfrak{s}, q_F^{1/2})\big) \text{ and } (\phi_\pi,\rho_\pi) \in \Phi_e(G,T).$$

We use the same notations with $w\mathfrak{s}$ instead of $\mathfrak{s}$. Proposition 2.8 implies that $\pi_{w\mathfrak{s}} = \pi_{\mathfrak{s}} \circ \mathrm{Ad}(\phi_w)$ where

$$\phi_w(fN_v) = (f \circ w) N_{w^{-1}vw} \quad \text{for } f \in \mathcal{O}(T_{w\mathfrak{s}}), v \in W_{w\mathfrak{s}}.$$

Now we consider w as element of $N_{G^\vee}(T^\vee \rtimes \mathbf{W}_F)/T^\vee$ via (5.2), and we define an algebra isomorphism

$$\begin{array}{cccl} \mathrm{Ad}(\phi_w)^\vee : & \mathcal{H}(w\mathfrak{s}^\vee, q_F^{1/2}) & \to & \mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2}) \\ & fN_v & \mapsto & (f\circ w)N_{w^{-1}vw} \qquad f \in \mathcal{O}(T_{w\mathfrak{s}^\vee}), v \in W_{w\mathfrak{s}^\vee}. \end{array}$$

With Theorem 5.4 we find that $\pi_{w\mathfrak{s}}$ is matched with $\pi_{w\mathfrak{s}^\vee} = \pi_{\mathfrak{s}^\vee} \circ \mathrm{Ad}(\phi_w)^\vee$. All the constructions behind Theorem 4.3 and Lemma 6.10 are equivariant for automorphisms of $(G^\vee \rtimes \mathbf{W}_F, T^\vee \rtimes \mathbf{W}_F)$ which preserve the projections to $\mathbf{W}_F$ and are algebraic on $G^\vee$. This means that $\pi_{\mathfrak{s}^\vee} \circ \mathrm{Ad}(\phi_w)^\vee$ is parametrized by $(w\phi_\pi w^{-1}, w \cdot \rho_\pi)$, for any representative of w in $N_{G^\vee}(T^\vee \rtimes \mathbf{W}_F)$. As $(w\phi_\pi w^{-1}, w\cdot\rho_\pi)$ equals (ϕ_π,ρ_π)

in $\Phi_e(G)$, we deduce that the bijection between $\mathrm{Irr}(G)^{\mathfrak{s}}$ and $\Phi_e(G)^{\mathfrak{s}^\vee}$ from (7.1) does not depend on the choice of an inertial equivalence class for T underlying $\mathfrak{s}$.

Knowing that, we can unambiguously take the union of the bijections (7.1) over all Bernstein components of $\mathrm{Irr}(G,T)$. $\square$

Remark 7.2. If we had used the isomorphism $\mathcal{H}(\mathfrak{s}) \cong \mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$ from Remark 5.5 instead of Theorem 5.4, then (7.1) would provide a canonical bijection between $\mathrm{Irr}(G)^{\mathfrak{s}}$ and $\mathrm{Irr}\big(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})^{op}\big)$. That could be more natural, depending on the point of view.

In the remainder of this section we will show that the bijection from Theorem 7.1 has many desirable properties.

The definition of $\tilde\phi$ in Lemma 6.10 applies to any Langlands parameter $\phi \in \Phi(G)$. The group $Z_{G^\vee}(\tilde\phi(\mathbf{W}_F))$ acts by conjugation on the variety

$$V_{\tilde\phi} = \big\{ v \in Z_{G^\vee}(\phi(\mathbf{I}_F)) : v \text{ is unipotent and } \tilde\phi(\mathrm{Frob}_F)^{-1} v \tilde\phi(\mathrm{Frob}_F) = v^{q_F} \big\}.$$

It is known from [CFMMX, Proposition 5.6.1] that $V_{\tilde\phi}$ is an affine space over $\mathbb{C}$ on which $Z_{G^\vee}(\tilde\phi(\mathbf{W}_F))$ acts with finitely many orbits, of which exactly one is open. Following [CFZ, §0.6], we call $\phi \in \Phi(G)$ open if $u_\phi \in V_{\tilde\phi}$ lies in the open $Z_{G^\vee}(\tilde\phi(\mathbf{W}_F))$-orbit.

