

GROUP ALGEBRAS OF REDUCTIVE p -ADIC GROUPS, THEIR REPRESENTATIONS AND THEIR NONCOMMUTATIVE GEOMETRY

Maarten Solleveld

IMAPP, Radboud Universiteit Nijmegen
Heyendaalseweg 135, 6525AJ Nijmegen, the Netherlands
email: m.solleveld@science.ru.nl

ABSTRACT. This is a survey paper about representation theory and noncommutative geometry of reductive p -adic groups G . The main focus points are:

1. The structure of the Hecke algebra $\mathcal{H}(G)$, the Harish-Chandra–Schwartz algebra $\mathcal{S}(G)$ and the reduced C^* -algebra $C_r^*(G)$.

2. The classification of irreducible G -representations in terms of supercuspidal representations.

3. The Hochschild homology and topological K-theory of these algebras.

In the final part we prove one new result, namely we compute $K_*(C_r^*(G))$ including torsion elements, in terms of equivariant K-theory of compact tori.

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Date: October 20, 2025.

2010 Mathematics Subject Classification. 20G25, 22E50, 16E40, 19K99, 20C08.

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INTRODUCTION

This survey paper is based on a series of lectures delivered by the author in February–March 2025, for the thematic trimester ”Representation theory and non-commutative geometry”. The lectures focused on the themes of the trimester, for reductive p -adic groups G . These notes serve the same general goals as the lectures (with less time pressure):

- Describe the structure of various (topological) versions of the group algebra of G .
- Explain the classification of irreducible G -representations in terms of super-cuspidal representations of Levi subgroups.
- Discuss the computation of the Hochschild homology and the topological K-theory of these group algebras.

The Hecke algebra $\mathcal{H}(G)$ of a reductive p -adic group G has been a popular object of study. A lot is known about this algebra and its representations, and the more abstract part of that theory has been consolidated in Renard’s textbook [Ren]. The Harish-Chandra–Schwartz algebra $\mathcal{S}(G)$ is important in harmonic analysis and for tempered G -representations. Further, the reduced C^* -algebra $C_r^*(G)$ is crucial in the noncommutative geometry of G . Although both topological algebras $\mathcal{S}(G)$ and $C_r^*(G)$ have been known for more than half a century, the results about them are scattered in the literature. We bring some of those results together in these notes

Roughly speaking, in Section 1 we will discuss the following relations between the three group algebras of G :

Group algebras		
$\mathcal{H}(G)$	$S(G)$	$C_r^*(G)$
Locally constant functions on group		
compactly supported	rapidly decaying	" C^* -norm bounded"
Representations		
smooth	tempered	tempered unitary
Fourier transform		
algebraic sections	differentiable sections	continuous sections

The classification of irreducible (smooth, complex) G -representations has two main parts:

- (i) supercuspidal G -representations,
- (ii) irreducible representations in the parabolic inductions of a given set of supercuspidal representations of a Levi subgroup of G .

Part (i) is rather arithmetic, and outside the scope of these notes. We refer the reader to the surveys [Afg, Fin]. Part (ii) is more geometric and highly relevant for the noncommutative geometry of G . Moreover there is a beautiful solution to (ii), which was already conjectured around 2012 by ABPS (Aubert, Baum, Plymen and the author). We recall that every Bernstein block $\text{Rep}(G)^\mathfrak{s}$ in the category of smooth G -representations comes with:

- a complex torus $T_\mathfrak{s}$ which parametrizes the underlying supercuspidal representations,
- a finite group $W_\mathfrak{s}$ such that the centre of $\text{Rep}(G)^\mathfrak{s}$ is $\mathcal{O}(T_\mathfrak{s})^{W_\mathfrak{s}} = \mathcal{O}(T_\mathfrak{s}/W_\mathfrak{s})$.

A simplified version of the ABPS conjecture says that there is a canonical bijection between

- the set $\text{Irr}(G)^\mathfrak{s}$ of irreducible representations in $\text{Rep}(G)^\mathfrak{s}$,
- the set of irreducible representations of the crossed product $\mathcal{O}(T_\mathfrak{s}) \rtimes W_\mathfrak{s}$.

Actually, in general one has to extend both $T_\mathfrak{s}$ and $W_\mathfrak{s}$ by a finite group, and one has to twist the group algebra of $W_\mathfrak{s}$ by a 2-cocycle. For the sake of presentation, we suppress those details in the introduction.

The proof of the ABPS conjecture [Sol9] entails several steps:

- $\text{Rep}(G)^\mathfrak{s}$ is made equivalent to the module category of the G -endomorphism algebra of a progenerator $\Pi_\mathfrak{s}$.
- General analysis of the structure of $\text{End}_G(\Pi_\mathfrak{s})$.
- Relate localized versions of $\text{End}_G(\Pi_\mathfrak{s})$ to graded Hecke algebras.
- Study the representation theory of graded Hecke algebras.
- Classify the irreducible representations of a graded Hecke algebra in terms of those of a crossed product algebra.
- Classify the irreducible representations of a crossed product algebra.

In Sections 2 and 3 we treat all these topics, in survey style (and in a different order).

After that we come to the noncommutative geometric part of the paper. Section 4 starts with an introduction to Hochschild homology for algebras, based on [Lod] and focussing on algebras that are close to commutative. Then we discuss the Hochschild homology of $\mathcal{H}(G)$ and of $\mathcal{H}(G)^\mathfrak{s}$, the summand of $\mathcal{H}(G)$ associated to

$\text{Rep}(G)^\mathfrak{s}$. The latter is computed via the Hochschild homology of graded Hecke algebras. It turns out that $\mathcal{H}(G)^\mathfrak{s}$ has the same Hochschild homology as the crossed product $\mathcal{O}(T_\mathfrak{s}) \rtimes W_\mathfrak{s}$, which is known from [Nis].

Section 5 is dedicated to the Hochschild homology of topological algebras related to $\mathcal{S}(G)$. There is a lot technique behind this, because it involves a mix of homological algebra and functional analysis. We briefly discuss the setup and some examples of such topological Hochschild homology groups. Let $\mathcal{S}(G)^\mathfrak{s}$ be the summand of $\mathcal{S}(G)$ associated to $\text{Rep}(G)^\mathfrak{s}$ and let $T_\mathfrak{s}^u$ be the compact real subtorus of $T_\mathfrak{s}$ that parametrizes the unitary supercuspidal representations underlying $\text{Rep}(G)^\mathfrak{s}$. Then

$$(1) \quad HH_n(\mathcal{S}(G)^\mathfrak{s}) \cong HH_n(C^\infty(T_\mathfrak{s}^u) \rtimes W_\mathfrak{s}),$$

and the right hand side had already been computed earlier [Bry]. We point out that $HH_*(C_r^*(G))$ is less interesting: it recovers the cocenter $C_r^*(G)/[C_r^*(G), C_r^*(G)]$ but nothing more.

The isomorphism (1) and its analogue for $\mathcal{H}(G)^\mathfrak{s}$ can be regarded as a version of the ABPS conjecture in Hochschild homology. From there it is only a small step to compute the periodic cyclic homology HP_* of $\mathcal{H}(G)$ and $\mathcal{S}(G)$, again in terms of crossed product algebras.

Finally we come to topological K-theory in Section 6, the most truly noncommutative geometric part of the paper. We need K-theory both for Banach algebras and for Fréchet algebras, and we recall some general results in the latter setting. Via Chern characters, K-theory is related to periodic cyclic homology, which leads to isomorphisms

$$(2) \quad K_*(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{C} \cong K_*(\mathcal{S}(G))_{\mathbb{Z}} \mathbb{C} \cong HP_*(\mathcal{S}(G)).$$

These results can be compared with the Baum–Connes conjecture, which was proven for G in [Laf]. We formulate the comparison with a completely algebraic counterpart to the Baum–Connes conjecture from [HiNi].

To determine $K_*(C_r^*(G))$ including torsion elements, we abandon the survey style and add some new results in the last three paragraphs. Our idea is to transfer the setup used for $\mathcal{H}(G)^\mathfrak{s}$ to the summand $C_r^*(G)^\mathfrak{s}$ of $C_r^*(G)$. We construct a progenerator $\Pi_\mathfrak{s}^c$ for the module category of $C_r^*(G)^\mathfrak{s}$, and an action of $C(T_\mathfrak{s}^u) \rtimes W_\mathfrak{s}$ on $\Pi_\mathfrak{s}^c$. We deduce isomorphisms

$$(3) \quad K_*(C_r^*(G)^\mathfrak{s}) \cong K_*(C(T_\mathfrak{s}^u) \rtimes W_\mathfrak{s}) \cong K_{W_\mathfrak{s}}^*(T_\mathfrak{s}^u),$$

where the rightmost term denotes the $W_\mathfrak{s}$ -equivariant K-theory of the topological space $T_\mathfrak{s}^u$. In Theorem 6.14, the precise version of (3), $T_\mathfrak{s}^u$ and $W_\mathfrak{s}$ are replaced by finite covers, and a twist by a 2-cocycle of the extension of $W_\mathfrak{s}$ is involved. In the end, that computes $K_*(C_r^*(G)^\mathfrak{s})$ in terms of "twisted" $W_\mathfrak{s}$ -equivariant vector bundles on $T_\mathfrak{s}^u$. That settles the ABPS conjecture in topological K-theory [ABPS2].

Acknowledgements

This paper was conceived when the author visited the Institut Henri Poincaré in Paris (France), for the trimester program "Representation theory and noncommutative geometry". It is a pleasure to thank all the organizers of this program for their efforts, and the Institut Henri Poincaré for the hospitality.

Part A. Representation theory of reductive p -adic groups

1. GROUP ALGEBRAS

1.1. Example: $GL_1(F)$.

For a nice introduction to p -adic numbers we refer to [Gou]. In this paragraph we mention several well-known aspects of p -adic numbers without further comments.

Let F be a non-archimedean local field. If F has characteristic zero, then it is a finite extension of the field of p -adic numbers $\mathbb{Q}_p$, for some prime p . On the other hand, if F has characteristic p , then it is a finite extension of the local function field $\mathbb{F}_p((T))$.

Let $v_F : F \rightarrow \mathbb{Z} \cup \{\infty\}$ be the discrete valuation and fix an element ϖ_F with $v_F(\varpi_F) = 1$. Let $\mathfrak{o}_F = v_F^{-1}(\mathbb{Z}_{\geq 0} \cup \{\infty\})$ be the ring of integers of F and let $\varpi_F \mathfrak{o}_F = v_F^{-1}(\mathbb{Z}_{> 0} \cup \{\infty\})$ be its unique maximal ideal. The residue field $k_F = \mathfrak{o}_F / \varpi_F \mathfrak{o}_F$ is a finite field of characteristic p , whose cardinality we denote by q_F .

Example 1.1. In $F = \mathbb{Q}_p$ we have $v_F(p^n a/b) = n$ for $a, b \in \mathbb{Z}$ not divisible by p . Further $\mathfrak{o}_F = \mathbb{Z}_p$ and we can take $\varpi_F = p$. Then $k_F = \mathbb{Z}_p / p\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$.

In $F = \mathbb{F}_p((T))$ we have $v_F(\sum_{n=N}^{\infty} a_n T^n) = N$ when $a_n \in \mathbb{F}_p$ and $a_N \neq 0$. Here $\mathfrak{o}_F = \mathbb{F}_p[[T]]$ and one takes $\varpi_F = T$. Then $k_F = \mathbb{F}_p[[T]]/T\mathbb{F}_p[[T]] = \mathbb{F}_p$.

On F one defines the absolute value $|x|_F = q_F^{-v_F(x)}$. We note that $v_F(0) = \infty$ and that $|0|_F = 0$. The absolute value defines a metric $d_F(x, y) = |x - y|_F$ on F , with respect to which F is complete. Since the metric takes values in $q_F^{-\mathbb{Z}} \cup \{0\}$, the image of d_F does not contain any interval in $\mathbb{R}$. Hence F with the metric topology is a totally disconnected Hausdorff space. It is not discrete though, for instance the sequence $(\varpi_F^n)_{n=1}^{\infty}$ converges to 0 in F . By the completeness of F with respect to d_F and the finiteness of k_F , F is locally compact. The ring $\mathfrak{o}_F$ is compact.

We shall discuss various group algebras of algebraic groups over F . Let us start with the simplest example of such a group: $F^\times = GL_1(F)$. Here and below, we denote the group of invertible elements in a unital ring R by $R^\times$. For $\mathfrak{o}_F$ that gives

$$\mathfrak{o}_F^\times = \mathfrak{o}_F \setminus \varpi_F \mathfrak{o}_F = v_F^{-1}(0).$$

Notice that this is strictly smaller than $\mathfrak{o}_F \cap F^\times = \mathfrak{o}_F \setminus \{0\}$. In fact $\mathfrak{o}_F^\times$ is the unique maximal compact subgroup of $F^\times$. We note that the valuation v_F induces a group isomorphism $F^\times / \mathfrak{o}_F^\times \cong \mathbb{Z}$.

For simplicity we look only at functions on $F^\times$ that are invariant under $\mathfrak{o}_F^\times$. As $F^\times$ is abelian, those are the same as $\mathfrak{o}_F^\times$ -biinvariant functions on $F^\times$. Three interesting algebras of such functions are:

- The Hecke algebra

$$(1.1) \quad \mathcal{H}(F^\times)^{\mathfrak{o}_F^\times} = \mathbb{C}[F^\times / \mathfrak{o}_F^\times] \cong \mathbb{C}[\mathbb{Z}] \cong \mathcal{O}(\mathbb{C}^\times),$$

where $\mathcal{O}$ means regular functions on an affine variety.

- The Harish-Chandra–Schwartz algebra

$$(1.2) \quad \mathcal{S}(F^\times)^{\mathfrak{o}_F^\times} = \mathcal{S}(F^\times / \mathfrak{o}_F^\times) \cong \mathcal{S}(\mathbb{Z}) = \{\text{Schwartz functions } f : \mathbb{Z} \rightarrow \mathbb{C}\} \cong C^\infty(S^1),$$

where S^1 is unit circle in $\mathbb{C}$.

- The reduced C^* -algebra

$$(1.3) \quad C_r^*(F^\times)^{\mathfrak{o}_F^\times} = C_r^*(F^\times / \mathfrak{o}_F^\times) \cong C_r^*(\mathbb{Z}) \cong C(S^1).$$

Remarkable here is that, although $F^\times$ is rather complicated and strange as a topological space (totally disconnected but not discrete), these three algebras of functions on $F^\times$ are very nice and well-behaved. They are among the standard examples of algebras from a course on (respectively) algebraic geometry, differential geometry and topology.

1.2. Definitions and first properties.

From now on $\mathcal{G}$ is a connected reductive algebraic group defined over F . We call $G = \mathcal{G}(F)$ a reductive p -adic group (even though F is not necessarily a p -adic field, it may have positive characteristic). Of course G can be endowed with the Zariski topology, but for the purposes of representation theory it is more useful to consider a topology on F which comes from the metric topology on F .

We spell out this topology. Since $\mathcal{G}$ is a linear algebraic group, it can be embedded in GL_n for some $n \in \mathbb{N}$. On the matrix ring $M_n(F)$ we define a norm by

$$\|(a_{ij})_{i,j=1}^n\|_F = \max_{i,j} |a_{ij}|_F.$$

That in turn yields a metric on $GL_n(F)$ by

$$(1.4) \quad d(A, B) = \max\{\|A - B\|_F, \|A^{-1} - B^{-1}\|_F\},$$

where the term with the inverses is needed to make inversion on $GL_n(F)$ continuous. We restrict this to a metric d on $G \subset GL_n(F)$. While d depends on the choice of an embedding $\mathcal{G} \rightarrow GL_n$, the resulting topology on G does not. This makes G into a totally disconnected locally compact group. (By convention locally compact groups are Hausdorff.)

Since the metric on G only takes values in $q_F^\mathbb{Z} \cup \{0\}$, every open ball in G is also a closed ball. By the local compactness every closed ball is compact, so G has lots of compact open subsets. Even better, G has many compact open subgroups.

Here is one construction of small compact open subgroups. The group

$$GL_n(\mathfrak{o}_F) = \{A \in M_n(\mathfrak{o}_F) : \det(A) \in \mathfrak{o}_F^\times\}$$

is compact and open in $GL_n(F)$. Namely, $M_n(\mathfrak{o}_F)$ and

$$M_n(\mathfrak{o}_F)^\sharp := \{A \in M_n(\mathfrak{o}_F) : \det(A) \in \varpi_F \mathfrak{o}_F\}$$

are compact open subsets of $M_n(F)$, so their difference is compact. Further

$$GL_n(\mathfrak{o}_F) = M_n(\mathfrak{o}_F) \setminus M_n(\mathfrak{o}_F)^\sharp = (M_n(\mathfrak{o}_F) \cap GL_n(F)) \setminus (M_n(\mathfrak{o}_F)^\sharp \cap GL_n(F)),$$

which is open in $GL_n(F)$. For any $m \in \mathbb{Z}_{\geq 0}$ we have the ring homomorphism $\mathfrak{o}_F \rightarrow \mathfrak{o}_F / \varpi_F^m \mathfrak{o}_F$. This induces a group homomorphism

$$\text{mod}_{\varpi_F^m} : GL_n(\mathfrak{o}_F) \rightarrow GL_n(\mathfrak{o}_F / \varpi_F^m \mathfrak{o}_F),$$

whose image is finite because

$$|\mathfrak{o}_F / \varpi_F^m \mathfrak{o}_F| = q_F^m < \infty.$$

Then $GL_n(F)_m := \ker(\text{mod}_{\varpi_F^m})$ is an open subgroup of the compact group $GL_n(\mathfrak{o}_F)$, so automatically closed and compact. In fact $GL_n(F)_m$ equals the closed ball of radius q_F^{-m} around I_n in $GL_n(F)$. It follows that

$$(1.5) \quad G_m := G \cap GL_n(F)_m$$

is a compact open subgroup of G . This is known as a congruence subgroup of G , because it consists of matrices that are equal to the identity modulo $\varpi_F^m \mathfrak{o}_F$. With

respect to the metric d from (1.4), the group G_m is the closed ball of radius q_F^{-m} and the open ball of radius $q_F^{1/2-m}$, both around the unit element I_n . In particular the decreasing sequence of compact open subgroup $\{G_m : m \in \mathbb{Z}_{\geq 0}\}$ is a neighborhood basis of I_n in G .

We denote the set of compact open subgroups of G by $\mathcal{CO}(G)$. A large supply of such subgroups comes from Bruhat–Tits theory, for which we refer to [KaPr, Tit]. For each $K \in \mathcal{CO}(G)$, the spaces G/K and $K \backslash G$ are discrete and countable. For the latter, notice that G is a countable union of compact balls, and that each compact subset of G is covered by finitely many cosets of the open group K .

The above provides, among others, ways to partition G as a disjoint union of compact open subsets, just take the left cosets of one of the subgroups G_m . The abundance of compact open subsets means that G admits many locally constant functions. For instance, any function on the discrete space G/G_m can be inflated to a locally constant function on G . We let $C^\infty(G)$ be the vector space of locally constant functions $f : G \rightarrow \mathbb{C}$. It is an algebra with respect to pointwise multiplication. The notation C^∞ comes from manifolds, even though on a totally disconnected space like G there is no good notion of differentiability for general functions. The reasoning is that locally constant functions are the only functions on G that one can differentiate for sure: all their partial derivatives are 0.

We fix a left Haar measure μ on G . The group G is unimodular (because it is reductive, see [Ren, §V.5.4]), so μ is also a right Haar measure. The convolution product of two integrable functions $f_1, f_2 : G \rightarrow \mathbb{C}$ is defined as

$$(f_1 * f_2)(x) = \int_G f_1(xg^{-1})f_2(g)dg = \int_G f_1(g)f_2(g^{-1}x)dg.$$

Here and later we suppress the Haar measures from the notations of integrals. We note that the convolution product generalizes the multiplication in the full group algebra $\mathbb{C}[G]$, with a modification for the measures of sets. For instance, let $f_i = 1_{g_i K_i}$ be the indicator function of a left coset of $K_i \in \mathcal{CO}(G)$. For $x = g_1 k_1 g_2 k_2$ with $k_i \in K_i$ one computes

$$\begin{aligned} (1_{g_1 K_1} * 1_{g_2 K_2})(x) &= \int_{g K_1} 1_{g_2 K_2}(g^{-1}x)dg = \int_{K_1} 1_{g_2 K_2}(h^{-1}g_1^{-1}x)dh \\ &= \int_{K_1} 1_{g_2 K_2}(h^{-1}k_1 g_2 k_2)dh = \int_{K_1} 1_{g_2 K_2}(k g_2 k_2)dk \\ &= \mu(K_1 g_2 k_2 \cap g_2 K_2) = \mu(K_1 \cap g_2 K_2 g_2^{-1}). \end{aligned}$$

The support of $f_1 * f_2$ is always contained in $\text{supp}(f_1) \cdot \text{supp}(f_2)$, so

$$(1.6) \quad 1_{g_1 K_1} * 1_{g_2 K_2} = \mu(K_1 \cap g_2 K_2 g_2^{-1}) 1_{g_1 K_1 g_2 K_2}.$$

This equation shows that the convolution product generalizes the multiplication in the full group algebra $\mathbb{C}[G]$, with a modification for the measures of sets. If we would replace μ by the counting measure on G and we would only use functions with finite support, then (1.6) would recover the multiplication in $\mathbb{C}[G]$.

Definition 1.2. The Hecke algebra $\mathcal{H}(G)$ is the vector space $C_c^\infty(G)$ of locally constant compactly supported functions $f : G \rightarrow \mathbb{C}$, endowed with the convolution product.

The $\mathbb{C}$ -algebra $\mathcal{H}(G)$ is associative but not unital. Namely, from (1.6) we see that a unit element of $\mathcal{H}(G)$ would have to be supported only at the identity of G . But $\{e\}$ is open in G if and only if G is finite, which happens only when $\mathcal{G} = \{e\}$.

For any $K \in \mathcal{CO}(G)$, the K -biinvariant functions form a subalgebra

$$\mathcal{H}(G, K) = C_c(K \backslash G / K)$$

of $\mathcal{H}(G)$. This subalgebra has

$$\langle K \rangle := \mu(K)^{-1} 1_K$$

as unit, as one can check like in (1.6).

- Proposition 1.3.** (a) $\mathcal{H}(G) = \bigcup_{K \in \mathcal{CO}(G)} \mathcal{H}(G, K) = \bigcup_{m=1}^{\infty} \mathcal{H}(G, G_m)$.
(b) The algebra $\mathcal{H}(G)$ has local units: for every finite subset $S \subset \mathcal{H}(G)$ there exists an idempotent $e_S \in \mathcal{H}(G)$ such that $e_S s = s = s e_S$ for all $s \in S$.
(c) $\mathcal{H}(G)$ has countable dimension.

Proof. (a) Every $f \in \mathcal{H}(G)$ is compactly supported and locally constant, so takes only finitely many values in $\mathbb{C}$. For each nonzero value z , $f^{-1}(z)$ is a compact open subset of G . Since the G_m form a neighborhood basis of e in G , there exists $m \in \mathbb{N}$ such that $f^{-1}(z)$ is a union of G_m -double cosets. As f takes only finitely many values, there exists an m that works for all values $z \neq 0$. Then $f \in \mathcal{H}(G, G_m)$.

(b) By part (a) there exist $K_s \in \mathcal{CO}(G)$ such that $s \in \mathcal{H}(G, K_s)$. Define $K = \bigcap_{s \in S} K_s$, this is a compact open subgroup because S is finite. Now $s \in \mathcal{H}(G, K)$ for all $s \in S$, so $e_S = \langle K \rangle$ has the required property.

(c) The space $G_m \backslash G / G_m$ is countable, so $\mathcal{H}(G, G_m)$ has countable dimension. Combine that with part (a). $\square$

On $C_c(G)$ we have the standard norms

$$\|f\|_r = \left(\int_G |f(g)|^r dg \right)^{1/r} \text{ with } r \in \mathbb{R}_{\geq 1} \quad \text{and} \quad \|f\|_{\infty} = \sup_{g \in G} |f(g)|.$$

Lemma 1.4. $\mathcal{H}(G)$ is dense in $C_c(G)$ for the norms $\|\cdot\|_r$ with $r \in \mathbb{R}_{\geq 1} \cup \{\infty\}$.

Proof. Let $f \in C_c(G)$. For $m \in \mathbb{Z}_{\geq 1}$ we define $f_m \in \mathcal{H}(G, G_m)$ as follows: pick a set of representatives $\{g_{m,i}\}_i$ for $G_m \backslash G / G_m$ and put $f_m(g) = f(g_{m,i})$ for $g \in G_m g_{m,i} G_m$. By the continuity of f and the compactness of its support, the sequence $(f_m)_{m=1}^{\infty}$ converges uniformly to f . Hence it also converges to f with respect to the norms $\|\cdot\|_r$. $\square$

The Banach algebra $L^1(G)$ acts continuously on the Hilbert space $L^2(G)$, by the convolution product. That yields an injective homomorphism from $L^1(G)$ to $\mathcal{B}(L^2(G))$, the C^* -algebra of bounded linear operators on $L^2(G)$. We make $\mathcal{H}(G)$ and $L^1(G)$ into $*$ -algebras by

$$f^*(g) = \overline{f(g^{-1})}.$$

Definition 1.5. The reduced C^* -algebra $C_r^*(G)$ is the closure of $L^1(G)$ in $\mathcal{B}(L^2(G))$, with respect to the operator norm.

By Lemma 1.4 $\mathcal{H}(G)$ is dense in $L^1(G)$, hence $\mathcal{H}(G)$ is also dense in $C_r^*(G)$. In other words, $C_r^*(G)$ can be regarded as the completion of $\mathcal{H}(G)$ for the operator norm of $\mathcal{B}(L^2(G))$. Lemma 1.4 also says that $\mathcal{H}(G)$ is dense in $L^2(G)$. Therefore the operator norm of $\mathcal{H}(G)$ acting on $L^2(G)$ equals the operator norm of $\mathcal{H}(G)$ acting

on itself by left multiplication. Thus $C_r^*(G)$ can be constructed entirely in terms of $\mathcal{H}(G)$, as a C^* -completion of that $*$ -algebra.

For $K \in \mathcal{CO}(G)$, we let

$$C_r^*(G, K) = \langle K \rangle C_r^*(G) \langle K \rangle$$

be the sub- C^* -algebra of K -biinvariant functions in $C_r^*(G)$. In contrast with $C_r^*(G)$, the algebra $C_r^*(G, K)$ has a unit, namely $\langle K \rangle$. One may identify $C_r^*(G, K)$ with the closure of $\mathcal{H}(G, K) = \langle K \rangle \mathcal{H}(G) \langle K \rangle$ in $C_r^*(G)$ or in $\mathcal{B}(L^2(G))$. It follows from Proposition 1.3 that

$$(1.7) \quad C_r^*(G) = \lim_{m \rightarrow \infty} C_r^*(G, G_m) = \lim_{K \in \mathcal{CO}(G)} C_r^*(G, K),$$

where the limit is meant in the category of Banach algebras.

The original definition of the Harish-Chandra–Schwartz algebra of G [HC] is rather complicated, we prefer the simpler construction in [Vig]. Using the embedding $G \rightarrow GL_n(F)$, we define a length function on G by

$$g \mapsto \log \left(\max \{ \|g\|_F, \|g^{-1}\|_F \} \right).$$

Then the function

$$\sigma : G \rightarrow \mathbb{R}_{\geq 1}, \quad \sigma(g) = 1 + \log \left(\max \{ \|g\|_F, \|g^{-1}\|_F \} \right)$$

is a scale, which means that it satisfies $\sigma(g^{-1}) = \sigma(g)$ and $\sigma(gg') \leq \sigma(g)\sigma(g')$. For $m \in \mathbb{Z}_{>0}$ we define a norm ν_m on $C_c(G)$ by $\nu_m(f) = \|\sigma^m f\|_2$.

Definition 1.6. For $K \in \mathcal{CO}(G)$, $\mathcal{S}(G, K)$ is the completion of $\mathcal{H}(G, K)$ with respect to the family of norms ν_m ($m \in \mathbb{Z}_{>0}$). The Harish-Chandra–Schwartz algebra of G is $\mathcal{S}(G) = \bigcup_{K \in \mathcal{CO}(G)} \mathcal{S}(G, K)$, endowed with the inductive limit topology.

Thus $\mathcal{S}(G)$ consists of locally constant functions on G that decay rapidly. These functions need not have compact support, but every one of them is biinvariant under some $K \in \mathcal{CO}(G)$. Some important properties of the algebras $\mathcal{S}(G, K)$ were proven by Vignéras:

Theorem 1.7. [Vig, Propositions 10, 13, 28]

- (a) $\mathcal{S}(G, K)$ is a nuclear Fréchet $*$ -algebra with unit $\langle K \rangle$.
- (b) $\mathcal{S}(G, K)$ is a dense subalgebra of $C_r^*(G, K)$, with a finer topology.
- (c) $\mathcal{S}(G, K) \cap C_r^*(G, K)^\times = \mathcal{S}(G, K)^\times$, and this set is open in $\mathcal{S}(G, K)$.

By Theorem 1.7.b and (1.7), $\mathcal{S}(G)$ is contained in $C_r^*(G)$. Like $\mathcal{H}(G)$ and $C_r^*(G)$, the Harish-Chandra–Schwartz algebra of G is not unital. The same argument as for Proposition 1.3.b shows that $\mathcal{S}(G)$ does have local units, for instance the idempotents $\langle K \rangle$ with $K \in \mathcal{CO}(G)$.

The multiplication in $\mathcal{S}(G)$ is separately continuous, that is, for any fixed $a \in \mathcal{S}(G)$ the maps $f \mapsto fa$ and $f \mapsto af$ are continuous [Wal, Lemme III.6.1]. However, $\mathcal{S}(G)$ is not a Fréchet algebra, because the topological vector space $\mathcal{S}(G)$ is not Fréchet. It is a strict inductive limit of Fréchet spaces, but such spaces are not metrizable [DiSc, Corollaire 4.2].

1.3. Classes of representations of reductive p -adic groups.

In these notes, all representations will by default be on complex vector spaces. The best notion of continuity for a representations of a reductive p -adic group is smoothness.

Definition 1.8. A G -representation (π, V) is smooth if for all $v \in V$ the stabilizer group $G_v = \{g \in G : \pi(g)v = v\}$ is open in G . Equivalently, π is smooth if the map $\pi : G \times V \rightarrow V$ is continuous with respect to the discrete topology on V .

This is a rather crude interpretation of smooth, like $C^\infty(G)$. It says that for any fixed $v \in V$ the map $G \rightarrow V : g \mapsto \pi(g)v$ is locally constant.

Example 1.9. Let G act on $\mathcal{H}(G)$ by left translations:

$$(\lambda(g)f)(x) = f(g^{-1}x) \quad \text{for } g, x \in G, f \in \mathcal{H}(G).$$

If $f \in \mathcal{H}(G, K)$, then $\lambda(k)f = f$ for all $k \in K$. Hence $(\lambda, \mathcal{H}(G))$ is a smooth G -representation.

The motivation for considering the class of smooth representations comes from profinite groups. Consider a projective limit of finite groups $H = \varprojlim H_i$. Then each finite group H_i is a quotient of H and $\ker(H \rightarrow H_i)$ is an open subgroup of H . It is natural to impose that every irreducible H -representation factors through H_i for some i . Smoothness of H -representations enforces that (at least under the small extra condition that H is its own profinite completion), and at the same time is sufficiently flexible to enable direct limits of H -representations.

Recall that every compact totally disconnected Hausdorff group is a profinite group, and conversely. In particular every compact subgroup of a reductive p -adic group is profinite. Thus smoothness of a G -representation (π, V) means that for every $K \in \mathcal{CO}(G)$, the restriction $\pi|_K$ belongs to the natural class of K -representations.

Every smooth G -representation extends to a representation of $\mathcal{H}(G)$ on V , by

$$(1.8) \quad \pi(f)v = \int_G f(g)\pi(g)v dg \quad f \in \mathcal{H}(G), v \in V.$$

Since $f \in C_c^\infty(G)$ and π is smooth, this integral boils down to a finite sum and there are no convergence issues.

It is not quite true every $\mathcal{H}(G)$ -module gives rise to a (smooth) G -representation, because $\mathcal{H}(G)$ is not unital. For instance, one could have a vector space W on which $\mathcal{H}(G)$ acts by $f \cdot w = 0$ for all $f \in \mathcal{H}(G), w \in W$. That does not correspond to any G -representation, because G would have to act on W by invertible linear operators.

Definition 1.10. We say that an $\mathcal{H}(G)$ -module is nondegenerate if for each $v \in V$ there exists a $K_v \in \mathcal{CO}(G)$ such that $\langle K_v \rangle v = v$. Let $\text{Rep}(G)$ be the category of smooth G -representations and let $\text{Mod}(\mathcal{H}(G))$ be the category of nondegenerate $\mathcal{H}(G)$ -modules. The morphisms in these categories are the $\mathbb{C}$ -linear maps that intertwine the action of G or $\mathcal{H}(G)$.

Lemma 1.11. (a) *The categories $\text{Rep}(G)$ and $\text{Mod}(\mathcal{H}(G))$ are naturally equivalent, via (1.8).*

(b) *Every finitely generated smooth G -representation has countable dimension.*

Proof. (a) Let $V \in \text{Mod}(\mathcal{H}(G))$ and let $v \in V$. For every compact open subgroup $K \subset K_v$ we have

$$\langle K \rangle v = \langle K \rangle \langle K_v \rangle v = \langle K_v \rangle v = v.$$

This enables us to define an action π of G on V by

$$(1.9) \quad \pi(g)v = \lim_{K \in \mathcal{CO}(G)} \mu(K)^{-1} 1_{gK} \cdot v.$$

As $\pi(g')v = \pi(g)v$ for all $g' \in gK_v$, π is smooth. The functor

$$\text{Mod}(\mathcal{H}(G)) \rightarrow \text{Rep}(G) : V \mapsto \pi$$

is inverse to (1.8).

