ENDOSCOPY FOR REPRESENTATIONS OF DISCONNECTED REDUCTIVE GROUPS OVER FINITE FIELDS

MAARTEN SOLLEVELD

Abstract. Let G be the group of rational points of a connected reductive group over a finite field. Based on work of Lusztig and Yun, we make the Jordan decomposition for irreducible G-representations canonical. It comes in the form of an equivalence between the category of G-representations with a fixed semisimple parameter s and the category of unipotent representations of an endoscopic group of G, enriched with an equivariant structure with respect to the component group of the centralizer of s.

Next we generalize these results, replacing the connected reductive group by a smooth group scheme with reductive neutral component. Again we establish canonical equivalences for both the rational and the geometric series of G-representations, in terms of unipotent representations of endoscopic groups.

1. Introduction

Let $\mathcal{G}$ be a connected reductive group over a finite field $\mathbb{F}_q$. We are interested in the representations of $G = \mathcal{G}(\mathbb{F}_q)$, always on vector spaces over $\overline{\mathbb{Q}}_\ell$ where q and ℓ are coprime.

Deligne and Lusztig [DeLu] showed that the irreducible G-representations can be partitioned in series, indexed by semisimple conjugacy classes in the $\mathbb{F}_q$-group $\mathcal{G}^\vee$ dual to $\mathcal{G}$. More concretely, starting from a maximal $\mathbb{F}_q$-torus $\mathcal{T}$ in $\mathcal{G}$ and a character θ of $T = \mathcal{T}(\mathbb{F}_q)$ they constructed a virtual G-representation $R_T^G(\theta)$. The data $(\mathcal{T}, \theta)$ determine a semisimple element $s \in G^\vee = \mathcal{G}^\vee(\mathbb{F}_q)$, up to $G^\vee$-conjugacy. Moreover, it was shown in [DeLu] that $R_T^G(\theta)$ and $R_{T'}^G(\theta')$ have no common irreducible constituents if their semisimple parameters not geometrically conjugate. That enables the following definition: the geometric series with parameter s consists of the irreducible G-representations that appear in $R_T^G(\theta)$ for some $(\mathcal{T}, \theta)$ whose associated element in $sG^\vee$ is geometrically conjugate to s.

Later Lusztig [Lus1] improved on that, by showing that two virtual representations $R_T^G(\theta)$ and $R_{T'}^G(\theta')$ are disjoint whenever their parameters in $G^\vee$ are not $G^\vee$-conjugate. This lead to the rational (or Lusztig) series $\mathrm{Irr}_s(G)$, where s represents a semisimple (rational) conjugacy class in $G^\vee(\mathbb{F}_q)$. The element s is called the semisimple parameter of a representation in $\mathrm{Irr}_s(G)$, uniquely determined up to $G^\vee$-conjugacy. The representations in $\mathrm{Irr}_1(G)$ are especially interesting, these are known as unipotent representations.

In [Lus1, Lus2], Lusztig proved that there is always a bijection

$$\mathrm{Irr}_s(G) \longleftrightarrow \mathrm{Irr}_1(Z_{G^\vee}(s)). \tag{1}$$

This works best when $Z_{\mathcal{G}^\vee}(s)$ is connected, then the bijection is canonical. Thus every irreducible G-representation is determined by its semisimple parameter and a

2010 *Mathematics Subject Classification.* 20C33, 20G40.

unipotent representation of another reductive $\mathbb{F}_q$-group. This is known as the Jordan decomposition of irreducible G-representations, we refer to [DiMi, GeMa] for much more background.

In [LuYu1, LuYu2], Lusztig and Yun set out to make (1) canonical, in geometric terms. This required several modifications. Firstly, they do not work with rational but with geometric series. A maximal $\mathbb{F}_q$-torus $\mathcal{T}$ in $\mathcal{G}$ is fixed, say with Weyl group W. Characters of T are replaced by rank one character sheaves on $\mathcal{T}$. These are not required to be stable under the action of a Frobenius element Fr, only the W-orbit of $\mathcal{L}$ must be Fr-stable. Such data $(W\mathcal{L}, \mathcal{T})$ correspond to precisely one Fr-stable geometric conjugacy class in $G^\vee$.

Secondly, the main results of Lusztig–Yun involve categories, not only irreducible representations. Instead of $\mathrm{Irr}_x(G)$ (given some data x for which this is defined) we have $\mathrm{Rep}_x(G)$, the category generated by $\mathrm{Irr}_x(G)$. In particular $\mathrm{Rep}_{W\mathcal{L}}(G)$ denotes the geometric series associated to $W\mathcal{L}$.

Thirdly, $Z_{\mathcal{G}^\vee}(s)$ is replaced by the endoscopic group $\mathcal{H}$ dual to $Z_{\mathcal{G}^\vee}(s)^\circ$ plus some group $\Omega_\mathcal{L}$ isomorphic to $\pi_0(Z_\mathcal{G}(s))$. In this way the use of $\mathcal{G}^\vee$ can be avoided in [LuYu1, LuYu2].

With impressive geometric machinery Lusztig and Yun obtain canonical equivalences between subcategories of $\mathrm{Rep}_{W\mathcal{L}}(G)$ (determined by cells) and subcategories of $\mathrm{Rep}_1(H)$ enriched with $\Omega_\mathcal{L}$-equivariant structures. This can be regarded as an endoscopic categorical version of Deligne–Lusztig theory.

The aims of this paper are threefold. Firstly, we want to explain the final results of [LuYu2] in simpler representation theoretic terms. Secondly, we want to find a version of these results for rational series, so a canonical Jordan decomposition of irreducible G-representations, like (1). This may have been known to Lusztig and Yun, but it is not stated in [LuYu1, LuYu2]. And thirdly, we want to generalize these canonical equivalences of categories to disconnected reductive $\mathbb{F}_q$-groups. In a sense, this is the natural generality for such results. Namely, (1) already shows that to analyse irreducible representations of connected reductive $\mathbb{F}_q$-groups, one has to involve unipotent representations of disconnected reductive $\mathbb{F}_q$-groups. If instead we start with a disconnected reductive $\mathbb{F}_q$-group, then we still need to study unipotent representations of other disconnected reductive $\mathbb{F}_q$-groups, but we stay within the same class of groups.

Now we discuss our main results in more detail. Let $\mathcal{G}$ be a smooth $\mathbb{F}_q$-group scheme whose neutral component $\mathcal{G}^\circ$ is reductive. Only the connected components of $\mathcal{G}$ which are stable under a Frobenius element Fr contribute to $G = \mathcal{G}(\mathbb{F}_q)$, so we may assume that all connected components of $\mathcal{G}$ are Fr-stable. We allow $\pi_0(\mathcal{G})$ to be infinite.

Let $\mathcal{L}$ be a rank one character sheaf on a maximal $\mathbb{F}_q$-torus $\mathcal{T}$ in $\mathcal{G}^\circ$, such that $W(\mathcal{G}^\circ, \mathcal{T})\mathcal{L}$ is Fr-stable. We define the geometric series $\mathrm{Rep}_{\mathcal{W}(\mathcal{G},\mathcal{T})\mathcal{L}}(G)$ as consisting of the G-representations π such that $\mathrm{Res}^G_{G^\circ}\pi$ lies in the subcategory generated by $\bigcup_{w \in W(\mathcal{G},\mathcal{T})} \mathrm{Rep}_{W(\mathcal{G}^\circ,\mathcal{T})w\mathcal{L}}(G^\circ)$.

Let $\mathcal{H}$ be the (connected) endoscopic group of $\mathcal{G}^\circ$ determined by $(\mathcal{L}, \mathcal{T})$. To $(\mathcal{L}, \mathcal{T})$ we also associate a group $\Omega_\mathcal{L}$, which plays the role of the component group of some disconnected version of $\mathcal{H}$. We need a certain set $\mathcal{B}^\circ_\mathcal{L}$ with an action of $\Omega^\circ_\mathcal{L}$ (the version of $\Omega_{\mathcal{L}}$ for $\mathcal{G}^\circ$) such that $\mathfrak{B}^\circ_{\mathcal{L}} / \Omega^\circ_{\mathcal{L}}$ parametrizes the rational conjugacy classes in one geometric conjugacy class in $G^{0\vee}$. Every $\beta \in \mathfrak{B}^\circ_{\mathcal{L}}$ gives rise to a $\mathbb{F}_q$-form of $\mathcal{H}$, say $\mathcal{H}^\beta$.