Lemma 7.3. *The representation $\pi(\phi,\rho) \in \mathrm{Irr}(G,T)$ is (U,ξ)-generic if and only if ϕ is open and ρ is trivial.*

Proof. By Theorem 3.4, $\pi(\phi,\rho)$ is (U,ξ)-generic if and only if the $\mathrm{End}_G(\Pi_{\mathfrak{s}})^{op}$-module $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi(\phi,\rho))$ contains St. Via Theorems 5.4 and 4.3 that becomes the analogous statement for $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$-representations. In Lemma 6.10 we showed the equivalence with the stated conditions on ϕ and ρ, except unipotency. The conditions in Lemma 6.10 imply that $\log u_\phi$ must be nilpotent. Hence u_ϕ must be unipotent (as is any case required for Langlands parameters). $\square$

We note that Lemma 7.3 agrees with the Reeder's findings [Ree1, Ree2] for generic unipotent representations and generic principal series representations, in both cases for split reductive p-adic groups with connected centre.

For the next properties of our LLC, the setup will be similar to [Sol6, §5].

Lemma 7.4. *Theorem 7.1 is compatible with direct products of quasi-split F-groups.*

Proof. If $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$, then all involved objects for $\mathcal{G}$ are naturally products of the analogous objects for G_1 and G_2. $\square$

Recall that the group of (smooth) characters $\mathrm{Hom}(G,\mathbb{C}^\times)$ is naturally isomorphic to $H^1(\mathbf{W}_F, Z(G^\vee))$ [Bor2, Lan1]. The former group acts on $\mathrm{Irr}(G)$ by tensoring, and that action commutes with the supercuspidal support map so stabilizes $\mathrm{Irr}(G,T)$.

On the other hand, $H^1(\mathbf{W}_F, Z(G^\vee))$ acts on $\Phi(G)$ by multiplication of maps $\mathbf{W}_F \times SL_2(\mathbb{C}) \to G^\vee$, where $H^1(\mathbf{W}_F, Z(G^\vee))$ gives maps that do not use $SL_2(\mathbb{C})$). That action does not change R_ϕ, so it induces an action of $H^1(\mathbf{W}_F, Z(G^\vee))$ on $\Phi_e(G)$ which does not change the enhancements. This last action commutes with the cuspidal support maps, so it stabilizes $\Phi_e(G,T)$.

Lemma 7.5. *The bijection in Theorem 7.1 is $H^1(\mathbf{W}_F, Z(G^\vee))$-equivariant.*

Proof. For the fourth bijection in (7.1), such equivariance was shown in [Sol6, Lemma 2.2.a]. Here $z \in H^1(\mathbf{W}_F, Z(G^\vee))$ acts via the algebra isomorphism

$$\begin{array}{llll} \mathcal{H}(z): & \mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2}) & \to & \mathcal{H}(z\mathfrak{s}^\vee, q_F^{1/2}) \\ & f N_w & \mapsto & (f \circ z^{-1}) N_w \qquad f \in \mathcal{O}(T_{\mathfrak{s}^\vee}), w \in W_{\mathfrak{s}^\vee}. \end{array}$$

In view of Theorem 5.4, the same formula also defines an algebra isomorphism

$$\mathcal{H}(z) : \mathcal{H}(\mathfrak{s})^{op} \to \mathcal{H}(z\mathfrak{s})^{op}.$$

We define an action of $H^1(\mathbf{W}_F, Z(G^\vee))$ on the union of the spaces $\mathrm{Irr}(\mathcal{H}(\mathfrak{s})^{op})$ by $z \cdot \tau = \tau \circ \mathcal{H}(z)^{-1}$. That renders the third bijection in (7.1) equivariant. Using Theorem 2.7 and the same argument we also make the second bijection in (7.1) equivariant for $H^1(\mathbf{W}_F, Z(G^\vee))$.