(b) Let $(\pi, V) \in \text{Rep}(G)$ be generated by m elements. By part (a) there exists a surjection of $\mathcal{H}(G)$ -modules $\mathcal{H}(G)^m \rightarrow V$. Combine that with Proposition 1.3.c. $\square$

A deeper result in the direction of Lemma 1.11.b is known as “uniform admissibility” [Ber]. It is usually stated for irreducible representations, and it extends to finite length representations because the restriction of a smooth G -representation to a compact subgroup is always completely reducible.

Theorem 1.12. [Ber]

Let $K \in \mathcal{CO}(G)$. There exists $N(G, K) \in \mathbb{N}$ such that, for all $(\pi, V) \in \text{Rep}(G)$ of finite length, $\dim V^K$ is at most $N(G, K)$ times the length of π .

For an arbitrary G -representation (ρ, W) , the space of smooth vectors is

$$W^\infty = \{w \in W : \langle K \rangle w = w \text{ for some } K \in \mathcal{CO}(G)\}.$$

Then ρ restricts to a smooth G -representation on W^∞ , called the smoothening of (ρ, W) . The group G acts on the space W^* of linear functions $\lambda : W \rightarrow \mathbb{C}$ by

$$(\rho^\vee(g)\lambda)(w) = \lambda(\rho(g^{-1})w) \quad w \in W.$$

We define $W^\vee = W^{*,\infty}$ to be the smooth part of the algebraic linear dual W^* , so the set of $\lambda : W \rightarrow \mathbb{C}$ that factor via $\rho(\langle K \rangle) : W \rightarrow W^K$ for some $K \in \mathcal{CO}(G)$.

Definition 1.13. Let (π, V) be a smooth G -representation. We call $(\pi^\vee, V^\vee)$ the (smooth) contragredient of π . A matrix coefficient of π is a function of the form

$$c_{\lambda,v} : G \rightarrow \mathbb{C}, \quad c_{\lambda,v}(g) = \lambda(\pi(g)v) \quad \text{for some } v \in V, \lambda \in V^\vee.$$

Definition 1.14. A finite length smooth G -representation (π, V) is tempered if it extends to a $\mathcal{S}(G)$ -module by the formula

$$\pi(f)v = \int_G f(g)\pi(g)v dg \quad f \in \mathcal{S}(G), v \in V.$$

By [SSZ, Appendix, Proposition 1], Definition 1.14 is equivalent with the more common definition of temperedness in terms of growth of matrix coefficients [Wal, §III.2]. For representations of infinite length the matrix coefficients do not say enough, we need a more subtle version of temperedness. Consider the category $\text{Mod}(\mathcal{S}(G))$ of nondegenerate $\mathcal{S}(G)$ -modules. Since $\mathcal{H}(G) \subset \mathcal{S}(G)$, every nondegenerate $\mathcal{S}(G)$ -module restricts to a nondegenerate $\mathcal{H}(G)$ -module, which by Lemma 1.11 can be regarded as a smooth G -representation.

Definition 1.15. The category of tempered smooth G -representations is the category $\text{Mod}(\mathcal{S}(G))$ of nondegenerate $\mathcal{S}(G)$ -modules.

Thus any tempered smooth G -representation is by definition endowed with an extension to a $\mathcal{S}(G)$ -module. Notice that we do not put any continuity condition on the action of $\mathcal{S}(G)$ on the module.

For a description of $C_r^*(G)$ -modules, we need to look at unitary G -representations.

Definition 1.16. Let V be a complex vector space with an inner product, and let π be a G -representation on V . We say that π is pre-unitary if $\langle \pi(g)v, v' \rangle = \langle v, \pi(g^{-1})v' \rangle$ for all $v, v' \in V$. We say that (π, V) is unitary if in addition V is a Hilbert space (so complete with respect to the norm from the inner product).

For any (pre-)unitary G -representation (π, V) , the smoothening (π, V^∞) is a smooth pre-unitary G -representation. The pre-unitarity on V^∞ is equivalent to requiring that

$$\langle \pi(f)v, v' \rangle = \langle v, \pi(f^*)v' \rangle \quad \text{for all } v, v' \in V^\infty.$$

In other words, $\pi(f^*) = \int_G \overline{f(g^{-1})} \pi(g) dg$ is the adjoint of $\pi(f) : V^\infty \rightarrow V^\infty$. We warn that a unitary G -representation on an infinite dimensional Hilbert space is typically not smooth, because smoothness and completeness of the Hilbert space fit badly together.

Every unitary G -representation (π, V) extends to a representation of the Banach $*$ -algebra $L^1(G)$. More precisely, the category of unitary $L^1(G)$ -modules is naturally equivalent to the category of unitary G -representations. We say that V is (topologically) irreducible if $\{0\}$ and V are the only G -invariant closed linear subspaces of V . There are functors

$$(1.10) \quad \{\text{smooth pre-unitary } G\text{-reps}\} \begin{array}{c} \xrightarrow{\text{completion}} \\ \xleftarrow{\text{smoothening}} \end{array} \{\text{unitary } G\text{-reps}\}.$$

These functors are bijective on irreducible representations, see [Sol6, §4.2] which is based on [Ber]. In view of the complete reducibility of finite length unitary representations, it follows that (1.10) restricts to an equivalence between the subcategories of finite length objects on both sides.

Definition 1.17. $\text{Mod}_{C_*}(C_r^*(G))$ is the category of those $C_r^*(G)$ -modules that are Hilbert spaces on which $C_r^*(G)$ acts unitarily.

Via the natural homomorphism $L^1(G) \rightarrow C_r^*(G)$, $C_r^*(G)$ -modules can also be regarded as unitary G -representations. However, not all unitary G -representations give rise to $C_r^*(G)$ -modules, only those that are weakly contained in $L^2(G)$.

We call unitary G -representations that extend to $C_r^*(G)$ -modules tempered. In view of the continuity of the involved unitary operators, such an extension is always given by the formula (1.8). Thus we have a natural identification

$$\{\text{tempered unitary } G\text{-representations}\} \longleftrightarrow \text{Mod}_{C_*}(C_r^*(G)).$$

By the next result this is compatible with our notion of temperedness for smooth G -representations.

Lemma 1.18. $\text{Mod}_{C_*}(C_r^*(G))$ consists precisely of the unitary G -representations that extend to $\mathcal{S}(G)$ -modules.

Proof. Since $\mathcal{S}(G)$ embeds in $C_r^*(G)$ (by Theorem 1.7), every $C_r^*(G)$ -module (π, V) is also a $\mathcal{S}(G)$ -module. Hence (π, V) is tempered in the sense of Definition 1.15, except that it need not be smooth as G -representation or nondegenerate as $\mathcal{S}(G)$ -module.

Consider a non-tempered unitary G -representation ρ . By Zorn's lemma it has at least one irreducible non-tempered subquotient ρ' . We will see in Lemma 1.24 that ρ' does not extend to a $\mathcal{S}(G)$ -module. Hence ρ cannot extend to $\mathcal{S}(G)$ either. $\square$

1.4. Normalized parabolic induction and Jacquet restriction.

Let P be a parabolic subgroup of G , that is, the F -points of a parabolic F -subgroup of $\mathcal{G}$. Let U_P be its unipotent radical and let L be a Levi factor of P . We briefly call L a Levi subgroup of G . Any $(\pi, V) \in \text{Rep}(L)$ may be regarded as a smooth P -representation via the quotient map $P \rightarrow L$. The smooth parabolic induction $\text{ind}_P^G(\pi)$ is the vector space

$$\text{ind}_P^G(V) = \{f : G \rightarrow V \mid f(pg) = \pi(p)f(g) \text{ for all } p \in P, g \in G, f \text{ is locally constant}\}$$

endowed with the G -action by right translations. Then $\text{ind}_P^G(\pi)$ is a smooth G -representation. The functor $\text{ind}_P^G : \text{Rep}(L) \rightarrow \text{Rep}(G)$ is exact, but does not preserve temperedness or pre-unitarity. To improve on that, one involves the modular function δ_P of P . By [Sil, §1.2.1], it can be computed as

$$(1.11) \quad \delta_P(lu) = |\det(\text{Ad}(l) : \text{Lie}(U_P) \rightarrow \text{Lie}(U_P))|_F \quad l \in L, u \in U_P.$$

Definition 1.19. The normalized parabolic induction of $(\pi, V) \in \text{Rep}(L)$ is $I_P^G(\pi) = \text{ind}_P^G(\pi \otimes \delta_P^{1/2})$, on the vector space

$$I_P^G(V) = \{f : G \rightarrow V \mid f(ulg) = \delta_P^{1/2}(l)\pi(l)f(g) \text{ for all } u \in U_P, l \in L, g \in G, \\ f \text{ is locally constant}\}.$$

Let $X_{\text{nr}}(L)$ be the group of unramified characters of L , ie. characters whose kernel contains every compact subgroup of L . For instance, δ_P and $\delta_P^{1/2}$ are unramified. All the representations $I_P^G(\pi \otimes \chi)$ with $\chi \in X_{\text{nr}}(L)$ can be realized on the same vector space, as follows. We pick a good maximal compact subgroup K_0 of G . The Iwasawa decomposition [Tit, §3.3.2] says that

$$(1.12) \quad G = PK_0 = K_0P.$$

This implies that restriction of functions to K_0 defines a linear bijection

$$(1.13) \quad \text{ind}_P^G(V_{\pi \otimes \chi}) \rightarrow \text{ind}_{K_0 \cap P}^{K_0}(V_{\pi \otimes \chi}).$$

As K_0 is compact, its action on $\text{ind}_{K_0 \cap P}^{K_0}(V_{\pi \otimes \chi})$ does not depend on χ , so that we can identify it with $\text{ind}_{K_0 \cap P}^{K_0}(V_\pi)$ as K_0 -representations. We call this vector space $I_{P_0 \cap P}^{K_0}(V_\pi)$, and we will always think of $I_P^G(\pi \otimes \chi)$ as defined on $I_{P_0 \cap P}^{K_0}(V_\pi)$.

The following properties can be found in [Ren, §IV.2.3, §VI.1.1, §VI.6.2, §VII.5] and [Wal, Lemme III.2.3].

Theorem 1.20. *The functor $I_P^G : \text{Rep}(L) \rightarrow \text{Rep}(G)$ is exact and preserves pre-unitarity, finite length, finite generation and temperedness.*

Suppose that π is pre-unitary with respect to an inner product $\langle, \rangle_V$. In the proof of Theorem 1.20 it is shown that $I_P^G(\pi)$ is pre-unitary with respect to the inner product

$$(1.14) \quad \langle f_1, f_2 \rangle = \int_{P \backslash G} \langle f_1(g), f_2(g) \rangle_V dg \quad f_1, f_2 \in I_P^G(V).$$

For any $g \in G$ there are canonical isomorphisms of G -representations

$$(1.15) \quad \begin{aligned} \text{ind}_P^G(\pi) &\rightarrow \text{ind}_{gPg^{-1}}^G(g \cdot \pi) \\ I_P^G(\pi) &\rightarrow I_{gPg^{-1}}^G(g \cdot \pi) \\ f &\mapsto [x \mapsto f(g^{-1}x)] \end{aligned}$$

The version of (1.15) for $I_P^G(\pi)$ follows from that for $\text{ind}_P^G(\pi)$ and the equality of modular functions $\delta_{gPg^{-1}} = g \cdot \delta_P : x \mapsto \delta_P(g^{-1}xg)$.

The parabolic induction of an L -representation depends only a little on the choice of a parabolic subgroup with Levi factor L :

Lemma 1.21. [ABPS1, Lemma 1.1]

Let P and P' be parabolic subgroups with Levi factor L and let $\pi \in \text{Rep}(L)$ have finite length. Then the finite length G -representations $I_P^G(\pi)$ and $I_{P'}^G(\pi)$ have the same irreducible subquotients, counted with multiplicity.

Let $(\rho, W) \in \text{Rep}(G)$. The Jacquet module of ρ with respect to P is

$$W_{U_P} = W/W(U_P) = W/\text{span}\{\rho(u)w - w : u \in U_P, w \in W\}.$$

This is the largest quotient of W on which U_P acts trivially. It is naturally a smooth L -representation, denoted ρ_{U_P} and called the Jacquet restriction (or parabolic restriction) of ρ . Like ind_P^G , it can be improved by a normalization.

Definition 1.22. The normalized Jacquet (or parabolic) restriction $J_P^G(\rho)$ is the vector space W_{U_P} with the L -action

$$l \cdot (w + W(U_P)) = \delta_P^{-1/2}(l)\rho(l)w + W(U_P).$$

The functor J_P^G has nice many properties, for instance exactness and preservation of finite length [Ren, §VI.6.4]. However, it does not preserve temperedness or pre-unitarity. Frobenius reciprocity provides adjointness relations

$$(1.16) \quad \begin{aligned} \text{Hom}_G(\rho, \text{ind}_P^G(\pi)) &\cong \text{Hom}_M(\rho_{U_P}, \pi), \\ \text{Hom}_G(\rho, I_P^G(\pi)) &\cong \text{Hom}_M(J_P^G(\rho), \pi). \end{aligned}$$

There is a much deeper second adjointness relation, due to Bernstein:

Theorem 1.23. [Ren, §VI.9.6]

Let $\bar{P}$ be the parabolic subgroup with Levi factor L that is opposite to P , ie. $P \cap \bar{P} = L$. There are natural bijections

$$\begin{aligned} \text{Hom}_G(\text{ind}_P^G(\pi), \rho) &\cong \text{Hom}_M(\pi, \delta_P \otimes \rho_{U_{\bar{P}}}), \\ \text{Hom}_G(I_P^G(\pi), \rho) &\cong \text{Hom}_M(\pi, J_{\bar{P}}^G(\rho)). \end{aligned}$$

1.5. Description of the tempered dual.

For any algebra A , we denote by $\text{Irr}(A)$ the set of irreducible A -representations up to isomorphism. We endow it with the Jacobson topology, whose closed subsets are

$$V(S) = \{\pi \in \text{Irr}(A) : S \subset \ker(\pi)\} \quad \text{for } S \subset A.$$

In particular we have the space $\text{Irr}(\mathcal{S}(G))$ of irreducible nondegenerate $\mathcal{S}(G)$ -modules. We call that the tempered dual $\text{Irr}_{\text{temp}}(G)$.

More analytically, $\text{Irr}(L^1(G))$ is the unitary dual $\text{Irr}_{\text{unit}}(G)$ of G . It contains $\text{Irr}(C_r^*(G))$, which is sometimes called the reduced (unitary) dual of G .

Lemma 1.24. *Smoothing of G -representations defines a bijection*

$$\text{Irr}(C_r^*(G)) \longrightarrow \text{Irr}(\mathcal{S}(G)).$$

In particular every element of $\text{Irr}_{\text{temp}}(G)$ is a pre-unitary smooth G -representation.

Proof. The Plancherel formula for G [Wal] implies that every irreducible tempered G -representation π belongs to the support of the Plancherel measure for G . That is another way of saying that π is weakly contained in $L^2(G)$ [Dix, §18.8]. Hence π is pre-unitary and its Hilbert space completion is an element $\tilde{\pi}$ of the reduced dual. By (1.10), π is the smoothening of $\tilde{\pi}$. $\square$

We warn that not every pre-unitary irreducible smooth G -representation belongs to $\text{Irr}_{\text{temp}}(G)$. For instance, the trivial G -representation is unitary but not tempered (unless $\mathcal{G} = \{e\}$). From now on we will be a little sloppy and call a pre-unitary smooth G -representation unitary, as is customary.

By Schur's lemma, the center $Z(G)$ acts by a character on any irreducible G -representation. If the representation is unitary, then its central character is unitary as well. In that case $|c_{\lambda,v}| : G \rightarrow \mathbb{R}_{\geq 0}$, the absolute value of the matrix coefficient $c_{\lambda,v}$, descends to a map $G/Z(G) \rightarrow \mathbb{R}_{\geq 0}$. This applies more generally to any G -representation on which $Z(G)$ acts by a unitary character.

Definition 1.25. Let π be a smooth G -representation that admits a unitary $Z(G)$ -character. We say that π is square-integrable modulo center if $|c_{\lambda,v}| \in L^2(G/Z(G))$ for every matrix coefficient $c_{\lambda,v}$ of π .

If π is moreover irreducible, then it is called a discrete series representation. We denote the set of discrete series G -representations (up to isomorphism) by $\text{Irr}_{\text{disc}}(G)$.

Every discrete series G -representation π is weakly contained in $L^2(G)$, so its completion extends to a $C_r^*(G)$ -module. By Lemma 1.24 π is tempered and unitary. If $Z(G)$ is compact, then every discrete series G -representation can be embedded in $L^2(G)$, which implies that it is an isolated point in $\text{Irr}(C_r^*(G)) \cong \text{Irr}_{\text{temp}}(G)$ [Dix, §18.4]. Sometimes discrete series are defined such that they can exist only when $Z(G)$ is compact.

Example 1.26. The most important example of a discrete series representation is the Steinberg representation. It exists for any G and is defined as

$$\text{St}_G = \text{ind}_B^G(\text{triv}) / \sum_{B \subseteq P \subseteq G} \text{ind}_P^G(\text{triv}),$$

where B is a minimal parabolic subgroup of G . This representation is unitary because $\text{triv} \in \text{Rep}(B)$ and $\text{ind}_B^G(\text{triv})$ are so. See [Cas] for a proof that St_G is irreducible and discrete series.

Recall that a character $\chi : G \rightarrow \mathbb{C}^\times$ is unramified if it is trivial on every compact subgroup of G . We will use the notations

$$\begin{aligned} X_{\text{nr}}(G) &= \{\text{unramified characters of } G\}, \\ X_{\text{nr}}^u(G) &= \{\text{unitary unramified characters of } G\}. \end{aligned}$$

Let $G^1 \subset G$ be the subgroup generated by all compact subgroups of G . It is an open normal subgroup that contains the derived group of G . The quotient G/G^1 is a free abelian group of the same rank as the F -split part of $Z(\mathcal{G})^\circ$, say rank d . Although the natural map $Z(G) \rightarrow G/G^1$ need not be surjective, its image is sublattice of the same rank d , so that image has finite index in G/G^1 .

There are isomorphisms of topological groups

$$(1.17) \quad \begin{aligned} \text{Hom}(G/G^1, \mathbb{C}^\times) &= X_{\text{nr}}(G) \cong \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^d, \mathbb{C}^\times) \cong (\mathbb{C}^\times)^d, \\ \text{Hom}(G/G^1, S^1) &= X_{\text{nr}}^u(G) \cong \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^d, S^1) \cong (S^1)^d. \end{aligned}$$

In this way $X_{\text{nr}}(G)$ acquires the structure of a complex algebraic torus and $X_{\text{nr}}^u(G)$ is the maximal compact real subtorus of $X_{\text{nr}}(G)$.

For every discrete series G -representation π and every $\chi \in X_{\text{nr}}^u(G)$, $\pi \otimes \chi$ is again discrete series. The group

$$X_{\text{nr}}(G, \pi) := \{\chi \in X_{\text{nr}}(G) : \pi \otimes \chi \cong \pi\}$$

consists of characters that are trivial on $Z(G)$, because $\pi \otimes \chi$ needs to have the same central character as π . Since $G/G^1 Z(G)$ is finite, the group $X_{\text{nr}}(G, \pi)$ is finite and contained in $X_{\text{nr}}^u(G)$. The bijection

$$(1.18) \quad X_{\text{nr}}^u(G)/X_{\text{nr}}(G, \pi) \rightarrow X_{\text{nr}}^u(G)\pi = \{\pi \otimes \chi \in \text{Irr}_{\text{temp}}(G) : \chi \in X_{\text{nr}}^u(G)\}$$

endows $X_{\text{nr}}^u(G)$ with the topological structure of a compact real torus (but $X_{\text{nr}}^u(G)\pi$ does not have a multiplication). We will soon see that every set of the form $X_{\text{nr}}^u(G)\pi$ with $\pi \in \text{Irr}_{\text{disc}}(G)$ is a connected component of $\text{Irr}_{\text{temp}}(G)$. These are the components of $\text{Irr}_{\text{temp}}(G)$ of minimal dimension, and when $Z(G)$ is compact, they constitute precisely the discrete part of $\text{Irr}_{\text{temp}}(G)$.

To describe the connected components of $\text{Irr}_{\text{temp}}(G)$ outside the discrete series, we need the following result of Harish-Chandra.

Theorem 1.27. [Wal, Proposition III.4.1]

Let $\pi \in \text{Irr}_{\text{temp}}(G)$.

- (a) *There exists a parabolic subgroup P with Levi factor L and a discrete series L -representation δ , such that π is a direct summand of $I_P^G(\delta)$.*
- (b) *The pair (L, δ) is uniquely determined by π , up to G -conjugation and isomorphism of L -representations.*

By Theorem 1.20 the G -representation $I_P^G(\delta)$ is tempered, unitary and of finite length, so completely reducible. The following result is known as Harish-Chandra's disjointness theorem. It says in particular that $I_P^G(\delta)$ does not depend on the choice of a parabolic subgroup $P \subset G$ with given Levi factor L .

Theorem 1.28. *Let $P = LU_P$ and $Q = MU_P$ be parabolic subgroups of G and let $\delta \in \text{Irr}_{\text{disc}}(L)$ and $\sigma \in \text{Irr}_{\text{disc}}(M)$. The following are equivalent:*

- (i) *There exists a $g \in G$ such that $M = gLg^{-1}$ and $\sigma \cong g \cdot \delta$.*
- (ii) *$I_P^G(\delta)$ and $I_Q^G(\sigma)$ are isomorphic.*
- (iii) *$I_P^G(\delta)$ and $I_Q^G(\sigma)$ have an irreducible subquotient in common.*

Proof. Clearly (ii) is stronger than (iii), which by Theorem 1.27.b implies (i). By Lemma 1.21 and the complete reducibility of $I_P^G(\delta)$ and $I_Q^G(\sigma)$, (i) implies (ii). $\square$

On the set $\{(L, \delta) : L \text{ Levi subgroup of } G, \delta \in \text{Irr}_{\text{disc}}(L)\}$ we put the equivalence relation $\sim$ generated by G -conjugation, isomorphism of L -representations and tensoring L -representations by elements of $X_{\text{nr}}^u(L)$. We write

$$\Delta(G) = \{(L, \delta) : L \text{ Levi subgroup of } G, \delta \in \text{Irr}_{\text{disc}}(L)\} / \sim$$

and we denote its elements by $[L, \delta]_G$. To any $\mathfrak{d} \in \Delta(G)$ we associate the set

$$\text{Irr}_{\text{temp}}(G)_{\mathfrak{d}} = \{\pi \in \text{Irr}_{\text{temp}}(G) : \pi \text{ is a summand of } I_P^G(\delta) \text{ for some } (L, \delta) \in \mathfrak{d}\}.$$

From Theorems 1.27 and 1.28 one deduces:

Corollary 1.29. *The sets $\text{Irr}_{\text{temp}}(G)_{\mathfrak{d}}$ with $\mathfrak{d} \in \Delta(G)$ are precisely the connected components of $\text{Irr}_{\text{temp}}(G)$, so*

$$\text{Irr}_{\text{temp}}(G) = \bigsqcup_{\mathfrak{d} \in \Delta(G)} \text{Irr}_{\text{temp}}(G)_{\mathfrak{d}}.$$

We call this the Harish-Chandra decomposition of $\text{Irr}_{\text{temp}}(G)$ and we call the sets $\text{Irr}_{\text{temp}}(G)_{\mathfrak{d}}$ Harish-Chandra components for G .

By the generic irreducibility of parabolically induced representations [Sau, Théorème 3.2], in the family $\{I_P^G(\delta \otimes \chi) : \chi \in X_{\text{nr}}^u(L)\}$ the irreducible representations form an open dense subset. Hence $\text{Irr}_{\text{temp}}(G)_{[L, \delta]}$ looks like the compact torus $X_{\text{nr}}^u(L)\delta$, with some points on submanifolds of smaller dimensions replaced by finite packets of inseparable points.

Example 1.30. $G = SL_2(F)$

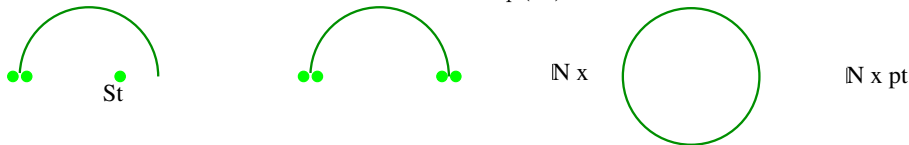
The set $\text{Irr}_{\text{disc}}(G)$ is countable, apart from St_G its elements are supercuspidal representations.

Up to conjugation, the only proper Levi subgroup of G is the diagonal torus $T \cong F^\times$. The set $\text{Irr}_{\text{disc}}(T)/X_{\text{nr}}^u(T)$ is naturally in bijection with $\text{Irr}(\mathfrak{o}_F^\times)$, which is countable because $\mathfrak{o}_F^\times$ is profinite. The only further equivalences in $\Delta(G)$ come from conjugation by $N_G(T) = T \cup s_\alpha T$, where s_α acts on T by inversion. Hence the non-discrete part of $\Delta(G)$ is $\text{Irr}(\mathfrak{o}_F^\times)$ modulo inversion.

Let B be the Borel subgroup of upper triangular matrices in G . A representation $I_B^G(\chi)$ with $\chi \in \text{Irr}_{\text{disc}}(T) = \text{Irr}_{\text{unit}}(T)$ is reducible if and only if χ has order two. There are three such characters (if $p > 2$): the quadratic unramified character χ_- and two ramified characters $\chi_r, \chi_r \chi_-$ (these fail when $p = 2$). Two representations $I_B^G(\chi)$ and $I_B^G(\chi')$ have common irreducible subquotients if and only if $\chi' \in \{\chi, s_\alpha(\chi) = \chi^{-1}\}$, and in that case $I_B^G(\chi) \cong I_B^G(\chi')$.

A class $[T, \chi]_G \in \Delta(G)$ with $\chi_{\mathfrak{o}} := \chi|_{\mathfrak{o}_F^\times}$ of order bigger than two gives rise to a circle of irreducible G -representations, which is equivalent to the analogous circle $\text{Irr}_{\text{temp}}(G)_{[T, \chi^{-1}]}$. When $\chi_{\mathfrak{o}}$ has order at most two, the isomorphisms from conjugation with s_α mean that $\text{Irr}(G)_{[T, \chi]}$ looks a half-circle, with one or two double points.

Altogether, the topological space $\text{Irr}_{\text{temp}}(G)$ looks like



Here "N x" just means countably many copies of something. We draw St_G close to $\text{Irr}_{\text{temp}}(G)_{[T, 1]}$ because it is a subquotient of $I_B^G(\chi)$ for some $\chi \in X_{\text{nr}}(T)$.

1.6. Structure of $\mathcal{S}(G)$ and $C_r^*(G)$.

The algebra $\mathcal{S}(G)$ can be described by its Fourier transform, which relates closely to the Plancherel formula for G [Wal]. By Theorem 1.27.a that means: $\mathcal{S}(G)$ is determined by how it acts on the representations $I_P^G(\delta)$ with $P = LU_P$ a parabolic subgroup of G and $\delta \in \text{Irr}_{\text{disc}}(L)$.

In view of Theorem 1.28, we need only one L and one P for each conjugacy class of Levi subgroups of G . We fix a set of Levi subgroups $\mathfrak{Lev}(G)$ representing the conjugacy classes of Levi subgroups, and for each $L \in \mathfrak{Lev}(G)$ we fix one parabolic

subgroup P with Levi factor L . We have to analyse how the images of $\mathcal{S}(G)$ in $\text{End}_{\mathbb{C}}(I_P^G(\delta))$ and in $\text{End}_{\mathbb{C}}(I_P^G(\delta'))$ are related when $I_P^G(\delta) \cong I_P^G(\delta')$.

First we look at $\chi_1 \in X_{\text{nr}}^u(L, \delta)$. By definition there exists an isomorphism of L -representations $\phi_{\chi_1} : \delta \rightarrow \delta \otimes \chi_1$, unique up to scalars. Since δ and $\delta \otimes \chi_1$ are unitary representations (on the same inner product space), we may assume that ϕ_{χ_1} is unitary. Then ϕ_{χ_1} defines a unitary L -isomorphism $\delta \otimes \chi \rightarrow \delta \otimes \chi_1 \otimes \chi$ for all $\chi \in X_{\text{nr}}^u(L)$. By the functoriality of I_P^G , we obtain a family of G -isomorphisms

$$(1.19) \quad I(\chi_1, P, \delta, \chi) : I_P^G(\delta \otimes \chi) \rightarrow I_P^G(\delta \otimes \chi_1 \otimes \chi) \quad \chi \in X_{\text{nr}}^u(L).$$

The formula (1.14) shows that $I(\chi_1, P, \delta, \chi)$ is unitary. Via (1.13), we can consider $I(\chi_1, P, \delta, \chi)$ as an operator on the vector space $I_{K_0 \cap P}^P(V_\delta)$, then it does not depend on χ .

Next we look at the other source of equivalences between parabolically induced representations in Theorem 1.27: conjugation by elements $w \in N_G(L)$. Harish-Chandra initiated the study of the integral operators

$$(1.20) \quad \begin{aligned} J(w, P, \delta, \chi) : I_P^G(\delta \otimes \chi) &\rightarrow I_P^G(w \cdot \delta \otimes w \cdot \chi) \\ f &\mapsto \int_{w^{-1}U_P w \cap U_{\bar{P}}} f(uw^{-1}g) du \end{aligned}$$

As a map from $I_{K_0 \cap P}^{K_0}(V_\delta)$ to itself, this integral depends rationally on $\chi \in X_{\text{nr}}(L)$ and converges when $|\chi|$ is large enough in a certain direction [Wal, Théorème IV.1.1]. Using [Hei2, Lemma 1.8] one can normalize $J(w, P, \delta, \chi)$ with $\chi \in X_{\text{nr}}^u(L)$ to a unitary G -isomorphism

$$(1.21) \quad J'(w, P, \delta, \chi) : I_P^G(\delta \otimes \chi) \rightarrow I_P^G(w \cdot \delta \otimes w \cdot \chi).$$

(This can be done so that $J'(w, P, \delta, \chi)$ depends continuously on χ , but it is only canonical up to functions from $X_{\text{nr}}^u(L)$ to S^1 .) The definition of $\Delta(G)$ entails that it suffices to look at δ in set of representatives for $\text{Irr}_{\text{disc}}(L)/(X_{\text{nr}}^u(L) \rtimes N_G(L))$. In combination with the union over $L \in \mathfrak{L}\mathfrak{e}\mathfrak{v}(L)$, that yields a set of representatives for $\Delta(G)$. In this setting we only need the isomorphisms (1.21) for $w \in N_G(L)$ such that $w \cdot \delta \in X_{\text{nr}}^u(L)\delta$ (as subsets of $\text{Irr}(L)$, so up to isomorphism of representations). Moreover w and wl with $l \in L$ give essentially the same operators (1.20) and (1.21), so we may let w run over a set of representatives for the finite group

$$W_{\mathfrak{d}} = \{w \in N_G(L)/L : w \cdot \delta \in X_{\text{nr}}^u(L)\delta\}.$$

For each w in this set we choose a unitary L -isomorphism $\phi_{w, \delta} : w \cdot \delta \rightarrow \delta \otimes \chi_{w, \delta}$, for some $\chi_{w, \delta} \in X_{\text{nr}}^u(L)$. Then we can compose (1.21) with $I_P^G(\phi_{w, \delta})$ to obtain a family of unitary G -isomorphisms

$$(1.22) \quad I(w, P, \delta, \chi) : I_P^G(\delta \otimes \chi) \rightarrow I_P^G(\delta \otimes \chi_{w, \delta} \otimes w \cdot \chi) \quad \chi \in X_{\text{nr}}^u(L).$$

Definition 1.31. Write $\mathfrak{d} = [L, \delta]_G$. We define $W_{\mathfrak{d}}^e = W_{[L, \delta]}^e$ as the finite group of diffeomorphisms of $\{\delta\} \times X_{\text{nr}}^u(L) \cong X_{\text{nr}}^u(L)$ generated by $\chi \mapsto \chi_1 \otimes \chi$ with $\chi_1 \in X_{\text{nr}}(L, \delta)$ and $\chi \mapsto \chi_{w, \delta} \otimes w \cdot \chi$ with $w \in W_{\mathfrak{d}}$.