Theorem A. *There exists a canonical equivalence of categories*

$$\mathrm{Rep}_{W(\mathcal{G},\mathcal{T})\mathcal{L}}(G) \cong \Big(\bigoplus\nolimits_{\beta \in \mathfrak{B}^\circ_{\mathcal{L}}} \mathrm{Rep}_1(H^\beta) \Big)^{\Omega_{\mathcal{L}}}.$$

The superscript $\Omega_{\mathcal{L}}$ means that we enrich the objects in the category between brackets with $\Omega_{\mathcal{L}}$-equivariant structures. Thus Theorem A describes G-representations with semisimple geometric parameter $W(\mathcal{G},\mathcal{T})\mathcal{L}$ in terms of unipotent representations of forms of an endoscopic group $\mathcal{H}$ plus a kind of action of the group $\Omega_{\mathcal{L}}$. When $\mathcal{G}$ is connected, Theorem A is a quick consequence of [LuYu2, Corollary 12.7].

Suppose now that $\mathcal{L}$ is Fr-stable. Then it determines a semisimple rational conjugacy class in $G^{\circ\vee}$, say represented by s. We define the rational series $\mathrm{Rep}_s(G)$ as the category of G-representations π such that $\mathrm{Res}^G_{G^\circ}\pi$ lies in the subcategory generated by $\bigcup_{g\in G} \mathrm{Ad}(g)^* \mathrm{Rep}_s(G^\circ)$.

Theorem B. *There exists a canonical equivalence of categories*

$$\mathrm{Rep}_s(G) \cong \mathrm{Rep}_1(H)^{\Omega_{\mathcal{L}}^{\mathrm{Fr}}}.$$

This can be regarded as a Jordan decomposition for representations of the possibly disconnected reductive group $G = \mathcal{G}(\mathbb{F}_q)$. It existed already when $\mathcal{G}$ is connected [Lus2], but even in that case Theorem B adds canonicity.

We expect that our results will have applications to depth zero representations of reductive p-adic groups. The main reason is that by [MoPr] every supercuspidal depth zero representation of such a group, say M, is obtained from an irreducible representation of a group of the form $\mathcal{G}(\mathbb{F}_q)$. In that setting $\mathcal{G}(\mathbb{F}_q)$ arises as a quotient of the M-stabilizer of a vertex in the reduced Bruhat–Tits building of M.

In the recent preprint [Fuj], depth zero supercuspidal representations of simple p-adic groups were analysed, with several techniques among which some version of a Jordan decomposition for $\mathrm{Irr}(\mathcal{G}(\mathbb{F}_q))$. With Theorem B it should be possible to make the results in [Fuj, §4] applicable in larger generality.

2. Connected groups

We start by recalling a part of the setup and the relevant results from [LuYu1, LuYu2]. Let $\mathcal{G}$ be a connected reductive group defined over the algebraic closure of a finite field $\mathbb{F}_q$. Let ℓ be a prime number different from $p = \mathrm{char}(\mathbb{F}_q)$. All our representations will be on vector spaces over $\overline{\mathbb{Q}_\ell}$.

Let ϵ be the action of a geometric Frobenius element $\mathrm{Fr} \in \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ for some rational structure of $\mathcal{G}$. We denote the corresponding $\mathbb{F}_q$-group by $\mathcal{G}^\epsilon$, and we write

$$G^\epsilon = \mathcal{G}^\epsilon(\mathbb{F}_q) = \mathcal{G}\big(\overline{\mathbb{F}_q}\big)^\epsilon.$$

We fix an ϵ-stable maximal torus $\mathcal{T}$ in $\mathcal{G}$, so $\mathcal{T}^\epsilon$ is a maximal $\mathbb{F}_q$-torus in $\mathcal{G}^\epsilon$. Let $\mathcal{B}$ be an ϵ-stable Borel subgroup of $\mathcal{G}$ containing $\mathcal{T}$ and let Φ^+ be the corresponding set of positive roots in $\Phi = \Phi(\mathcal{G},\mathcal{T})$. We write $W = W(\mathcal{G},\mathcal{T})$ and for each $w \in W$ we fix a representative $\dot{w} \in N_{\mathcal{G}}(\mathcal{T})$.

Let $\mathcal{L}$ be a rank one character sheaf on $\mathcal{T}$, and assume that $W\mathcal{L}$ is ϵ-stable. Whenever $w\epsilon\mathcal{L} = \mathcal{L}$, $\mathcal{L}$ can be regarded as a character sheaf on $\mathcal{T}^{w\epsilon}$, stable under the Frobenius action $\mathrm{Ad}(w) \circ \epsilon$ of that $\mathbb{F}_q$-torus. Then $\mathcal{L}$ is equivalent to the data of a character $T^{w\epsilon} \to \overline{\mathbb{Q}_\ell}^\times$.

The orbit $W\mathcal{L}$ corresponds to a Fr-stable semisimple conjugacy class in the reductive $\mathbb{F}_q$-group $\mathcal{G}^\vee$ dual to $\mathcal{G}$ [DeLu, §5]. Deligne–Lusztig theory yields the category $\mathrm{Rep}_{W\mathcal{L}}(G^\epsilon)$ generated by irreducible G^ϵ-representations with semisimple parameter $W\mathcal{L}$. The set of irreducible representations in $\mathrm{Rep}_{W\mathcal{L}}(G^\epsilon)$ is called the geometric series with parameter $\mathcal{L}$ or $W\mathcal{L}$.

The stabilizer of $\mathcal{L}$ in W can be written as

$$W_{\mathcal{L}} = W_{\mathcal{L}}^\circ \rtimes \Omega_{\mathcal{L}}.$$

Here $W_{\mathcal{L}}^\circ$ is the Weyl group of the root system

$$\Phi_{\mathcal{L}} = \{\alpha \in \Phi : (\alpha^\vee)^*\mathcal{L} \text{ is trivial}\}$$

and $\Omega_{\mathcal{L}}$ is the $W_{\mathcal{L}}$-stabilizer of $\Phi_{\mathcal{L}}^+ = \Phi^+ \cap \Phi_{\mathcal{L}}$. Let $\mathcal{H}$ be the connected reductive $\overline{\mathbb{F}_q}$-group with maximal torus $\mathcal{T}$ and root system $\Phi_{\mathcal{L}}$. It is called the endoscopic group of $\mathcal{G}$ associated to $\mathcal{L}$. It can be endowed with a pinning relative to $\mathcal{G}$, as in [LuYu2, §10.2]. Consider the set

$$\mathfrak{B}_{\mathcal{L}} = W_{\mathcal{L}}^\circ \backslash \{w \in W : w\epsilon\mathcal{L} = \mathcal{L}\}.$$

The group $\Omega_{\mathcal{L}} \cong W_{\mathcal{L}}^\circ \backslash W_{\mathcal{L}}$ acts on $\mathcal{B}_{\mathcal{L}}$ by

$$\mathrm{Ad}_\epsilon(\omega) w = \omega w \epsilon(\omega)^{-1}. \tag{2.1}$$

Any other ϵ-stable maximal torus in $\mathcal{G}$ has the form $g\mathcal{T}g^{-1}$ for some $g \in \mathcal{G}$. There are bijections

$$\begin{array}{ccccccc} W_{\mathcal{L}} & \to & W_{g\mathcal{L}} & : & w & \mapsto & gwg^{-1}, \\ \mathfrak{B}_{\mathcal{L}} & \to & \mathfrak{B}_{g\mathcal{L}} & : & w & \mapsto & gw\epsilon(g)^{-1}, \end{array}$$

which induce a bijection

$$\mathfrak{B}_{\mathcal{L}}/\mathrm{Ad}_\epsilon(\Omega_{\mathcal{L}}) \to \mathfrak{B}_{g\mathcal{L}}/\mathrm{Ad}_\epsilon(\Omega_{g\mathcal{L}}).$$

The data $(\mathcal{L}, \mathcal{T}, \beta)$ are considered equivalent to $(g\mathcal{L}, g\mathcal{T}g^{-1}, g\beta\epsilon(g)^{-1})$, and all the below constructions from these two sets of data related by conjugation by g.

Each class $\beta \in \mathfrak{B}_{\mathcal{L}}$ contains a unique element w^β of minimal length, which sends $\Phi_{\mathcal{L}}^+$ to Φ^+. As explained in [LuYu2, §12.1], the relative pinning and the lift $\dot{w}^\beta$ determine an automorphism $\sigma_{\beta\epsilon}$ of $(\mathcal{H}, \mathcal{T}, \Phi_{\mathcal{L}}^+)$, whose action on $\mathcal{T}$ and on the root subgroups in $\mathcal{H}$ corresponds to the action of $\mathrm{Ad}(\dot{w}^\beta) \circ \epsilon$ on $\mathcal{T}$ and on the root subgroups in $\mathcal{G}$. This provides a Frobenius action on $\mathcal{H}$, and hence a $\mathbb{F}_q$-form $\mathcal{H}^{\sigma_{\beta\epsilon}}$ of $\mathcal{H}$. Up to a canonical isomorphism, it does not depend on the choice of $\dot{w}^\beta$.