Finally, consider $\pi \in \mathrm{Irr}(G)^{\mathfrak{s}}$ and $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi) \in \mathrm{Irr}(\mathrm{End}_G(\Pi_{\mathfrak{s}})^{op})$. Then $z \otimes \pi$ lies in $\mathrm{Irr}(G)^{z\mathfrak{s}}$ and

$$\begin{aligned} \mathrm{Hom}_G(\Pi_{z\mathfrak{s}}, z \otimes \pi) = \mathrm{Hom}_G\big(I_B^G \mathrm{ind}_{T_{\mathrm{cpt}}}^T (z \otimes \chi), z \otimes \pi\big) \cong \\ \mathrm{Hom}_G\big(z \otimes I_B^G \mathrm{ind}_{T_{\mathrm{cpt}}}^T (\chi), z \otimes \pi\big) = \mathrm{Hom}_G\big(I_B^G \mathrm{ind}_{T_{\mathrm{cpt}}}^T (\chi), \pi\big) = \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi). \end{aligned}$$

The isomorphism (from bottom to top) is given by translation by z on $\mathrm{Irr}(T)$. In terms of modules over $\mathcal{H}(\mathfrak{s})$ and $\mathcal{H}(z\mathfrak{s})$, that isomorphism is implemented by composition with $\mathcal{H}(z)^{-1}$. Hence the first bijection in (7.1) is equivariant as well. □

It is clear that a principal series G-representation is supercuspidal if and only if G is a torus. Similarly, the discussion at the start of Section 4 entails that a principal series enhanced L-parameter for G is cuspidal if and only if G is a torus. The next result relates the cuspidal support maps on both sides, when G is not a torus.

Lemma 7.6. *Theorem 7.1 and the cuspidal support maps make a commutative diagram*

$$\begin{array}{ccc} \mathrm{Irr}(G,T) & \longleftrightarrow & \Phi_e(G,T) \\ \downarrow \mathrm{Sc} & & \downarrow \mathrm{Sc} \\ \mathrm{Irr}(T)/N_G(T) & \xrightarrow{LLC} & \Phi(T)/N_{G^\vee}(T \rtimes \mathbf{W}_F) \end{array}.$$

Proof. From the formula for the cuspidal support (4.1) and Theorem 4.2.a, we see that the central character of $\overline{M}(\phi, \rho, q_F^{1/2})$ is given by $\mathrm{Sc}(\phi, \rho)/W_{\mathfrak{s}^\vee} \in \Phi_e(T)/W_{\mathfrak{s}^\vee}$. Hence the central character of $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi(\phi, \rho))$ is the image $W_{\mathfrak{s}}\chi_\phi$ of $\mathrm{Sc}(\phi, \rho)/W_{\mathfrak{s}^\vee}$ in $\mathrm{Irr}(T)/W_{\mathfrak{s}}$.

More explicitly, $\mathcal{O}(T_{\mathfrak{s}})^{W_{\mathfrak{s}}}$ acts on $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi(\phi, \rho))$ via $W_{\mathfrak{s}}\chi_\phi$. Then a glance at the construction of $\Pi_{\mathfrak{s}}$ reveals that $W_{\mathfrak{s}}\chi_\phi$ represents the supercuspidal support of $\pi(\phi, \rho)$. □

We turn to more analytic properties of G-representations.

Lemma 7.7. $\pi \in \mathrm{Irr}(G,T)$ *is tempered if and only if* $\phi_\pi \in \Phi(G)$ *is bounded.*

Proof. Theorem 4.2.b says that the fourth bijection in (7.1) has the desired property. By Lemma 5.3 and Theorem 5.4, the third bijection in (7.1) preserves temperedness. By [Sol5, Theorem 9.6.a], so does the composition of the first and the second bijections in (7.1). □

Lemma 7.8. $\pi \in \mathrm{Irr}(G,T)$ *is essentially square-integrable if and only if* ϕ *is discrete.*

Proof. Suppose first that $R_{\mathfrak{s},\mu}$ has smaller rank that $R(\mathcal{G},\mathcal{S})$. By [Sol5, Theorem 9.6.b], $\mathrm{Rep}(G)^{\mathfrak{s}}$ contains no essentially square-integrable representations. As $\mathrm{rk}(R(\mathcal{G},\mathcal{S}))$ equals the F-split rank of $\mathcal{G}$ and

$$\mathrm{rk}(R_{\mathfrak{s},\mu}) = \mathrm{rk}(R_{\mathfrak{s}}^{\vee}) = \mathrm{rk}(R_{\mathfrak{s}^\vee}),$$

Theorem 4.2.c says that $\Phi_e(G)^{\mathfrak{s}^\vee}$ contains no discrete enhanced L-parameters.