Thus $W_{\mathfrak{d}}^e$ is an extension of $W_{\mathfrak{d}}$ by $X_{\text{nr}}(L, \delta)$. We note that $X_{\text{nr}}(L, \delta)$ is really needed here, in general the action of $W_{\mathfrak{d}}$ on $X_{\text{nr}}^u(L)\delta$ does not lift to a group action on $X_{\text{nr}}^u(L)$. Combining (1.19) and (1.22), we obtain operators as in (1.22) for all $w \in W_{\mathfrak{d}}^e$, canonical up to functions $X_{\text{nr}}^u(L) \rightarrow S^1$. It follows that, for all $w, w' \in W_{\mathfrak{d}}^e$, the operators

$$(1.23) \quad I(w, P, \delta, w'(\chi)) \circ I(w', P, \delta, \chi) \quad \text{and} \quad I(ww', P, \delta, \chi)$$

differ only by a function $X_{\text{nr}}^u(L) \rightarrow S^1$. This enables us to define a group action of W_δ^e on the algebra of functions from $X_{\text{nr}}^u(L)$ to $\text{End}_{\mathbb{C}}(I_{K_0 \cap P}^P(V_\delta))$ by

$$(1.24) \quad (w \cdot \mathcal{F})(w(\chi)) = I(w, P, \delta, \chi) \circ \mathcal{F}(\chi) \circ I(w, P, \delta, \chi)^{-1}$$

$$\chi \in X_{\text{nr}}^u(L), \mathcal{F} : X_{\text{nr}}^u(L) \rightarrow \text{End}_{\mathbb{C}}(I_{K_0 \cap P}^{K_0}(V_\delta)).$$

This action stabilizes various subalgebras, for instance $C(X_{\text{nr}}^u(L)) \otimes \text{End}_{\mathbb{C}}^\infty(I_{K_0 \cap P}^{K_0}(V_\delta))$. Here $\text{End}_{\mathbb{C}}^\infty(W)$, for a G -representation W , means the smooth vectors in $\text{End}_{\mathbb{C}}(W)$ as $G \times G$ -representation. The algebra $\text{End}_{\mathbb{C}}^\infty(W)$ has W as unique irreducible module, and it is Morita equivalent to $\mathbb{C}$ via the bimodules W and $W^\vee$.

For each $f \in \mathcal{S}(G)$, the intertwining property of $I(w, P, \delta, \chi)$ says that

$$(1.25) \quad I(w, P, \delta, \chi) \circ I_P^G(\delta \otimes \chi)(f) = I_P^G(\delta \otimes w(\chi))(f) \circ I(w, P, \delta, \chi).$$

Hence the operators $I_P^G(\delta \otimes \chi)(f)$ are invariant for the W_δ^e -action from (1.24). Further, since $\mathcal{S}(G)$ is a smooth $G \times G$ -representation, $I_P^G(\delta \otimes \chi)(f) \in \text{End}_{\mathbb{C}}^\infty(I_{K_0 \cap P}^{K_0}(V_\delta))$. The example (1.2) shows that we can expect that $I_P^G(\delta \otimes \chi)(f)$ is a smooth function of $\chi \in X_{\text{nr}}^u(L)$. We are ready to state the Plancherel isomorphism for G , which is due to Harish-Chandra.

Theorem 1.32. [Wal]

(a) *The Fourier transform defines an isomorphism of topological $*$ -algebras*

$$\mathcal{S}(G) \cong \bigoplus_{[L, \delta]_G \in \Delta(G)} \left(C^\infty(X_{\text{nr}}^u(L)) \otimes \text{End}_{\mathbb{C}}^\infty(I_{K_0 \cap P}^{K_0}(V_\delta)) \right)^{W_{[L, \delta]}^e}.$$

(b) *For any $K \in \mathcal{CO}(G)$ with $K \subset K_0$, part (a) restricts to an isomorphism of unital Fréchet $*$ -algebras*

$$\mathcal{S}(G, K) \cong \bigoplus_{[L, \delta]_G \in \Delta(G)} \left(C^\infty(X_{\text{nr}}^u(L)) \otimes \text{End}_{\mathbb{C}}(I_{K_0 \cap P}^{K_0}(V_\delta)^K) \right)^{W_{[L, \delta]}^e}.$$

Here each space $I_{K_0 \cap P}^{K_0}(V_\delta)^K$ has finite dimension, and it is nonzero for only finitely many $[L, \delta]_G \in \Delta(G)$.

Theorem 1.32.b applies arbitrarily small $K \in \mathcal{CO}(G)$, but not to all $K \in \mathcal{CO}(G)$. We may also replace K_0 by another good maximal compact subgroup of G , so that Theorem 1.32.b applies to other $K \in \mathcal{CO}(G)$. We call

$$\mathcal{S}(G)_{[L, \delta]} \cong \left(C^\infty(X_{\text{nr}}^u(L)) \otimes \text{End}_{\mathbb{C}}^\infty(I_{K_0 \cap P}^{K_0}(V_\delta)) \right)^{W_{[L, \delta]}^e}$$

a Harish-Chandra block of $\mathcal{S}(G)$. Then we can think of Theorem 1.32.a as an explicit “Harish-Chandra decomposition”

$$(1.26) \quad \mathcal{S}(G) = \bigoplus_{[L, \delta]_G \in \Delta(G)} \mathcal{S}(G)_{[L, \delta]}.$$

Example 1.33. $G = SL_2(G)$.

We extend the description of $\text{Irr}_{\text{temp}}(G)$ in Example 1.33 to $\mathcal{S}(G)$. We take $K_0 = SL_2(\mathfrak{o}_F)$ and we represent $W(G, T) \cong S_2$ in K_0 . For $[G, \delta]_G \in \Delta(G)$ we have $W_{G, \delta} = \{1\}$ because $X_{\text{nr}}(G) = \{1\}$. Hence $\mathcal{S}(G)_{[G, \delta]} \cong \text{End}_{\mathbb{C}}^\infty(V_\delta)$.

For every $[T, \chi] \in \Delta(G)$, the group $X_{\text{nr}}(T, \chi)$ is trivial because $T = Z(T)$. When the order of $\chi_\mathfrak{o} = \chi|_{\mathfrak{o}_F^\times}$ is larger than two, the group $W_{[T, \chi_0]}^e$ is trivial and

$$\mathcal{S}(G)_{[T, \chi_0]} \cong C^\infty(S^1) \otimes \text{End}_{\mathbb{C}}^\infty(I_{K_0 \cap B}^{K_0} \mathbb{C}).$$

For χ_o of order at most two we get a summand

$$\mathcal{S}(G)_{[T, \chi]} \cong (C^\infty(S^1) \otimes \text{End}_{\mathbb{C}}^\infty(I_{K_0 \cap B}^{K_0} \mathbb{C}))^{S_2},$$

but the S_2 -actions differ for $\chi_o = 1$ and χ_o ramified quadratic (this case does not occur when $p = 2$). It becomes easier if we are satisfied with a description up to Morita equivalence. Namely, $\mathcal{S}(G)$ is Morita equivalent to

$$(C^\infty(S^1) \otimes M_2(\mathbb{C}))^{S_2} \oplus \mathbb{C}_{St} \oplus C^\infty(S^1) \rtimes S_2 \oplus \bigoplus_{j=1}^{\infty} C^\infty(S^1) \oplus \bigoplus_{i=1}^{\infty} \mathbb{C},$$

where $\mathbb{C}_{St}$ comes from the Steinberg representation. This compares well with the picture of $\text{Irr}_{\text{temp}}(G)$ in Example 1.30. See (2.5) for the definition of crossed products like $C^\infty(S^1) \rtimes S_2$.

The reduced C^* -algebra $C_r^*(G)$ admits a description similar to Theorem 1.32. By Lemma 1.18 and (1.10) we have to replace all pre-unitary tempered representations by their Hilbert space completions, then they become $C_r^*(G)$ -modules. We note that, unlike Theorem 1.32.b, $C_r^*(G)$ does not have to act on the Hilbert space completion of $I_P^G(V_\delta)$ via one of the finite dimensional subspaces $I_P^G(V_\delta)^K$. Instead it acts by compact operators, because the compact operators form the closure of the algebra of finite rank operators on a Hilbert space (with respect to the operator norm). We denote the C^* -algebra of compact operators on the Hilbert space completion of an inner product space V by $\mathfrak{K}(V)$. A study of the C^* -norm on both sides of Theorem 1.32 leads to:

Theorem 1.34. [Ply]

The Fourier transform (or equivalently Theorem 1.32) induces an isomorphism of C^ -algebras*

$$C_r^*(G) \cong \bigoplus_{[L, \delta]_G \in \Delta(G)} \left(C(X_{\text{nr}}^u(L)) \otimes \mathfrak{K}(I_{K_0 \cap P}^{K_0}(V_\delta)) \right)^{W_{[L, \delta]}^e}.$$

Here the tensor products and the direct sum are taken in the sense of C^ -algebras. For any open subgroup $K \subset K_0$, this restricts to an isomorphism of unital C^* -algebras*

$$C_r^*(G, K) \cong \bigoplus_{[L, \delta]_G \in \Delta(G)} \left(C(X_{\text{nr}}^u(L)) \otimes \text{End}_{\mathbb{C}}(I_{K_0 \cap P}^{K_0}(V_\delta)^K) \right)^{W_{[L, \delta]}^e}.$$

Notice that $C(X_{\text{nr}}^u(L))$ is the C^* -completion of $C^\infty(X_{\text{nr}}^u(L))$, and that it generalizes the example (1.10) for $GL_1(F)$. For $[L, \delta]_G \in \Delta(G)$ we write

$$C_r^*(G)_{[L, \delta]} = \left(C(X_{\text{nr}}^u(L)) \otimes \mathfrak{K}(I_{K_0 \cap P}^{K_0}(V_\delta)) \right)^{W_{[L, \delta]}^e}.$$

Like (1.26), Theorem 1.34 gives a Harish-Chandra decomposition

$$(1.27) \quad C_r^*(G) \cong \bigoplus_{[L, \delta]_G \in \Delta(G)} C_r^*(G)_{[L, \delta]}.$$

In many cases $C_r^*(G)_{[L, \delta]}$ is Morita equivalent to the crossed product $C(X_{\text{nr}}^u(L)) \rtimes W_{[L, \delta]}^e$, see [AfAu, Theorem 1.4].

1.7. Description of the smooth dual.

By Lemma 1.11, $\text{Irr}(\mathcal{H}(G))$ is the space of irreducible smooth G -representations, which we write simply as $\text{Irr}(G)$. We endow it with the Jacobson topology from $\mathcal{H}(G)$. The space $\text{Irr}(\mathcal{S}(G)) = \text{Irr}_{\text{temp}}(G)$ of irreducible nondegenerate $\mathcal{S}(G)$ -modules injects in $\text{Irr}(\mathcal{H}(G))$ [SSZ, Appendix, Proposition 3]. However, the Jacobson topology from $\mathcal{S}(G)$ on $\text{Irr}_{\text{temp}}(G)$ is finer than the subspace topology from $\text{Irr}(\mathcal{H}(G))$. The typical example is $S^1 \subset \mathbb{C}^\times$, where S^1 carries the Euclidean topology and $\mathbb{C}^\times$ is endowed with the Jacobson topology.

Definition 1.35. Let $(\pi, V) \in \text{Rep}(G)$. We say that π is

- compact modulo center if all its matrix coefficients are compactly supported modulo $Z(G)$,
- cuspidal if $J_P^G(\pi) = 0$ for all proper parabolic subgroups P of G ,
- supercuspidal if π is irreducible and not isomorphic to any subquotient of $I_P^G(\rho)$, for any proper parabolic subgroup $P = LU_P \subset G$ and any $\rho \in \text{Rep}(L)$.

From Frobenius reciprocity (1.16) one sees that every supercuspidal G -representation π is cuspidal:

$$\text{Hom}_L(J_P^G(\pi), J_P^G(\pi)) \cong \text{Hom}_G(\pi, I_P^G(J_P^G(\pi))) = 0,$$

so $J_P^G(\pi) = 0$. Jacquet and Bernstein proved that the three notions in Definition 1.35 coincide for irreducible representations:

Theorem 1.36. [Jac], [Ren, Théorème VI.2.1]

A smooth G -representation is compact modulo center if and only if it is cuspidal.

Moreover every irreducible cuspidal G -representation is supercuspidal.

Proof. We provide an argument for the second claim. Suppose that $\pi \in \text{Irr}(G)$ is cuspidal but not supercuspidal. Then it is isomorphic to a subquotient of $I_P^G(\rho)$, for some smooth representation ρ of a proper Levi subgroup $L \subset G$. By [Ren, Lemme VI.3.6], π is also a subrepresentation of $I_P^G(\rho)$.¹ Frobenius reciprocity shows that

$$0 \neq \text{Hom}_L(\pi, I_P^G(\rho)) \cong \text{Hom}_G(J_P^G(\pi), \rho),$$

so $J_P^G(\pi) \neq 0$, contradicting the cuspidality of π . $\square$

Motivated by Theorem 1.36, we denote the set of supercuspidal G -representations (up to isomorphism) by $\text{Irr}_{\text{cusp}}(G)$.

If $Z(G)$ is compact, then every $\pi \in \text{Irr}_{\text{cusp}}(G)$ is an isolated point of $\text{Irr}(G)$. In general, if $\pi \in \text{Irr}_{\text{cusp}}(G)$ and $\chi \in X_{\text{nr}}(G)$, then $\pi \otimes \chi$ is again supercuspidal. Let $X_{\text{nr}}(G, \pi)$ be the stabilizer of π for the action of $X_{\text{nr}}(G)$ by tensoring. Like in (1.18), we have a bijection

$$(1.28) \quad X_{\text{nr}}(G)/X_{\text{nr}}(G, \pi) \rightarrow X_{\text{nr}}(G)\pi = \{\pi \otimes \chi \in \text{Irr}(G) : \chi \in X_{\text{nr}}(G)\}.$$

This endows $X_{\text{nr}}(G)\pi$ with the structure of an algebraic variety, namely a complex algebraic torus. It will turn out that $X_{\text{nr}}(G)\pi$ is a connected component of $\text{Irr}(G)$, of minimal dimension.

A supercuspidal representation need not be tempered or unitary, but in a sense it is not far off. By Lemma 1.24 every tempered supercuspidal representation is

¹“Supercuspidale” in [Ren] is cuspidal in our terminology.

unitary, and conversely every unitary supercuspidal representation is discrete series so in particular tempered. In particular

$$\mathrm{Irr}_{\mathrm{cusp}, \mathrm{temp}}(G) := \mathrm{Irr}_{\mathrm{cusp}}(G) \cap \mathrm{Irr}_{\mathrm{temp}}(G) \quad \text{is a subset of } \mathrm{Irr}_{\mathrm{disc}}(G).$$

The group of smooth characters $\mathrm{Hom}(G, \mathbb{R}_{>0}^\times)$ consists of unramified characters, because $\mathbb{R}_{>0}^\times$ has no compact subgroups apart from $\{1\}$. We write

$$X_{\mathrm{nr}}^+(G) = \mathrm{Hom}(G, \mathbb{R}_{>0}^\times),$$

a group isomorphic to $\mathrm{Hom}(\mathbb{Z}^d, \mathbb{R}_{>0}^\times) \cong (\mathbb{R}_{>0}^\times)^d$. Then $X_{\mathrm{nr}}(G)$ admits the polar decomposition

$$(1.29) \quad X_{\mathrm{nr}}(G) = X_{\mathrm{nr}}^u(G) \times X_{\mathrm{nr}}^+(G).$$

Lemma 1.37. [FlSo, Lemma 4.2]

Tensoring provides a bijection

$$\mathrm{Irr}_{\mathrm{cusp}, \mathrm{temp}}(G) \times X_{\mathrm{nr}}^+(G) \rightarrow \mathrm{Irr}_{\mathrm{cusp}}(G).$$

By Lemma 1.37 and (1.29), we may identify

$$(1.30) \quad \mathrm{Irr}_{\mathrm{cusp}}(G)/X_{\mathrm{nr}}(G) = \mathrm{Irr}_{\mathrm{cusp}, \mathrm{temp}}(G)/X_{\mathrm{nr}}^u(G).$$

Like all irreducible tempered G -representations arise from discrete series representations via parabolic induction, all irreducible smooth G -representations arise from supercuspidal representations via parabolic induction. That was shown by Jacquet [Jac], while Bernstein proved the uniqueness in the next result.

Theorem 1.38. [Ren, Théorème VI.5.4]

Let $\pi \in \mathrm{Irr}(G)$. There exists a parabolic subgroup $P = LU_P$ of G and a $\sigma \in \mathrm{Irr}_{\mathrm{cusp}}(L)$ such that π is isomorphic to a subquotient of $I_P^G(\sigma)$. Moreover the pair (L, σ) is uniquely determined up to G -conjugation.

By (1.15) and Lemma 1.21, every pair (M, τ) which is G -conjugate to (L, σ) yields a parabolically induced representation $I_Q^G(\tau)$ with exactly the same irreducible constituents as $I_P^G(\sigma)$. Nevertheless $I_Q^G(\tau)$ need not be isomorphic to $I_P^G(\sigma)$.

The G -conjugacy class of (L, σ) in Theorem 1.38 is called the supercuspidal support of π , denoted $\mathrm{Sc}(\pi)$. This can be regarded as a map

$$\mathrm{Sc} : \mathrm{Irr}(G) \longrightarrow \{(L, \sigma) : L \subset G \text{ Levi subgroup}, \sigma \in \mathrm{Irr}_{\mathrm{cusp}}(L)\}/G.$$

On the set of pairs (L, σ) as above we put the equivalence relation generated by G -conjugation, by isomorphism of L -representations by and $(L, \sigma \otimes \chi) \sim (L, \sigma)$ for $\chi \in X_{\mathrm{nr}}(G)$. We write

$$\mathfrak{B}(G) = \{(L, \sigma) : L \subset G \text{ Levi subgroup}, \sigma \in \mathrm{Irr}_{\mathrm{cusp}}(L)\}/\sim.$$

By (1.30), $\mathfrak{B}(G)$ is a subset of $\Delta(G)$. The elements of $\mathfrak{B}(G)$, denoted $[L, \sigma]_G$, are called inertial equivalence classes for G . To any $\mathfrak{s} = [L, \sigma]_G$ one associates the set

$$\begin{aligned} \mathrm{Irr}(G)^\mathfrak{s} &= \{\pi \in \mathrm{Irr}(G) : \mathrm{Sc}(\pi)/\sim \in \mathfrak{s}\} \\ &= \{\pi \in \mathrm{Irr}(G) : \pi \text{ is a subquotient of } I_P^G(\sigma \otimes \chi) \text{ for some } \chi \in X_{\mathrm{nr}}(L)\}. \end{aligned}$$

Theorem 1.38 is the main step towards the Bernstein decomposition of the smooth dual:

Theorem 1.39. [Ren, Théorème VI.7.1]

The sets $\text{Irr}(G)^\mathfrak{s}$ with $\mathfrak{s} \in \mathfrak{B}(G)$ are precisely the connected components of $\text{Irr}(G)$, so $\text{Irr}(G) = \bigsqcup_{\mathfrak{s} \in \mathfrak{B}(G)} \text{Irr}(G)^\mathfrak{s}$.

The set $\text{Irr}(G)^\mathfrak{s}$ endowed with the Jacobson topology from $\text{Irr}(\mathcal{H}(G))$ is called a Bernstein component of $\text{Irr}(G)$. By [Sau, Théorème 3.2], the representations $I_P^G(\sigma \otimes \chi)$ are irreducible for χ in a Zariski-open dense subset of $X_{\text{nr}}(G)$. Further, by Theorem 1.38 two representations $I_P^G(\sigma \otimes \chi)$ and $I_{P'}^G(\sigma \otimes \chi')$ have common irreducible subquotients if and only if $\sigma \otimes \chi' \cong n \cdot (\sigma \otimes \chi)$ for some $n \in N_G(L)$. Hence $\text{Irr}(G)^{[L, \sigma]}$ is a possibly nonseparated algebraic variety with maximal separable quotient $X_{\text{nr}}(L)\sigma/N_G(L)$, and such that the inseparable points live only over some lower dimensional subvarieties of $X_{\text{nr}}(L)\sigma/N_G(L)$.

Example 1.40. $G = SL_2(F)$

Every supercuspidal representation gives an isolated point in $\text{Irr}(G)$.

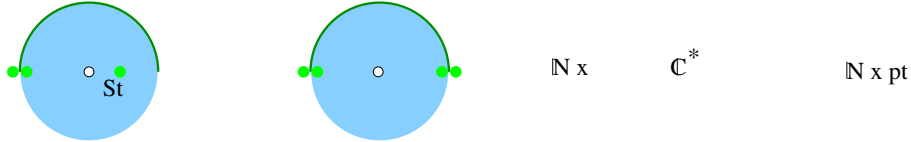
Every inertial equivalence class $[T, \chi]_G$ is determined by $\chi_\circ = \chi|_{\mathfrak{o}_F^\times}$. If $\text{ord}(\chi_\circ) > 2$, then all the representations $I_B^G(\chi')$ with $\chi' \in X_{\text{nr}}(T)\chi$ are irreducible and mutually inequivalent. There are still isomorphisms

$$(1.31) \quad I_B^G(\chi') \cong I_B^G(s \cdot \chi') = I_B^G(\chi'^{-1}) \text{ for } s \in N_G(T) \setminus T.$$

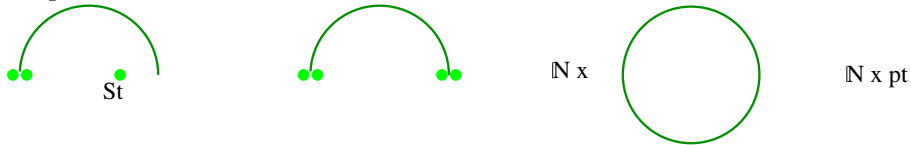
This gives countably many Bernstein components homeomorphic to $\mathbb{C}^\times$, indexed by

$$\{\chi_\circ \in \text{Irr}(\mathfrak{o}_F^\times) : \text{ord}(\chi_\circ) > 2\} / W(G, T).$$

When $\chi_\circ$ has order two, the representations $I_B^G(\chi')$ with $\chi' \in X_{\text{nr}}(T)\chi$ are reducible if χ' is quadratic and irreducible otherwise, while the only equivalences among them are (1.31). When $\chi_\circ = 1$, characters χ' in $X_{\text{nr}}(T)\chi$ are precisely the unramified characters of T . The representation $I_B^G(\chi_-)$ with $\text{ord}(\chi_-) = 2$ is a direct sum of two irreducibles. If $\chi \in X_{\text{nr}}(T)$ sends a generator of $T/T^1 \cong \mathbb{Z}$ to q_F or to q_F^{-1} , then $I_B^G(\chi)$ has two irreducible constituents: St_G and the trivial G -representation. All other representations $I_B^G(\chi)$ with $\chi \in X_{\text{nr}}(T)$ are irreducible and satisfy (1.31). Altogether, we find that $\text{Irr}(SL_2(F))$ is homeomorphic to



The two discs with a hole in the middle represent $\mathbb{C}^\times/S^2$ with S^2 acting by inversion. It is interesting to compare this with the picture of $\text{Irr}_{\text{temp}}(SL_2(F))$ from Example 1.30:



We see that $\text{Irr}(SL_2(F))$ is some sort of complexification of $\text{Irr}_{\text{temp}}(SL_2(F))$. More precisely $\text{Irr}_{\text{temp}}(SL_2(F))$ is built from circles and points, and if we replace each circle by $\mathbb{C}^\times$, we obtain a space $\text{Irr}_{\text{temp}}(SL_2(F))_{\mathbb{C}}$ with a natural bijection to $\text{Irr}(SL_2(F))$. However, the topology of $\text{Irr}_{\text{temp}}(SL_2(F))_{\mathbb{C}}$ is finer than that of $\text{Irr}(SL_2(F))$. For instance, $I_B^G(\text{triv})$ and St_G are in different connected components of $\text{Irr}_{\text{temp}}(SL_2(F))_{\mathbb{C}}$ but in the same Bernstein component.

The relation between Examples 1.30 and 1.40 generalizes to arbitrary reductive p -adic groups [ABPS1]. Namely, $\text{Irr}_{\text{temp}}(G)$ is built from copies of $X_{\text{nr}}^u(L)$ for Levi subgroups $L \subset G$. If we replace each occurrence of $X_{\text{nr}}^u(L)$ by its complexification $X_{\text{nr}}(L)$, then we obtain a space $\text{Irr}_{\text{temp}}(G)_{\mathbb{C}}$. By an extension of the Langlands classification, $\text{Irr}_{\text{temp}}(G)_{\mathbb{C}}$ maps bijectively to $\text{Irr}(G)$. This can be stated (rather imprecisely) as

$$(1.32) \quad \text{Irr}(G) \text{ is canonically in bijection with a complexification of } \text{Irr}_{\text{temp}}(G).$$

Hence, for any reductive p -adic group the smooth dual and the tempered dual are equally difficult to determine.

1.8. Rough structure of $\mathcal{H}(G)$.

There are two stronger versions of Theorem 1.39, which are very useful to understand $\text{Rep}(G)$ and $\mathcal{H}(G)$. For $\mathfrak{s} \in \mathfrak{B}(G)$, we define

$$\text{Rep}(G)^{\mathfrak{s}} = \{\pi \in \text{Rep}(G) : \text{every irreducible subquotient of } \pi \text{ lies in } \text{Irr}(G)^{\mathfrak{s}}\}.$$

In other words, $\text{Rep}(G)^{\mathfrak{s}}$ is the full subcategory of $\text{Rep}(G)$ generated by $\text{Irr}(G)^{\mathfrak{s}}$. Further, we define a two-sided ideal of $\mathcal{H}(G)$ by

$$\mathcal{H}(G)^{\mathfrak{s}} = \{f \in \mathcal{H}(G) : \pi(h) = 0 \text{ for all } \pi \in \text{Irr}(G) \setminus \text{Irr}(G)^{\mathfrak{s}}\}.$$

The next theorem is known as the Bernstein decomposition.

Theorem 1.41. [BeDe, Ren]

Each $\text{Rep}(G)^{\mathfrak{s}}$ is a block of $\text{Rep}(G)$, that is, an indecomposable direct summand of $\text{Rep}(G)$. The category $\text{Rep}(G)$ admits an orthogonal decomposition

$$\text{Rep}(G) = \prod_{\mathfrak{s} \in \mathfrak{B}(G)} \text{Rep}(G)^{\mathfrak{s}}.$$

Each two-sided ideal $\mathcal{H}(G)^{\mathfrak{s}}$ of $\mathcal{H}(G)$ is indecomposable and $\mathcal{H}(G) = \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} \mathcal{H}(G)^{\mathfrak{s}}$.

By Lemma 1.11.a, $\text{Rep}(G)^{\mathfrak{s}}$ can be identified with $\text{Mod}(\mathcal{H}(G)^{\mathfrak{s}})$. These subcategories are called Bernstein blocks of $\text{Rep}(G)$ or for G .

To understand $\mathcal{H}(G)$, it suffices to classify $\mathfrak{B}(G)$ and to understand each $\mathcal{H}(G)^{\mathfrak{s}}$. However, in spite of Theorem 1.41 and (1.32), the structure of $\text{Rep}(G)$ is substantially more complicated than that of $\text{Mod}(\mathcal{S}(G))$. The main reason is that the building blocks $I_P^G(\delta)$ for $\text{Mod}(\mathcal{S}(G))$ are unitary, while for $\mathcal{H}(G)$ the building blocks $I_P^G(\sigma)$ with $\sigma \in \text{Irr}_{\text{cusp}}(L)$ need not be completely reducible.

Like the Plancherel isomorphism for $\mathcal{S}(G)$, one would like to understand $\mathcal{H}(G)$ by the Fourier transform, which in this case means its action on the representations $I_P^G(\sigma)$ with $\sigma \in \text{Irr}_{\text{cusp}}(L)$. We proceed as in Paragraph 1.6. By Lemma 1.37 we may assume that $\sigma \in \text{Irr}_{\text{cusp,temp}}(L)$, so that it is a discrete series representation.

For $\chi_1 \in X_{\text{nr}}(L, \sigma)$ we have the operator

$$I(\chi_1, P, \sigma, \chi) : I_P^G(\sigma \otimes \chi) \rightarrow I_P^G(\sigma \otimes \chi_1 \otimes \chi).$$

For $w \in N_G(L)$ there is the intertwining operator

$$J(w, P, \sigma, \chi) : I_P^G(\sigma \otimes \chi) \rightarrow I_P^G(w \cdot \sigma \otimes w \cdot \chi)$$

from (1.20), which depends rationally $\chi \in X_{\text{nr}}(L)$ once we identify the underlying vector spaces with $I_{K_0 \cap P}^{K_0}(V_{\sigma})$ as in (1.13). The operator $J(w, P, \sigma, \chi)$ may have poles and zeros at some nonunitary $\chi \in X_{\text{nr}}(L)$, and that complicates things. Even when we normalize it to $J'(w, P, \sigma, \chi)$ as in (1.21), it need not be injective for some

$\chi \in X_{\text{nr}}(L)\sigma \setminus X_{\text{nr}}^u(L)\sigma$. This means that the action (1.24) of $W_{[L,\sigma]}$ on $C(X_{\text{nr}}^u(L)) \otimes \text{End}_{\mathbb{C}}^{\infty}(I_{K_0 \cap P}^{K_0} V_{\sigma})$ does not stabilize the subalgebra $\mathcal{O}(X_{\text{nr}}(L)) \otimes \text{End}_{\mathbb{C}}^{\infty}(I_{K_0 \cap P}^{K_0} V_{\sigma})$. The (weaker) analogue of Theorem 1.32 for $\mathcal{H}(G)$ is:

Theorem 1.42. [Hei1]

(a) *The Fourier transform (the action of $\mathcal{H}(G)$ on the representations $I_P^G(\sigma')$ with $\sigma' \in \text{Irr}_{\text{cusp}}(L)$) determines an isomorphism of $*$ -algebras $\mathcal{H}(G) \cong$*

$$\bigoplus_{[L,\sigma]_G \in \mathfrak{B}(G)} \left(C^{\infty}(X_{\text{nr}}^u(L)) \otimes \text{End}_{\mathbb{C}}^{\infty}(I_{K_0 \cap P}^{K_0}(V_{\sigma})) \right)^{W_{[L,\sigma]}^e} \cap \mathcal{O}(X_{\text{nr}}(L)) \otimes \text{End}_{\mathbb{C}}^{\infty}(I_{K_0 \cap P}^{K_0}(V_{\sigma})).$$

(b) *For every $K \in \mathcal{CD}(G)$ with $K \subset K_0$, part (a) restricts to an isomorphism*

$$\mathcal{H}(G, K) \cong \bigoplus_{[L,\sigma]_G \in \mathfrak{B}(G)} \left(C^{\infty}(X_{\text{nr}}^u(L)) \otimes \text{End}_{\mathbb{C}}(I_{K_0 \cap P}^{K_0}(V_{\sigma})^K) \right)^{W_{[L,\sigma]}^e} \cap \mathcal{O}(X_{\text{nr}}(L)) \otimes \text{End}_{\mathbb{C}}(I_{K_0 \cap P}^{K_0}(V_{\sigma})^K).$$

Here $I_{K_0 \cap P}^{K_0}(V_{\sigma})^K$ has finite dimension, and it is nonzero for only finitely many $[L,\sigma]_G \in \mathfrak{B}(G)$.

Notice that in Theorem 1.42 the regular functions on $X_{\text{nr}}(L)$ appear in the same way as $\mathcal{H}(F^{\times})^{\circ_F^{\times}} \cong \mathcal{O}(\mathbb{C}^{\times})$ in (1.1).