Next we want to define $\Omega_{\mathcal{L}}$-equivariant structures on $\bigoplus_{\beta \in \mathfrak{B}_{\mathcal{L}}} \mathrm{Rep}_1(H^{\sigma_{\beta\epsilon}})$. For every $\pi_\beta \in \mathrm{Rep}_1(H^{\sigma_{\beta\epsilon}})$ and every $\omega \in \Omega_{\mathcal{L}}$ we have

$$\mathrm{Ad}(\dot{\omega})\pi_\beta \in \mathrm{Rep}_1\big(H^{\mathrm{Ad}(\dot{\omega})\sigma_{\beta\epsilon}\mathrm{Ad}(\dot{\omega})^{-1}}\big).$$

For another representatives $\ddot{\omega}$, $\mathrm{Ad}(\ddot{\omega})\pi_\beta$ differs from $\mathrm{Ad}(\dot{\omega})\pi_\beta$ by a canonical isomorphism, namely $\mathrm{Ad}(\ddot{\omega}\dot{\omega}^{-1})$ with $\ddot{\omega}\dot{\omega}^{-1} \in \mathcal{T}$. Lang's theorem ensures the existence of an isomorphism

$$H^{\mathrm{Ad}(\dot{\omega})\sigma_{\beta\epsilon}\mathrm{Ad}(\dot{\omega})^{-1}} \cong H^{\sigma_{\mathrm{Ad}_\epsilon(\omega)\beta\epsilon}}.$$

It is conjugation by an element of $\mathcal{T}$, which is unique up to $T^{\sigma_{\mathrm{Ad}_\epsilon(\omega)\beta\epsilon}}$. Via this isomorphism we regard $\mathrm{Ad}(\dot\omega)\pi_\beta$ as an element of $\mathrm{Rep}_1(H^{\sigma_{\mathrm{Ad}_\epsilon(\omega)\beta\epsilon}})$. Equivalently, this construction means that we select a representative $n(\omega,\beta) \in N_{\mathcal{G}}(\mathcal{T})$ for ω (unique up $T^{\sigma_{\mathrm{Ad}_\epsilon(\omega)\beta\epsilon}}$ from the left and up to $T^{\sigma_{\beta\epsilon}}$ from the right), such that

$$\sigma_{\mathrm{Ad}_\epsilon(\omega)\beta\epsilon} = \mathrm{Ad}(n(\omega,\beta))\sigma_{\beta\epsilon}\mathrm{Ad}(n(\omega,\beta))^{-1}.$$

Then we define

$$\mathrm{Ad}(\omega)\pi_\beta = \mathrm{Ad}(n(\omega,\beta))\pi_\beta \in \mathrm{Rep}_1(H^{\sigma_{\mathrm{Ad}_\epsilon(\omega)\beta\epsilon}}).$$

Here $\mathrm{Ad}(\omega)\pi$ is independent of the choices, up to canonical isomorphisms (namely inner automorphisms, from $T^{\sigma_{\mathrm{Ad}_\epsilon(\omega)\beta\epsilon}}$). That defines an action of $\Omega_{\mathcal{L}}$ on $\bigoplus_{\beta\in\mathfrak{B}_{\mathcal{L}}} \mathrm{Rep}_1(H^{\sigma_{\beta\epsilon}})$ modulo inner automorphisms.

Now an $\Omega_{\mathcal{L}}$-equivariant structure on $\pi = \bigoplus_{\beta\in\mathfrak{B}_{\mathcal{L}}} \pi_\beta$ consists of a morphism from π to $\mathrm{Ad}(\omega)\pi$ for each $\omega\in\Omega_{\mathcal{L}}$, multiplicative in ω up to canonical isomorphisms. To make that explicit we need, for each representative $n(\omega,\beta)$, a morphism

$$\pi(n(\omega,\beta)) : \pi_\beta \to \mathrm{Ad}(n(\omega,\beta))\pi_\beta \tag{2.2}$$

such that:

- $\pi(n(1,\beta)) = \pi_\beta(n(1,\beta))$ for all $n(1,\beta) \in T^{\sigma_{\beta\epsilon}}$,
- for all eligible $n(\omega',\mathrm{Ad}_\epsilon(\omega)\beta)$:

$$\pi\big(n(\omega',\mathrm{Ad}_\epsilon(\omega)\beta)n(\omega,\beta)\big) = \pi\big(n(\omega',\mathrm{Ad}_\epsilon(\omega)\beta)\big)\,\pi\big(n(\omega,\beta)\big).$$

The category $\bigoplus_{\beta\in\mathfrak{B}_{\mathcal{L}}} \mathrm{Rep}_1(H^{\sigma_{\beta\epsilon}})$ enriched with $\Omega_{\mathcal{L}}$-equivariant structures is denoted

$$\Big(\bigoplus\nolimits_{\beta\in\mathfrak{B}_{\mathcal{L}}} \mathrm{Rep}_1(H^{\sigma_{\beta\epsilon}})\Big)^{\Omega_{\mathcal{L}}}.$$

The following result is a version of [LuYu2, Corollary 12.7] without cells.

Theorem 2.1. *Let $\mathcal{G}^\epsilon$ be a connected reductive $\mathbb{F}_q$-group and let $W\mathcal{L}$ be an ϵ-stable orbit of character sheaves on $\mathcal{T}$. There is a canonical equivalence of categories*

$$\mathrm{Rep}_{W\mathcal{L}}(G^\epsilon) \cong \Big(\bigoplus\nolimits_{\beta\in\mathfrak{B}_{\mathcal{L}}} \mathrm{Rep}_1(H^{\sigma_{\beta\epsilon}})\Big)^{\Omega_{\mathcal{L}}}.$$

Proof. Let $\mathbf{c}$ be a cell in $W_{\mathcal{L}}^\circ$ and let $[\mathbf{c}] \subset W\times W\mathcal{L}$ be the associated two-sided cell, as in [LuYu2, §11.4]. They are related by

$$[\mathbf{c}] \cap (W_{\mathcal{L}}^\circ \times \{\mathcal{L}\}) = \Omega_{\mathcal{L}}\mathbf{c}. \tag{2.3}$$

By construction

$$\mathrm{Rep}_{W\mathcal{L}}(G^\epsilon) = \bigoplus\nolimits_{[\mathbf{c}]} \mathrm{Rep}_{W\mathcal{L}}^{[\mathbf{c}]}(G^\epsilon). \tag{2.4}$$

Let $\Omega_{\mathbf{c}}$ be the stabilizer of $\mathbf{c}$ in $\Omega_{\mathcal{L}}$ and write

$$\mathfrak{B}_{\mathbf{c}} = \{\beta\in\mathfrak{B}_{\mathcal{L}} : w^\beta\epsilon \text{ stabilizes } \mathbf{c}\}.$$

By [LuYu2, Corollary 12.7] there are canonical equivalences of categories

$$\mathrm{Rep}_{W\mathcal{L}}^{[\mathbf{c}]}(G^\epsilon) \cong \bigoplus_{\beta\in\mathfrak{B}_{\mathbf{c}}/\mathrm{Ad}_\epsilon(\Omega_{\mathbf{c}})} \mathrm{Rep}_1^{\mathbf{c}}\big(H^{\sigma_{\beta\epsilon}}\big)^{\Omega_{\mathbf{c},\beta}} \cong \Big(\bigoplus_{\beta\in\mathfrak{B}_{\mathbf{c}}} \mathrm{Rep}_1^{\mathbf{c}}\big(H^{\sigma_{\beta\epsilon}}\big)\Big)^{\Omega_{\mathcal{L}}}. \tag{2.5}$$

When $\beta \in \mathfrak{B}_{\mathcal{L}} \setminus \mathfrak{B}_{\mathbf{c}}$, $H^{\sigma_{\beta\epsilon}}$ has no nonzero representations associated to the cell $\mathbf{c}$ (because the cell of an irreducible representation is unique). Hence the right hand side of (2.5) can be rewritten as

$$\Big(\bigoplus\nolimits_{\beta \in \mathfrak{B}_{\mathcal{L}}} \mathrm{Rep}_1^{\Omega_{\mathcal{L}} \mathbf{c}} \big(H^{\sigma_{\beta\epsilon}} \big) \Big)^{\Omega_{\mathcal{L}}}.$$