Now we suppose that $\mathrm{rk}(R_{\mathfrak{s},\mu}) = \mathrm{rk}(R(\mathcal{G},\mathcal{S}))$. Then [Sol5, Theorem 9.6.c] says that (1.1) restricts to a bijection between essentially square-integrable representations in $\mathrm{Irr}(G)^{\mathfrak{s}}$ and essentially discrete series representations in $\mathrm{Irr}(\mathcal{H}(\mathfrak{s})^{op})$. By Lemma 5.3 and Theorem 5.4, the latter set is naturally in bijection with the set of essentially discrete series representations in $\mathrm{Irr}\big(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})\big)$. Combine that with Theorem 4.2.c. $\square$

Recall from [Lan1, p. 20–23] and [Bor2, §10] that every $\phi \in \Phi(G)$ determines in a canonical way a character χ_ϕ of $Z(G)$.

Lemma 7.9. *For any $(\phi,\rho) \in \Phi_e(G,T)$, the central character of $\pi(\phi,\rho)$ equals χ_ϕ.*

Proof. For any subquotient π of $I_B^G(\chi) = \mathrm{ind}_B^G(\chi \otimes \delta_B^{1/2})$, the central character of π equals $(\chi \otimes \delta_B^{1/2})|_{Z(G)} = \chi|_{Z(G)}$. In particular the central character of $\pi(\phi,\rho)$ equals $\mathrm{Sc}(\pi(\phi,\rho))|_{Z(G)}$. By Lemma 7.6 that is $\pi(\mathrm{Sc}(\phi,\rho))|_{Z(G)}$. With (4.1) we write it as

$$\chi|_{Z(G)} \quad \text{where} \quad \hat{\chi}|_{\mathbf{I}_F} = \phi|_{\mathbf{I}_F} \quad \text{and} \quad \hat{\chi}(\mathrm{Frob}_F) = \phi\Big(\mathrm{Frob}_F, \begin{pmatrix} q_F^{-1/2} & 0 \\ 0 & q_F^{1/2} \end{pmatrix}\Big).$$

It remains to show that χ_ϕ equals $\chi|_{Z(G)}$, and to that end we revisit the construction from [Bor2, Lan1]. Let $\overline{G}$ be a quasi-split reductive F-group with connected centre, such that $\mathcal{G}_{\mathrm{der}} = \overline{\mathcal{G}}_{\mathrm{der}}$. Let $\overline{\phi} \in \Phi(\overline{G})$ be a lift of $\phi \in \Phi(G)$. With the canonical map $\overline{p} : {}^L\overline{G} \to {}^L Z(\overline{G})$ we obtain $\overline{p}(\overline{\phi}) \in \Phi(Z(\overline{G}))$. Via the LLC for tori that gives $\chi_{\overline{p}(\overline{\phi})} \in \mathrm{Irr}(Z(\overline{G}))$, and by definition $\chi_\phi = \chi_{\overline{p}(\overline{\phi})}|_{Z(G)}$.

Let $\overline{T} = Z_{\overline{G}}(S) = Z_{\overline{G}}(\overline{T})$. From (4.1) we see that, for any enhancement $\overline{\rho}$ of $\overline{\phi}$ such that $(\overline{\phi},\overline{\rho}) \in \Phi_e(\overline{G},\overline{T})$, we have $\mathrm{Sc}(\overline{\phi},\overline{\rho}) = (\overline{\psi},\overline{\epsilon})$, where $\overline{\psi} \in \Phi(\overline{T})$ is a lift of $\hat{\chi} \in \Phi(T)$. As $\overline{\phi}$ and $\overline{\psi}$ differ only by elements of ${\overline{G}^\vee}_{\mathrm{der}} \subset \ker(\overline{p})$, we have $\overline{p}\overline{\phi} = \overline{p}\overline{\psi}$. By the naturality of the LLC for tori, $\chi_{\overline{\psi}}$ extends both $\chi \in \mathrm{Irr}(T)$ and $\chi_{\overline{p}\overline{\psi}} = \chi_{\overline{p}\overline{\phi}} \in \mathrm{Irr}(Z(\overline{G}))$. Hence $\chi|_{Z(G)} = \chi_{\overline{p}\overline{\phi}}|_{Z(G)} = \chi_\phi$. $\square$

Suppose that $P = MR_u(P)$ is a parabolic subgroup of G, where M is a Levi factor of P and $T \subset M$. We can use the normalized parabolic induction functor I_P^G to relate representations of M and of G.