Example 1.43. Every $\sigma \in \text{Irr}_{\text{cusp}}(G)$ gives a summand

$$\mathcal{H}(G)^{[G,\sigma]} = \text{End}_{\mathbb{C}}^{\infty}(V_{\sigma}) \cong \bigcup_{n \in \mathbb{N}} M_n(\mathbb{C})$$

of $\mathcal{H}(G)$. For an inertial equivalence class $\mathfrak{s} = [T, \chi]_G$ with $\chi_{\mathfrak{o}} = \chi|_{\mathfrak{o}_F^{\times}}$ of order bigger than two, $W_{[T,\chi]}^e$ is trivial and

$$\mathcal{H}(G)^{\mathfrak{s}} \cong \mathcal{O}(X_{\text{nr}}(T)) \otimes \text{End}_{\mathbb{C}}^{\infty}(I_{B \cap K_0}^{K_0}(\mathbb{C}_{\chi})),$$

where $K_0 = SL_2(\mathfrak{o}_F)$. Let $\chi_r \in \text{Irr}(T)$ be ramified quadratic, and represent $S_2 \cong W(G, T)$ in K_0 , then

$$\mathcal{H}(G)^{[T,\chi_r]} \cong (\mathcal{O}(X_{\text{nr}}(T)) \otimes \text{End}_{\mathbb{C}}(\mathbb{C}[S_2]))^{S_2} \otimes \text{End}_{\mathbb{C}}^{\infty}(I_{B \cap K_0}^{K_0}(\mathbb{C}_{\chi_r})^{S_2}).$$

Finally, for the trivial character of T there is an affine Hecke algebra $\mathcal{H}_{\text{aff}}$ such that

$$\mathcal{H}(G)^{[T,1]} \cong \mathcal{H}_{\text{aff}} \otimes \text{End}_{\mathbb{C}}^{\infty}(I_{B \cap K_0}^{K_0}(\text{triv}_T)^{S_2}).$$

We find that $\mathcal{H}(G)$ is Morita equivalent to

$$\mathcal{H}_{\text{aff}} \oplus \mathcal{O}(X_{\text{nr}}(T)) \rtimes S_2 \oplus \bigoplus_{j=1}^{\infty} \mathcal{O}(\mathbb{C}^{\times}) \oplus \bigoplus_{i=1}^{\infty} \mathbb{C}.$$

This can be compared with the description of $\mathcal{S}(G)$ up to Morita equivalence, from Example 1.33:

$$\left((C^{\infty}(S^1) \otimes M_2(\mathbb{C}))^{S_2} \oplus \mathbb{C}_{S^1} \right) \oplus C^{\infty}(S^1) \rtimes S_2 \oplus \bigoplus_{j=1}^{\infty} C^{\infty}(S^1) \oplus \bigoplus_{i=1}^{\infty} \mathbb{C}.$$

1.9. Decompositions of $\mathcal{S}(G)$ and $C_r^*(G)$.

The Bernstein decomposition of $\mathcal{H}(G)$ induces decompositions of $\mathcal{S}(G)$ and $C_r^*(G)$. Namely, let $\mathcal{S}(G)^\mathfrak{s}$ be the closure of $\mathcal{H}(G)^\mathfrak{s}$ in $\mathcal{S}(G)$ and let $C_r^*(G)^\mathfrak{s}$ be the closure of $\mathcal{H}(G)^\mathfrak{s}$ in $C_r^*(G)$. Theorem 1.41 and the density of $\mathcal{H}(G)$ in $\mathcal{S}(G)$ and in $C_r^*(G)$ imply the Bernstein decompositions

$$(1.33) \quad \begin{aligned} \mathcal{S}(G) &= \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} \mathcal{S}(G)^\mathfrak{s}, \\ C_r^*(G) &= \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} C_r^*(G)^\mathfrak{s}, \end{aligned}$$

where for $C_r^*(G)$ the direct sum is meant in the sense of Banach algebras. The two-sided ideals $\mathcal{S}(G)^\mathfrak{s} \subset \mathcal{S}(G)$ and $C_r^*(G)^\mathfrak{s} \subset C_r^*(G)$ are often decomposable. Hence the decompositions (1.33) are coarser than the Harish-Chandra decompositions in (1.26) and (1.27), which are block decompositions. For $\mathfrak{d} \in \Delta(G)$ and $\mathfrak{s} \in \mathfrak{B}(G)$, we write

$$\mathfrak{d} \in \Delta(G, \mathfrak{s}) \quad \text{when} \quad \text{Irr}_{\text{temp}}(G)_\mathfrak{d} \subset \text{Irr}(G)^\mathfrak{s}.$$

We note that $\mathfrak{s}$ itself is an element of $\Delta(G, \mathfrak{s})$, in fact the only element that can be represented by a supercuspidal representation of a Levi subgroup of G . From (1.33) and Theorems 1.32 and 1.34 we obtain

$$(1.34) \quad \begin{aligned} \mathcal{S}(G)^\mathfrak{s} &= \bigoplus_{\mathfrak{d} \in \Delta(G, \mathfrak{s})} \mathcal{S}(G)_\mathfrak{d}, \\ C_r^*(G)^\mathfrak{s} &= \bigoplus_{\mathfrak{d} \in \Delta(G, \mathfrak{s})} C_r^*(G)_\mathfrak{d}. \end{aligned}$$

The finite groups indexing the intertwining operators in Theorems 1.32 and 1.42 for $\text{Rep}(G)^\mathfrak{s}$ are related:

Lemma 1.44. *For $\mathfrak{d} \in \Delta(G, \mathfrak{s})$, the group $W_\mathfrak{d}^e$ is a subquotient of $W_\mathfrak{s}^e$.*

Proof. We write $\mathfrak{d} = [M, \delta]_G$ and $\mathfrak{s} = [L, \sigma]_G$, where $M \supset L$. Recall from Definition 1.31 that

$$W_{[M, \delta]}^e / X_{\text{nr}}(M, \delta) \cong W_{[M, \delta]} \quad \text{and} \quad W_{[L, \sigma]}^e / X_{\text{nr}}(L, \sigma) \cong W_{[L, \sigma]}.$$

The group $W_{[M, \delta]} \subset N_G(M)/M$ stabilizes $\text{Irr}_{\text{temp}}(G)_{[M, \delta]} \subset \text{Irr}(G)^{[L, \sigma]}$, so stabilizes $\text{Rep}(G)^{[L, \sigma]}$. Hence $W_{[M, \delta]}$ can be represented in $\text{Stab}_{N_G(M, L)/L}(X_{\text{nr}}(L)\sigma)$, which is a subgroup of $W_{[L, \sigma]}$.

Suppose that $(L, \sigma \otimes \chi_L)$ represents $\text{Sc}(\delta)$ and that $\chi_M \in X_{\text{nr}}(M, \delta)$. Then $(L, \sigma \otimes \chi_L(\chi_M \chi)|_L)$ represents $\text{Sc}(\delta \otimes \chi)$, for all $\chi \in X_{\text{nr}}^u(M)$. By the uniqueness of cuspidal supports up to G -conjugation, there exists $w \in W_{[L, \sigma]}^e$ such that $w(\sigma \otimes \chi_L \chi|_L) = \sigma \otimes \chi_L(\chi_M \chi)|_L$ for all $\chi_M \in X_{\text{nr}}^u(M)$.

Hence the action of any element of $W_{[M, \delta]}^e$ on $X_{\text{nr}}(M)\delta$ arises from an element of $W_{[L, \sigma]}^e$. It follows that $W_{[M, \delta]}^e$ is isomorphic to the quotient of $\text{Stab}_{W_{[L, \sigma]}^e}(X_{\text{nr}}^u(M)\delta)$ by the elements that act trivially on $X_{\text{nr}}(M)\delta$. $\square$

For $K \in \mathcal{C}\mathcal{O}(G)$ we put

$$(1.35) \quad \begin{aligned} \mathcal{H}(G, K)^\mathfrak{s} &= \langle K \rangle \mathcal{H}(G)^\mathfrak{s} \langle K \rangle &= \mathcal{H}(G, K) \cap \mathcal{H}(G)^\mathfrak{s}, \\ \mathcal{S}(G, K)^\mathfrak{s} &= \langle K \rangle \mathcal{S}(G)^\mathfrak{s} \langle K \rangle &= \mathcal{S}(G, K) \cap \mathcal{S}(G)^\mathfrak{s}, \\ C_r^*(G, K)^\mathfrak{s} &= \langle K \rangle C_r^*(G)^\mathfrak{s} \langle K \rangle &= C_r^*(G, K) \cap C_r^*(G)^\mathfrak{s}. \end{aligned}$$

Then $\mathcal{S}(G, K)^\mathfrak{s}$ and $C_r^*(G, K)^\mathfrak{s}$ are closures of $\mathcal{H}(G, K)^\mathfrak{s}$, and they are generated by $\mathcal{H}(G, K)^\mathfrak{s}$ as two-sided ideals in, respectively, $\mathcal{S}(G, K)$ and $C_r^*(G, K)$. To understand $\mathcal{H}(G)^\mathfrak{s}$, it suffices to consider $\mathcal{H}(G, K)^\mathfrak{s}$ for $K \in \mathcal{C}\mathcal{O}(G)$ in a countably family:

Theorem 1.45. [BeDe, §2.2 and Corollaire 3.9]

There exists a decreasing sequence $(K_n)_{n=1}^\infty$ of compact open subgroups of G with the following properties:

- $\bigcap_{n=1}^\infty K_n = \{1\}$
- Each K_n is a normal subgroup of the good maximal compact subgroup K_0 .
- For each $\mathfrak{s} \in \mathfrak{B}(G)$ and each $n \in \mathbb{Z}_{>0}$, $\mathcal{H}(G, K_n)^\mathfrak{s}$ is either 0 or Morita equivalent to $\mathcal{H}(G)^\mathfrak{s}$.
- The bimodules for such a Morita equivalence are $\langle K_n \rangle \mathcal{H}(G)^\mathfrak{s}$ and $\mathcal{H}(G)^\mathfrak{s} \langle K_n \rangle$.

From Theorems 1.42 and 1.45 one concludes:

Corollary 1.46. There exists a unique finite subset $\mathfrak{B}(G, K_n) \subset \mathfrak{B}(G)$ such that

$$\mathcal{H}(G, K_n) \text{ equals } \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G, K_n)} \mathcal{H}(G, K_n)^\mathfrak{s}$$

and is Morita equivalent to $\bigoplus_{\mathfrak{s} \in \mathfrak{B}(G, K_n)} \mathcal{H}(G)^\mathfrak{s}$.

The sequence of sets $(\mathfrak{B}(G, K_n))_{n=1}^\infty$ increases and its union is $\mathfrak{B}(G)$.

There are versions of Theorem 1.45 and Corollary 1.46 for $\mathcal{S}(G)$ and $C_r^*(G)$:

Proposition 1.47. Let K_n be as in Theorem 1.45.

- For any $\mathfrak{s} \in \mathfrak{B}(G)$, $\mathcal{S}(G, K_n)^\mathfrak{s}$ is either 0 or Morita equivalent to $\mathcal{S}(G)^\mathfrak{s}$. In the latter case, the Morita bimodules are $\langle K_n \rangle \mathcal{S}(G)^\mathfrak{s}$ and $\mathcal{S}(G)^\mathfrak{s} \langle K_n \rangle$.
- The algebra $\mathcal{S}(G, K_n)$ equals $\bigoplus_{\mathfrak{s} \in \mathfrak{B}(G, K_n)} \mathcal{S}(G, K_n)^\mathfrak{s}$ and is Morita equivalent to $\bigoplus_{\mathfrak{s} \in \mathfrak{B}(G, K_n)} \mathcal{S}(G)^\mathfrak{s}$.
- Parts (a) and (b) also hold for $C_r^*(G, K_n)$.

Proof. (a) If $\mathcal{H}(G, K_n)^\mathfrak{s} = 0$, then also $\mathcal{S}(G, K_n)^\mathfrak{s} = 0$. Therefore we may assume that $\mathcal{S}(G, K_n)^\mathfrak{s}$ is nonzero. We consider $\langle K_n \rangle \mathcal{S}(G)^\mathfrak{s}$ and $\mathcal{S}(G)^\mathfrak{s} \langle K_n \rangle$, which are bimodules for $\mathcal{S}(G)^\mathfrak{s}$ and $\mathcal{S}(G, K_n)^\mathfrak{s}$. Multiplication provides an isomorphism of $\mathcal{S}(G, K_n)^\mathfrak{s}$ -bimodules

$$(1.36) \quad \langle K_n \rangle \mathcal{S}(G)^\mathfrak{s} \otimes_{\mathcal{S}(G)^\mathfrak{s}} \mathcal{S}(G)^\mathfrak{s} \langle K_n \rangle \rightarrow \mathcal{S}(G, K_n)^\mathfrak{s},$$

and an isomorphism of $\mathcal{S}(G)^\mathfrak{s}$ -bimodules

$$(1.37) \quad \mathcal{S}(G)^\mathfrak{s} \langle K_n \rangle \otimes_{\mathcal{S}(G, K_n)^\mathfrak{s}} \langle K_n \rangle \mathcal{S}(G)^\mathfrak{s} \rightarrow \mathcal{S}(G)^\mathfrak{s} \langle K_n \rangle \mathcal{S}(G)^\mathfrak{s}.$$

The right hand side of (1.37) is an $\mathcal{S}(G)^\mathfrak{s}$ -sub-bimodule of $\mathcal{S}(G)^\mathfrak{s}$, and by Theorem 1.45 it contains $\mathcal{H}(G)^\mathfrak{s}$. Since $\mathcal{H}(G)^\mathfrak{s}$ generates $\mathcal{S}(G)^\mathfrak{s}$ as an ideal, we deduce that

$$(1.38) \quad \mathcal{S}(G)^\mathfrak{s} \langle K_n \rangle \mathcal{S}(G)^\mathfrak{s} = \mathcal{S}(G)^\mathfrak{s}.$$

From (1.36), (1.37) and (1.38) we see that $\langle K_n \rangle \mathcal{S}(G)^\mathfrak{s}$ and $\mathcal{S}(G)^\mathfrak{s} \langle K_n \rangle$ implement a Morita equivalence between $\mathcal{S}(G)^\mathfrak{s}$ and $\mathcal{S}(G, K_n)^\mathfrak{s}$.

(b) This follows from Corollary 1.46 and part (a).

(c) This can be shown in the same way as parts (a) and (b). $\square$

2. TWISTED GRADED HECKE ALGEBRAS

In this section we survey some algebras which will play an important role in the analysis of the Hecke algebra of a reductive p -adic group.

2.1. Twisted crossed products.

Let Γ be a finite group. A 2-cocycle for Γ is a map $\natural : \Gamma \times \Gamma \rightarrow \mathbb{C}^\times$ such that

$$(2.1) \quad \natural(\gamma_1\gamma_2, \gamma_3)\natural(\gamma_1, \gamma_2) = \natural(\gamma_1, \gamma_2\gamma_3)\natural(\gamma_2, \gamma_3) \quad \text{for all } \gamma_1, \gamma_2, \gamma_3 \in \Gamma.$$

To these data one associates the twisted group algebra $\mathbb{C}[\Gamma, \natural]$, which has a $\mathbb{C}$ -basis $\{T_\gamma : \gamma \in \Gamma\}$ and multiplication rules

$$T_{\gamma_1}T_{\gamma_2} = \natural(\gamma_1, \gamma_2)T_{\gamma_1\gamma_2} \quad \text{for all } \gamma_1, \gamma_2 \in \Gamma.$$

The condition (2.1) means precisely that $\mathbb{C}[\Gamma, \natural]$ is an associative algebra. Replacing T_e by $\natural(e, e)^{-1}T_e$, we can achieve that $T_e \cdot T_e = T_e$ (at the cost of modifying $\natural$). Therefore we may and will always assume that $\natural(e, e) = 1$. Then (2.1) for $(\gamma_1, \gamma_2, \gamma_3) = (e, e, \gamma)$ shows that $\natural(e, \gamma) = 1$ while (2.1) for $(\gamma_1, \gamma_2, \gamma_3) = (\gamma, e, e)$ shows that $\natural(\gamma, e) = 1$. In other words, the condition $\natural(e, e) = 1$ implies that T_e is the unit element of $\mathbb{C}[\Gamma, \natural]$.

For any function $f : \Gamma \rightarrow \mathbb{C}^\times$, one may pass to new basis elements $T'_\gamma = f(\gamma)T_\gamma$. In those terms, the multiplication rules read

$$(2.2) \quad T'_{\gamma_1}T'_{\gamma_2} = f(\gamma_1)f(\gamma_2)f(\gamma_1\gamma_2)^{-1}\natural(\gamma_1, \gamma_2)T'_{\gamma_1\gamma_2}.$$

The map

$$b(f) : (\gamma_1, \gamma_2) \mapsto f(\gamma_1)f(\gamma_2)f(\gamma_1\gamma_2)^{-1}$$

is called the coboundary of f . Hence the algebra $\mathbb{C}[\Gamma, \natural]$ depends, up to rescaling, only on $\natural$ modulo coboundaries, that is, on the image of $\natural$ in $H^2(\Gamma, \mathbb{C}^\times)$. This construction yields a map from the second group cohomology $H^2(\Gamma, \mathbb{C}^\times)$ to twisted versions of $\mathbb{C}[\Gamma]$ up to isomorphism.

There exists a finite central extension Γ^* of Γ , such that the inflation to $\Gamma^* \times \Gamma^*$ of any 2-cocycle $\natural$ for Γ represents the trivial class in $H^2(\Gamma^*, \mathbb{C}^\times)$. It is known as the Schur extension or multiplier of Γ [CuRe, §53]. Let Z^* be the kernel of $\Gamma^* \rightarrow \Gamma$. Then $\natural$ determines a character $c_\natural$ of Z^* , with central idempotent $e_\natural \in \mathbb{C}[Z^*] \subset \mathbb{C}[\Gamma^*]$, such that

$$(2.3) \quad \mathbb{C}[\Gamma, \natural] \cong e_\natural \mathbb{C}[\Gamma^*].$$

From (2.3) one can recover $\natural$ (up to some coboundary), as follows. Pick representatives $\{\gamma^* : \gamma \in \Gamma\}$ for Γ in Γ^* . Then $\{e_\natural \gamma^* : \gamma \in \Gamma\}$ is a $\mathbb{C}$ -basis of $e_\natural \mathbb{C}[\Gamma^*]$, and $\natural$ can be defined by the formula

$$e_\natural \gamma_1^* \cdot e_\natural \gamma_2^* = \natural(\gamma_1, \gamma_2)e_\natural (\gamma_1\gamma_2)^*.$$

Moreover $\mathbb{C}[\Gamma^*] = \bigoplus_{c_\natural \in \text{Irr}(Z^*)} e_\natural \mathbb{C}[\Gamma^*]$, so each $\mathbb{C}[\Gamma, \natural]$ is isomorphic to a direct summand of $\mathbb{C}[\Gamma^*]$. As $\mathbb{C}[\Gamma^*]$ is semisimple, so is $\mathbb{C}[\Gamma, \natural]$.

Example 2.1. For $\Gamma = S_2 \times S_2$, the Schur multiplier is Q_8 , the quaternion group of order eight. The group Q_8 has four irreducible representations of dimension one, and one of dimension two, so $\mathbb{C}[Q_8] \cong \mathbb{C}^4 \oplus M_2(\mathbb{C})$.

We have $Z^* = Z(Q_8) = \{\pm 1\}$, so there are precisely two inequivalent twisted group algebras of Γ . The first comes from $c_\natural = 1 \in \text{Irr}(Z^*)$, it is just $\mathbb{C}[\Gamma]$. As $\mathbb{C}[\Gamma] \cong \mathbb{C}^4$, this corresponds to the direct summand $\mathbb{C}^4$ of $\mathbb{C}[Q_8]$.

The second comes from $c_\natural = \text{sign} \in \text{Irr}(Z^*)$. It corresponds to the remaining direct summand of $\mathbb{C}[Q_8]$, so $\mathbb{C}[S_2 \times S_2, \natural] \cong M_2(\mathbb{C})$ for any 2-cocycle $\natural$ whose image in $H^2(S_2 \times S_2, \mathbb{C}^\times)$ is nontrivial.

For $g \in \Gamma$ we introduce the map

$$(2.4) \quad \mathfrak{h}^g : \Gamma \rightarrow \mathbb{C}^\times, \quad \mathfrak{h}^g(\gamma) = T_\gamma T_g T_\gamma^{-1} T_{\gamma g^{-1} \gamma^{-1}}^{-1}.$$

One checks that the restriction $\mathfrak{h}^g|_{Z_\Gamma(g)}$ is a character of $Z_\Gamma(g)$. These characters measure the difference between $\mathbb{C}[\Gamma, \mathfrak{h}]$ and $\mathbb{C}[\Gamma]$. For instance, they can be used to count the number of irreducible representations of $\mathbb{C}[\Gamma, \mathfrak{h}]$:

Lemma 2.2. [Sol10, Lemma 1.1]

The cardinality of $\{g \in \Gamma : \mathfrak{h}^g|_{Z_\Gamma(g)} = 1\} / \Gamma\text{-conjugacy}$ equals $|\text{Irr}(\mathbb{C}[\Gamma, \mathfrak{h}])|$.

More precisely, for every conjugacy class C with the above property, define a trace ν_C on $\mathbb{C}[\Gamma, \mathfrak{h}]$ by $\nu_C(T_g) = 1$ if $g \in C$ and $\nu_C(T_g) = 0$ otherwise. Then the ν_C form a basis of the space of all traces on the semisimple algebra $\mathbb{C}[\Gamma, \mathfrak{h}]$.

Notice that Lemma 2.2 generalizes the well-known equality between the number of conjugacy classes and the number irreducible representations (over $\mathbb{C}$) of a finite group.

Suppose now that Γ acts on a $\mathbb{C}$ -algebra A , by automorphisms. Then we can form the crossed product algebra $A \rtimes \Gamma$, which is $A \otimes_{\mathbb{C}} \mathbb{C}[\Gamma]$ with multiplication rules

$$(2.5) \quad (a_1 \otimes \gamma_1)(a_2 \otimes \gamma_2) = a_1 \gamma_1(a_2) \otimes \gamma_1 \gamma_2 \quad \text{for all } a_i \in A, \gamma_i \in \Gamma.$$

More generally, for any 2-cocycle $\mathfrak{h}$ of Γ there is a twisted crossed product $A \rtimes_{\mathfrak{h}} \mathbb{C}[\Gamma, \mathfrak{h}]$. It is the same vector space $A \otimes_{\mathbb{C}} \mathbb{C}[\Gamma]$, but now the multiplication is given by

$$(a_1 \otimes T_{\gamma_1})(a_2 \otimes T_{\gamma_2}) = a_1 \gamma_1(a_2) \otimes \mathfrak{h}(\gamma_1, \gamma_2) T_{\gamma_1 \gamma_2} \quad \text{for all } a_i \in A, \gamma_i \in \Gamma.$$

This is an associative algebra that contains A and $\mathbb{C}[\Gamma, \mathfrak{h}]$ as subalgebras (for the latter we need to assume that A is unital). There is a close relation between $\text{Irr}(A)$ and $\text{Irr}(A \rtimes_{\mathfrak{h}} \mathbb{C}[\Gamma, \mathfrak{h}])$, known as Clifford theory. We refer to [RaRa, p. 24] for the cases with $\mathfrak{h} = 1$, and to [AMS1, §1] for how to handle nontrivial $\mathfrak{h}$.

In the remainder of this paragraph we focus on the cases where A is commutative and unital. More concretely, consider the algebra $A = \mathcal{O}(X)$ of regular functions on some complex affine variety X . The upcoming arguments also work for smooth functions on a closed manifold and for continuous functions on a compact Hausdorff space, the main point is that A can be regarded as an algebra of $\mathbb{C}$ -valued functions on $\text{Irr}(A)$.

So, we assume that Γ acts on X by homeomorphisms, and we form the twisted crossed product

$$(2.6) \quad B = A \rtimes_{\mathfrak{h}} \mathbb{C}[\Gamma, \mathfrak{h}] = \mathcal{O}(X) \rtimes_{\mathfrak{h}} \mathbb{C}[\Gamma, \mathfrak{h}].$$

The crossed product $A \rtimes \Gamma$ is self-opposite, so its left and right module categories are equivalent. The analogue for the twisted crossed product B is more subtle. The opposite algebra B^{op} can be studied via the isomorphism

$$(2.7) \quad \begin{array}{ccc} B^{op} = (A \rtimes_{\mathfrak{h}} \mathbb{C}[\Gamma, \mathfrak{h}])^{op} & \xrightarrow{\sim} & A \rtimes_{\mathfrak{h}^{-1}} \mathbb{C}[\Gamma, \mathfrak{h}^{-1}] \\ a T_\gamma & \mapsto & T_\gamma^{-1} a \quad a \in A, \gamma \in \Gamma. \end{array}$$

This enables us to identify right B -modules with left modules over $A \rtimes_{\mathfrak{h}^{-1}} \mathbb{C}[\Gamma, \mathfrak{h}^{-1}]$. We work out the classification of the irreducible left modules of the algebra B , a simple case of Clifford theory.

Lemma 2.3. (a) B has finite rank as a module over its centre $Z(B)$.

(b) All irreducible B -representations have finite dimension.

Proof. (a) Firstly, $\mathcal{O}(X)^\Gamma = \mathcal{O}(X/\Gamma)$ is a central subalgebra of B . Since $\mathcal{O}(X)$ is finitely generated and integral over $\mathcal{O}(X)^\Gamma$, $\mathcal{O}(X)$ has finite rank as $\mathcal{O}(X)^\Gamma$ -module. As $Z(B) \supset \mathcal{O}(X)^\Gamma$, B also has finite rank over $Z(B)$.

(b) Let V be an irreducible B -module. As B has countable dimension, so has V . Hence Schur's lemma applies, and says that $Z(B)$ acts by scalars on V . With part (a) it follows that the image of B in $\text{End}_{\mathbb{C}}(V)$ has finite dimension. Therefore $B \cdot v$ has finite dimension for any $v \in V$. But by the irreducibility $B \cdot v = V$ whenever $v \neq 0$. $\square$

Lemma 2.3 implies that the restriction of any irreducible B -module (π, V) to $\mathcal{O}(X)$ has an $\mathcal{O}(X)$ -eigenvector, say v_x with weight $x \in X$. By Frobenius reciprocity

$$(2.8) \quad \text{Hom}_B(\text{ind}_{\mathcal{O}(X)}^B(\mathbb{C}_x), \pi) \cong \text{Hom}_{\mathcal{O}(X)}(\mathbb{C}_x, \pi) \neq 0,$$

so π is a quotient of $\text{ind}_{\mathcal{O}(X)}^B(\mathbb{C}_x)$. For any $\gamma \in \Gamma$ and $f \in \mathcal{O}(X)$, $\pi(T_\gamma f)v_x \in V$ is an $\mathcal{O}(X)$ -eigenvector with weight γx . If π is irreducible, then it cannot have more weights than these γx . It follows that the irreducible quotients of $\text{ind}_{\mathcal{O}(X)}^B(\mathbb{C}_x)$ are precisely the irreducible B -modules with $\mathcal{O}(X)^\Gamma$ -character Γx .

Let Γ_x be the stabilizer of x in Γ . For $\rho \in \text{Irr}(\mathbb{C}[\Gamma_x, \mathfrak{h}])$, we form the irreducible $\mathcal{O}(X) \rtimes \mathbb{C}[\Gamma_x, \mathfrak{h}]$ -module $x \otimes \rho$, on which $\mathcal{O}(X)$ acts by evaluation at x and $\mathbb{C}[\Gamma_x, \mathfrak{h}]$ acts by ρ . We write

$$\pi(x, \rho) = \text{ind}_{\mathcal{O}(X) \rtimes \mathbb{C}[\Gamma_x, \mathfrak{h}]}^{\mathcal{O}(X) \rtimes \mathbb{C}[\Gamma, \mathfrak{h}]}(x \otimes \rho).$$

We let Γ act on

$$\{(x, \rho) : x \in X, \rho \in \text{Irr}(\mathbb{C}[\Gamma_x, \mathfrak{h}])\}$$

by $\gamma(x, \rho) = (\gamma x, \gamma \rho)$, where $\gamma \rho(h) = \rho(T_\gamma^{-1} h T_\gamma)$ for $h \in \mathbb{C}[\Gamma_{\gamma x}, \mathfrak{h}]$. The space

$$(2.9) \quad (X//\Gamma)_{\mathfrak{h}} = \{(x, \rho) : x \in X, \rho \in \text{Irr}(\mathbb{C}[\Gamma_x, \mathfrak{h}])\} / \Gamma$$

is called a twisted extended quotient.

Theorem 2.4. *Recall that $B = \mathcal{O}(X) \rtimes \mathbb{C}[\Gamma, \mathfrak{h}]$.*

(a) *The B -module $\pi(x, \rho)$ is irreducible.*

(b) *There is canonical a bijection*

$$\begin{aligned} (X//\Gamma)_{\mathfrak{h}} &\rightarrow \text{Irr}(B) \\ (x, \rho) / \sim &\mapsto \pi(x, \rho) \end{aligned}$$

Proof. (a) Consider the ideal $I_{\Gamma x} = \{f \in \mathcal{O}(X)^\Gamma : f(\Gamma x) = 0\}$ of $\mathcal{O}(X)^\Gamma$. By the Chinese remainder theorem

$$\mathcal{O}(X)/I_{\Gamma x}\mathcal{O}(X) = \mathcal{O}(X)/\{f \in \mathcal{O}(X) : f|_{\Gamma x} = 0\} \cong \bigoplus_{\gamma \in \Gamma/\Gamma_x} \mathbb{C}_{\gamma x}.$$

The irreducible B -modules with $\mathcal{O}(X)^\Gamma$ -character Γx descend to modules of

$$B/I_{\Gamma x}B \cong \left(\bigoplus_{\gamma \in \Gamma/\Gamma_x} \mathbb{C}_{\gamma x} \right) \rtimes \mathbb{C}[\Gamma, \mathfrak{h}].$$

Let $p \in \bigoplus_{\gamma \in \Gamma/\Gamma_x} \mathbb{C}_{\gamma x}$ be the idempotent which is 1 in $\mathbb{C}_x$ and 0 in all other summands, then as bimodules:

$$(2.10) \quad \begin{aligned} p(B/I_{\Gamma x}B) &= \mathbb{C}_x \otimes_{\mathbb{C}} \mathbb{C}[\Gamma, \mathfrak{h}], \\ (B/I_{\Gamma x}B)p &= \mathbb{C}[\Gamma, \mathfrak{h}] \otimes_{\mathbb{C}} \mathbb{C}_x, \\ p(B/I_{\Gamma x}B)p &= \mathbb{C}_x \otimes_{\mathbb{C}} \mathbb{C}[\Gamma_x, \mathfrak{h}] \cong \mathbb{C}[\Gamma_x, \mathfrak{h}], \\ (B/I_{\Gamma x}B)p(B/I_{\Gamma x}B) &= B/I_{\Gamma x}B. \end{aligned}$$

The first three of these equalities are straightforward, the last follows because $(B/I_{\Gamma x}B)p(B/I_{\Gamma x}B)$ contains $T_\gamma p T_\gamma^{-1} = \gamma(p)$ and $\sum_{\gamma \in \Gamma/\Gamma_x} \gamma(p) = 1$. From (2.10) we see that the bimodules $p(B/I_{\Gamma x}B)$ and $(B/I_{\Gamma x}B)p$ provide a Morita equivalence between $B/I_{\Gamma x}B$ and $\mathbb{C}[\Gamma_x, \mathfrak{h}]$. This equivalence sends $x \otimes \rho \in \text{Irr}(\mathbb{C}_x \otimes_{\mathbb{C}} \mathbb{C}[\Gamma_x, \mathfrak{h}])$ to

$$(B/I_{\Gamma x}B)p \otimes_{\mathbb{C}_x \otimes_{\mathbb{C}} \mathbb{C}[\Gamma_x, \mathfrak{h}]} (x \otimes \rho) = \text{ind}_{\bigoplus_{\gamma \in \Gamma/\Gamma_x} \mathbb{C}_{\gamma x} \rtimes \mathbb{C}[\Gamma_x, \mathfrak{h}]}^{B/I_{\Gamma x}B} (x \otimes \rho),$$

which is therefore an irreducible $B/I_{\Gamma x}B$ -module. Via inflation to B , we find that $\text{ind}_{\mathcal{O}(X) \rtimes \mathbb{C}[\Gamma_x, \mathfrak{h}]}^{\mathcal{O}(X) \rtimes \mathbb{C}[\Gamma, \mathfrak{h}]} (x \otimes \rho) = \pi(x, \rho)$ is irreducible.

(b) By the remarks before the theorem, every irreducible B -module has a unique $\mathcal{O}(X)^\Gamma$ -character Γx . The set of irreducible B -modules with that $\mathcal{O}(X)^\Gamma$ -character is naturally in bijection with $\text{Irr}(B/I_{\Gamma x}B)$. The proof of part (a) yields a bijection

$$\begin{array}{ccc} \text{Irr}(\mathbb{C}_x \otimes \mathbb{C}[\Gamma_x, \mathfrak{h}]) & \rightarrow & \text{Irr}(B/I_{\Gamma x}B) \\ x \otimes \rho & \mapsto & \pi(x, \rho) \end{array}.$$

Hence every element of $\text{Irr}(B)$ has the form $\pi(x, \rho)$. The only freedom in the above construction is the choice of x in Γx . Suppose that instead we pick $\gamma x \in \Gamma x$. Then conjugation by T_γ can be pulled through the entire construction, and we end up with $\gamma\rho$ instead of ρ . Thus $\pi(x, \rho) \cong \pi(\gamma x, \gamma\rho)$ and these are the only possible equivalences between modules of this form. $\square$

Twisted crossed products for free group actions are considerably easier.

Proposition 2.5. *Suppose that the action of Γ on X is free.*

- (a) *The categories of finite length modules of $\mathcal{O}(X/\Gamma)$ and of $B = \mathcal{O}(X) \rtimes \mathbb{C}[\Gamma, \mathfrak{h}]$ are naturally equivalent.*
- (b) *If $\mathfrak{h}$ is trivial in $H^2(\Gamma, \mathbb{C}^\times)$, then $\mathcal{O}(X) \rtimes \mathbb{C}[\Gamma, \mathfrak{h}]$ is Morita equivalent to $\mathcal{O}(X/\Gamma)$.*

Proof. (a) The $\mathcal{O}(X/\Gamma)$ -weights provide a decomposition of the category of finite length modules

$$\text{Mod}_{\mathfrak{h}}(\mathcal{O}(X) \rtimes \mathbb{C}[\Gamma, \mathfrak{h}]) = \bigoplus_{\Gamma x \in X/\Gamma} \text{Mod}_{\mathfrak{h}}(\mathcal{O}(X) \rtimes \mathbb{C}[\Gamma, \mathfrak{h}])_{\Gamma x},$$

and similarly for $\mathcal{O}(X/\Gamma)$. Therefore it suffices to fix a Γ -orbit Γx in X and to consider finite length $\mathcal{O}(X) \rtimes \mathbb{C}[\Gamma, \mathfrak{h}]$ -modules V such that all irreducible subquotients of $V|_{\mathcal{O}(X/\Gamma)}$ are isomorphic to $\mathbb{C}_{\Gamma x}$.