Now we take the direct sum over all two-sided cells $[\mathbf{c}]$ and with (2.4) and (2.3) we obtain

$$\mathrm{Rep}_{W\mathcal{L}}(G^\epsilon) \cong \bigoplus_{[\mathbf{c}]} \Big(\bigoplus_{\beta \in \mathfrak{B}_{\mathcal{L}}} \mathrm{Rep}_1^{\Omega_{\mathcal{L}} \mathbf{c}} \big(H^{\sigma_{\beta\epsilon}} \big) \Big)^{\Omega_{\mathcal{L}}} = \Big(\bigoplus_{\beta \in \mathfrak{B}_{\mathcal{L}}} \mathrm{Rep}_1 \big(H^{\sigma_{\beta\epsilon}} \big) \Big)^{\Omega_{\mathcal{L}}}. \qquad \square$$

We turn our attention to rational series for G^ϵ. Let $\mathcal{G}^\vee$ be the dual group of $\mathcal{G}$. It is endowed with the $\mathbb{F}_q$-structure from the Frobenius action dual to ϵ, which we shall also denote ϵ. Recall that every rational series is parametrized by the (rational) conjugacy class of a semisimple element s of $G^{\vee\epsilon} = \mathcal{G}^\vee(\mathbb{F}_q)$. In terms of $(\mathcal{L}, \mathcal{T}^\epsilon)$, the condition $s \in G^{\vee\epsilon}$ means that $\mathcal{L}$ must be ϵ-stable. That can always be arranged:

Lemma 2.2. *Suppose that $W\mathcal{L}$ is ϵ-stable. Then $(\mathcal{L}, \mathcal{T})$ is $\mathcal{G}$-conjugate to an ϵ-stable pair $(\mathcal{L}_w, \mathcal{T}_w)$ and* $\mathrm{Rep}_{W\mathcal{L}}(G^\epsilon) = \mathrm{Rep}_{W(\mathcal{G},\mathcal{T}_w)\mathcal{L}_w}(G^\epsilon)$.

Proof. Select $w \in W$ such that $w\epsilon\mathcal{L} = \mathcal{L}$. By the surjectivity of Lang's map on $\mathcal{G}$ [GeMa, Theorem 1.4.8], there exists $g_w \in \mathcal{G}$ such that $g_w^{-1}\epsilon(g_w)$ represents $w \in N_{\mathcal{G}}(\mathcal{T})/\mathcal{T}$. Then

$$\begin{aligned} & g_w \mathcal{T} g_w^{-1} = g_w w \mathcal{T} w^{-1} g_w^{-1} = \epsilon(g_w)\mathcal{T}\epsilon(g_W)^{-1} = \epsilon(g_w)\epsilon(\mathcal{T})\epsilon(g_w)^{-1} = \epsilon(g_w \mathcal{T} g_w^{-1}), \\ & g_w \mathcal{L} = g_w w \epsilon \mathcal{L} = \epsilon(g_w)\epsilon\mathcal{L} = \epsilon g_w \mathcal{L}, \end{aligned}$$

so $(\mathcal{L}_w, \mathcal{T}_w) := (g_w\mathcal{L}, g_w\mathcal{T}g_w^{-1})$ is ϵ-stable. The equality of categories holds because $W\mathcal{L}$ and $W(\mathcal{G}, \mathcal{T}_w)\mathcal{L}_w$ correspond to the same geometric conjugacy in $G^{\vee\epsilon}$. $\square$

With Lemma 2.2 in mind, we assume that $(\mathcal{L}, \mathcal{T})$ is ϵ-stable. Recall from [DeLu, §5.2] or [GeMa, Corollary 2.5.14] that there is a natural bijection between

- the set of G^ϵ-conjugacy classes of ϵ-stable pairs $(\mathcal{L}, \mathcal{T})$,
- the set of $G^{\vee\epsilon}$-conjugacy classes of ϵ-stable pairs in $G^{\vee\epsilon}$ formed by a maximal torus and an element thereof.

We record that as

$$(\mathcal{L}, \mathcal{T}) \longleftrightarrow (s, \mathcal{T}^\vee). \tag{2.6}$$

The intersection of the geometric conjugacy class $\mathrm{Ad}(\mathcal{G}^\vee)s$ with $\mathcal{T}^\vee$ is Ws with $W \cong W(\mathcal{G}^\vee, \mathcal{T}^\vee)$. By [DeLu, §5] or [GeMa, Propositions 2.4.28, 2.5.5 and p. 155], this corresponds precisely to $(W\mathcal{L}, \mathcal{T})$. Then $\mathrm{Rep}_{W\mathcal{L}}(G^\epsilon)$ is a direct sum of finitely many rational series, which we want to parametrize in terms of $\mathcal{L}$.

Lemma 2.3. *The quotient set $\mathfrak{B}_{\mathcal{L}}/\mathrm{Ad}_\epsilon(\Omega_{\mathcal{L}})$ naturally parametrizes the $G^{\vee\epsilon}$-conjugacy classes in $(\mathrm{Ad}(\mathcal{G}^\vee)s)^\epsilon$. The parametrization can be realized as follows. For $w \in W_{\mathcal{L}}$ representing $\beta \in \mathfrak{B}_{\mathcal{L}}$, find $h_w \in \mathcal{G}^\vee$ such that $h_w^{-1}\epsilon(h_w)$ represents w as element of $W \cong N_{\mathcal{G}^\vee}(\mathcal{T}^\vee)/\mathcal{T}^\vee$. Then the bijection sends $\beta = [w] \in \mathfrak{B}_{\mathcal{L}}/\mathrm{Ad}_\epsilon(\Omega_{\mathcal{L}})$ to* $\mathrm{Ad}(G^{\vee\epsilon})(h_w s h_w^{-1})$.

Proof. As $\mathrm{Fr}(s) = s$, the $\mathcal{G}^\vee$-conjugacy class of s is isomorphic to $\mathcal{G}^\vee / Z_{\mathcal{G}^\vee}(s)$ as $\mathbb{F}_q$-variety. Galois cohomology provides an exact sequence

$$Z_{\mathcal{G}^\vee}(s)^{\mathrm{Fr}} \to (\mathcal{G}^\vee)^{\mathrm{Fr}} \to \big(\mathcal{G}^\vee / Z_{\mathcal{G}^\vee}(s)\big)^{\mathrm{Fr}} \to H^1(\mathrm{Fr}, Z_{\mathcal{G}^\vee}(s)) \to H^1(\mathrm{Fr}, \mathcal{G}^\vee). \tag{2.7}$$

Here $H^1(\mathrm{Fr}, \mathcal{G}^\vee)$ is a short notation for the cohomology of $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ with coefficients in $\mathcal{G}^\vee$. By the surjectivity of Lang's map on connected $\mathbb{F}_q$-groups, $H^1(\mathrm{Fr}, \mathcal{G}^\vee)$ is trivial and

$$H^1(\mathrm{Fr}, Z_{\mathcal{G}^\vee}(s)) \cong H^1\big(\mathrm{Fr}, \pi_0(Z_{\mathcal{G}^\vee}(s))\big).$$

Thus (2.7) says that the set we are looking for is parametrized naturally by $H^1\big(\mathrm{Fr}, \pi_0(Z_{\mathcal{G}^\vee}(s))\big)$. This bijection sends

$$gsg^{-1} \in (\mathrm{Ad}(\mathcal{G}^\vee)s)^{\mathrm{Fr}} \quad \text{to} \quad [\mathrm{Fr} \mapsto g^{-1}\mathrm{Fr}(g)] \in H^1(\mathrm{Fr}, Z_{\mathcal{G}^\vee}(s)). \tag{2.8}$$

By Steinberg's description of the centralizers of semisimple elements in reductive groups [Ste], $\pi_0(Z_{\mathcal{G}^\vee}(s)) \cong W_s^\circ \backslash W_s$, where $W_s^\circ \subset W \cong W(\mathcal{G}^\vee, \mathcal{T}^\vee)$ is generated by the reflections s_α with $\alpha^\vee(s) = 1$. In terms of $(\mathcal{L}, \mathcal{T})$, this becomes

$$\pi_0(Z_{\mathcal{G}^\vee}(s)) \cong W_s^\circ \backslash W_s \cong W_{\mathcal{L}}^\circ \backslash W_{\mathcal{L}} = \mathcal{B}_{\mathcal{L}} = W_{\mathcal{L}}^\circ \backslash W_{\mathcal{L}}^\circ \rtimes \Omega_{\mathcal{L}} \cong \Omega_{\mathcal{L}}. \tag{2.9}$$