The restriction of ξ to $U \cap M$ is a nondegenerate character ξ_M. We use $(U \cap M, \xi_M)$ to define genericity of M-representations and to normalize the LLC for $\mathrm{Irr}(M,T)$.

Suppose furthermore that $\phi \in \Phi(G)$ factors via $\Phi(M)$. By [AMS1, Theorem 7.10.a] the group $R_\phi^M = \pi_0(Z_{M^\vee}(\phi)/Z(M^\vee))$ injects naturally into R_ϕ. Hence any enhancement of $\phi \in \Phi(G)$ can be considered as (possibly reducible) representation of R_ϕ^M.

Lemma 7.10. *Let $(\phi,\rho^M) \in \Phi_e(M,T)$ be bounded. Then*

$$I_P^G \pi(\phi,\rho^M) \cong \bigoplus\nolimits_\rho \mathrm{Hom}_{R_\phi^M}(\rho^M,\rho) \otimes \pi(\phi,\rho),$$

where the sum runs over all $\rho \in \mathrm{Irr}(R_\phi)$ with $\mathrm{Sc}(\phi,\rho) = \mathrm{Sc}(\phi,\rho^M)$.

Proof. By [AMS3, Theorem 3.18.f and Lemma 3.19.a], the analogous statement holds for $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$-modules. Theorem 5.4 (which is compatible with parabolic induction) entails it also holds for $\mathcal{H}(\mathfrak{s})^{op}$-modules. Then (1.2) enables us to transfer the desired statement from $\mathcal{H}(\mathfrak{s})^{op}$ to $\mathrm{Rep}(G)^{\mathfrak{s}}$. $\square$

Recall that the Langlands classification for irreducible G-representations [Lan1, Ren] associates to any $\pi \in \mathrm{Irr}(G)$ a unique standard parabolic subgroup $P = MR_u(P)$, a unique tempered $\tau \in \mathrm{Irr}(M)$ and a unique strictly positive $z \in \mathrm{Hom}(M, \mathbb{R}_{>0})$, such that π is the unique irreducible quotient of the standard module $I_P^G(\tau \otimes z)$. It has a counterpart for (enhanced) L-parameters [SiZi]. Let $(\phi, \rho) \in \Phi_e(G, T)$ and let $(P = MR_u(P), \phi_b, z)$ be the triple associated to ϕ by [SiZi, Theorem 4.6]. Here $\phi_b \in \Phi(M)$ is bounded and $z \in X_{\mathrm{nr}}(M) \cong (Z(M^\vee)^{\mathbf{I}_F})_{\mathbf{W}_F}^\circ$ is "strictly positive with respect to P". By [AMS1, Theorem 7.10.b] there are natural isomorphisms

$$R^M_{\phi_b} \cong R^M_{z\phi_b} = R^M_\phi \cong R_\phi .$$

Hence ρ can also be regarded as enhancement of $\phi \in \Phi(M)$ or $\phi_b \in \Phi(M)$.

Lemma 7.11. *In the above setting:*

(a) $\pi(\phi, \rho)$ is the unique irreducible quotient of $I_P^G \pi^M(\phi, \rho)$.
(b) $\pi^M(\phi, \rho) = \pi^M(z\phi_b, \rho) = z \otimes \pi^M(\phi_b, \rho)$ with $\pi^M(\phi_b, \rho) \in \mathrm{Irr}(M)$ tempered.
(c) The triple associated to $\pi(\phi, \rho)$ by the Langlands classification for $\mathrm{Irr}(G)$ *is $(P, \pi^M(\phi_b, \rho), z)$.*

Proof. (a) By [Sol6, Proposition 2.3], the analogue in $\mathrm{Rep}(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2}))$ holds. As in the proof of Lemma 7.10, that can be transferred to $\mathrm{Rep}(G)^{\mathfrak{s}}$ via (1.2).
(b) This is a direct consequence of Lemmas 7.5 and 7.7.
(c) This follows from parts (a) and (b) and the uniqueness in the Langlands classification. $\square$

Suppose that F'/F is a finite extension inside the fixed separable closure F_s. Let $\mathcal{G}'$ be a quasi-split F'-group and put $\mathcal{G} = \mathrm{Res}_{F'/F}(\mathcal{G}')$. Then $\mathcal{G}(F) = \mathcal{G}'(F')$, so there is a natural bijection $\mathrm{Irr}(\mathcal{G}(F)) \to \mathrm{Irr}(\mathcal{G}'(F'))$. On the other hand, Shapiro's lemma provides a natural isomorphism

$$\mathrm{Sh} : \Phi_e(\mathcal{G}(F)) \to \Phi_e(\mathcal{G}'(F')),$$

see [FOS1, Lemma A.3]. Let $\mathcal{T}'$ be the centralizer of a maximal F'-split torus of $\mathcal{G}'$ and put $\mathcal{T} = \mathrm{Res}_{F'/F}(\mathcal{T}')$.