With the notations from the proof of Theorem 2.4, let

$$\begin{aligned} \widehat{\mathcal{O}(X/\Gamma)}_{\Gamma x} &= \varprojlim_n \mathcal{O}(X/\Gamma)/I_{\Gamma x}^n \\ \widehat{\mathcal{O}(X)}_{\Gamma x} &= \varprojlim_n \mathcal{O}(X)/I_{\Gamma x}^n \mathcal{O}(X) \cong \bigoplus_{x' \in \Gamma x} \widehat{\mathcal{O}(X)}_{x'} \\ \hat{B}_{\Gamma x} &= B \otimes_{\mathcal{O}(X/\Gamma)} \widehat{\mathcal{O}(X/\Gamma)}_{\Gamma x} = \varprojlim_n B/I_{\Gamma x}^n B \end{aligned}$$

be the formal completions of $\mathcal{O}(X/\Gamma)$, $\mathcal{O}(X)$ and B with respect to the ideal generated by $I_{\Gamma x}$. This completion operation does not change the category of finite length modules whose only $\mathcal{O}(X/\Gamma)$ -weight is Γx . As in (2.10), one checks that the bimodules $p\hat{B}_{\Gamma x}$ and $\hat{B}_{\Gamma x}p$ provide a Morita equivalence between

$$p\hat{B}_{\Gamma x}p \cong \widehat{\mathcal{O}(X/\Gamma)}_{\Gamma x} \quad \text{and} \quad \hat{B}_{\Gamma x}p\hat{B}_{\Gamma x} = \hat{B}_{\Gamma x}.$$

(The freedom of the action is used in the last equality.) That yields equivalences of categories

$$\text{Mod}_{\mathfrak{h}}(B)_{\Gamma x} \cong \text{Mod}_{\mathfrak{h}}(\hat{B}_{\Gamma x}) \cong \text{Mod}_{\mathfrak{h}}(\widehat{\mathcal{O}(X/\Gamma)}_{\Gamma x}) \cong \text{Mod}_{\mathfrak{h}}(\mathcal{O}(X/\Gamma))_{\Gamma x}.$$

(b) Suppose that $\mathfrak{h} \in H^2(\Gamma, \mathbb{C}^\times)$ is trivial. As explained after (2.2), $B = \mathcal{O}(X) \rtimes \mathbb{C}[\Gamma, \mathfrak{h}]$ is isomorphic to $\mathcal{O}(X) \rtimes \Gamma$. This enables us to construct the idempotent

$$e_\Gamma := |\Gamma|^{-1} \sum_{\gamma \in \Gamma} \gamma \in \mathbb{C}[\Gamma] \subset B.$$

We claim that $BeB = B$. For this, it suffices to check that $\widehat{BeB}_{\Gamma x} = \hat{B}_{\Gamma x}$ for all $x \in X$. Since Γ acts freely on X , $1_x |\Gamma| e_\Gamma 1_x \in \widehat{BeB}_{\Gamma x}$ equals $1_x \in \widehat{\mathcal{O}(X)}_x$. As in the proof of Theorem 2.4, it follows that $\widehat{BeB}_{\Gamma x} = \hat{B}_{\Gamma x}$, proving our claim.

Now the isomorphisms of bimodules

$$eB \cong \mathcal{O}(X), \quad Be \cong \mathcal{O}(X), \quad eBe \cong \mathcal{O}(X/\Gamma), \quad BeB = B$$

show that eB and Be provide a Morita equivalence between B and $\mathcal{O}(X/\Gamma)$. $\square$

For nontrivial $\mathfrak{h} \in H^2(\Gamma, \mathbb{C}^\times)$ one cannot say directly that Proposition 2.5.b fails, it depends on X . In the cases where X is irreducible as an algebraic variety, one can improve on Proposition 2.5, as follows.

Suppose that there is a Morita equivalence between B and $\mathcal{O}(X/\Gamma)$. Let $\mathbb{C}(X)$ be the quotient field of $\mathcal{O}(X)$ and tensor the Morita bimodules with $\mathbb{C}(X/\Gamma)$ over $\mathcal{O}(X/\Gamma)$. That yields a Morita equivalence between $\mathbb{C}(X)^\Gamma = \mathbb{C}(X/\Gamma)$ and

$$(2.11) \quad \mathbb{C}(X/\Gamma) \otimes_{\mathcal{O}(X/\Gamma)} B \cong \mathbb{C}(X/\Gamma) \otimes_{\mathcal{O}(X/\Gamma)} \mathcal{O}(X) \rtimes \mathbb{C}[\Gamma, \mathfrak{h}] = \mathbb{C}(X) \rtimes \mathbb{C}[\Gamma, \mathfrak{h}].$$

By the next result and Proposition 2.5.b, (2.11) is only possible if $\mathfrak{h} \in H^2(\Gamma, \mathbb{C}^\times)$ is trivial.

Theorem 2.6. *Let X be an irreducible algebraic variety, endowed with a faithful action of a finite group Γ . Then $\mathbb{C}(X) \rtimes \mathbb{C}[\Gamma, \mathfrak{h}]$ is a central simple $\mathbb{C}(X)^\Gamma$ -algebra. It is Morita equivalent to $\mathbb{C}(X) \rtimes \Gamma$ if and only if $\mathfrak{h}$ is trivial in $H^2(\Gamma, \mathbb{C}^\times)$.*

Proof. By the general theory of central simple algebras, $\mathbb{C}(X) \rtimes \mathbb{C}[\Gamma, \mathfrak{h}]$ is one, see for instance [Ker, §7.5]. By a result of Noether [Ker, §7.7], $\mathbb{C}(X) \rtimes \mathbb{C}[\Gamma, \mathfrak{h}]$ is Morita equivalent to $\mathbb{C}(X) \rtimes \Gamma$ if and only if $\mathfrak{h}$ is trivial in $H^2(\Gamma, \mathbb{C}(X/\Gamma)^\times)$.

We claim that the natural map

$$H^2(\Gamma, \mathbb{C}^\times) \rightarrow H^2(\Gamma, \mathbb{C}(X/\Gamma)^\times) \quad \text{is injective.}$$

Suppose that $c \in Z^2(\Gamma, \mathbb{C}^\times)$ is a 2-cocycle, which equals a 2-coboundary $b(f)$ for some $f : \Gamma \rightarrow \mathbb{C}(X/\Gamma)^\times$. Write $f = f_1/f_2$ for some $f_i \in \mathcal{O}(X/\Gamma) \setminus \{0\}$, and pick $x \in X$ such that $f_1(\Gamma x)f_2(\Gamma x) \neq 0$. Then f and $b(f)$ can be evaluated at Γx , which gives $b(f(x)) = b(f)(x) = c(x) = c$. Hence c is trivial in $H^2(\Gamma, \mathbb{C}^\times)$. $\square$

2.2. Definitions of graded Hecke algebras.

We will analyse the modules over the Hecke algebra of a reductive p -adic group in terms of modules of graded Hecke algebras. Here we provide a short introduction to those algebras, which were discovered by Lusztig [Lus].

We need the following data:

- a finite dimensional real vector space $\mathfrak{t}_{\mathbb{R}}$,
- the linear dual space $\mathfrak{t}_{\mathbb{R}}^\vee$,
- a reduced integral root system Φ in $\mathfrak{t}_{\mathbb{R}}^\vee$, with a basis Δ ,

- the Weyl group $W = W(\Phi)$, which acts on $\mathfrak{t}_{\mathbb{R}}$ and on $\mathfrak{t}_{\mathbb{R}}^{\vee}$,
- a W -invariant parameter function $k : \Phi \rightarrow \mathbb{C}$,
- a formal variable $\mathbf{r}$,
- the complexifications $\mathfrak{t}$ of $\mathfrak{t}_{\mathbb{R}}$ and $\mathfrak{t}^{\vee}$ of $\mathfrak{t}_{\mathbb{R}}^{\vee}$.

Definition 2.7. The graded Hecke algebra $\mathbb{H}(\mathfrak{t}, W, k, \mathbf{r})$ is the vector space $\mathcal{O}(\mathfrak{t}) \otimes \mathbb{C}[\mathbf{r}] \otimes \mathbb{C}[W]$ with multiplication rules

- $\mathbb{C}[W]$ and $\mathcal{O}(\mathfrak{t}) \otimes \mathbb{C}[\mathbf{r}] = \mathcal{O}(\mathfrak{t} \oplus \mathbb{C})$ are embedded as subalgebras,
- $\mathbb{C}[\mathbf{r}]$ is central,
- for $\alpha \in \Delta$ and $f \in \mathcal{O}(\mathfrak{t})$:

$$(2.12) \quad f \cdot s_{\alpha} - s_{\alpha} \cdot s_{\alpha}(f) = k(\alpha)\mathbf{r}(f - s_{\alpha}(f))/\alpha.$$

The grading on $\mathbb{H}(\mathfrak{t}, W, k, \mathbf{r})$ is twice the usual grading on the polynomial algebra $\mathcal{O}(\mathfrak{t} \oplus \mathbb{C})$, while all nonzero elements of $\mathbb{C}[W]$ have degree 0.

It is easy to check that $f - s_{\alpha}(f)$ is divisible by α in $\mathcal{O}(\mathfrak{t})$, so that (2.12) is really a relation in $\mathbb{H}(\mathfrak{t}, W, k, \mathbf{r})$. For $f = x \in \mathfrak{t}^{\vee}$, (2.12) simplifies to

$$(2.13) \quad x \cdot s_{\alpha} - s_{\alpha} \cdot s_{\alpha}(x) = k(\alpha)\mathbf{r}x(\alpha^{\vee}),$$

where $\alpha^{\vee} \in \mathfrak{t}_{\mathbb{R}}$ denotes the coroot of α . The elements $\alpha \in \mathfrak{t}^{\vee}$ and $\mathbf{r}$ have degree two, so the relation (2.12) is homogeneous. It follows that $\mathbb{H}(\mathfrak{t}, W, k, \mathbf{r})$ is a graded algebra:

$$\deg(xy) = \deg(x) + \deg(y) \text{ when } x, y \in \mathbb{H}(\mathfrak{t}, W, k, \mathbf{r}) \text{ are homogeneous.}$$

Example 2.8. If Φ is empty, then $\mathbb{H}(\mathfrak{t}, W, k, \mathbf{r})$ reduces to $\mathcal{O}(\mathfrak{t}) \otimes \mathbb{C}[\mathbf{r}]$. For $k = 0$ we have

$$\mathbb{H}(\mathfrak{t}, W, 0, \mathbf{r}) = (\mathcal{O}(\mathfrak{t}) \rtimes W) \otimes \mathbb{C}[\mathbf{r}].$$

In practice the central element $\mathbf{r}$ is often specialized to a complex number. In view of the identity $\mathbb{H}(\mathfrak{t}, W, k, z\mathbf{r}) = \mathbb{H}(\mathfrak{t}, W, zk, \mathbf{r})$ for $z \in \mathbb{C}$, it suffices to consider the specialization of $\mathbf{r}$ to 1. We define

$$(2.14) \quad \mathbb{H}(\mathfrak{t}, W, k) = \mathbb{H}(\mathfrak{t}, W, k, \mathbf{r})/(\mathbf{r} - 1).$$

The vector space $\mathbb{H}(\mathfrak{t}, W, k) = \mathcal{O}(\mathfrak{t}) \otimes \mathbb{C}[W]$ is graded in the same way as $\mathbb{H}(\mathfrak{t}, W, k, \mathbf{r})$. However, the multiplication relation (2.12) becomes

$$f \cdot s_{\alpha} - s_{\alpha} \cdot s_{\alpha}(f) = k(\alpha)(f - s_{\alpha}(f))/\alpha,$$

in $\mathbb{H}(\mathfrak{t}, W, k)$, which is usually not homogeneous. Therefore $\mathbb{H}(\mathfrak{t}, W, k)$ is not graded as an algebra. Instead it is a filtered algebra, which means that for homogeneous elements $x, y \in \mathbb{H}(\mathfrak{t}, W, k)$, the product xy is a sum of terms of degrees at most $\deg(x) + \deg(y)$.

Multiplication by $z \in \mathbb{C}^{\times}$ on $\mathfrak{t}$ induces an algebra isomorphism

$$(2.15) \quad \begin{array}{ccc} \mathbb{H}(\mathfrak{t}, W, k) & \rightarrow & \mathbb{H}(\mathfrak{t}, W, zk) \\ fw & \mapsto & (f \circ z)w \end{array}.$$

For $z = 0$ this becomes a projection

$$\mathbb{H}(\mathfrak{t}, W, k) \rightarrow \mathbb{C}[W] \subset \mathbb{H}(\mathfrak{t}, W, 0).$$

In addition to the data at the start of this paragraph, let Γ be a finite group acting linearly on $\mathfrak{t}_{\mathbb{R}}^{\vee}$, such that it stabilizes Φ, Δ and k . Then Γ acts on W by conjugation in $GL(\mathfrak{t}_{\mathbb{R}}^{\vee})$ and on $\mathbb{H}(\mathfrak{t}, W, k)$ by

$$\gamma(fw) = (f \circ \gamma^{-1})(\gamma w \gamma^{-1}) \quad f \in \mathcal{O}(\mathfrak{t}), w \in W, \gamma \in \Gamma.$$

Definition 2.9. Let $\mathfrak{h} : \Gamma \times \Gamma \rightarrow \mathbb{C}^\times$ be a 2-cocycle. We call

$$\mathbb{H}(\mathfrak{t}, W\Gamma, k, \mathfrak{h}) = \mathbb{H}(\mathfrak{t}, W, k) \rtimes \mathbb{C}[\Gamma, \mathfrak{h}]$$

a twisted graded Hecke algebra.

Notice that we allow $\Gamma = \{e\}$, in which case $\mathbb{H}(\mathfrak{t}, W\Gamma, k, \mathfrak{h})$ reduces to $\mathbb{H}(\mathfrak{t}, W, k)$. By (2.3), $\mathbb{H}(\mathfrak{t}, W, k, \mathfrak{h})$ is a direct summand of the algebra $\mathbb{H}(\mathfrak{t}, W, k) \rtimes \Gamma^*$. Therefore most results that have been proven for $\mathbb{H}(\mathfrak{t}, W, k) \rtimes \Gamma^*$ apply automatically to $\mathbb{H}(\mathfrak{t}, W, k, \mathfrak{h})$. We will tacitly use that several times.

We shall also want to consider the opposite algebras of twisted graded Hecke algebras. These are of the same kind, because there is an algebra isomorphism

$$(2.16) \quad \begin{array}{ccc} \mathbb{H}(\mathfrak{t}, W\Gamma, k, \mathfrak{h})^{op} & \xrightarrow{\sim} & \mathbb{H}(\mathfrak{t}, W\Gamma, k, \mathfrak{h}^{-1}) \\ f w N_\gamma & \mapsto & N_\gamma^{-1} w^{-1} f \quad f \in \mathcal{O}(\mathfrak{t}), w \in W, \gamma \in \Gamma. \end{array}$$

The centre of $\mathbb{H}(\mathfrak{t}, W, k)$ is known from [Lus, Proposition 3.11]:

$$Z(\mathbb{H}(\mathfrak{t}, W, k)) = \mathcal{O}(\mathfrak{t})^W = \mathcal{O}(\mathfrak{t}/W).$$

We have $Z(\mathbb{H}(\mathfrak{t}, W\Gamma, k, \mathfrak{h})) \supset \mathcal{O}(\mathfrak{t})^{W\Gamma}$, with equality if Γ acts faithfully on $\mathfrak{t}_{\mathbb{R}}^\vee$. By the same arguments as in Lemma 2.3.a,

$$(2.17) \quad \mathbb{H}(\mathfrak{t}, W\Gamma, k, \mathfrak{h}) \text{ has finite rank as module over its centre and over } \mathcal{O}(\mathfrak{t})^{W\Gamma}.$$

For another parameter function k' , the algebras $\mathbb{H}(\mathfrak{t}, W\Gamma, k', \mathfrak{h})$ and $\mathbb{H}(\mathfrak{t}, W\Gamma, k, \mathfrak{h})$ are usually not isomorphic. Nevertheless they are always very similar, in several ways. The clearest relation between these algebras can be seen when we include the quotient field $\mathbb{C}(\mathfrak{t}/W\Gamma) = \mathbb{C}(\mathfrak{t})^{W\Gamma}$ of $\mathcal{O}(\mathfrak{t}/W\Gamma)$. There are field isomorphisms

$$\begin{array}{ccc} \mathcal{O}(\mathfrak{t}) \otimes_{\mathcal{O}(\mathfrak{t}/W\Gamma)} \mathbb{C}(\mathfrak{t}/W\Gamma) & \cong & \mathbb{C}(\mathfrak{t}) \\ f_1 \otimes f_2 / f_3 & \mapsto & f_1 f_2 / f_3 \\ g_1 \prod_{w \in W\Gamma \setminus \{e\}} w(g_2) \otimes \prod_{w \in W\Gamma} w(g_2)^{-1} & \leftarrow & g_1 / g_2 \end{array}$$

In particular $\mathbb{C}(\mathfrak{t})$ is naturally a subalgebra of $\mathbb{C}(\mathfrak{t}/W\Gamma) \otimes_{\mathcal{O}(\mathfrak{t}/W\Gamma)} \mathbb{H}(\mathfrak{t}, W\Gamma, k, \mathfrak{h})$. For $\alpha \in \Delta$ we define

$$(2.18) \quad \tau_{s_\alpha} = (1 + s_\alpha) \frac{\alpha}{\alpha + k(\alpha)} - 1 \in \mathbb{C}(\mathfrak{t}/W\Gamma) \otimes_{\mathcal{O}(\mathfrak{t}/W\Gamma)} \mathbb{H}(\mathfrak{t}, W\Gamma, k, \mathfrak{h}).$$

A direct calculation shows that $\tau_{s_\alpha}^2 = 1$. For $\gamma \in \Gamma$ we put $\tau_\gamma = T_\gamma \in \mathbb{C}[\Gamma, \mathfrak{h}]$.

Theorem 2.10. [Lus, §5] and [Sol5, Proposition 1.5.1]

(a) The map $s_\alpha \mapsto \tau_{s_\alpha}$ extends to a group homomorphism

$$\tau : W \rightarrow (\mathbb{C}(\mathfrak{t}/W\Gamma) \otimes_{\mathcal{O}(\mathfrak{t}/W\Gamma)} \mathbb{H}(\mathfrak{t}, W\Gamma, k, \mathfrak{h}))^\times.$$

(b) There is an isomorphism of $\mathbb{C}(\mathfrak{t}/W\Gamma)$ -algebras

$$\begin{array}{ccc} \mathbb{C}(\mathfrak{t}) \rtimes \mathbb{C}[W\Gamma, \mathfrak{h}] & \rightarrow & \mathbb{C}(\mathfrak{t}/W\Gamma) \otimes_{\mathcal{O}(\mathfrak{t}/W\Gamma)} \mathbb{H}(\mathfrak{t}, W\Gamma, k, \mathfrak{h}) \\ f \otimes w & \mapsto & f \tau_w \end{array}.$$

We note that

$$\mathbb{C}(\mathfrak{t}) \rtimes \mathbb{C}[W\Gamma, \mathfrak{h}] = \mathbb{C}(\mathfrak{t}/W\Gamma) \otimes_{\mathcal{O}(\mathfrak{t}/W\Gamma)} \mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W\Gamma, \mathfrak{h}] = \mathbb{C}(\mathfrak{t}/W\Gamma) \otimes_{\mathcal{O}(\mathfrak{t}/W\Gamma)} \mathbb{H}(\mathfrak{t}, W\Gamma, 0, \mathfrak{h}).$$

Theorem 2.10 shows that, upon including $\mathbb{C}(\mathfrak{t}/W\Gamma)$, the dependence of $\mathbb{H}(\mathfrak{t}, W\Gamma, k, \mathfrak{h})$ on k disappears.

2.3. Representation theory.

In this paragraph we survey representations of a (twisted) graded Hecke algebra $\mathbb{H} = \mathbb{H}(\mathfrak{t}, W, k, \mathfrak{q})$, in analogy with representations of reductive groups over local fields. For more background on $\mathbb{H}$ -representations we refer to [KrRa, Sol4].

The starting point is always representations of the maximal commutative subalgebra $\mathcal{O}(\mathfrak{t})$.

Definition 2.11. Let (π, V) be a $\mathbb{H}$ -representation and let $\lambda \in \mathfrak{t}$. We write

$$V_\lambda = \{v \in V : \exists N \in \mathbb{N} \text{ such that } (\pi(x) - x(\lambda))^N v = 0 \text{ for all } x \in \mathfrak{t}^\vee\}.$$

We say that λ is an $\mathcal{O}(\mathfrak{t})$ -weight (or simply a weight) of π if V_λ is nonzero. The set of $\mathcal{O}(\mathfrak{t})$ -weights of π is denoted $\text{Wt}(\pi)$.

With simple linear algebra one checks that:

Lemma 2.12. (a) *Every nonzero weight space V_λ contains an $\mathcal{O}(\mathfrak{t})$ -eigenvector, that is, a $v \neq 0$ such that $\pi(x)v = x(\lambda)v$ for all $x \in \mathfrak{t}^\vee$.*
 (b) *If $\dim V$ is finite, then $V = \bigoplus_{\lambda \in \text{Wt}(\pi)} V_\lambda$.*

Let $\mathbb{C}_\lambda$ be the onedimensional $\mathcal{O}(\mathfrak{t})$ -module with weight λ . With induction one constructs the $\mathbb{H}$ -module

$$I(\lambda) = \text{ind}_{\mathcal{O}(\mathfrak{t})}^{\mathbb{H}}(\mathbb{C}_\lambda).$$

By [BaMo2, Theorem 6.4], $\text{Wt}(I(\lambda)) = W\Gamma\lambda$.

Lemma 2.13. *Let (π, V) be an irreducible $\mathbb{H}$ -representation.*

- (a) *$\dim V$ is finite.*
- (b) *π is a quotient of $I(\lambda)$, for some $\lambda \in \mathfrak{t}$.*

Proof. (a) This is shown in the same way as Lemma 2.3.

(b) Lemma 2.12 ensures that π has an $\mathcal{O}(\mathfrak{t})$ -weight, say λ , with an eigenvector. Then Frobenius reciprocity shows that

$$\text{Hom}_{\mathbb{H}}(I(\lambda), \pi) \cong \text{Hom}_{\mathcal{O}(\mathfrak{t})}(\mathbb{C}_\lambda, \pi) \neq 0.$$

Hence there exists a nonzero $\mathbb{H}$ -homomorphism from $I(\lambda)$ to π , which by the irreducibility of π must be surjective. $\square$

In simplest noncommutative case, one can classify the irreducible $\mathbb{H}$ -representations by hand.

Example 2.14. We take $\mathfrak{t}_{\mathbb{R}} = \mathfrak{t}_{\mathbb{R}}^\vee = \mathbb{R}$, $\Phi = \{\pm 1\}$ and $W = S_2$. We write $k(\alpha) = k \in \mathbb{C}$ and we consider $\mathbb{H} = \mathbb{H}(\mathfrak{t}, S_2, k)$. There are two onedimensional $\mathbb{H}$ -representations:

- The trivial representation triv defined by $\text{triv}|_{\mathcal{O}(\mathfrak{t})} = \mathbb{C}_k$ and $\text{triv}|_{\mathbb{C}[S_2]} = \text{triv}$.
- The Steinberg representation St defined by $\text{St}|_{\mathcal{O}(\mathfrak{t})} = \mathbb{C}_{-k}$ and $\text{St}|_{\mathbb{C}[S_2]} = \text{sign}$.

All other irreducible representations are of the form $I(\lambda)$ with $\lambda \in \mathbb{C}$. As vector spaces $\mathbb{H} = \mathbb{C}[S_2] \otimes \mathcal{O}(\mathfrak{t})$, so $\text{Res}_{\mathbb{C}[S_2]}^{\mathbb{H}} I(\lambda)$ is the regular representation $\mathbb{C}[S_2]$. The only proper S_2 -invariant subspaces are

$$\mathbb{C}(1 + s_\alpha) \quad \text{and} \quad \mathbb{C}(1 - s_\alpha),$$

so if $I(\lambda)$ is reducible, then at least one of these two is an $\mathbb{H}$ -subrepresentation. For $x \in \mathfrak{t}^\vee$ (corresponding to $x(1) \in \mathbb{C}$) one computes in $I(\lambda)$, using (2.13) with $\mathbf{r} = 1$

and $\alpha^\vee = 2$:

$$\begin{aligned} x \cdot (1 + s_\alpha) &= x + s_\alpha s_\alpha(x) + k(\alpha)x(\alpha^\vee) = (\lambda + 2k - \lambda s_\alpha)x(1) \\ x \cdot (1 - s_\alpha) &= x - s_\alpha s_\alpha(x) - k(\alpha)x(\alpha^\vee) = (\lambda - 2k + \lambda s_\alpha)x(1). \end{aligned}$$

Hence $\mathbb{C}(1 + s_\alpha)$ is an $\mathbb{H}$ -submodule of $I(\lambda)$ if and only if $\lambda = -k$, while $\mathbb{C}(1 - s_\alpha)$ is an $\mathbb{H}$ -submodule of $I(\lambda)$ if and only if $\lambda = k$. One can check that $I(k)$ and $I(-k)$ both have length two, with irreducible subquotients triv and St . All the representations $I(\lambda)$ with $\lambda \in \mathbb{C} \setminus \{k, -k\}$ are irreducible. By Lemma 2.13.b, this exhausts $\text{Irr}(\mathbb{H})$.

Frobenius reciprocity tells us that

$$\text{Hom}_{\mathbb{H}}(I(\lambda), I(\lambda')) \cong \text{Hom}_{\mathcal{O}(\mathfrak{t})}(\mathbb{C}_\lambda, I(\lambda')).$$

This is nonzero if and only if $\lambda' = \pm\lambda$, because $\text{Wt}(I(\lambda')) = S_2\lambda' = \{\lambda', -\lambda'\}$. Therefore the irreducible representations $I(\lambda)$ and $I(\lambda')$ are isomorphic if and only if $\lambda' = \pm\lambda$.

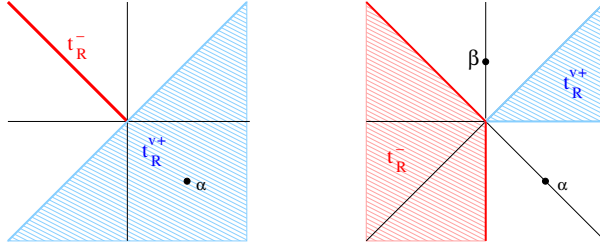
Temperedness of $\mathbb{H}$ -representations is defined via their weights. Consider the positive Weyl chamber

$$(\mathfrak{t}_{\mathbb{R}}^\vee)^+ = \{x \in \mathfrak{t}_{\mathbb{R}}^\vee : x(\alpha^\vee) \geq 0 \text{ for all } \alpha \in \Delta\}$$

and the obtuse negative cone

$$\mathfrak{t}_{\mathbb{R}}^- = \{\lambda \in \mathfrak{t}_{\mathbb{R}} : x(\lambda) \leq 0 \text{ for all } x \in (\mathfrak{t}_{\mathbb{R}}^\vee)^+\}.$$

Example 2.15. Suppose that $\mathfrak{t}_{\mathbb{R}} = \mathfrak{t}_{\mathbb{R}}^\vee = \mathbb{R}^2$ with the standard inner product. The positive Weyl chamber and the obtuse negative cone for $R = A_1$ (from GL_2) and



$R = B_2$ look like

Definition 2.16. Let (π, V) be a finite dimensional $\mathbb{H}$ -representation. We call π tempered if $\text{Wt}(\pi) \subset \mathfrak{t}_{\mathbb{R}}^- + i\mathfrak{t}_{\mathbb{R}}$. More restrictively, we say that π is discrete series if it is irreducible and $\text{Wt}(\pi) \subset \text{int}(\mathfrak{t}_{\mathbb{R}}^-) + i\mathfrak{t}_{\mathbb{R}}$. Here $\text{int}(\mathfrak{t}_{\mathbb{R}}^-)$ denotes the interior of $\mathfrak{t}_{\mathbb{R}}^-$ as a subset of the $\mathbb{R}$ -span of the root system in $\mathfrak{t}_{\mathbb{R}}$ dual to Φ .

In terms of the canonical real part map $\Re : \mathfrak{t} \rightarrow \mathfrak{t}_{\mathbb{R}}$, we can reformulate these conditions as, respectively, $\Re \text{Wt}(\pi) \subset \mathfrak{t}_{\mathbb{R}}^-$ and $\Re \text{Wt}(\pi) \subset \text{int}(\mathfrak{t}_{\mathbb{R}}^-)$.

It is known from [Sol9, §9]² that the notions of temperedness and discrete series for representations of twisted graded Hecke algebras correspond to the synonymous notions for representations of reductive p -adic groups.

Example 2.17. The representation $I(\lambda)$ is tempered if and only if $\lambda \in i\mathfrak{t}_{\mathbb{R}}$, because $\text{Wt}(I(\lambda)) = W\lambda$. It is never discrete series, because $W\Re(\lambda) \subset \mathfrak{t}_{\mathbb{R}}$ cannot be contained in the interior of $\mathfrak{t}_{\mathbb{R}}^-$.

Let $\mathbb{H}(\mathbb{C}, S_2, k)$ be as in Example 2.14. The Steinberg representation is tempered if and only if $\Re(k(\alpha)) \geq 0$. It is discrete series if and only if $\Re(k(\alpha)) > 0$.

²Our discrete series representations are called tempered essentially discrete series in [Sol9].

Definition 2.18. $\text{Irr}_{\text{temp}}(\mathbb{H})$ is the set of irreducible tempered $\mathbb{H}$ -representations. For a subset $E \subset \mathfrak{t}$, we let $\text{Irr}(\mathbb{H})_E$ be the set of irreducible $\mathbb{H}$ -representations π such that $\text{Wt}(\pi) \subset E$. The same condition determines the subset $\text{Irr}_{\text{temp}}(\mathbb{H})_E$. We define $\text{Mod}(\mathbb{H})_E$ as the category of finite length $\mathbb{H}$ -representations all whose weights lie in E . We say that an $\mathbb{H}$ -representation (π, V) has real weights if it lies in $\text{Mod}(\mathbb{H})_{\mathfrak{t}_{\mathbb{R}}}$.

Example 2.19. Assume that $k = 0$. Then $\mathbb{H}(\mathfrak{t}, W, 0) = \mathcal{O}(\mathfrak{t}) \rtimes W$, which we discussed in Paragraph 2.1. From Theorem 2.4 we see that the weights of any irreducible $\mathcal{O}(\mathfrak{t}) \rtimes W$ -representation always form one full W -orbit $W\lambda \subset \mathfrak{t}$. As $W\mathfrak{R}(\lambda)$ cannot be contained in $\text{int}(\mathfrak{t}_{\mathbb{R}}^-)$, $\mathbb{H}(\mathfrak{t}, W, 0)$ does not have discrete series representations.

The conditions $\text{Wt}(\pi) \subset i\mathfrak{t}_{\mathbb{R}}$ and $\text{Wt}(\pi) \subset \mathfrak{t}_{\mathbb{R}}$ together imply $\text{Wt}(\pi) = \{0\}$, so $\text{Irr}_{\text{temp}}(\mathcal{O}(\mathfrak{t}) \rtimes W)_{\mathfrak{t}_{\mathbb{R}}} = \text{Irr}(\mathcal{O}(\mathfrak{t}) \rtimes W)_{\{0\}}$. This set is naturally identified with $\text{Irr}(W)$.

In general the representations in $\text{Irr}_{\text{temp}}(\mathbb{H})_{\mathfrak{t}_{\mathbb{R}}}$ can have more weights than just 0, but not many, the conditions tempered and real weights are very restrictive. For an algebra or group A , let $R(A)$ be the Grothendieck group of the category of finite length A -representations. When A is a group or a group algebra, $R(A)$ is called the representation ring of A (with the tensor product as multiplication).

Theorem 2.20. [Sol3, Theorem 6.5.c] and [Sol8, Theorem 6.2.a]

Suppose that k is real-valued. The set $\{\pi|_{\mathbb{C}[W\Gamma, \mathfrak{h}]} : \pi \in \text{Irr}_{\text{temp}}(\mathbb{H})_{\mathfrak{t}_{\mathbb{R}}}\}$ is $\mathbb{Z}$ -basis of $R(\mathbb{C}[W\Gamma, \mathfrak{h}])$.

Example 2.21. Let $\mathbb{H} = \mathbb{H}(\mathbb{C}, S_2, k)$ be as in Example 2.14, with $k > 0$. Then $\text{Irr}_{\text{temp}}(\mathbb{H}) = \{\text{St}, I(0)\}$. We recall that $\text{St}|_{\mathbb{C}[S_2]} = \text{sign}$ and $I(0)|_{\mathbb{C}[S_2]} = \mathbb{C}[S_2]$. These form a basis of the representation ring of S_2 , for instance $\text{triv} = \mathbb{C}[S_2] - \text{sign}$ in $R(S_2)$.

2.4. Parametrization of irreducible representations.

The best method to produce new $\mathbb{H}$ -representations is parabolic induction. For the moment we do this without Γ , so for $\mathbb{H}(\mathfrak{t}, W, k)$.

Definition 2.22. Let $P \subset \Delta$ and let $\Phi_P \subset \Phi$ be the associated parabolic root subsystem. Then $\mathbb{H}^P = \mathbb{H}(\mathfrak{t}, W(\Phi_P), k)$ is called a parabolic subalgebra of $\mathbb{H}(\mathfrak{t}, W, k)$. Parabolic induction is the functor $\text{ind}_{\mathbb{H}^P}^{\mathbb{H}(\mathfrak{t}, W, k)}$.

Example 2.23. If P is empty, then $\mathbb{H}^P = \mathcal{O}(\mathfrak{t})$. If $P = \Delta$, then $\mathbb{H}^P = \mathbb{H}(\mathfrak{t}, W, k)$.

It is known [BaMo1, Corollary 6.5] that parabolic induction preserves temperedness. The vector space $\mathfrak{t}$ decomposes as $P^\perp \oplus \mathbb{C}P^\vee$, where $P^\vee = \{\alpha^\vee : \alpha \in P\}$. Accordingly $\mathbb{H}^P$ decomposes as a tensor product of algebras

$$(2.19) \quad \mathbb{H}^P = \mathbb{H}(\mathbb{C}P^\vee, W(\Phi_P), k) \otimes \mathcal{O}(P^\perp) = \mathbb{H}_P \otimes \mathcal{O}(P^\perp).$$

When we think of $\mathbb{H}^P$ as corresponding to a Levi subgroup M of a reductive group G , $\mathbb{H}_P$ corresponds to the derived group of M .