It follows that $H^1\big(\mathrm{Fr}, \pi_0(Z_{\mathcal{G}^\vee}(s))\big)$ is naturally isomorphic to

$$H^1(\mathrm{Fr}, \Omega_{\mathcal{L}}) = (\Omega_{\mathcal{L}})_{\mathrm{Fr}} = \mathfrak{B}_{\mathcal{L}} / \mathrm{Ad}_\epsilon(\Omega_{\mathcal{L}}). \tag{2.10}$$

The map from (2.10) to conjugacy classes of s is essentially the inverse of (2.8). $\square$

Lemmas 2.2 and 2.3 provide a bijection between

- the set of semisimple conjugacy classes in $G^{\vee\epsilon}$,
- the W-orbits of pairs $(\mathcal{L}, \beta)$, where $\mathcal{L}$ is a character sheaf on $\mathcal{T}$ with ϵ-stable W-orbit and $\beta \in \mathfrak{B}_{\mathcal{L}} / \mathrm{Ad}_\epsilon(\mathcal{L})$,
- triples $(\mathcal{L}', \mathcal{T}', \beta')$ where $\mathcal{T}'$ is an ϵ-stable maximal torus in $\mathcal{G}$, $(\mathcal{L}', \beta')$ is as in the previous bullet and any such triple of the form $(g\mathcal{L}', g\mathcal{T}'g^{-1}, g\beta'\epsilon(g)^{-1})$ with $g \in \mathcal{G}^\circ$ is considered as equivalent to $(\mathcal{L}', \mathcal{T}', \beta')$.

Let us make it more explicit, starting from $W(\mathcal{L}, \beta)$. As in Lemma 2.2, pick $g \in \mathcal{G}$ such that $(g\mathcal{L}, g\mathcal{T}g^{-1})$ is ϵ-stable, and replace β by $g\beta\epsilon(g)^{-1} \in \mathfrak{B}_{g\mathcal{L}}$. Associate $s \in G^{\vee\epsilon}$ to $(g\mathcal{L}, g\mathcal{T}g^{-1})$ by (2.6). From s and $g\beta\epsilon(g)^{-1}$, Lemma 2.3 produces $s_\beta = h_w s h_w^{-1}$, where w represents $g\beta\epsilon(g)^{-1}$. We record this as

$$W(\mathcal{L}, \beta) \longleftrightarrow s_\beta. \tag{2.11}$$

The explicit description of the correspondence (2.6) in [GeMa, §2.5.12 and Corollary 2.4.14] is in terms of elements h_w and g_w as in the proof of Lemma 2.2. Consequently

$$(s_\beta, h_w \mathcal{T}^\vee h_w^{-1}) \text{ corresponds to } (\mathcal{L}_w, \mathcal{T}_w). \tag{2.12}$$

We denote the rational series (in $\mathrm{Rep}(G^\epsilon)$) associated to $s_\beta \in G^{\vee\epsilon}$ by $\mathrm{Rep}_{s_\beta}(G^\epsilon)$. By the correspondence (2.11) we may write $\mathrm{Rep}_{s_\beta}(G^\epsilon) = \mathrm{Rep}_{W(\mathcal{L},\beta)}(G^\epsilon)$. Lemma 2.3 shows how to write a geometric series as a direct sum of rational series, with s_β as in (2.11):

$$\mathrm{Rep}_{W\mathcal{L}}(G^\epsilon) = \bigoplus_{\beta \in \mathfrak{B}_{\mathcal{L}} / \mathrm{Ad}_\epsilon(\Omega_{\mathcal{L}})} \mathrm{Rep}_{W(\mathcal{L},\beta)}(G^\epsilon) = \bigoplus_{\beta \in \mathfrak{B}_{\mathcal{L}} / \mathrm{Ad}_\epsilon(\Omega_{\mathcal{L}})} \mathrm{Rep}_{s_\beta}(G^\epsilon), \tag{2.13}$$

Let $\mathcal{G} \to \mathcal{G}_c$ be a regular embedding of reductive $\mathbb{F}_q$-groups [GeMa, §1.7], so $Z(\mathcal{G}_c)$ is connected and $\mathcal{G}_{\mathrm{der}} = \mathcal{G}_{1,\mathrm{der}}$. As is common, we will analyse series for G^ϵ in terms of those for G_c^ϵ. Let $\mathcal{T}_c \subset \mathcal{G}_c$ be the maximal torus with $\mathcal{T}_c \cap \mathcal{G} = \mathcal{T}$. We fix an extension $\mathcal{L}_c$ of $\mathcal{L}$ to a character sheaf on $\mathcal{T}_c^\epsilon$, and we let $s_c \in T_c^\epsilon$ be the corresponding element. The canonical quotient map $\mathcal{G}_c^\vee \to \mathcal{G}^\vee$ sends $\mathcal{T}_c^\vee \to \mathcal{T}^\vee$ and s_c to s. The group $W_c = W(\mathcal{G}_c, \mathcal{T}_c)$ can be identified with W. Since $\mathcal{G}_c$ has connected centre, $\mathcal{H}_c = Z_{\mathcal{G}_c}(s_c)$ is connected and

$$W_{s_c} = W_{s_c}^\circ \cong W_{\mathcal{L}_c} = W_{\mathcal{L}_c}^\circ .$$

The rational and geometric series for G_c^ϵ coincide, and by [Lus2]:

$$\mathrm{Rep}_s(G^\epsilon) = \mathrm{Res}^{G_c^\epsilon}_{G^\epsilon}\big(\mathrm{Rep}_{s_c}(G_c^\epsilon)\big) = \mathrm{Res}^{G_c^\epsilon}_{G^\epsilon}\big(\mathrm{Rep}_{W\mathcal{L}_c}(G_c^\epsilon)\big). \tag{2.14}$$

Since $\mathfrak{B}_{\mathcal{L}_c} = \{1\}$ and $\Omega_{\mathcal{L}_c} = \{1\}$, Theorem 2.1 for G_c^ϵ simplifies to

$$\mathrm{Rep}_{W\mathcal{L}_c}(G_c^\epsilon) \cong \mathrm{Rep}_1(H_c^\epsilon). \tag{2.15}$$

The group $\mathcal{H}_c$ is generated by $\mathcal{T}_c$ and the root subgroups U_α with $\alpha^\vee(s_c) = 1$. The latter condition depends only on s, so these are precisely the roots in $\Phi_{\mathcal{L}}$ and $\mathcal{H}_c$ is the full preimage of $\mathcal{H}$ in $\mathcal{G}_c$.

Theorem 2.4. *Let $(\mathcal{L}, \mathcal{T})$ be ϵ-stable, corresponding to $(s, \mathcal{T}^\vee)$ with $s \in G^{\vee\epsilon}$. There is a canonical equivalence of categories*

$$\mathrm{Rep}_s(G^\epsilon) = \mathrm{Rep}_{W(\mathcal{L},1)}(G^\epsilon) \cong \mathrm{Rep}_1(H^\epsilon)^{\Omega_{\mathcal{L}}^\epsilon},$$

where $\Omega_{\mathcal{L}}^\epsilon \cong \pi_0(Z_{\mathcal{G}^\vee}(s))^\epsilon$.

Proof. We will use the functorial properties of the constructions of Lusztig and Yun. The functor

$$\mathrm{Res}^{G_c^\epsilon}_{G^\epsilon} : \mathrm{Rep}_{W\mathcal{L}_c}(G_c^\epsilon) \to \mathrm{Rep}_{W\mathcal{L}}(G^\epsilon)$$

corresponds to restriction of sheaves from $\mathcal{G}_c$ to $\mathcal{G}$ in [LuYu2]. Via (2.15) and Theorem 2.1, it becomes a functor

$$\mathrm{Rep}_1(H_c^\epsilon) \longrightarrow \bigoplus\nolimits_{\beta \in \mathfrak{B}_{\mathcal{L}}/\mathrm{Ad}_\epsilon(\Omega_{\mathcal{L}})} \mathrm{Rep}_1\big(H^{\sigma_\beta \epsilon}\big)^{\Omega_{\mathcal{L},\beta}}. \tag{2.16}$$

From [LuYu2, §12.10] one can see what (2.16) does. Namely, first it restricts representations of H_c^ϵ to H^ϵ, and then it endows them with some equivariant structure for $\Omega_{\mathcal{L},\beta} = \mathrm{Stab}_{\Omega_{\mathcal{L}}}(\beta)$ with $\beta = 1 \in \mathfrak{B}_{\mathcal{L}}$. In particular the image of (2.16) lies entirely in the summand $\beta = 1$ of the codomain. Together with (2.14), it follows that the image of $\mathrm{Rep}_s(G^\epsilon)$ under Theorem 2.1 is contained in $\mathrm{Rep}_1(H^\epsilon)^{\Omega_{\mathcal{L},1}}$.