Lemma 7.12. *The bijection in Theorem 7.1 is compatible with restriction of scalars, in the sense that the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Irr}(\mathcal{G}(F), \mathcal{T}(F)) & \to & \Phi_e(\mathcal{G}(F), \mathcal{T}(F)) \\ \downarrow & & \downarrow \mathrm{Sh} \\ \mathrm{Irr}(\mathcal{G}'(F'), \mathcal{T}'(F')) & \to & \Phi_e(\mathcal{G}'(F'), \mathcal{T}'(F')) \end{array} .$$

Proof. By [Sol6, (26)], Sh induces a bijection from the set of Bernstein components of $\Phi_e(\mathcal{G}(F))$ to the analogous set for $\mathcal{G}'(F')$. This bijection commutes with the cuspidal support maps, so it also applies to $\Phi_e(\mathcal{G}(F), \mathcal{T}(F))$ and $\Phi_e(\mathcal{G}'(F'), \mathcal{T}'(F'))$. When Sh sends $\mathfrak{s}^\vee$ to $\mathfrak{s}'^\vee$, there is a natural algebra isomorphism $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2}) \cong \mathcal{H}(\mathfrak{s}'^\vee, q_{F'}^{1/2})$ [Sol6, Lemma 2.4]. Combine that with (7.1). $\square$

Finally we investigate in what sense our (enhanced) L-parameters are unique.

Lemma 7.13. *Let $\pi \in \mathrm{Irr}(G,T)$. Then the ϕ_π from Theorem 7.1 is uniquely determined by Lemmas 7.5, 7.6, 7.7 and 7.11.*

Proof. Suppose that π is tempered. Lemma 7.6 determines $\mathrm{Sc}(\phi_\pi, \rho_\pi) = \tilde{\phi}$ up to $N_{G^\vee}(T^\vee \rtimes \mathbf{W}_F)$. Lemma 7.8 says that ϕ_π must be bounded, so according to [CFZ, §0.6] ϕ_π is an open Langlands parameter. In other words, u_{ϕ_π} is uniquely determines (up to $Z_{G^\vee}(\tilde{\phi_\pi}(\mathbf{W}_F))$-conjugacy) as an element of the open orbit in $V_{\tilde{\phi_\pi}}$. Thus ϕ_π is unique up to $G^\vee$-conjugacy.

Suppose now that π is not tempered. Let (P, τ, z) be the triple associated to π by the Langlands classification. Here τ is tempered, so the above determines $\phi_\tau \in \Phi(P/R_u(P), T)$ uniquely. Then Lemma 7.5 forces $\phi_{\tau \otimes z} = z \cdot \phi_\tau$ and Lemma 7.11 says that ϕ_π equals $z\phi_\tau$ up to $G^\vee$-conjugacy. $\square$

It is less clear to what extent the enhancement ρ_π of ϕ_π is uniquely specified. Lemma 7.11 reduces this issue to tempered $\pi \in \mathrm{Irr}(G,T)$. Then ϕ_π is bounded, so open. By Lemma 7.3 the L-packet $\Pi_{\phi_\pi}(G)$ contains a unique generic member, namely $\pi(\phi_\pi, \mathrm{triv})$. That fixes the normalization of the intertwining operators from elements of R_{ϕ_π}, which then determines $\pi(\phi_\pi, \rho)$ for any $\rho \in \mathrm{Irr}(R_{\phi_\pi})$ such that $(\phi_\pi, \rho) \in \Phi_e(G,T)$. However, to make that precise one has to say on which module these intertwining operators acts. That involves the constructions with Hecke algebras in Sections 5 and 6. Those are canonical, but they may not be unique, see Remarks 5.5 and 7.2.

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