For any $\mathbb{H}_P$ -representation (π_P, V_P) and any $\lambda^P \in P^\perp$, we can form the $\mathbb{H}^P$ -representation $(\pi_P \otimes \lambda^P, V_P)$. By varying λ^P , this construction yields continuous or algebraic families of $\mathbb{H}^P$ -representations.

We sketch how the classification of $\text{Irr}(\mathbb{H})$ is obtained. A crucial role is played by Theorem 2.20, and therefore we assume throughout this paragraph that

$$(2.20) \quad \text{the parameter function } k \text{ is real-valued.}$$

The condition 2.20 implies [Slo, Lemma 2.13]:

$$(2.21) \quad \text{every discrete series representation } \pi \text{ of } \mathbb{H}^P \text{ has } \text{Wt}(\pi) \subset \mathbb{R}P^\vee + i(\mathfrak{t}_\mathbb{R} \cap P^\perp).$$

By completion with respect to central characters, as in [Lus, §7–8] and [Sol5, §2.1], one shows that for any $\lambda, \mu \in \mathfrak{t}_\mathbb{R}$ there is an equivalence of categories

$$(2.22) \quad \text{Mod}(\mathbb{H}(\mathfrak{t}, W, k))_{W(i\lambda+\mu)} \cong \text{Mod}(\mathbb{H}(\mathfrak{t}, W_{i\lambda}, k))_{W_{i\lambda} \cdot \mu}.$$

Notice that $W_{i\lambda} = W_\lambda$ is again a Weyl group, from the root system $\{\alpha \in \Phi : \alpha(\lambda) = 0\}$. With (2.22) one can reduce the issues in this paragraph from $\text{Mod}(\mathbb{H})$ to $\mathbb{H}$ -representations with real weights. By [AMS2, Proposition 2.7], (2.22) restricts to a bijection

$$\text{Irr}_{\text{temp}}(\mathbb{H}(\mathfrak{t}, W, k))_{W(i\lambda+\mu)} \longleftrightarrow \text{Irr}_{\text{temp}}(\mathbb{H}(\mathfrak{t}, W_{i\lambda}, k))_{W_{i\lambda} \cdot \mu}.$$

Taking the union over all $\mu \in \mathfrak{t}_\mathbb{R}$ yields a bijection

$$\text{Irr}_{\text{temp}}(\mathbb{H}(\mathfrak{t}, W, k))_{W_{i\lambda} + \mathfrak{t}_\mathbb{R}} \longleftrightarrow \text{Irr}_{\text{temp}}(\mathbb{H}(\mathfrak{t}, W_{i\lambda}, k))_{\mathfrak{t}_\mathbb{R}}.$$

By Theorem 2.20, restriction to $\mathbb{C}[W_{i\lambda}]$ sends the last set to a basis of $R(\mathbb{C}[W_{i\lambda}])$. Theorem 2.4 provides a bijection

$$R(\mathbb{C}[W_{i\lambda}]) \longrightarrow R(\mathcal{O}(\mathfrak{t}) \rtimes W)_{W_{i\lambda}} = R(\mathbb{H}(\mathfrak{t}, W, 0))_{W_{i\lambda}},$$

and we note that this consists entirely of tempered representations. The composition of all these maps is

$$(2.23) \quad \begin{array}{ccc} \text{Irr}_{\text{temp}}(\mathbb{H}(\mathfrak{t}, W, k))_{W_{i\lambda} + \mathfrak{t}_\mathbb{R}} & \rightarrow & R(\mathcal{O}(\mathfrak{t}) \rtimes W)_{W_{i\lambda}} \\ \text{ind}_{\mathbb{H}(\mathfrak{t}, W_{i\lambda}, k)}^{\mathbb{H}(\mathfrak{t}, W, k)}(\pi) & \mapsto & \text{ind}_{\mathcal{O}(\mathfrak{t}) \rtimes W_{i\lambda}}^{\mathcal{O}(\mathfrak{t}) \rtimes W}(\pi|_{\mathbb{C}[W_{i\lambda}]} \otimes i\lambda) \end{array}.$$

Now we let λ run over $\mathfrak{t}_\mathbb{R}$, and we conclude that $\text{Irr}_{\text{temp}}(\mathbb{H})$ maps canonically to a $\mathbb{Z}$ -basis of $R(\mathcal{O}(\mathfrak{t}) \rtimes W)_{i\mathfrak{t}_\mathbb{R}} = R(\mathcal{O}(\mathfrak{t}) \rtimes W)_{\text{temp}}$.

Next we involve the Langlands classification for graded Hecke algebras, from [Eve]. We write

$$P^{\perp+} = \{\lambda^P \in P^\perp : \alpha(\Re(\lambda^P)) > 0 \text{ for all } \alpha \in \Delta \setminus P\}.$$

For each $\lambda^P \in P^{\perp+}$ and each $\pi_P \in \text{Irr}_{\text{temp}}(\mathbb{H}_P)$, the $\mathbb{H}$ -representation $\text{ind}_{\mathbb{H}_P}^{\mathbb{H}}(\pi_P \otimes \lambda_P)$ has a unique irreducible quotient. That sets up a bijection [Eve]

$$(2.24) \quad \bigcup_{P \subset \Delta} \text{Irr}_{\text{temp}}(\mathbb{H}_P) \times P^{\perp+} \longrightarrow \text{Irr}(\mathbb{H}(\mathfrak{t}, W, k)) = \text{Irr}(\mathbb{H}).$$

This also applies to $\mathcal{O}(\mathfrak{t}) \rtimes W = \mathbb{H}(\mathfrak{t}, W, 0)$. The Langlands classification and (2.23) lead to a version of [Sol5, Theorem 2.3.1] for graded instead of affine Hecke algebras:

Theorem 2.24. *Suppose that k is real-valued. There exists a natural bijection*

$$\zeta_0 : R(\mathbb{H}(\mathfrak{t}, W, k)) \rightarrow R(\mathbb{H}(\mathfrak{t}, W, 0)) = R(\mathcal{O}(\mathfrak{t}) \rtimes W)$$

satisfying the following properties.

- (i) *For $\rho \in \text{Irr}_{\text{temp}}(\mathbb{H}(\mathfrak{t}, W, k))_{\mathfrak{t}_\mathbb{R}}$ we have $\zeta_0(\rho) = \mathbb{C}_0 \otimes \rho|_{\mathbb{C}[W]}$.*
- (ii) *ζ_0 commutes with parabolic induction and with character twists:*

$$\zeta_0(\text{ind}_{\mathbb{H}_P}^{\mathbb{H}(\mathfrak{t}, W, k)}(\pi_P \otimes \lambda^P)) = \text{ind}_{\mathcal{O}(\mathfrak{t}) \rtimes W(\Phi_P)}^{\mathcal{O}(\mathfrak{t}) \rtimes W}(\zeta_0(\pi_P) \otimes \lambda^P) \text{ for } \pi_P \in \text{Irr}(\mathbb{H}_P), \lambda^P \in P^\perp.$$
- (iii) *ζ_0 preserves the underlying (virtual) $\mathbb{C}[W]$ -representations.*
- (iv) *For any $\lambda \in \mathfrak{t}_\mathbb{R}$, ζ_0 sends virtual representations with $\mathcal{O}(\mathfrak{t})$ -weights in $i\lambda + \mathfrak{t}_\mathbb{R}$ to virtual representations with $\mathcal{O}(\mathfrak{t})$ -weights in $i\lambda + \mathfrak{t}_\mathbb{R}$.*
- (v) *ζ_0 sends tempered representations to tempered representations, and restricts to a bijection between the tempered parts on both sides.*

In Theorem 2.24.(iv) ζ_0 may adjust the weights by elements of $\mathfrak{t}_{\mathbb{R}}$, and these changes are always weights of a discrete series representation.

Further, with [Sol8, §6] ζ_0 can be refined to a bijection

$$(2.25) \quad \zeta_{\text{Irr}} : \text{Irr}(\mathbb{H}(\mathfrak{t}, W, k)) \rightarrow \text{Irr}(\mathcal{O}(\mathfrak{t}) \rtimes W).$$

Both Theorem 2.24 and (2.25) confirm what we already saw in Theorem 2.10: that $\mathbb{H}(\mathfrak{t}, W, k)$ and $\mathbb{H}(\mathfrak{t}, W, 0)$ are very similar.

Example 2.25. We consider $\mathbb{H} = \mathbb{H}(\mathbb{C}, S_2, k)$ with $k = k(\alpha) > 0$. With the classification of $\text{Irr}(\mathbb{H})$ from Example 2.14 at hand, the maps ζ_0 and ζ_{Irr} can be tabulated.

$\text{Irr}(\mathbb{H})$	$R(\mathcal{O}(\mathfrak{t}) \rtimes W)$	$\text{Irr}(\mathcal{O}(\mathfrak{t}) \rtimes W)$	
$I(\lambda)$	$\pi(\lambda) := \text{ind}_{\mathcal{O}(\mathfrak{t})}^{\mathcal{O}(\mathfrak{t}) \rtimes W}(\lambda)$	$\pi(\lambda)$	$\lambda \in \mathbb{C} \setminus \{k, -k\}$
$I(0)$	$\pi(0)$	$\mathbb{C}_0 \otimes \text{triv}$	
St	$\mathbb{C}_0 \otimes \text{sign}$	$\mathbb{C}_0 \otimes \text{sign}$	
triv	$\pi(k) - \mathbb{C}_0 \otimes \text{sign}$	$\pi(k)$	

Notice that for St and triv, ζ_0 changes the $\mathcal{O}(\mathfrak{t})$ -weights, while ζ_{Irr} for $I(0) = \text{ind}_{\mathcal{O}(\mathfrak{t})}^{\mathbb{H}}(\mathbb{C}_0)$ is not compatible with parabolic induction. Further $\zeta_0(\text{triv})$ is not an actual representation, and it is certainly not irreducible.

Now we generalize to the twisted graded Hecke algebras

$$(2.26) \quad \mathbb{H}(\mathfrak{t}, W\Gamma, k, \mathfrak{h}) = \mathbb{H}(\mathfrak{t}, W, k) \rtimes \mathbb{C}[\Gamma, \mathfrak{h}].$$

The equivalence of categories (2.22) and the arguments leading up to (2.23) also work for these algebras, so there is a canonical map

$$\text{Irr}_{\text{temp}}(\mathbb{H}(\mathfrak{t}, W\Gamma, k, \mathfrak{h})) \rightarrow R(\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W\Gamma, \mathfrak{h}])_{\text{temp}}$$

whose image is a $\mathbb{Z}$ -basis of its range.

A parabolic subalgebra of $\mathbb{H}(\mathfrak{t}, W\Gamma, k, \mathfrak{h})$ has the form $\mathbb{H}(\mathfrak{t}, W(\Phi_P)\Gamma_P, k, \mathfrak{h})$, where $\Gamma_P \subset \Gamma$ is a subgroup stabilizing P . (In general there are several choices for Γ_P , and in principle they are all feasible. Sometimes specific circumstances determine Γ_P .) The tensor product decomposition (2.19) generalizes only in the weaker form

$$\mathbb{H}(\mathfrak{t}, W(\Phi_P)\Gamma_P, k, \mathfrak{h}) = (\mathbb{H}(\mathfrak{t}, W(\Phi_P), k) \otimes \mathcal{O}(P^\perp)) \rtimes \mathbb{C}[\Gamma_P, \mathfrak{h}].$$

This creates complications for the Langlands correspondence, the version for $\mathbb{H}(\mathfrak{t}, W\Gamma, k, \mathfrak{h})$ is more cumbersome, see [Sol8, Corollary 6.8].

Theorem 2.26. *Theorem 2.24 and (2.25) hold also for $\mathbb{H}(\mathfrak{t}, W\Gamma, k, \mathfrak{h})$.*

To retain property (iii), we can either start with representations of the parabolic subalgebra $\mathbb{H}^P = \mathbb{H}(\mathfrak{t}, W(\Phi_P), k)$ (so with $\Gamma_P = \{e\}$), or we have to put an extra condition on λ^P .

Proof. For representations with real weights, this is proven in [Sol8, Proposition 6.10.a]. The general case follows from that with (2.22). $\square$

The property (2.21) can be used to obtain some information about families of discrete series representations of parabolic subalgebras of (2.26).

Lemma 2.27. *For $P \subset \Delta$ and $\lambda \in i\mathfrak{t}_{\mathbb{R}}$, let $\Delta(P, \lambda) \subset \text{Irr}(\mathbb{H}(\mathfrak{t}, W(\Phi_P)\Gamma_P, k, \mathfrak{h}))$ be the set of discrete series representations occurring in $\text{ind}_{\mathcal{O}(\mathfrak{t})}^{\mathbb{H}(\mathfrak{t}, W(\Phi_P)\Gamma_P, k, \mathfrak{h})} \mathbb{C}_{\lambda+\nu}$ for some $\nu \in \mathfrak{t}_{\mathbb{R}}$. Let C be a contractible subset of $i\mathfrak{t}_{\mathbb{R}}$, such that $(W(\Phi_P)\Gamma_P)_\lambda$ is the*

same for all $\lambda \in C$. Then all the sets $\Delta(P, \lambda)$ with $\lambda \in C$ are canonically in bijection with each other.

Proof. First we look at $\mathbb{H}(\mathfrak{t}, W, k)$ and its parabolic subalgebra $\mathbb{H}^P$. By (2.21), $\Delta(P, \lambda)$ is empty unless $\lambda \in i(\mathfrak{t}_{\mathbb{R}} \cap P^{\vee})$. From (2.19) we see that

$$(2.27) \quad \Delta(P, \lambda) = \Delta(P, 0) \otimes \mathbb{C}_{\lambda} \text{ for } \lambda \in i(\mathfrak{t}_{\mathbb{R}} \cap P^{\vee}).$$

The set $i(\mathfrak{t}_{\mathbb{R}} \cap P^{\vee})$ is precisely the fixed point set of $W(\Phi_P)$ in $i\mathfrak{t}_{\mathbb{R}}$. Therefore either $C \subset i(\mathfrak{t}_{\mathbb{R}} \cap P^{\vee})$ or $C \cap i(\mathfrak{t}_{\mathbb{R}} \cap P^{\vee})$ is empty. In the latter case $\Delta(P, \lambda)$ is empty, while in the former case (2.27) provides a canonical bijection from $\Delta(P, \lambda)$ to $\Delta(P, \lambda')$ for any $\lambda, \lambda' \in C$.

Now we consider $\mathbb{H}_+^P := \mathbb{H}(\mathfrak{t}, W(\Phi_P)\Gamma_P, k, \mathfrak{h})$, and we denote $\Delta(P, \lambda)$ for that algebra by $\Delta(P, \lambda)_+$. Notice that the discrete series condition from Definition 2.16 is stable under the action of Γ_P on $\mathfrak{t}$. Therefore the restriction to $\mathbb{H}^P$ of any $\delta_+ \in \Delta(P, \lambda)_+$ has all irreducible subquotients in $\Delta(P, \lambda)$. It follows that we can exhaust $\Delta(P, \lambda)_+$ with the irreducible subquotients of $\text{ind}_{\mathbb{H}^P}^{\mathbb{H}_+^P}(\delta)$ for $\delta \in \Delta(P, \lambda)$.

By Clifford theory, as for instance in [RaRa, p. 24] or [AMS1, §1], $\text{ind}_{\mathbb{H}^P}^{\mathbb{H}_+^P}(\delta)$ is completely reducible and its decomposition into irreducible representations is governed by its algebra of self-intertwiners, which is a twisted group algebra $\mathbb{C}[\Gamma_{P, \delta}, \mathfrak{h}_{\delta}]$. Write $\delta = \delta_0 \otimes \lambda$ as in (2.27). For $\gamma \in \Gamma_P$ we have

$$\gamma(\delta_0 \otimes \lambda) = \gamma(\delta_0) \otimes \gamma(\lambda).$$

Consequently $\Gamma_{P, \delta_0 \otimes \lambda}$ is the same for all $\lambda \in C$. The 2-cocycles $\mathfrak{h}_{\delta_0 \otimes \lambda}$ come a choice of intertwining operators

$$I_{\delta_0 \otimes \lambda}^{\gamma} \in \text{Hom}_{\mathbb{H}^P}(\gamma(\delta_0) \otimes \gamma(\lambda), \delta_0 \otimes \lambda),$$

see [AMS1, (4)–(5)]. We can choose these independently of $\lambda \in C$, so $\mathfrak{h}_{\delta_0 \otimes \lambda}$ does not depend on $\lambda \in C$ either. Hence the entire decomposition of $\text{ind}_{\mathbb{H}^P}^{\mathbb{H}_+^P}(\delta_0 \otimes \lambda)$ depends continuously on $\lambda \in C$. Together with the contractibility of C , that yields a canonical bijections between the sets of irreducible subquotients of $\text{ind}_{\mathbb{H}^P}^{\mathbb{H}_+^P}(\delta_0 \otimes \lambda)$ and of $\text{ind}_{\mathbb{H}^P}^{\mathbb{H}_+^P}(\delta_0 \otimes \lambda')$, for any $\lambda, \lambda' \in C$. When we carry this out for all $\delta \in \Delta(P, \lambda)$, we obtain the desired bijection $\Delta(P, \lambda)_+ \rightarrow \Delta(P, \lambda')_+$. $\square$

3. PROGENERATORS AND THEIR ENDOMORPHISM ALGEBRAS

We look in more detail at the structure of the Hecke algebra of a reductive p -adic group G . In Paragraph 1.8 we surveyed it in terms of harmonic analysis, but that description does not suffice to say much about the representation theory of $\mathcal{H}(G)$.

The Bernstein decomposition (Theorem 1.41) reduces the issue to understanding each Bernstein block $\text{Rep}(G)^s$, so we will focus on one such block. We will study it by means of a finitely generated projective generator, a progenerator for short. This strategy is very general, it can be employed whenever one has a ring A and a progenerator P for $\text{Mod}(A)$. The main point is the following result from category theory:

Proposition 3.1. [Roc, Theorem 1.8.2.1]

There are equivalences of categories

$$\begin{array}{ccc} \text{Mod}(A) & \longleftrightarrow & \text{End}_A(P) - \text{Mod} \cong \text{Mod}(\text{End}_A(P)^{op}) \\ M & \mapsto & \text{Hom}_A(P, M) \\ V \otimes_{\text{End}_A(P)} P & \xleftarrow{\quad} & V \end{array}.$$

Here $\text{End}_A(P) - \text{Mod}$ denotes the category of right $\text{End}_A(P)$ -modules. The bimodules for the Morita equivalence between A and $\text{End}_A(P)^{op}$ are P and $\text{Hom}_A(P, A)$.

Progenerators are quite common in the representation theory of reductive p -adic groups, although sometimes implicitly. Namely, suppose that (K, ρ) is a type for $\text{Rep}(G)^s$, in the sense of Bushnell and Kutzko [BuKu]. Then $\text{ind}_K^G(\rho)$ is a progenerator of $\text{Rep}(G)^s$. Moreover the Hecke algebra $\mathcal{H}(G, K, \rho)$, as defined in [BuKu, §2], is the opposite algebra of $\text{End}_G(\text{ind}_K^G(\rho))$, and the equivalence of categories $\text{Rep}(G)^s \cong \text{Mod}(\mathcal{H}(G, K, \rho))$ from [BuKu, Theorem 4.3] is just an instance of Proposition 3.1.

In this paper we will not use types, because we want to treat all Bernstein blocks, whereas types are not always available. Whenever one has a type (K, ρ) , the algebra $\mathcal{H}(G, K, \rho)$ is Morita equivalent to the G -endomorphism algebra of any other progenerator for $\text{Rep}(G)^s$, so in that sense the choice of a progenerator does not really matter.

3.1. The cuspidal case.

Let $L = \mathcal{L}(F)$ be a reductive p -adic group (which in the next paragraphs will be a Levi subgroup of G). Let $\sigma \in \text{Irr}(L)$ be supercuspidal and consider the Bernstein block $\text{Rep}(L)^{s_L}$ with $s_L = [L, \sigma]_L$. Recall from Theorem 1.36 that $\text{Res}_{L^1}^L \sigma$ is a compact L^1 -representation. Compact representations are always projective [Ren, Proposition IV.1.6], because they behave like representations of compact groups.

We note that $\text{Res}_{L^1}^L \sigma$ has finite length because $[L : Z(L)L^1]$ is finite. Let

$$\Pi_{s_L} := \text{ind}_{L^1}^L(\text{Res}_{L^1}^L \sigma)$$

be the smooth compact induction, from L^1 to L , of σ . Here compact means that the underlying vector space is

$$\{f : G \rightarrow V_\sigma \mid f(kl) = \sigma(k)f(l) \text{ for all } k \in L^1, l \in L, \text{ supp}(f) \text{ is compact in } L/L^1\}.$$

Proposition 3.2. (Bernstein, see [Ren, Proposition VI.4.1])

The L -representation Π_{s_L} is a progenerator of $\text{Rep}(L)^{s_L}$.

We note that Π_{s_L} is canonical, in the sense that it depends only on $[L, \sigma]_L$, or equivalently only on $X_{\text{nr}}(L)\sigma$. Thus Propositions 3.1 and 3.2 give a canonical equivalence of categories

$$(3.1) \quad \text{Rep}(L)^{[L, \sigma]} \cong \text{End}_L(\Pi_{[L, \sigma]}) - \text{Mod}.$$

There are isomorphisms of L -representations

$$\Pi_{s_L} = \text{ind}_{L^1}^L(\text{Res}_{L^1}^L \sigma) \cong \mathbb{C}[L/L^1] \otimes \sigma \cong \mathcal{O}(X_{\text{nr}}(L)) \otimes \sigma,$$

where L acts diagonally on the tensor products. Since L/L^1 is commutative, the multiplication action of $\mathbb{C}[L/L^1] \cong \mathcal{O}(X_{\text{nr}}(L))$ on $\mathbb{C}[L/L^1]$ is by L -intertwiners. This gives an embedding

$$(3.2) \quad \mathcal{O}(X_{\text{nr}}(L)) \rightarrow \text{End}_L(\mathcal{O}(X_{\text{nr}}(L)) \otimes \sigma) \cong \text{End}_L(\Pi_{s_L}).$$

Recall the finite group $X_{\text{nr}}(L, \sigma) = \{\chi \in X_{\text{nr}}(L) : \sigma \otimes \chi \cong \sigma\}$. It acts on $X_{\text{nr}}(L)$ by translations, and there is a homeomorphism

$$(3.3) \quad X_{\text{nr}}(L)/X_{\text{nr}}(L, \sigma) \longrightarrow \text{Irr}(L)^{\mathfrak{s}_L}.$$

For $\chi \in X_{\text{nr}}(L, \sigma)$, the choice of a nonzero element of $\text{Hom}_L(\sigma, \sigma \otimes \chi)$ gives rise to an element $T_\chi \in \text{End}_L(\Pi_{\mathfrak{s}_L})$, which lifts the multiplication action of χ on $X_{\text{nr}}(L)$. We define a 2-cocycle $\mathfrak{d}_{\mathfrak{s}_L}$ of $X_{\text{nr}}(L, \sigma)$ by

$$(3.4) \quad T_{\chi_1} T_{\chi_2} = \mathfrak{d}_{\mathfrak{s}_L}(\chi_1, \chi_2) T_{\chi_1 \chi_2}.$$

Theorem 3.3. [Sol9, Proposition 2.2 and (2.25)]

The elements T_χ with $\chi \in X_{\text{nr}}(L, \sigma)$ determine an algebra isomorphism

$$\text{End}_L(\Pi_{\mathfrak{s}_L}) \cong \mathcal{O}(X_{\text{nr}}(L)) \rtimes \mathbb{C}[X_{\text{nr}}(L, \sigma), \mathfrak{d}_{\mathfrak{s}_L}].$$

From Theorem 3.3 one sees that the centre is

$$(3.5) \quad Z(\text{End}_L(\Pi_{\mathfrak{s}_L}) \cong \mathcal{O}(X_{\text{nr}}(L))^{X_{\text{nr}}(L, \sigma)} = \mathcal{O}(X_{\text{nr}}(L)/X_{\text{nr}}(L, \sigma)) \cong \mathcal{O}(\text{Irr}(L)^{\mathfrak{s}_L}).$$

This is also the centre of the category $\text{Rep}(L)^{\mathfrak{s}_L}$. By Proposition 2.5 there are equivalences of categories of finite length representations

$$(3.6) \quad \text{Rep}_{\mathfrak{fl}}(L)^{\mathfrak{s}_L} \cong \text{End}_L(\Pi_{\mathfrak{s}_L}) - \text{Mod}_{\mathfrak{fl}} \cong \mathcal{O}(\text{Irr}(L)^{\mathfrak{s}_L}) - \text{Mod}_{\mathfrak{fl}}.$$

By Proposition 2.5.b the restriction to finite length can be omitted if $\mathfrak{d}_{\mathfrak{s}_L}$ is trivial, but by (2.11) and Theorem 2.6 it is necessary if $\mathfrak{d}_{\mathfrak{s}_L}$ is nontrivial in $H^2(X_{\text{nr}}(L, \sigma), \mathbb{C}^\times)$. An example of the latter situation is [Sol9, Example 2.G].

3.2. The non-cuspidal case.

Let $P = LU_P$ be a parabolic subgroup of G with Levi factor L . Recall that $\mathfrak{s} = [L, \sigma]_G$ and $\mathfrak{s}_L = [L, \sigma]_L$. The following result of Bernstein is quite deep, in particular it uses the second adjointness relation (Theorem 1.23).

Proposition 3.4. *The G -representation*

$$\Pi_{\mathfrak{s}} := I_P^G(\Pi_{\mathfrak{s}_L}) = I_P^G(\text{ind}_{L^1}^L(\text{Res}_{L^1}^L \sigma))$$

is a progenerator of $\text{Rep}(G)^{\mathfrak{s}}$. It is canonical, in the sense that up to isomorphism it depends only $\mathfrak{s}$.

From Propositions 3.1 and 3.2 we obtain a canonical equivalence of categories

$$(3.7) \quad \text{Rep}(G)^{\mathfrak{s}} \cong \text{End}_G(\Pi_{\mathfrak{s}}) - \text{Mod}.$$

Example 3.5. Suppose that G is quasi-split and that $\mathfrak{s} = [T, 1]_G$ for a maximal torus T of G . Then, by [Blo, Théorème 2]

$$\Pi_{\mathfrak{s}} = I_B^G(\text{ind}_{T^1}^T(\text{triv})) = I_B^G(\mathbb{C}[T/T^1]) \cong \mathbb{C}[G/I]$$

for an Iwahori subgroup I of G . In this case

$$\text{End}_G(\Pi_{\mathfrak{s}}) \cong \text{End}_G(\mathbb{C}[G/I]) \cong \mathcal{H}(G, I),$$

so in words $\text{End}_G(\Pi_{\mathfrak{s}})$ is isomorphic to the Iwahori–Hecke algebra of G . It is known from [IwMa] that $\mathcal{H}(G, I)$ has the structure of an affine Hecke algebra. The equivalence between the module category of $\mathcal{H}(G, I)$ and the category of Iwahori-spherical G -representations from [Bor] is a special case of the equivalence obtained from Propositions 3.1 and 3.2, combined with an isomorphism between $\mathcal{H}(G, I)$ and its opposite algebra.

However, in general the structure of $\text{End}_G(\Pi_{\mathfrak{s}})$ is considerably more involved, and we will only approach it in several steps. The first observation in this direction is that (3.2) and the functor I_P^G provide an embedding

$$(3.8) \quad \mathcal{O}(X_{\text{nr}}(L)) \rightarrow \text{End}_G(\Pi_{\mathfrak{s}}).$$

Recall that

$$W_{\mathfrak{s}} = \text{Stab}_{N_G(L)/L}([L, \sigma]_L) = \{w \in N_G(L) : w \cdot \sigma \in X_{\text{nr}}(L)\sigma\}/L.$$

The action of $N_G(L)$ on $\text{Irr}(L)$ induces an action of $W_{\mathfrak{s}}$ on $\text{Irr}(L)^{\mathfrak{s}L}$.

Example 3.6. We consider the special case $W_{\mathfrak{s}} = \{e\}$, which is very common. Let $\pi, \rho \in \text{Rep}(L)^{\mathfrak{s}L}$. Recall that Bernstein's geometric lemma [Ren, Théorème VI.5.1] provides a filtration of the L -representation $J_P^G I_P^G(\pi)$. The condition $W_{\mathfrak{s}} = \{e\}$ implies that from the irreducible subquotients of this filtration only π itself belongs to $\text{Rep}(L)^{\mathfrak{s}L}$. From that and Frobenius reciprocity we obtain

$$\text{Hom}_G(I_P^G(\pi), I_P^G(\rho)) \cong \text{Hom}_L(J_P^G I_P^G(\pi), \rho) \cong \text{Hom}_L(\pi, \rho).$$

Therefore the functor $I_P^G : \text{Rep}(L)^{\mathfrak{s}L} \rightarrow \text{Rep}(G)^{\mathfrak{s}}$ is an equivalence of categories. In particular I_P^G induces an algebra isomorphism

$$\text{End}_L(\Pi_{\mathfrak{s}}) \rightarrow \text{End}_G(I_P^G(\Pi_{\mathfrak{s}L})) = \text{End}_G(\Pi_{\mathfrak{s}}).$$

This is a very satisfactory outcome, but of course things are more complicated (and more interesting) when $W_{\mathfrak{s}} \neq \{e\}$.

Recall that the Bernstein centre of G [BeDe] is the centre of the category $\text{Rep}(G)$. It can also be expressed in terms of distributions on G [MoTa, Proposition 3.2g]. Some aspects of the Bernstein decomposition involve the Bernstein centre:

Theorem 3.7. [BeDe]

There are natural isomorphisms

$$Z(\text{Rep}(G)^{\mathfrak{s}}) \cong Z(\text{End}_G(\Pi_{\mathfrak{s}})) \cong \mathcal{O}(\text{Irr}(L)^{\mathfrak{s}L})^{W_{\mathfrak{s}}} = \mathcal{O}(\text{Irr}(L)^{\mathfrak{s}L}/W_{\mathfrak{s}}).$$

Recall from Definition 1.31 that $W_{\mathfrak{s}}^e$ is an extension of $W_{\mathfrak{s}}$ by $X_{\text{nr}}(L, \sigma)$, which acts on $X_{\text{nr}}(L)$. By construction, the quotient map

$$X_{\text{nr}}(L) \rightarrow \text{Irr}(L)^{\mathfrak{s}L} : \chi \mapsto \sigma \otimes \chi$$

induces a homeomorphism

$$X_{\text{nr}}(L)/W_{\mathfrak{s}}^e \rightarrow \text{Irr}(L)^{\mathfrak{s}L}/W_{\mathfrak{s}}.$$

Knowing this, Theorem 3.7 says that

$$(3.9) \quad Z(\text{End}_G(\Pi_{\mathfrak{s}})) \cong \mathcal{O}(X_{\text{nr}}(L))^{W_{\mathfrak{s}}^e} = \mathcal{O}(X_{\text{nr}}(L)/W_{\mathfrak{s}}^e).$$

As we saw in (1.22) and Theorem 1.42, the group $W_{\mathfrak{s}}^e$ acts on the family of representations $I_P^G(\sigma \otimes \chi)$ with $\chi \in X_{\text{nr}}(L)$, but by operators that depend rationally on χ and may have poles. Moreover, from (1.23) and (3.4) we see that in general this is only a projective action of $W_{\mathfrak{s}}^e$. Still, from these one can construct, as done in [Sol9, §4], an embedding

$$(3.10) \quad \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}] \rightarrow \text{End}_G(\Pi_{\mathfrak{s}}) \otimes_{\mathcal{O}(X_{\text{nr}}(L)/W_{\mathfrak{s}}^e)} \mathbb{C}(X_{\text{nr}}(L)/W_{\mathfrak{s}}^e),$$

for a suitable 2-cocycle $\mathfrak{h}_{\mathfrak{s}}$ generalizing (1.23). This and the next result can be compared with Theorem 2.10.

Theorem 3.8. [Sol9, Corollary 5.8]

The embeddings (3.8) and (3.10) combine to an algebra isomorphism

$$\mathrm{End}_G(\Pi_{\mathfrak{s}}) \otimes_{\mathcal{O}(X_{\mathrm{nr}}(L)/W_{\mathfrak{s}}^e)} \mathbb{C}(X_{\mathrm{nr}}(L)/W_{\mathfrak{s}}^e) \cong \mathbb{C}(X_{\mathrm{nr}}(L)) \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}].$$

In Theorem 3.8 it is necessary to include the quotient field $\mathbb{C}(X_{\mathrm{nr}}(L)/W_{\mathfrak{s}}^e)$, unlike in Theorem 3.3, without that Theorem 3.8 would only hold in special cases. Since $W_{\mathfrak{s}}^e$ acts faithfully on $X_{\mathrm{nr}}(L)$, $\mathbb{C}(X_{\mathrm{nr}}(L))$ has dimension $|W_{\mathfrak{s}}^e|$ over $\mathbb{C}(X_{\mathrm{nr}}(L)/W_{\mathfrak{s}}^e)$. Theorem 3.8 shows that

$$\dim_{\mathbb{C}(X_{\mathrm{nr}}(L)/W_{\mathfrak{s}}^e)} \mathrm{End}_G(\Pi_{\mathfrak{s}}) \otimes_{\mathcal{O}(X_{\mathrm{nr}}(L)/W_{\mathfrak{s}}^e)} \mathbb{C}(X_{\mathrm{nr}}(L)/W_{\mathfrak{s}}^e) = |W_{\mathfrak{s}}^e|^2.$$

This can be stated more precisely in terms of central simple algebras. Namely, it follows from Theorems 2.6 and 3.8 that:

Corollary 3.9. $\mathrm{End}_G(\Pi_{\mathfrak{s}}) \otimes_{\mathcal{O}(X_{\mathrm{nr}}(L)/W_{\mathfrak{s}}^e)} \mathbb{C}(X_{\mathrm{nr}}(L)/W_{\mathfrak{s}}^e)$ is a central simple algebra over $\mathbb{C}(X_{\mathrm{nr}}(L)/W_{\mathfrak{s}}^e)$. It is Morita equivalent to $\mathbb{C}(X_{\mathrm{nr}}(L)) \rtimes W_{\mathfrak{s}}^e$ if and only if $\mathfrak{h}_{\mathfrak{s}}$ is trivial in $H^2(W_{\mathfrak{s}}^e, \mathbb{C}^\times)$.