Now we consider another ϵ-fixed element in the geometric conjugacy class of s, namely $s_\beta = h_w s h_w^{-1}$ from (2.13) with $\beta \in \mathfrak{B}_{\mathcal{L}} = W_{\mathcal{L}}^\circ \backslash W_{\mathcal{L}}$ represented by $w = w^\beta$ of minimal length in its class. We saw in (2.12) that it corresponds to an ϵ-stable pair $(\mathcal{L}_w, \mathcal{T}_w) = (g_w \mathcal{L}, g_w \mathcal{T} g_w^{-1})$. The above constructions can also be applied to $(s_\beta, \mathcal{L}_w, \mathcal{T}_w)$, and they show that

$$\mathrm{Res}^{G_c^\epsilon}_{G^\epsilon} : \mathrm{Rep}_{W\mathcal{L}_{w,1}}(G_c^\epsilon) \to \mathrm{Rep}_{(g_w W g_w^{-1})\mathcal{L}_w}(G^\epsilon) = \mathrm{Rep}_{W\mathcal{L}}(G^\epsilon)$$

corresponds to a functor

$$\mathrm{Rep}_1\big((g_w H g_w^{-1})_c^\epsilon\big) \to \mathrm{Rep}_1\big((g_w H g_w^{-1})^\epsilon\big)^{\Omega_{\mathcal{L}_w,1}}. \tag{2.17}$$

When we restore the previous bookkeeping by conjugation with g_w^{-1}, the Frobenius action ϵ is replaced by $\mathrm{Ad}(g_w^{-1}\epsilon(g_w))\epsilon$, which equals $\mathrm{Ad}(\dot{w}^\beta) \circ \epsilon$ for some lift $\dot{w}^\beta \in N_{\mathcal{G}}(\mathcal{T})$ of w^β. Following the conventions in [LuYu2, §12.1] as mentioned after (2.1), (2.17) becomes a functor

$$\mathrm{Rep}_1\big(H_c^{\sigma_\beta \epsilon}\big) \to \mathrm{Rep}_1\big(H^{\sigma_\beta \epsilon}\big)^{\Omega_{\mathcal{L},\beta}}.$$

Thus Theorem 2.1 sends

$$\mathrm{Rep}_{s_\beta}(G^\epsilon) = \mathrm{Res}^{G_c^\epsilon}_{G^\epsilon}\big(\mathrm{Rep}_{W\mathcal{L}_{w,1}}(G_c^\epsilon)\big) \text{ to } \mathrm{Rep}_1\big(H^{\sigma_\beta \epsilon}\big)^{\Omega_{\mathcal{L},\beta}}. \tag{2.18}$$

If we take the direct sum over $\beta \in \mathfrak{B}_\mathcal{L}/\mathrm{Ad}_\epsilon(\Omega_\mathcal{L})$ of the functors (2.18), then by (2.13) we recover the equivalence from Theorem 2.1. Hence (2.18) is an equivalence of categories for every $\beta \in \mathfrak{B}_\mathcal{L}$.

In the special case $\beta = 1$, we have

$$\Omega_{\mathcal{L},1} = \{\omega \in \Omega_\mathcal{L} : \omega\epsilon(\omega)^{-1} \in W_\mathcal{L}^\circ\} = \{\omega \in \Omega_\mathcal{L} : \omega\epsilon(\omega)^{-1} = 1\} = \Omega_\mathcal{L}^\epsilon. \tag{2.19}$$

From (2.9) we see that this group is canonically isomorphic to $\pi_0(Z_{\mathcal{G}^\vee}(s))^\epsilon$. □

3. Disconnected groups

We no longer require that $\mathcal{G}$ is connected. Instead, we assume the following throughout this section:

Condition 3.1. $\mathcal{G}^\epsilon$ is a smooth $\mathbb{F}_q$-group scheme with reductive neutral component, such that every connected component of $\mathcal{G}$ is ϵ-stable.

If $\mathcal{G}$ would have connected components that are not ϵ-stable, then they would not contribute to G^ϵ and could be ignored for the study of G^ϵ-representations. Hence the last part of Condition 3.1 is not a restriction, it merely serves to simplify the notations. Notice that $\pi_0(\mathcal{G})$ may be infinite.

Let $\pi_0^\epsilon(\mathcal{G})$ be the group of connected components of $\mathcal{G}$. If $x \in \mathcal{G}$ represents an element of $\pi_0(\mathcal{G})$, then $x^{-1}\epsilon(x) \in \mathcal{G}^\circ$. By Lang's theorem there exist $g \in \mathcal{G}^\circ$ with $g^{-1}\epsilon(g) = x^{-1}\epsilon(x)$. Then $xg^{-1} \in x\mathcal{G}^\circ$ is fixed by ϵ, so $x\mathcal{G}^\circ$ has $\mathbb{F}_q$-points.

For each $\omega \in \pi_0(\mathcal{G})$ we pick a representative $\dot\omega \in G^\epsilon = \mathcal{G}^\epsilon(\mathbb{F}_q)$. That gives a map

$$\mathrm{Ad}(\dot\omega) : \mathrm{Rep}(G^{\circ\epsilon}) \to \mathrm{Rep}(G^{\circ\epsilon}).$$

It depends on the choice of $\dot\omega$, but the map $\mathrm{Ad}(\ddot\omega)$ for another representative $\ddot\omega$ differs only by $\mathrm{Ad}(\ddot\omega\dot\omega^{-1})$ with $\ddot\omega\dot\omega^{-1} \in G^{\circ\epsilon}$. Hence $\mathrm{Ad}(\dot\omega)$ is unique up to canonical isomorphisms.

A $\pi_0(\mathcal{G})$-equivariant structure on $\pi^\circ \in \mathrm{Rep}(G^{\circ\epsilon})$ consists of a morphism $\pi(\dot\omega) : \pi^\circ \to \mathrm{Ad}(\dot\omega)\pi^\circ$ for each $\omega \in \pi_0(\mathcal{G})$. These must be multiplicative up to canonical isomorphisms, which means that

$$\pi(\dot\omega_1)\pi(\dot\omega_2) = \pi^\circ(\dot\omega_1\dot\omega_2(\dot\omega_3)^{-1})\pi(\dot\omega_3)$$

whenever $\omega_1\omega_2 = \omega_3 \in \pi_0(\mathcal{G})$.

Lemma 3.2. *The category $\mathrm{Rep}(G^{\circ\epsilon})^{\pi_0(\mathcal{G})}$ of $G^{\circ\epsilon}$-representations enriched with a $\pi_0(\mathcal{G})$-equivariant structure is naturally equivalent with $\mathrm{Rep}(G^\epsilon)$.*

Proof. For any G^ϵ-representation π, write $\pi^\circ = \mathrm{Res}^{G^\epsilon}_{G^{\circ\epsilon}}\pi$. We obtain a $\pi_0(\mathcal{G})$-equivariant structure on π° by defining the required morphism $\pi^\circ \to \mathrm{Ad}(\dot\omega)\pi^\circ$ as $\pi(\dot\omega)$, for any representative $\dot\omega$ of $\omega \in \pi_0(\mathcal{G})$.

Conversely, suppose that $\pi^\circ \in \mathrm{Rep}(G^{\circ\epsilon})$ has a $\pi_0(\mathcal{G})$-equivariant structure. Then we have a morphism of $G^{\circ\epsilon}$-representations $\pi(\dot\omega) : \pi^\circ \to \mathrm{Ad}(\dot\omega)\pi^\circ$, for any representative $\dot\omega \in G^\epsilon$ of $\omega \in \pi_0(\mathcal{G})$. By the multiplicativity of equivariant structures, these $\pi(\dot\omega)$ combine to a representation of

$$\bigcup\nolimits_{\omega \in \pi_0(\mathcal{G})} \omega G^{\circ\epsilon} = G^\epsilon$$

on the vector space underlying π°. It follows from $\pi(1) = \mathrm{id}$ that $\pi(g) = \pi^\circ(g)$ for all $g \in G^{\circ\epsilon}$, so π extends π°.

It is clear that the functors $\pi \mapsto \pi^\circ$ and $\pi^\circ \mapsto \pi$ are mutually inverse, so they define equivalences of categories. $\square$

Let $\mathcal{T}$ be a maximal $\mathbb{F}_q$-torus in $\mathcal{G}^\circ$ and let $\mathcal{T}_0$ be a maximally split maximal $\mathbb{F}_q$-torus in $\mathcal{G}^\circ$. Let $\mathcal{B}$ and $\mathcal{B}_0$ be Borel $\mathbb{F}_q$-subgroup of $\mathcal{G}^\circ$ containing, respectively, $\mathcal{T}$ and $\mathcal{T}_0$. Although $\mathcal{T}$ may equal to $\mathcal{T}_0$, we remain more flexible by not requiring that.

Lemma 3.3. *The group G^ϵ equals $G^{\circ\epsilon} N_{G^\epsilon}(T_0^\epsilon, B_0^\epsilon)$.*

Proof. For $x \in G^\epsilon$, $x\mathcal{B}_0 x^{-1}$ is an ϵ-stable Borel subgroup of $\mathcal{G}^\circ$. All such subgroups are $G^{0\epsilon}$-conjugate [Spr, 15.4.6.ii], so there exists $g \in G^{\circ\epsilon}$ such that $gx\mathcal{B}_0 x^{-1} g^{-1} = \mathcal{B}_0$.

Let $\mathcal{S}$ be the maximal $\mathbb{F}_q$-split subtorus of $\mathcal{T}_0$. Since $\mathcal{G}^\circ$ is quasi-split over $\mathbb{F}_q$, the centralizer $Z_{\mathcal{G}^\circ}(\mathcal{S})$ equals $\mathcal{T}_0$. As $gx \in N_{G^\epsilon}(B_0^\epsilon)$, $gx\mathcal{S}x^{-1}g^{-1}$ is a maximal $\mathbb{F}_q$-split torus in $\mathcal{B}_0$. All such tori are B^ϵ-conjugate [Spr, 14.4.3], so we can find $b \in B^\epsilon$ such that $bgx\mathcal{S}(bgx)^{-1} = \mathcal{S}$. Then

$$bgx\mathcal{T}_0(bgx)^{-1} = bgxZ_{\mathcal{G}^\circ}(\mathcal{S})(bgx)^{-1} = Z_{\mathcal{G}^\circ}\big(bgx\mathcal{S}(bgx)^{-1}\big) = Z_{\mathcal{G}^\circ}(\mathcal{S}) = \mathcal{T}_0,$$

so $bgx \in N_{G^\epsilon}(B_0^\epsilon, T_0^\epsilon)$. It follows that $x \in G^{\circ\epsilon} N_{G^\epsilon}(B_0^\epsilon, T_0^\epsilon)$. $\square$

Lemma 3.3 implies that the natural map $N_{G^\epsilon}(T_0^\epsilon, B_0^\epsilon)/T_0^\epsilon \to \pi_0(\mathcal{G})$ is a group isomorphism. That yields split short exact sequences

$$\begin{array}{ccccccccc} 1 & \to & W^\circ = W(\mathcal{G}^\circ, \mathcal{T}) & \to & W = W(\mathcal{G}, \mathcal{T}) & \to & \pi_0(\mathcal{G}) & \to & 1, \\ 1 & \to & W(G^{\circ\epsilon}, T_0^\epsilon) & \to & W(G^\epsilon, T_0^\epsilon) & \to & \pi_0(\mathcal{G}) & \to & 1. \end{array} \tag{3.1}$$

Condition 3.1 enables us to pick representatives $\dot{\omega} \in N_{G^\epsilon}(T_0^\epsilon, B_0^\epsilon)$ for $\omega \in \pi_0(\mathcal{G})$, unique up to T_0^ϵ. The action of $\pi_0(\mathcal{G})$ on $\mathrm{Rep}(G^{\circ\epsilon})$ permutes the subcategories $\mathrm{Rep}_{W^\circ\mathcal{L}}(G^{\circ\epsilon})$ and stabilizes the subcategory

$$\mathrm{Rep}_{W\mathcal{L}}(G^{\circ\epsilon}) = \bigoplus_{w \in W^\circ \backslash W / W_{\mathcal{L}}} \mathrm{Rep}_{W^\circ w\mathcal{L}}(G^{\circ\epsilon}) \tag{3.2}$$

generated by irreducible $G^{\circ\epsilon}$-representations with semisimple parameter in $W\mathcal{L}$. Indeed, for $\mathcal{T} = \mathcal{T}_0$ we see this with the representatives of $\pi_0(\mathcal{G})$ in $N_{G^\epsilon}(T_0^\epsilon, B_0^\epsilon)$. For other $\mathcal{T}$ it follows because $(\mathcal{T}, \mathcal{B})$ and $(\mathcal{T}_0, \mathcal{B}_0)$ are $\mathcal{G}^\circ$-conjugate and $\mathrm{Rep}_{W^\circ\mathcal{L}}(G^{\circ\epsilon})$ depends only on $W^\circ\mathcal{L}$ up to $\mathcal{G}^\circ$-conjugacy.

We define the geometric series $\mathrm{Rep}_{W\mathcal{L}}(G^\epsilon)$ as the category of G^ϵ-representations whose restriction to $G^{\circ\epsilon}$ lies in $\mathrm{Rep}_{W\mathcal{L}}(G^{\circ\epsilon})$. Thus Lemma 3.2 provides a natural equivalence of categories

$$\mathrm{Rep}_{W\mathcal{L}}(G^\epsilon) \cong \mathrm{Rep}_{W\mathcal{L}}(G^{\circ\epsilon})^{\pi_0(\mathcal{G})}. \tag{3.3}$$

The group $\pi_0(\mathcal{G}) \cong W/W^\circ$ permutes the direct summands in (3.2) transitively, which leads to a natural equivalence

$$\mathrm{Rep}_{W\mathcal{L}}(G^{\circ\epsilon})^{\pi_0(\mathcal{G})} \cong \mathrm{Rep}_{W^\circ\mathcal{L}}(G^{\circ\epsilon})^{\pi_0(\mathcal{G})_{W^\circ\mathcal{L}}}. \tag{3.4}$$

We define $\Omega_{\mathcal{L}} = \mathrm{Stab}_{W_{\mathcal{L}}}(\Phi_{\mathcal{L}}^+) \cong W_{\mathcal{L}}/W_{\mathcal{L}}^\circ$ as before, and we let $\Omega_{\mathcal{L}}^\circ \cong (W^\circ)_{\mathcal{L}}/W_{\mathcal{L}}^\circ$ be its version for $\mathcal{G}^\circ$. Then

$$\pi_0(\mathcal{G})_{W^\circ\mathcal{L}} \cong \Omega_{\mathcal{L}}/\Omega_{\mathcal{L}}^\circ. \tag{3.5}$$

Let $\mathfrak{B}^\circ_{\mathcal{L}}$ be $\mathfrak{B}_{\mathcal{L}}$ for $\mathcal{G}^\circ$. With Theorem 2.1 we can transfer the action of $\pi_0(\mathcal{G})_{W^\circ\mathcal{L}}$ on $\mathrm{Rep}_{W^\circ\mathcal{L}}(G^{\circ\epsilon})$ to an action on

$$\Big(\bigoplus\nolimits_{\beta \in \mathfrak{B}^\circ_{\mathcal{L}}} \mathrm{Rep}_1\big(H^{\sigma_\beta \epsilon}\big)\Big)^{\Omega^\circ_{\mathcal{L}}}, \tag{3.6}$$

unique up to canonical isomorphisms. Let us make that more explicit. The action of $\pi_0(\mathcal{G})_{W^\circ\mathcal{L}}$ on $\bigoplus_{\beta \in \mathfrak{B}^\circ_{\mathcal{L}}} \mathrm{Rep}_1(H^{\sigma_\beta \epsilon})$ can be described in the same way as the action of $\Omega^\circ_{\mathcal{L}}$ in (2.2), now only with representatives $n(\omega,\beta)$ from $N_{\mathcal{G}}(\mathcal{T},\mathcal{B})$. The effect on a morphism $\lambda_{\dot w} : \pi_\beta \to \mathrm{Ad}(\dot w)\pi_\beta$, from an $\Omega^\circ_{\mathcal{L}}$-equivariant structure λ on $\pi = \bigoplus_\beta \pi_\beta$, is the morphism

$$\mathrm{Ad}(n(\omega,\beta))\pi_\beta \to \mathrm{Ad}(\dot w)\mathrm{Ad}(n(\omega,\beta))\pi_\beta$$

given by

$$\lambda_{n(\omega,\beta)^{-1}\dot w n(\omega,\beta)} : \pi_\beta \to \mathrm{Ad}(n(\omega,\beta)^{-1})\mathrm{Ad}(\dot w)\mathrm{Ad}(n(w,\beta))\pi_\beta .$$

Via this transfer of actions, every $\pi_0(\mathcal{G})_{W^\circ\mathcal{L}}$-equivariant structure on (3.6) can be identified with one $\pi_0(\mathcal{G})_{W^\circ\mathcal{L}}$-equivariant structure on $\mathrm{Rep}_{W^\circ\mathcal{L}}(G^{\circ\epsilon})$.