3.3. Localization on the Bernstein centre.

For any $W_{\mathfrak{s}}$ -stable subset $U \subset \mathrm{Irr}(L)^{\mathfrak{s}L}$, one can consider

$$\mathrm{Rep}(G)_U^{\mathfrak{s}} = \{\pi \in \mathrm{Rep}(G)^{\mathfrak{s}} : \mathrm{Sc}(\pi') \in (L, U) \text{ for all irreducible subquotients } \pi' \text{ of } \pi\}.$$

In view of Theorem 3.7, this category can be obtained from $\mathrm{Rep}(G)^{\mathfrak{s}}$ by imposing conditions on how the Bernstein centre $Z(\mathrm{Rep}(G)^{\mathfrak{s}})$ may act on the representations.

Often it is more prudent to restrict to finite length representations. That will be indicated by a subscript fl , so $\mathrm{Rep}_{\mathrm{fl}}(G)_U^{\mathfrak{s}}$. When U_i , for i in some (possibly infinite) index set, are disjoint $W_{\mathfrak{s}}$ -invariant subsets of $\mathrm{Irr}(L)^{\mathfrak{s}L}$, there is a decomposition

$$(3.11) \quad \mathrm{Rep}_{\mathrm{fl}}(G)_{\cup_i U_i}^{\mathfrak{s}} = \bigoplus_i \mathrm{Rep}_{\mathrm{fl}}(G)_{U_i}^{\mathfrak{s}},$$

This does not work with representations of arbitrary length, and it is an important reason why it is easier to work with representations of finite length.

To proceed, we make the relation between $\mathrm{End}_G(\Pi_{\mathfrak{s}})$ and supercuspidal supports explicit. Let $Q = MU_Q$ be a parabolic subgroup of G containing $P = LU_P$, so that $\mathfrak{s}_M = [L, \sigma]_M$ is defined. Then the functor I_Q^G provides an embedding

$$\mathrm{End}_M(\Pi_{\mathfrak{s}_M}) \rightarrow \mathrm{End}_G(\Pi_{\mathfrak{s}}).$$

Lemma 3.10. [Sol7, Lemma 5.1]

(a) The equivalences of categories (3.7) are compatible with parabolic induction: they form a commutative diagram

$$\begin{array}{ccc} \mathrm{Rep}(G)^{\mathfrak{s}} & \cong & \mathrm{End}_G(\Pi_{\mathfrak{s}}) - \mathrm{Mod} \\ \uparrow I_Q^G & & \uparrow \mathrm{ind} \\ \mathrm{Rep}(M)^{\mathfrak{s}_M} & \cong & \mathrm{End}_M(\Pi_{\mathfrak{s}_M}) - \mathrm{Mod} \end{array}.$$

(b) The equivalences of categories (3.7) are compatible with parabolic restriction: they form a commutative diagram

$$\begin{array}{ccc} \mathrm{Rep}(G)^{\mathfrak{s}} & \cong & \mathrm{End}_G(\Pi_{\mathfrak{s}}) - \mathrm{Mod} \\ \downarrow \mathrm{pr}_{\mathfrak{s}_M} \circ J_Q^G & & \downarrow \mathrm{Res} \\ \mathrm{Rep}(M)^{\mathfrak{s}_M} & \cong & \mathrm{End}_M(\Pi_{\mathfrak{s}_M}) - \mathrm{Mod} \end{array}.$$

Here $\mathrm{pr}_{\mathfrak{s}_M} : \mathrm{Rep}(M) \rightarrow \mathrm{Rep}(M)^{\mathfrak{s}_M}$ is the projection from the Bernstein decomposition.

From Lemma 3.10, (3.5) and Theorem 3.7, we see that the equivalence of categories (3.7) is compatible with supercuspidal supports, in the following sense.

Corollary 3.11. *Suppose that $\pi \in \text{Irr}(G)^{\mathfrak{s}}$ has supercuspidal support $(L, \sigma \otimes \chi)$. Then $\text{Hom}_G(\Pi_{\mathfrak{s}}, \pi) \in \text{Irr}(\text{End}_G(\Pi_{\mathfrak{s}}))$ has central character $W_{\mathfrak{s}}(\sigma \otimes \chi) \in \text{Irr}(L)^{\mathfrak{s}_L}/W_{\mathfrak{s}}$, or equivalently $W_{\mathfrak{s}}^e \chi \in X_{\text{nr}}(L)/W_{\mathfrak{s}}^e$.*

Corollary 3.11 enables us to analyse the representation theory of $\text{End}_G(\Pi_{\mathfrak{s}})$ by putting conditions on the supercuspidal supports. For a $W_{\mathfrak{s}}^e$ -stable subset $U' \subset X_{\text{nr}}(L)$ we define

$$\text{End}_G(\Pi_{\mathfrak{s}})\text{-Mod}_{U'} = \{V \in \text{End}_G(\Pi_{\mathfrak{s}})\text{-Mod} : \text{all } \mathcal{O}(X_{\text{nr}}(L))\text{-weights of } V \text{ lie in } U'\},$$

and its subcategory $\text{End}_G(\Pi_{\mathfrak{s}}) - \text{Mod}_{\text{fl}, U'}$.

If U equals $\{\sigma \otimes \chi : \chi \in U'\}$, then (3.7) and Corollary 3.11 provide equivalences of categories

$$(3.12) \quad \text{Rep}(G)_U^{\mathfrak{s}} \cong \text{End}_G(\Pi_{\mathfrak{s}}) - \text{Mod}_{U'} \quad \text{and} \quad \text{Rep}_{\text{fl}}(G)_U^{\mathfrak{s}} \cong \text{End}_G(\Pi_{\mathfrak{s}}) - \text{Mod}_{\text{fl}, U'}.$$

When $U' \subset X_{\text{nr}}(L)$ is open (with respect to the Zariski topology or with respect to the analytic topology), we can analyse $\text{End}_G(\Pi_{\mathfrak{s}}) - \text{Mod}_{\text{fl}, U'}$ by localizing $\text{End}_G(\Pi_{\mathfrak{s}})$ with respect to an ideal of $Z(\text{End}_G(\Pi_{\mathfrak{s}})) \cong \mathcal{O}(X_{\text{nr}}(L)/W_{\mathfrak{s}}^e)$ or by involving complex analytic functions on U' . For specific U' , this localization may be Morita equivalent to a localization of a simpler algebra.

From now we assume

$$(3.13) \quad \sigma \in \text{Irr}_{\text{cusp}}(L) \text{ is unitary (or equivalently tempered).}$$

By Lemma 1.37, that is no restriction on $\mathfrak{s}$ or $\mathfrak{s}_L$. We are interested in the category $\text{Rep}(G)_{X_{\text{nr}}^+(L)W_{\mathfrak{s}}\sigma}^{\mathfrak{s}}$, which will be related to a twisted graded Hecke algebra.

Let $X^*(Z^\circ(L))$ be the lattice of F -rational characters $Z^\circ(\mathcal{L}) = Z(\mathcal{L})^\circ \rightarrow GL_1$ and let $\Phi(G, Z^\circ(L))$ be the set of $\alpha \in X^*(Z^\circ(L))$ that appear in the adjoint action of $Z^\circ(L)$ on the Lie algebra of G . This is not necessarily a root system, but it is always a generalized root system in the sense of [DiFi].

For a reduced root $\alpha \in \Phi(G, Z^\circ(L))$, let L_α be the reductive group generated by L and the root subgroups of G associated to the multiples of α . Then L is a maximal proper Levi subgroup of L_α . We say that α belongs to $\Phi(G, Z^\circ(L))_\sigma$ if, for some $\chi_\sigma \in \text{Hom}(L/Z(L_\alpha), \mathbb{R}_{>0}) \setminus \{1\}$, the representation $I_{L(P \cap L_\alpha)}^{L_\alpha}(\sigma \otimes \chi_\sigma)$ is reducible. By [Sol12, Corollary 1.3 and (1.8)] and [Hei2, Proposition 2.13],

$$\Phi(G, Z^\circ(L))_\sigma \subset X^*(Z^\circ(L)) \text{ is a reduced root system.}$$

For $\alpha \in \Phi(G, Z^\circ(L))_\sigma \subset X^*(Z^\circ(L))$, the above unramified character χ_σ is unique up to inversion. It can be captured with one real number $q_{\sigma, \alpha}$, as follows. Put $L_\sigma^2 = \bigcap_{\chi \in X_{\text{nr}}(L, \sigma)} \ker \chi$ and let $h_\alpha^\vee$ (a version of the coroot $\alpha^\vee$) be the generator of $(L_\sigma^2 \cap L_\alpha^1)/L^1 \cong \mathbb{Z}$ from [Sol9, (A.2)]. Then $q_{\sigma, \alpha} \in \mathbb{R}_{>1}$ is the unique number such that $\chi_\sigma(h_\alpha^\vee) \in \{q_{\sigma, \alpha}, q_{\sigma, \alpha}^{-1}\}$. For explicit computations of the numbers $q_{\sigma, \alpha}$ we refer to [Sol12, Oha].

We introduce the data for our twisted graded Hecke algebra.

- $\mathfrak{t} = \text{Lie}(X_{\text{nr}}(L)) = \text{Hom}(L, \mathbb{C}) \cong \text{Hom}(Z(L), \mathbb{C}) \cong X^*(Z(L)) \otimes_{\mathbb{Z}} \mathbb{C}$.
- $R_\sigma = \{h_\alpha^\vee : \alpha \in \Phi(G, Z^\circ(L))_\sigma\} \subset L_\sigma^2/L^1$ is a root system by [Sol9, Proposition 3.1]. The set R_σ^+ of $h_\alpha^\vee$ for which α appears in $\text{Lie } U_P$ is a positive system in R_σ .

- The parameter function $k_\sigma : R_\sigma \rightarrow \mathbb{R}_{>0}$ given by $k_\sigma(h_\alpha^\vee) = \log(q_{\sigma,\alpha})$.
- The finite group $W_{\mathfrak{s},\sigma}$ acts on R_σ . It can be written as

$$(3.14) \quad W_{\mathfrak{s},\sigma} = W(R_\sigma) \rtimes \Gamma_\sigma, \quad \text{where } \Gamma_\sigma \text{ is the stabilizer of } R_\sigma^+.$$

- The 2-cocycle $\natural_\sigma$ given by the multiplication rules for the intertwining operators in (1.23). It records how much $w \mapsto I(w, L(P \cap L_\alpha), \sigma, \chi = 1)$ deviates from a group homomorphism. We note that $\natural_\sigma$ depends on a normalization of these intertwining operators.

Let $\mathbb{H}(\mathfrak{t}, W_{\mathfrak{s},\sigma}, k_\sigma, \natural_\sigma)$ be the algebra as in Definition 2.9, determined by the above data.

Example 3.12. Consider $G = SL_2(F)$, the diagonal torus $T = L$ and $\sigma = \text{triv}_T$. In this setting $\mathfrak{t} = \text{Hom}(T, \mathbb{C}^\times) \cong \mathbb{C}$, $W_{\mathfrak{s}} = N_G(T)/T \cong S_2$ and the group $\Gamma_{\sigma'}$ is trivial for all $\sigma' \in X_{\text{nr}}(T)$. For various σ' , the algebras $\mathbb{H}(\mathfrak{t}, W_{\mathfrak{s},\sigma'}, k_{\sigma'})$ are:

- $\mathbb{H}(\mathbb{C}, S_2, \log(q_F))$ for $\sigma' = \text{triv}_T$,
- $\mathbb{H}(\mathbb{C}, S_2, 0) = \mathcal{O}(\mathbb{C}) \rtimes S_2$ for $\sigma' \in X_{\text{nr}}(T)$ quadratic,
- $\mathbb{H}(\mathbb{C}, \{e\}, 0) = \mathcal{O}(\mathbb{C})$ for other $\sigma' \in X_{\text{nr}}(T)$.

For open $U \subset X_{\text{nr}}(L)$, let $C^{an}(U)$ be the algebra of complex analytic functions on U . We define the analytic localization of $\text{End}_G(\Pi_{\mathfrak{s}})$ on U as

$$(3.15) \quad \text{End}_G(\Pi_{\mathfrak{s}}) \otimes_{\mathcal{O}(X_{\text{nr}}(L))} C^{an}(U).$$

This is an algebra if U is $W_{\mathfrak{s}}^e$ -stable.

We note that the map

$$\exp_\sigma : \mathfrak{t} \rightarrow \text{Irr}(L)^{s_L}, \quad x \mapsto \exp(x) \otimes \sigma$$

restricts to a diffeomorphism from $\mathfrak{t}_{\mathbb{R}} := X^*(Z^\circ(L))$ to $X_{\text{nr}}^+(L)\sigma$. Analytic localization of $\text{End}_G(\Pi_{\mathfrak{s}})$ on a small tubular open neighborhood of $X_{\text{nr}}^+(L)W_{\mathfrak{s}}$ in $\text{Irr}(L)^{s_L}$ can be compared with analytic localization of $\mathbb{H}(\mathfrak{t}, W_{\mathfrak{s},\sigma}, k_\sigma, \natural_\sigma)$ on a small tubular open neighborhood U_σ of $\mathfrak{t}_{\mathbb{R}}$ in $\mathfrak{t}$, an algebra of the form

$$(3.16) \quad \mathbb{H}(\mathfrak{t}, W_{\mathfrak{s},\sigma}, k_\sigma, \natural_\sigma) \otimes_{\mathcal{O}(\mathfrak{t}/W_{\mathfrak{s},\sigma})} C^{an}(U_\sigma)^{W_{\mathfrak{s},\sigma}}.$$

Arguments involving these localizations lead to:

Theorem 3.13. [Sol9, Corollary 8.1]

There are equivalences of categories

$$\text{Rep}_{\mathfrak{fl}}(G)_{X_{\text{nr}}^+(L)W_{\mathfrak{s}}\sigma}^s \cong \text{End}_G(\Pi_{\mathfrak{s}}) - \text{Mod}_{\mathfrak{fl}, X_{\text{nr}}^+(L)W_{\mathfrak{s}}\sigma} \cong \mathbb{H}(\mathfrak{t}, W_{\mathfrak{s},\sigma}, k_\sigma, \natural_\sigma) - \text{Mod}_{\mathfrak{fl}, \mathfrak{t}_{\mathbb{R}}}$$

such that

- Once a 2-cocycle $\natural_\sigma$ has been fixed (by a normalization of the involved intertwining operators), the equivalences are canonical.
- The equivalences preserve temperedness.
- $\pi \in \text{Irr}(G)_{X_{\text{nr}}^+(L)W_{\mathfrak{s}}\sigma}^s$ is discrete series if and only if the image of π in $\text{Irr}(\mathbb{H}(\mathfrak{t}, W_{\mathfrak{s},\sigma}, k_\sigma, \natural_\sigma))$ is discrete series and $\text{rk}(R_\sigma)$ equals $\dim_F(Z(L)/Z(G))$.
- The equivalences are compatible with parabolic induction and restriction, in the same sense as Lemma 3.10.
- For $\pi \in \text{Irr}(G)_{X_{\text{nr}}^+(L)W_{\mathfrak{s}}\sigma}^s$ with $\text{Sc}(\pi)$ represented by $\sigma \otimes \chi \in X_{\text{nr}}^+(L)\sigma$, the associated $\mathbb{H}(\mathfrak{t}, W_{\mathfrak{s},\sigma}, k_\sigma, \natural_\sigma)$ -module has central character $W_{\mathfrak{s},\sigma} \log \chi \in \mathfrak{t}_{\mathbb{R}}/W_{\mathfrak{s},\sigma}$. Equivalently, $\exp_\sigma$ translates central characters into supercuspidal supports.

Via Lemma 1.11, $\text{Rep}_{\text{fl}}(G)_{X_{\text{nr}}^+(L)W_{\mathfrak{s}}\sigma}^{\mathfrak{s}}$ is equivalent to the category of finite length $\mathcal{H}(G)^{\mathfrak{s}}$ -modules all whose supercuspidal supports lie in $X_{\text{nr}}^+(L)W_{\mathfrak{s}}\sigma/W_{\mathfrak{s}}$. Theorem 3.13 is our final answer to the question about the structure of $\mathcal{H}(G)^{\mathfrak{s}}$ and $\text{End}_G(\Pi_{\mathfrak{s}})$, in terms of their module categories.

3.4. The ABPS conjecture.

The big advantage of Theorem 3.13 is that it allows us to study G -representations via graded Hecke algebras. In particular all the results from Section 2 can now be applied to $\text{Rep}_{\text{fl}}(G)_{X_{\text{nr}}^+(L)W_{\mathfrak{s}}\sigma}^{\mathfrak{s}}$. Let $R(G)_{X_{\text{nr}}^+(L)W_{\mathfrak{s}}\sigma}^{\mathfrak{s}}$ be the Grothendieck group of $\text{Rep}_{\text{fl}}(G)_{X_{\text{nr}}^+(L)W_{\mathfrak{s}}\sigma}^{\mathfrak{s}}$. From (2.16) and Theorems 2.24, 2.26 and 3.13 we conclude:

Theorem 3.14. *Fix a 2-cocycle $\mathfrak{h}_{\sigma}$ as in Theorem 3.13.*

(a) *There exist canonical group isomorphisms*

$$R(G)_{X_{\text{nr}}^+(L)W_{\mathfrak{s}}\sigma}^{\mathfrak{s}} \longleftrightarrow R(\text{End}_G(\Pi_{\mathfrak{s}})^{\text{op}})_{X_{\text{nr}}^+(L)\sigma} \longleftrightarrow R(\mathbb{H}(\mathfrak{t}, W_{\mathfrak{s},\sigma}, k_{\sigma}, \mathfrak{h}_{\sigma}^{-1}))_{\mathfrak{t}_{\mathbb{R}}}.$$

These isomorphisms are compatible with parabolic induction and they preserve temperedness.

(b) *Part (a) can be refined canonically to a bijection*

$$\text{Irr}(G)_{X_{\text{nr}}^+(L)W_{\mathfrak{s}}\sigma}^{\mathfrak{s}} \longleftrightarrow \text{Irr}(\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W_{\mathfrak{s},\sigma}, \mathfrak{h}_{\sigma}^{-1}])_{\mathfrak{t}_{\mathbb{R}}}.$$

It preserves temperedness, but it need not respect parabolic induction or supercuspidal supports/central characters.

Next we want to combine instances of Theorem 3.14 for all $\sigma' = \chi_u \otimes \sigma \in \text{Irr}_{\text{temp}}(L)^{\mathfrak{s}L}$. Recall from Theorem 3.8 and (2.16) that

$$\text{End}_G(\Pi_{\mathfrak{s}})^{\text{op}} \otimes_{\mathcal{O}(X_{\text{nr}}(L)/W_{\mathfrak{s}}^e)} \mathbb{C}(X_{\text{nr}}(L)/W_{\mathfrak{s}}^e) \cong \mathbb{C}(X_{\text{nr}}(L)) \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}].$$

By [Sol9, Lemma 7.1] and Clifford theory, there are equivalences of categories

$$(3.17) \quad \begin{aligned} \mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W_{\mathfrak{s},\sigma'}, \mathfrak{h}_{\sigma'}] - \text{Mod}_{\text{fl}, \mathfrak{t}_{\mathbb{R}}} &\cong \mathcal{O}(X_{\text{nr}}(L) \rtimes \mathbb{C}[W_{\mathfrak{s},\sigma'}, \mathfrak{h}_{\sigma'}] - \text{Mod}_{\text{fl}, \chi_u X_{\text{nr}}^+(L)} \\ &\cong \mathcal{O}(X_{\text{nr}}(L) \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}] - \text{Mod}_{\text{fl}, W_{\mathfrak{s}}^e \chi_u X_{\text{nr}}^+(L)}). \end{aligned}$$

Theorem 3.15. [Sol11, Theorem 2.5]

Theorem 3.14.a and (3.17) induce a canonical group isomorphism

$$\zeta^{\vee} : R(G)^{\mathfrak{s}} \rightarrow R(\mathcal{O}(X_{\text{nr}}(L)) \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}])$$

with the following properties:

- (a) $\zeta^{\vee}$ and its inverse preserve temperedness. Moreover $\zeta^{\vee}$ sends tempered representations to tempered representations (so not to virtual representations).
- (b) If $\chi_u \in X_{\text{nr}}^u(L)$ and all irreducible subquotients of $\pi \in R(G)^{\mathfrak{s}}$ have cuspidal support in $\sigma \otimes W_{\mathfrak{s}}^e \chi_u X_{\text{nr}}^+(L)$, then all $\mathcal{O}(X_{\text{nr}}(L))$ -weights of $\zeta^{\vee}(\pi)$ lie in $W_{\mathfrak{s}}^e \chi_u X_{\text{nr}}^+(L)$.
- (c) In the setting of (b), suppose that $\pi \in \text{Irr}_{\text{temp}}(G)^{\mathfrak{s}}$. Then

$$\zeta^{\vee}(\pi) = \text{ind}_{\mathcal{O}(X_{\text{nr}}(L)) \rtimes \mathbb{C}[(W_{\mathfrak{s}}^e)_{\chi_u}, \mathfrak{h}_{\mathfrak{s}}^{-1}]}^{\mathcal{O}(X_{\text{nr}}(L)) \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}]} (\chi_u \otimes \pi_{\chi_u}),$$

where $\pi_{\chi_u} \in \text{Mod} - \mathbb{C}[(W_{\mathfrak{s}}^e)_{\chi_u}, \mathfrak{h}_{\mathfrak{s}}^{-1}]$ is obtained like in Theorem 2.24.a.

- (d) $\zeta^{\vee}$ commutes with parabolic induction and with unramified twists, in the sense that for a parabolic subgroup $Q = MU_Q \supset P$ and $\chi \in X_{\text{nr}}(M)$:

$$\zeta^{\vee}(I_Q^G(\tau \otimes \chi)) = \text{ind}_{\mathcal{O}(X_{\text{nr}}(L)) \rtimes \mathbb{C}[W_{\mathfrak{s}_M}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}]}^{\mathcal{O}(X_{\text{nr}}(L)) \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}]} (\zeta_M^{\vee}(\tau) \otimes \chi|_L).$$

We point out that in Theorem 3.15 the canonicity holds after $\mathfrak{h}_s$ has been fixed, the choice of the 2-cocycle $\mathfrak{h}_s$ is (in general) not canonical.

Recall from (2.9) that the irreducible representations of twisted crossed product algebras can be parametrized by twisted extended quotients. Combining Theorems 3.15, 3.14.b and Theorem 2.4.b leads to:

Theorem 3.16. [Sol9, Theorem 9.9]

(a) *There exists a family $\mathfrak{h}$ of 2-cocycles $\mathfrak{h}_\sigma^{-1}$ ($\sigma \in \text{Irr}_{\text{temp}}(L)^{s_L}$) and bijections*

$$\begin{aligned} \text{Irr}(G)^s_{X_{\text{nr}}^+(L)W_s\sigma} &\longleftrightarrow (\mathfrak{t}_{\mathbb{R}}//W_{s,\sigma})_{\mathfrak{h}_\sigma^{-1}} \\ \text{Irr}(G)^s &\longleftrightarrow (\text{Irr}(L)^{s_L} // W_s)_{\mathfrak{h}} \\ \text{Irr}_{\text{temp}}(G)^s &\longleftrightarrow (\text{Irr}_{\text{temp}}(L)^{s_L} // W_s)_{\mathfrak{h}} \end{aligned}$$

(b) *For $\sigma' \in X_{\text{nr}}^+(L)\sigma$ we write $\mathfrak{h}_{\sigma'} = \mathfrak{h}_\sigma$. The bijections from part (a) combine to a bijection from $\text{Irr}(G)$ to the set of G -orbits in*

$$\{(L, \sigma', \rho) : L \subset G \text{ Levi subgroup}, \sigma' \in \text{Irr}_{\text{cusp}}(L), \rho \in \text{Irr}(\mathbb{C}[\text{Stab}_G(L, \sigma')/L, \mathfrak{h}_{\sigma'}^{-1}])\}.$$

Theorem 3.16 proves a version of the ABPS conjecture, as formulated in [ABPS2, §2.3]. This was built upon several earlier versions of the conjecture, starting with [ABP, Conjecture 1].

Notice that the last space in Theorem 3.16.b projects naturally onto the variety of supercuspidal supports for G , by forgetting the ρ 's. The fibers of that map are finite, and parametrized by $\text{Irr}(\mathbb{C}[\text{Stab}_G(L, \sigma')/L, \mathfrak{h}_{\sigma'}^{-1}])$. However, like in Theorem 3.14.b, the bijections in Theorem 3.16 do not always translate the supercuspidal support map for G -representations into the natural projection $(L, \sigma', \rho) \mapsto (L, \sigma')$. In general it has to be corrected by some element of $X_{\text{nr}}^+(L)$, the absolute value of a $Z^\circ(L)$ -weight of some discrete series representation of L .

We may regard $\text{Stab}_G(L, \sigma')/L$ as $\pi_0(\text{Stab}_G(L, \sigma'))$, where π_0 is meant as algebraic varieties over F . We can modify the notion of a twisted extended quotient by replacing stabilizers by component groups of stabilizers. (That does not make a difference if we divide by finite groups like we did so far.) Let us denote such twisted extended quotients by $//_{\mathfrak{h}}$. Then Theorem 3.16.b can be reformulated as a bijection

$$(3.18) \quad \text{Irr}(G) \longleftrightarrow \left(\bigsqcup_{L \subset G \text{ Levi subgroup}} \text{Irr}_{\text{cusp}}(L) \right) //_{\mathfrak{h}} G.$$

This is a remarkably simple way to parametrize all irreducible smooth G -representations in terms of the supercuspidal representations of its Levi subgroups.

Part B. Noncommutative geometry of reductive p -adic groups

4. HOCHSCHILD HOMOLOGY FOR ALGEBRAS

Hochschild (co)homology, which appeared first in [Hoc], is a (co)homology theory for associative algebras over commutative rings. (Nowadays there are notions of Hochschild homology in categorical settings, but they are outside the scope of this paper.) We focus on algebras over fields, to make the definitions easier.

Let A be a unital algebra over a field k and let A^{op} be the opposite algebra. Let M be an A -bimodule, or equivalently a $A \otimes_k A^{op}$ -module.

Definition 4.1. Let $n \in \mathbb{Z}_{\geq 0}$. The n th homology of A with coefficients in M is

$$H_n(A, M) = \mathrm{Tor}_n^{A \otimes_k A^{op}}(A, M).$$

The n -th Hochschild homology of A is

$$HH_n(A) = H_n(A, A) = \mathrm{Tor}_n^{A \otimes_k A^{op}}(A, A).$$

One can compute $HH_n(A)$ as the homology of an explicit differential complex $(A^{\otimes m}, d_m)_{m=1}^{\infty}$ [Lod, §1.1]. This shows that HH_n is a functor from unital k -algebras to k -vector spaces. We abbreviate

$$HH_*(A) = \bigoplus_{n=0}^{\infty} HH_n(A).$$

There are two good ways to generalize Definition 4.1 to non-unital algebras.

- Assume that for every finite set $S \subset A$ there exists an idempotent $e_S \in A$ such that $e_S a = a = se_S$ for all $a \in S$. Such an algebra is called locally unital. For locally unital algebras we can still use Definition 4.1, without significant changes. This will be useful for Hecke algebras of reductive p -adic groups.
- Let A_+ be the vector space $A \oplus k$ with multiplication

$$(4.1) \quad (a_1, k_1)(a_2, k_2) = (a_1 a_2 + k_1 a_2 + k_2 a_1, k_1 k_2).$$

This algebra has unit $(0, 1)$, contains A as an ideal and is called the unitization of A . We note that $A \mapsto A_+$ is a functor. We define

$$HH_n(A) = \mathrm{coker}(HH_n(k) \rightarrow HH_n(A_+)),$$

where the map is induced by the inclusion $k \rightarrow A_+$.

4.1. Basic properties of HH_* .

In this paragraph A is a locally unital algebra and M is an A -bimodule.

1. Degree zero. One can compute the zeroth Hochschild homology groups directly from the definition:

$$H_0(A, M) = A \otimes_{A \otimes_k A^{op}} M = M/[M, A],$$

where $[M, A] \subset M$ is the k -span of $\{ma - am : m \in M, a \in A\}$. In particular

$$HH_0(A) = A/[A, A],$$

a vector space known as the cocenter of A .

Let $R_f(A)$ be the Grothendieck group of the category of finite dimensional A -representations. There is a natural pairing

$$\begin{array}{ccc} HH_0(A) \times R_f(A) & \rightarrow & k \\ (a, \pi) & \mapsto & \mathrm{tr} \pi(a) \end{array}.$$

This is the main use of Hochschild homology in representation theory.

2. HH_* as derived functor. Let $A \leftarrow P_*$ be a projective $A \otimes_k A^{op}$ -module resolution. By Definition 4.1

$$(4.2) \quad H_n(A, M) = H_n(P_* \otimes_{A \otimes_k A^{op}} M).$$

For example, if $A = k$, then we can take the resolution $k \leftarrow P_0 = k \leftarrow P_1 = 0$. With that we find

$$(4.3) \quad HH_n(k) = H_n(P_0 \leftarrow 0) = \begin{cases} k & n = 0 \\ 0 & n > 0 \end{cases}.$$

By 1. above, the Hochschild homology functors HH_n can be regarded as derived functors of the cocenter functor. Loosely speaking, that means that HH_* is the universal derived functor with the tracial property that it kills all commutators. Thus $HH_*(A)$ is a rather subtle invariant of A , which contains a lot of information. On the other hand, it is often quite difficult to determine the Hochschild homology of an algebra.

3. Additivity. For another locally unital algebra B , there is a natural isomorphism [Lod, Theorem 1.2.15]

$$HH_n(A \oplus B) \cong HH_n(A) \oplus HH_n(B).$$

4. Continuity. Consider a direct limit of locally unital algebras $\lim_i A_i$. The properties of Tor yield a natural isomorphism

$$HH_n(\lim_i A_i) \cong \lim_i HH_n(A_i).$$

5. Module structure over the centre. When A is unital, $H_*(A, M)$ and $HH_n(A)$ are naturally modules over the centre $Z(A)$. This follows for instance from the $Z(A)$ -module structure of $P_* \otimes_{A \otimes_k A^{op}} M$ in (4.2).

6. Morita invariance. The Hochschild homology of A depends only on the category of left A -modules. More precisely, suppose we have an equivalence of categories $\text{Mod}(A) \xrightarrow{\sim} \text{Mod}(B)$. Then A and B are Morita equivalent and there exist projective bimodules P and Q implementing the equivalence of categories. These also yield an equivalence of categories

$$\begin{array}{ccc} \text{Mod}(A \otimes_k A^{op}) & \xrightarrow{\sim} & \text{Mod}(B \otimes_k B^{op}) \\ M & \mapsto & P \otimes_A M \otimes_A Q \end{array},$$

which sends A to $P \otimes_A Q \cong B$. This induces an isomorphism

$$(4.4) \quad HH_n(A) \xrightarrow{\sim} HH_n(B),$$

see [Lod, Theorem 1.2.7]. If A is unital, then B is naturally a $Z(A)$ -algebra and (4.4) is $Z(A)$ -linear.

Example 4.2. Let Γ be a finite group and consider $A = \mathbb{C}[\Gamma]$. With additivity, Morita invariance and (4.3) we compute

$$\begin{aligned} HH_n(\mathbb{C}[\Gamma]) &\cong HH_n\left(\bigoplus_{\pi \in \text{Irr}(\Gamma)} \text{End}_{\mathbb{C}}(V_{\pi})\right) \cong \bigoplus_{\pi \in \text{Irr}(\Gamma)} HH_n(\text{End}_{\mathbb{C}}(V_{\pi})) \\ &\cong \bigoplus_{\pi \in \text{Irr}(\Gamma)} HH_n(\mathbb{C}) \cong \begin{cases} \bigoplus_{\pi \in \text{Irr}(\Gamma)} \mathbb{C} & n = 0 \\ 0 & n > 0 \end{cases}. \end{aligned}$$

More concisely: $HH_0(\mathbb{C}[\Gamma]) \cong Z(\mathbb{C}[\Gamma])$ and $HH_n(\mathbb{C}[\Gamma]) = 0$ for $n > 0$. The $Z(A)$ -module structure is just multiplication.