The group $\Omega_{\mathcal{L}}$ acts on $\bigoplus_{\beta \in \mathfrak{B}^\circ_{\mathcal{L}}} \mathrm{Rep}_1(H^{\sigma_\beta \epsilon})$, by the same constructions as in the case of a connected $\mathcal{G}$. Again this action is well-defined up to canonical isomorphisms, and there is a notion of $\Omega_{\mathcal{L}}$-equivariant structures on $\bigoplus_{\beta \in \mathfrak{B}^\circ_{\mathcal{L}}} \mathrm{Rep}_1(H^{\sigma_\beta \epsilon})$.

Theorem 3.4. *Suppose that $\mathcal{G}^\epsilon$ satisfies Condition 3.1, and let $\mathcal{L}$ be a rank one character sheaf on $\mathcal{T}$, such that $W^\circ\mathcal{L}$ is ϵ-stable. There exists a canonical equivalence of categories*

$$\mathrm{Rep}_{W\mathcal{L}}(G^\epsilon) \cong \Big(\bigoplus\nolimits_{\beta \in \mathfrak{B}^\circ_{\mathcal{L}}} \mathrm{Rep}_1\big(H^{\sigma_\beta \epsilon}\big)\Big)^{\Omega_{\mathcal{L}}}.$$

Proof. By Lemma 3.2, (3.3), (3.4) and Theorem 2.1, there are canonical equivalences of categories

$$\mathrm{Rep}_{W\mathcal{L}}(G^\epsilon) \cong \mathrm{Rep}_{W^\circ\mathcal{L}}(G^{\circ\epsilon})^{\pi_0(\mathcal{G})_{W^\circ\mathcal{L}}} \cong \Big(\big(\bigoplus\nolimits_{\beta \in \mathfrak{B}^\circ_{\mathcal{L}}} \mathrm{Rep}_1(H^{\sigma_\beta \epsilon})\big)^{\Omega^\circ_{\mathcal{L}}}\Big)^{\pi_0(\mathcal{G})_{W^\circ\mathcal{L}}}. \tag{3.7}$$

In view of (3.5), a $\pi_0(\mathcal{G})_{W^\circ\mathcal{L}}$-equivariant structure on top of a $\Omega^\circ_{\mathcal{L}}$-equivariant structure contains precisely the same information as an $\Omega_{\mathcal{L}}$-equivariant structure. Hence the right hand side of (3.7) is naturally equivalent to $\bigoplus_{\beta \in \mathfrak{B}^\circ_{\mathcal{L}}} \mathrm{Rep}_1(H^{\sigma_\beta \epsilon})$ enriched with $\Omega_{\mathcal{L}}$-equivariant structures. $\square$

We turn to rational series for G^ϵ. First we have to define them properly. By (2.11), we can parametrize the rational series in $\mathrm{Rep}(G^{\circ\epsilon})$ by W°-orbits of pairs $(\mathcal{L},\beta)$. If we set that up with $\mathcal{T}_0$, (3.1) shows that the groups $\pi_0(\mathcal{G}) \cong N_{G^\epsilon}(T_0^\epsilon, B_0^\epsilon)/T_0^\epsilon$ and $W(\mathcal{G},\mathcal{T}_0) \cong W$ and act naturally on the set of such parameters. Then the definition of $\mathrm{Rep}_{W^\circ(\mathcal{L},\beta)}(G^{\circ\epsilon})$ in terms of generators obtained by Deligne–Lusztig induction shows that $\pi_0(\mathcal{G})$ and W and permute the various rational series according to the actions on their parameters. It follows that the category

$$\mathrm{Rep}_{W(\mathcal{L},\beta)}(G^{\circ\epsilon}) := \bigoplus\nolimits_{\gamma \in W^\circ W_{\mathcal{L},\beta} \backslash W} \mathrm{Rep}_{W^\circ \gamma\mathcal{L}, \gamma\beta\epsilon(\gamma)^{-1}}(G^{\circ\epsilon})$$

is stable under the action of G^ϵ on $\mathrm{Rep}(G^\circ)$. Via conjugation in $\mathcal{G}^\circ$, these considerations also apply to any ϵ-stable maximal torus $\mathcal{T}$ instead of $\mathcal{T}_0$.

We define $\mathrm{Rep}_{W(\mathcal{L},\beta)}(G^\epsilon)$ as the category of G^ϵ-representations whose restriction to $G^{\circ\epsilon}$ lies in $\mathrm{Rep}_{W(\mathcal{L},\beta)}(G^{\circ\epsilon})$. When $W^\circ(\mathcal{L},\beta)$ corresponds to $\mathrm{Ad}(G^{\circ\vee\epsilon})s_\beta$, we write

$$\mathrm{Rep}_{W(\mathcal{L},\beta)}(G^\epsilon) = \mathrm{Rep}_{s_\beta}(G^\epsilon).$$

We call this the rational series of G^ϵ associated to s_β or to $W(\mathcal{L},\beta)$. It actually corresponds to a set of semisimple elements in $G^{\circ\vee\epsilon}$ larger than the rational conjugacy class of s_β, but it is difficult to express that precisely because we do not have a group dual to $\mathcal{G}$.

As the analysis of $\mathrm{Rep}_{W(\mathcal{L},\beta)}(G^\epsilon)$ proceeds like for $G^{\circ\epsilon}$, we will carry it out more briefly. Like in (3.3) and (3.4) we have

$$\mathrm{Rep}_{W(\mathcal{L},\beta)}(G^\epsilon) \cong \mathrm{Rep}_{W(\mathcal{L},\beta)}(G^{\circ\epsilon})^{\pi_0(\mathcal{G})} \cong \mathrm{Rep}_{W^\circ(\mathcal{L},\beta)}(G^{\circ\epsilon})^{\pi_0(\mathcal{G})_{W^\circ(\mathcal{L},\beta)}}. \tag{3.8}$$

There are isomorphisms

$$\pi_0(\mathcal{G})_{W^\circ(\mathcal{L},\beta)} \cong W_{\mathcal{L},\beta}/(W^\circ)_{\mathcal{L},\beta} = (W_{\mathcal{L}}^\circ \rtimes \Omega_{\mathcal{L},\beta})/(W_{\mathcal{L}}^\circ \rtimes \Omega_{\mathcal{L},\beta}^\circ) \cong \Omega_{\mathcal{L},\beta}/\Omega_{\mathcal{L},\beta}^\circ. \tag{3.9}$$

By Lemma 2.2 it suffices to consider the cases where $\mathcal{L}$ is ϵ-stable and $\beta = 1$.

Theorem 3.5. *Suppose that $\mathcal{G}^\epsilon$ satisfies Condition 3.1. Let $\mathcal{L}$ be an ϵ-stable rank one character sheaf on $\mathcal{T}$ and suppose that $W^\circ(\mathcal{L},1)$ corresponds to $s \in G^{\circ\vee\epsilon}$ via* (2.11). *There exists a canonical equivalence of categories*

$$\mathrm{Rep}_s(G^\epsilon) = \mathrm{Rep}_{W(\mathcal{L},1)}(G^\epsilon) \cong \mathrm{Rep}_1(H^\epsilon)^{\Omega_{\mathcal{L}}^\epsilon}.$$

Proof. This is entirely analogous to Theorem 3.4. By Lemma 3.2, (3.8) and Theorem 2.4, there are canonical equivalences of categories

$$\mathrm{Rep}_{W(\mathcal{L},1)}(G^\epsilon) \cong \mathrm{Rep}_{W^\circ(\mathcal{L},1)}(G^{\circ\epsilon})^{\pi_0(\mathcal{G})_{W^\circ(\mathcal{L},1)}} \cong \Big(\mathrm{Rep}_1(H^\epsilon)^{\Omega_{\mathcal{L},1}^\circ}\Big)^{\pi_0(\mathcal{G})_{W^\circ(\mathcal{L},1)}}.$$

In the rightmost term, the two equivariant structures combine to one $\Omega_{\mathcal{L},1}$-equivariant structure, by (3.9). We recall from (2.19) that $\Omega_{\mathcal{L},1} = \Omega_{\mathcal{L}}^\epsilon$. $\Box$

Acknowledgement. The author thanks Tasho Kaletha for many enlightening discussions on this and related topics.

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IMAPP, Radboud Universiteit Nijmegen, Heyendaalseweg 135, 6525AJ Nijmegen, the Netherlands

Email address: m.solleveld@science.ru.nl