7. Localization. We suppose that A is unital. Let $S \subset Z(A)$ be subset which is closed under multiplication, and contains 1 but not 0. Recall that the localization of A with respect to S is

$$S^{-1}A = \{s^{-1}a : s \in S, a \in A\},$$

where $s^{-1}a$ means (s^{-1}, a) modulo the equivalence relation $(s_1^{-1}s_2, b) \sim (s_1^{-1}, s_2b)$ for $s_1, s_2 \in S$ and $b \in A$. The operations in $S^{-1}A$ are like in the quotient field of a domain. Then $S^{-1}A$ is a k -algebra and $S^{-1}M$ is an $S^{-1}A$ -bimodule. Since localization is an exact functor, there are natural isomorphisms [Lod, Proposition 1.1.17]

$$(4.5) \quad \begin{aligned} S^{-1}H_n(A, M) &\cong H_n(S^{-1}A, S^{-1}M), \\ S^{-1}HH_n(A) &\cong HH_n(S^{-1}A). \end{aligned}$$

This can be used to reduce the determination of $HH_*(A)$ to a local problem on the spectrum of $Z(A)$.

4.2. Hochschild homology of some commutative algebras.

We start with an example that shows very regular behaviour.

Example 4.3. Take $A = k[x]$, so that $A \otimes_k A^{op} \cong k[x, y]$. There is a projective bimodule resolution

$$k[x, y] \xleftarrow{\text{mod } (x-y)} P_0 = k[x, y] \xleftarrow{\text{mult } x-y} P_1 = k[x, y] \leftarrow P_2 = 0.$$

With (4.2) we find

$$HH_n(k[x]) = H_n(P_* \otimes_{k[x, y]} k[x]) = H_n(k[x] \xleftarrow{0} k[x]) = \begin{cases} k[x] & n = 0, 1 \\ 0 & n > 1 \end{cases}$$

However, in general the Hochschild homology of A may be nonzero in degrees far above the Krull dimension of A .

Example 4.4. Consider $B = k[x]/(x^2)$ and the B -bimodules $P_n = k[x, y]/(x^2, y^2) \cong B \otimes_k B^{op}$ for all $n \geq 0$. They form a projective resolution

$$B \xleftarrow{\text{mod } (x-y)} P_0 \xleftarrow{\text{mult } x-y} P_1 \xleftarrow{\text{mult } x+y} P_2 \xleftarrow{\text{mult } x-y} P_3 \xleftarrow{\text{mult } x+y} \dots$$

We compute, assuming that the characteristic of k is not 2:

$$HH_n(B) = H_n(B \xleftarrow{0} B \xleftarrow{2x} B \xleftarrow{0} B \xleftarrow{2x} \dots) = \begin{cases} k[x]/(x^2) & n = 0 \\ k[x]/(x) = k & n \text{ odd} \\ xk[x]/(x^2) = kx & n > 0 \text{ even} \end{cases}.$$

The difference between Examples (4.3) and (4.4) is that $A = k[x]$ is the coordinate ring of a non-singular algebraic variety, while $B = k[x]/(x^2)$ is not reduced. Example (4.3) is a simple case of the Hochschild–Kostant–Rosenberg theorem:

Theorem 4.5. [HKR]

Let V be a nonsingular affine variety over an algebraically closed field k . Let $\mathcal{O}(V)$ be the k -algebra of regular functions on V and let $\Omega^n(V)$ be the $\mathcal{O}(V)$ -module of differential n -forms on V .

There is a natural isomorphism of $\mathcal{O}(V)$ -modules $HH_n(\mathcal{O}(V)) \cong \Omega^n(V)$.

Theorem 4.5 provides an interpretation of $HH_*(A)$: it is a kind of differential forms on $\text{Irr}(A)$. For commutative algebras that can be made quite precise [Lod, §1.3], while for non-commutative algebras it means that $HH_*(A)$ can be regarded as some "non-commutative differential forms". However, this interpretation only applies well to nice algebras, for Example 4.3 shows that $HH_*(A)$ is very sensitive to changes like non-reducedness.

4.3. Hochschild homology of finite type algebras.

Besides commutative algebras, there are some classes of algebras which are close to commutative and whose Hochschild homology can be computed reasonably well.

Let V be a complex affine variety. Recall that an $\mathcal{O}(V)$ -algebra is a $\mathbb{C}$ -algebra A with an algebra homomorphism from $\mathcal{O}(V)$ to the centre of the multiplier algebra of A . In other words, $\mathcal{O}(V)$ acts on A and

$$f(a_1a_2) = f(a_1)a_2 = a_1f(a_2) \quad \text{for all } f \in \mathcal{O}(V), a_i \in A.$$

For any unital $\mathcal{O}(V)$ -algebra A , the action of $\mathcal{O}(V)$ comes from an algebra homomorphism $\mathcal{O}(V) \rightarrow Z(A)$. If B is a nonunital $\mathcal{O}(V)$ -algebra, then $B \oplus \mathcal{O}(V)$ has the structure of a unital $\mathcal{O}(V)$ -algebra, like B_+ in (4.1).

Definition 4.6. We say that an $\mathcal{O}(V)$ -algebra A has finite type if A has finite rank as $\mathcal{O}(V)$ -module. An arbitrary $\mathbb{C}$ -algebra has finite type if it is a finite type $\mathcal{O}(V)$ -algebra for some complex affine variety V .

Example 4.7. $M_n(\mathcal{O}(V))$ is a finite type $\mathcal{O}(V)$ -algebra.

Let $K \in \mathbf{CO}(G)$ and let $\mathfrak{s} = [L, \sigma]_G$. By Theorem 1.42, the unital algebra $\mathcal{H}(G, K)^\mathfrak{s}$ from (1.35) has finite rank over

$$\mathcal{O}(X_{\text{nr}}(L)/W_\mathfrak{s}^e) \cong \mathcal{O}(\text{Irr}(L)^{\mathfrak{s}L}) \cong Z(\text{Rep}(G)^\mathfrak{s}).$$

The algebra $\mathcal{H}(G, K)$ is a finite type $\mathcal{O}(V)$ -algebra for $V = \sqcup_\mathfrak{s} \text{Irr}(L)^{\mathfrak{s}L}/W_\mathfrak{s}$, where $\mathfrak{s}$ runs over the finite subset of $\mathfrak{B}(G)$ such that $\mathcal{H}(G, K)^\mathfrak{s}$ is nonzero.

Standard algebraic techniques like those described in Paragraph 4.1 can be used to study Hochschild homology of finite type algebras, and to some extent reduce it to the case of commutative algebras [KNS, §2–3]. If A is a unital finite type algebra and $n \in \mathbb{Z}_{\geq 0}$, then $HH_n(A)$ is an $\mathcal{O}(V)$ -module of finite rank.

Consider a non-singular affine $\mathbb{C}$ -variety V , and let Γ be a finite group acting on V by automorphisms. Brylinski [Bry] and Nistor [Nis] generalized the Hochschild–Kostant–Rosenberg theorem to the crossed product $\mathcal{O}(V) \rtimes \Gamma = \mathcal{O}(V) \rtimes \mathbb{C}[\Gamma]$ from Paragraph 2.1.

More generally we may involve a 2-cocycle $\mathfrak{h} : \Gamma \times \Gamma \rightarrow \mathbb{C}^\times$. By Lemma 2.3, the twisted crossed product $\mathcal{O}(V) \rtimes \mathbb{C}[\Gamma, \mathfrak{h}]$, is a finite type $\mathcal{O}(V/\Gamma)$ -algebra. We recall the map $\mathfrak{h}^\gamma : \Gamma \rightarrow \mathbb{C}^\times$ from (2.4).

Theorem 4.8. [Sol10, Theorem 1.2 and (1.17)]

There exists an isomorphism of $\mathcal{O}(V)^\Gamma$ -modules

$$HH_n(\mathcal{O}(V) \rtimes \mathbb{C}[\Gamma, \mathfrak{h}]) \cong \bigoplus_{\gamma \in \Gamma/\text{conjugacy}} (\Omega^n(V^\gamma) \otimes \mathfrak{h}^\gamma)^{Z_\Gamma(\gamma)} \cong \left(\bigoplus_{\gamma \in \Gamma} \Omega^n(V^\gamma) \otimes \mathfrak{h}^\gamma \right)^\Gamma.$$

This generalizes (and relies on) the aforementioned result of Brylinski–Nistor, which can be recovered by setting $\mathfrak{h} = 1$. In view of Theorems 2.4, 4.5 and 4.8, we may regard $\left(\bigoplus_{\gamma \in \Gamma} \Omega^n(V^\gamma) \otimes \mathfrak{h}^\gamma \right)^\Gamma$ as a kind of differential forms on the twisted extended quotient $(V//\Gamma)_\mathfrak{h}$.

4.4. Hochschild homology of graded Hecke algebras.

Let $\mathbb{H}(\mathfrak{t}, W, k)$ be a graded Hecke algebra, as in (2.14), and let $\mathbb{H}(\mathfrak{t}, W\Gamma, k, \mathfrak{h}) = \mathbb{H}(\mathfrak{t}, W, k) \rtimes \mathbb{C}[\Gamma, \mathfrak{h}]$ be a twisted graded Hecke algebra, as in Definition 2.9. Recall from Theorem 2.10 that the algebras $\mathbb{H}(\mathfrak{t}, W\Gamma, k', \mathfrak{h})$ with varying parameters $k' : \Phi \rightarrow \mathbb{C}$ are all very similar, and that for $k' = 0$ we recover the simpler algebra $\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W\Gamma, \mathfrak{h}]$. That can also be seen in Hochschild homology:

Theorem 4.9. [Sol3, Theorem 3.4] and [Sol10, (2.5)–(2.6)]

There exist isomorphisms of vector spaces

$$\begin{aligned} HH_*(\mathbb{H}(\mathfrak{t}, W, k)) &\cong HH_*(\mathbb{H}(\mathfrak{t}, W, 0)) &\cong \left(\bigoplus_{w \in W} \Omega^*(\mathfrak{t}^w) \right)^W \\ HH_*(\mathbb{H}(\mathfrak{t}, W\Gamma, k, \mathfrak{h})) &\cong HH_*(\mathbb{H}(\mathfrak{t}, W\Gamma, 0, \mathfrak{h})) &\cong \left(\bigoplus_{w \in W\Gamma} \Omega^*(\mathfrak{t}^w) \otimes \mathfrak{h}^w \right)^{W\Gamma} \end{aligned}$$

Theorem 4.9 comes from the filtration of $\mathbb{H}(\mathfrak{t}, W\Gamma, k, \mathfrak{h})$ by degrees, and from an associated spectral sequence that converges to $HH_*(\mathbb{H}(\mathfrak{t}, W\Gamma, k, \mathfrak{h}))$. We point out that the isomorphisms in Theorem 4.9 are usually not linear over $\mathcal{O}(\mathfrak{t})^W$ or $\mathcal{O}(\mathfrak{t})^{W\Gamma}$.

From now on we assume that k is real-valued, like in Paragraph 2.4. Then we have the natural bijection

$$\zeta_0 : R(\mathbb{H}(\mathfrak{t}, W\Gamma, k, \mathfrak{h})) \rightarrow R(\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W\Gamma, \mathfrak{h}])$$

from Theorems 2.24 and 2.26.

Theorem 4.10. [Sol10, Corollary 2.10 and Proposition 2.11]

ζ_0 induces a natural $\mathbb{C}$ -linear bijection

$$HH_*(\zeta_0) : HH_*(\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W\Gamma, \mathfrak{h}]) \rightarrow HH_*(\mathbb{H}(\mathfrak{t}, W\Gamma, k, \mathfrak{h})).$$

The map $HH_*(\zeta_0)$ can be characterized as follows. For any tempered $\sigma \in \text{Irr}(\mathbb{H}_P)$ there is an algebraic family of $\mathbb{H}$ -representations

$$\mathfrak{F}_{P,\sigma} = \{ \text{ind}_{\mathbb{H}_P}^{\mathbb{H}}(\sigma \otimes \mathbb{C}_\lambda) : \lambda \in P^{\vee\perp} \subset \mathfrak{t} \}.$$

This family gives rise to an algebra homomorphism

$$\mathcal{F}_{P,\sigma} : \mathbb{H} = \mathbb{H}(\mathfrak{t}, W\Gamma, k, \mathfrak{h}) \rightarrow \mathcal{O}(P^{\vee\perp}) \otimes \text{End}_{\mathbb{C}}(\text{ind}_{\mathbb{H}_P}^{\mathbb{H}}(\sigma \otimes \mathbb{C}_0)).$$

By Morita invariance and Theorem 4.5 we have

$$HH_*(\mathcal{O}(P^{\vee\perp}) \otimes \text{End}_{\mathbb{C}}(\text{ind}_{\mathbb{H}_P}^{\mathbb{H}}(\sigma \otimes \mathbb{C}_0))) \cong HH_*(\mathcal{O}(P^{\vee\perp})) \cong \Omega^*(P^{\vee\perp}).$$

The same can be done with $\mathcal{F}_{P,\zeta_0(\sigma)}$, and in Theorem 4.10 there is an equality

$$(4.6) \quad HH_*(\mathcal{F}_{P,\sigma}) \circ HH_*(\zeta_0) = HH_*(\mathcal{F}_{P,\zeta_0(\sigma)}) : HH_*(\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W\Gamma, \mathfrak{h}]) \rightarrow \Omega^*(P^{\vee\perp}).$$

Conversely, imposing (4.6) for all families of the form $\mathfrak{F}_{P,\sigma}$ determines $HH_*(\zeta_0)$.

Example 4.11. Consider $\mathfrak{t} = \mathbb{C}$ and $R = \{\pm 1\}$. By Theorem 4.9

$$HH_*(\mathbb{H}(\mathfrak{t}, W, k)) \cong HH_*(\mathcal{O}(\mathbb{C}) \rtimes S_2) \cong \Omega^*(\mathbb{C})^{S_2} \oplus \Omega^*(\{0\}) = \begin{cases} \mathbb{C}[z^2] \oplus \mathbb{C} & n = 0 \\ \mathbb{C}[z^2]z \, dz & n = 1 \\ 0 & n > 1 \end{cases}.$$

With Theorem 4.10 and (4.6) we can make this more explicit. The first family of representations to consider comes from $P = \emptyset$ and the module $\mathbb{C}_0$ of $\mathbb{H}_\emptyset = \mathcal{O}(\mathfrak{t})$. Then $\zeta_0(\mathbb{C}_0) = \mathbb{C}_0$ and

$$\mathcal{F}_{\emptyset, \mathbb{C}_0} : HH_*(\mathcal{O}(\mathbb{C}) \rtimes S_2) \rightarrow \Omega^*(\mathbb{C})$$

can be identified with the projection $\Omega^*(\mathbb{C})^{S_2} \oplus \Omega^*(\{0\}) \rightarrow \Omega^*(\mathbb{C})^{S_2}$. The Steinberg representation of $\mathbb{H}(\mathfrak{t}, W, k)$ forms another family of representations. It satisfies $\zeta_0(\text{St}) = \text{sign}_{S_2} \otimes \mathbb{C}_0$ and $HH_*(\zeta_0(\text{St}))$ identifies with the projection $\Omega^*(\mathbb{C})^{S_2} \oplus \Omega^*(\{0\}) \rightarrow \Omega^*(\{0\})$. Hence

$$(4.7) \quad HH_*(\mathcal{F}_{\emptyset, \mathbb{C}_0}) \oplus HH_*(\zeta_0(\text{St})) : HH_n(\mathcal{O}(\mathbb{C}) \rtimes S_2) \rightarrow \Omega^*(\mathbb{C})^{S_2} \oplus \Omega^*(\{0\})$$

is a $\mathbb{C}$ -linear bijection. With (4.6) it follows that

$$(4.8) \quad HH_*(\mathcal{F}_{\emptyset, \mathbb{C}_0}) \oplus HH_*(\text{St}) : HH_n(\mathbb{H}(\mathfrak{t}, W, k)) \rightarrow \Omega^*(\mathbb{C})^{S_2} \oplus \Omega^*(\{0\})$$

is also a $\mathbb{C}$ -linear bijection. Via (4.7) and (4.8), $HH_n(\zeta_0)$ corresponds to the identity on $\Omega^*(\mathbb{C})^{S_2} \oplus \Omega^*(\{0\})$. The summand $\mathbb{C} \cong \Omega^*(\{0\})$ in $HH_*(\mathbb{H}(\mathfrak{t}, W, k))$ has $\mathcal{O}(\mathbb{C})^{S_2}$ -weight $\pm k$ because it comes from St , while $\Omega^*(\{0\})$ has $\mathcal{O}(\mathbb{C})^{S_2}$ -weight 0 in $HH_*(\mathcal{O}(\mathbb{C}) \rtimes S_2)$. Therefore $HH_n(\zeta_0)$ is not $\mathcal{O}(\mathbb{C})^{S_2}$ -linear.

4.5. $HH_*(\mathcal{H}(G))$: cuspidal Bernstein blocks.

Let G be a reductive p -adic group and let $\mathcal{H}(G)$ be its Hecke algebra, as in Section 1. Our goal is to compute $HH_*(\mathcal{H}(G))$ in terms of the representation theory of G . By Proposition 1.3.b $\mathcal{H}(G)$ has local units, so in relation to Hochschild homology we may treat it as a unital algebra. Recall that by the Bernstein decomposition (Theorem 1.41) $\mathcal{H}(G) = \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} \mathcal{H}(G)^\mathfrak{s}$. By the additivity and the continuity of HH_* :

$$(4.9) \quad \begin{aligned} HH_*(\mathcal{H}(G)) &= HH_*\left(\lim_{S \subset \mathfrak{B}(G), S \text{ finite}} \bigoplus_{\mathfrak{s} \in S} \mathcal{H}(G)^\mathfrak{s}\right) \cong \lim_{S \subset \mathfrak{B}(G), S \text{ finite}} HH_*\left(\bigoplus_{\mathfrak{s} \in S} \mathcal{H}(G)^\mathfrak{s}\right) \\ &\cong \lim_{S \subset \mathfrak{B}(G), S \text{ finite}} \bigoplus_{\mathfrak{s} \in S} HH_*(\mathcal{H}(G)^\mathfrak{s}) = \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} HH_*(\mathcal{H}(G)^\mathfrak{s}). \end{aligned}$$

This means that to compute $HH_n(\mathcal{H}(G))$, it suffices to classify $\mathfrak{B}(G)$ and to determine $HH_*(\mathcal{H}(G)^\mathfrak{s})$ for each Bernstein block $\text{Rep}(G)^\mathfrak{s}$ of $\text{Rep}(G)$. We will treat $\mathfrak{B}(G)$ as a black box and we will focus on $HH_*(\mathcal{H}(G)^\mathfrak{s})$ for one arbitrary $\mathfrak{s} \in \mathfrak{B}(G)$. We write $\mathfrak{s} = [L, \sigma]_G$ and let $\Pi_\mathfrak{s} = I_P^G(\text{ind}_{L_1}^L(\text{Res}_{L_1}^L \sigma))$ be the projective generator of $\text{Rep}(G)^\mathfrak{s} = \text{Mod}(\mathcal{H}(G)^\mathfrak{s})$ from Paragraph 3.2. By Morita invariance, the equivalence of categories (3.7) induces a natural isomorphism

$$(4.10) \quad HH_*(\mathcal{H}(G)^\mathfrak{s}) \cong HH_*(\text{End}_G(\Pi_\mathfrak{s})^{op}).$$

In the cuspidal cases this quickly leads to a nice description.

Proposition 4.12. *Suppose that $\mathfrak{s}$ is cuspidal, that is, $L = G$. There is a natural isomorphism of $\mathcal{O}(\text{Irr}(G)^\mathfrak{s})$ -modules*

$$HH_n(\mathcal{H}(G)^\mathfrak{s}) \cong \Omega^n(\text{Irr}(G)^\mathfrak{s}).$$

Proof. From Theorem 3.3 and (2.16) we know that

$$\mathrm{End}_G(\Pi_{\mathfrak{s}})^{op} \cong \mathcal{O}(X_{\mathrm{nr}}(G)) \rtimes \mathbb{C}[X_{\mathrm{nr}}(G, \sigma), \mathfrak{h}_{\mathfrak{s}}^{-1}].$$

Then Theorem 4.8 says that

$$(4.11) \quad HH_n(\mathrm{End}_G(\Pi_{\mathfrak{s}})^{op}) \cong \left(\bigoplus_{\chi \in X_{\mathrm{nr}}(G, \sigma)} \Omega^n(X_{\mathrm{nr}}(G)^{\chi}) \otimes (\mathfrak{h}_{\mathfrak{s}}^{-1})^{\chi} \right)^{X_{\mathrm{nr}}(G, \sigma)}.$$

Here $X_{\mathrm{nr}}(G, \sigma)$ acts on $X_{\mathrm{nr}}(G)$ by multiplication, so $X_{\mathrm{nr}}(G)^{\chi}$ is empty unless $\chi = 1$. As $(\mathfrak{h}_{\mathfrak{s}}^{-1})^1 = 1$, the right hand side of (4.11) reduces to $\Omega^n(X_{\mathrm{nr}}(G))^{X_{\mathrm{nr}}(G, \sigma)}$. Then (4.11) (from right to left) can be written as

$$(4.12) \quad \Omega^n(X_{\mathrm{nr}}(G))^{X_{\mathrm{nr}}(G, \sigma)} \rightarrow HH_n(X_{\mathrm{nr}}(G)) \rightarrow HH_n(\mathrm{End}_G(\Pi_{\mathfrak{s}})^{op}),$$

where the first arrow comes from Theorem 4.5 and is natural, while the second arrow is induced by the inclusion $\mathcal{O}(X_{\mathrm{nr}}(G)) \rightarrow \mathrm{End}_G(\Pi_{\mathfrak{s}})^{op}$. By (3.2) that inclusion is natural once $\sigma \in \mathrm{Irr}(G)^{\mathfrak{s}}$ has been chosen.

The action of $X_{\mathrm{nr}}(G, \sigma)$ on $X_{\mathrm{nr}}(G)$ is free, and $X_{\mathrm{nr}}(G)/X_{\mathrm{nr}}(G, \sigma) \cong \mathrm{Irr}(G)^{\mathfrak{s}}$ by (3.3), so

$$(4.13) \quad \Omega^n(X_{\mathrm{nr}}(G))^{X_{\mathrm{nr}}(G, \sigma)} = \Omega^n(X_{\mathrm{nr}}(G)/X_{\mathrm{nr}}(G, \sigma)) \cong \Omega^n(\mathrm{Irr}(G)^{\mathfrak{s}}).$$

The composition of (4.10), (4.11) and (4.13) is the required isomorphism

$$(4.14) \quad HH_n(\mathcal{H}(G)^{\mathfrak{s}}) \xrightarrow{\sim} \Omega^n(\mathrm{Irr}(G)^{\mathfrak{s}}).$$

The isomorphism (4.13) depends only on the choice of σ in $\mathrm{Irr}(G)^{\mathfrak{s}}$. That cancels out with the same choice in (4.12), so the composition of (4.11) and (4.13) is natural. As (4.10) is also natural, so is (4.14).

Recall from (3.1) that

$$(4.15) \quad Z(\mathrm{End}_G(\Pi_{\mathfrak{s}})^{op}) = Z(\mathrm{End}_G(\Pi_{\mathfrak{s}})) \cong \mathcal{O}(X_{\mathrm{nr}}(G))^{X_{\mathrm{nr}}(G, \sigma)} \cong \mathcal{O}(\mathrm{Irr}(G)^{\mathfrak{s}}).$$

We saw in (4.4) that (4.10) intertwines the actions of (4.15), and by Theorem 4.8 the same holds for (4.11). Further (4.13) is by definition $\mathcal{O}(\mathrm{Irr}(G)^{\mathfrak{s}})$ -linear, and we conclude that (4.14) is $\mathcal{O}(\mathrm{Irr}(G)^{\mathfrak{s}})$ -linear as well. $\square$

4.6. $HH_*(\mathcal{H}(G))$: non-cuspidal Bernstein blocks.

In this paragraph $\mathfrak{s} = [L, \sigma]_G$ with $L \neq G$. We want to determine $HH_*(\mathcal{H}(G)^{\mathfrak{s}})$, where $\mathrm{Mod}(\mathcal{H}(G)^{\mathfrak{s}})$ is a non-cuspidal Bernstein block of $\mathrm{Rep}(G)$. It is isomorphic to $HH_*(\mathrm{End}_G(\Pi_{\mathfrak{s}})^{op})$, but we do not understand $\mathrm{End}_G(\Pi_{\mathfrak{s}})$ well enough to handle this directly. Instead, we will approach it via localization on the Bernstein centre, as in Paragraph 3.3. Recall from Theorem 3.7 that the Bernstein centre for $\mathcal{H}(G)^{\mathfrak{s}}$ is

$$Z(\mathrm{Rep}(G)^{\mathfrak{s}}) \cong \mathcal{O}(\mathrm{Irr}(L)^{\mathfrak{s}_L})^{W_{\mathfrak{s}}} \cong \mathcal{O}(X_{\mathrm{nr}}(L)/W_{\mathfrak{s}}^e).$$

We may and will assume that $\sigma \in \mathrm{Irr}_{\mathrm{cusp}}(L)^{\mathfrak{s}_L}$ is tempered. It is known from [Sol9, §7] that the “analytic localization” of $\mathrm{End}_G(\Pi_{\mathfrak{s}})$ at $X_{\mathrm{nr}}^+(L)W_{\mathfrak{s}}\sigma$ is isomorphic to the “analytic localization” at $\mathfrak{t}_{\mathbb{R}}$ of a twisted graded Hecke algebra, denoted $\mathbb{H}(\mathfrak{t}, W_{\mathfrak{s}, \sigma}, k_{\sigma}, \mathfrak{h}_{\sigma})$ in Theorem 3.13. However, since $X_{\mathrm{nr}}^+(L)W_{\mathfrak{s}}\sigma$ is not Zariski-closed in $\mathrm{Irr}(L)^{\mathfrak{s}_L}/W_{\mathfrak{s}}$, we cannot localize there by means of subsets of $Z(\mathrm{Rep}(G)^{\mathfrak{s}})$.

The best we can achieve in that way is: for any $\chi \in X_{\mathrm{nr}}^+(L)$, take

$$S_{\chi} = \{s \in Z(\mathrm{Rep}(G)^{\mathfrak{s}}) : s(\sigma \otimes \chi) \neq 0\}$$

and consider $S_\chi^{-1}HH_*(\mathcal{H}(G)^\mathfrak{s})$. This can be related to the localization of $\mathbb{H}(\mathfrak{t}, W_{\mathfrak{s}, \sigma}, k_\sigma, \mathfrak{d}_\sigma)$ with respect to a maximal ideal $I_{\log \chi}$ of its centre. By (4.5) and Theorem 3.13, that relates

$$S_\chi^{-1}HH_*(\mathcal{H}(G)^\mathfrak{s}) \cong HH_*(S_\chi^{-1}\text{End}_G(\Pi_\mathfrak{s})^{op})$$

to a localization of $HH_*(\mathbb{H}(\mathfrak{t}, W_{\mathfrak{s}, \sigma}, k_\sigma, \mathfrak{d}_\sigma))$ at $I_{\log \chi}$. That provides a description of the localization of $HH_*(\mathcal{H}(G)^\mathfrak{s})$ at one arbitrary point of $\text{Irr}(L)^{\mathfrak{s}L}/W_\mathfrak{s}$.

Thus $HH_*(\mathcal{H}(G)^\mathfrak{s})$ gives a sheaf over $\text{Irr}(Z(\text{Rep}(G)^\mathfrak{s}))$, whose stalks we understand. In [Sol11], this entire sheaf is reconstructed by a convoluted glueing procedure. The main tools for that are algebraic families of G -representations like in (4.6).

Let $M \subset G$ be a Levi subgroup containing L , and let $(\tau, V_\tau) \in \text{Irr}_{\text{temp}}(M)^{\mathfrak{s}M}$. This gives a family of representations

$$\mathfrak{F}_{M, \tau} = \{I_{PM}^G(\tau \otimes \chi_M) : \chi_M \in X_{\text{nr}}(M)\}$$

and a homomorphism of $\mathcal{O}(\text{Irr}(L)^\mathfrak{s})^{W_\mathfrak{s}}$ -algebras

$$(4.16) \quad \mathcal{F}_{M, \tau} : \mathcal{H}(G)^\mathfrak{s} \rightarrow \mathcal{O}(X_{\text{nr}}(M)) \otimes \text{End}_{\mathbb{C}}^\infty(I_{PM}^G(V_\tau)).$$

By the Morita equivalence of $\mathbb{C}$ with $\text{End}_{\mathbb{C}}^\infty(I_{PM}^G(V_\tau))$, the smooth part of the $G \times G$ -representation $\text{End}_{\mathbb{C}}(I_{PM}^G(V_\tau))$, $\mathcal{F}_{M, \tau}$ induces a $\mathcal{O}(\text{Irr}(L)^{\mathfrak{s}L})^{W_\mathfrak{s}}$ -linear map

$$(4.17) \quad HH_*(\mathcal{F}_{M, \tau}) : HH_*(\mathcal{H}(G)^\mathfrak{s}) \rightarrow HH_*(\mathcal{O}(X_{\text{nr}}(M))) \cong \Omega^*(X_{\text{nr}}(M)).$$

Applying Theorem 3.15 to $\mathfrak{F}_{M, \tau}$ yields a family of $\mathcal{O}(X_{\text{nr}}(L)) \rtimes \mathbb{C}[W_\mathfrak{s}^e, \mathfrak{d}_\mathfrak{s}^{-1}]$ -modules

$$\mathfrak{F}_{M, \zeta_M^\vee(\tau)} = \left\{ \text{ind}_{\mathcal{O}(X_{\text{nr}}(L)) \rtimes \mathbb{C}[W_{\mathfrak{s}M}^e, \mathfrak{d}_{\mathfrak{s}M}^{-1}]}^{\mathcal{O}(X_{\text{nr}}(L)) \rtimes \mathbb{C}[W_\mathfrak{s}^e, \mathfrak{d}_\mathfrak{s}^{-1}]} (\zeta_M^\vee(\tau) \otimes \chi_M) : \chi_M \in X_{\text{nr}}(M) \right\}.$$

As in (4.17), this induces a $\mathcal{O}(X_{\text{nr}}(L))^{W_\mathfrak{s}^e}$ -linear map

$$HH_*(\mathcal{F}_{M, \zeta_M^\vee(\tau)}) : HH_*(\mathcal{O}(X_{\text{nr}}(L)) \rtimes \mathbb{C}[W_\mathfrak{s}^e, \mathfrak{d}_\mathfrak{s}^{-1}]) \rightarrow HH_*(\mathcal{O}(X_{\text{nr}}(M))).$$

Theorem 4.13. [Sol11, Theorem 2.14]

There exists a unique $\mathbb{C}$ -linear bijection

$$HH_n(\zeta^\vee) : HH_*(\mathcal{O}(X_{\text{nr}}(L)) \rtimes \mathbb{C}[W_\mathfrak{s}^e, \mathfrak{d}_\mathfrak{s}^{-1}]) \rightarrow HH_n(\mathcal{H}(G)^\mathfrak{s})$$

such that, for all families $\mathcal{F}_{M, \tau}$ as above,

$$HH_n(\mathcal{F}_{M, \tau}) \circ HH_n(\zeta^\vee) = HH_n(\mathcal{F}_{M, \zeta_M^\vee(\tau)}).$$

In general $HH_n(\zeta^\vee)$ does not respect the actions of $\mathcal{O}(X_{\text{nr}}(L))^{W_\mathfrak{s}^e} \cong \mathcal{O}(\text{Irr}(L)^{\mathfrak{s}L})^{W_\mathfrak{s}}$, but it can be described precisely how it deviates from $\mathcal{O}(\text{Irr}(L)^{\mathfrak{s}L})^{W_\mathfrak{s}}$ -linearity [Sol11, Theorem B]. On the part of $HH_n(\mathcal{H}(G)^\mathfrak{s})$ that comes from $\text{Irr}_{\text{temp}}(G)_{[M, \delta]}$ with $[M, \delta]_G \in \Delta(G, \mathfrak{s})$, the deviation is given by an element $r_\delta \in X_{\text{nr}}^+(M)$ such that δ is a constituent of $I_P^G(\sigma' \otimes r_\delta)$ for a unitary $\sigma' \in X_{\text{nr}}(L)\sigma$.

From Theorems 4.8 and 4.13 we obtain a $\mathbb{C}$ -linear bijection

$$(4.18) \quad HH_n(\mathcal{H}(G)^\mathfrak{s}) \rightarrow \left(\bigoplus_{w \in W_\mathfrak{s}^e} \Omega^n(X_{\text{nr}}(L)^w \otimes (\mathfrak{d}_\mathfrak{s}^{-1})^w) \right)^{W_\mathfrak{s}^e}.$$

This map can also be constructed via families of (virtual) G -representations, see [Sol11, Theorem 2.13.b].

Example 4.14. We consider $G = SL_2(F)$ and $\mathfrak{s} = [T, \text{triv}]_G$. Then $W_{\mathfrak{s}}^e = S_2$, $\mathfrak{h}_{\mathfrak{s}} = 1$ and Theorem 4.13 says that

$$HH_n(\mathcal{H}(G)^{\mathfrak{s}}) \cong HH_*(\mathcal{O}(X_{\text{nr}}(T)) \rtimes S_2).$$

We use three families of representations to construct this isomorphism:

$$\mathfrak{F}_{T,1} = \{I_B^G(\chi) : \chi \in X_{\text{nr}}(T)\},$$

$\mathfrak{F}_{G,\text{St}} = \{\text{St}\}$ and $\mathfrak{F}_{G,2} = \{\pi_+ - \pi_-\}$, where $I_B^G(\chi_-) = \pi_+ \oplus \pi_-$ for the unique $\chi_- \in X_{\text{nr}}(T)$ of order two. These induce $\mathcal{O}(X_{\text{nr}}(T))^{S_2}$ -linear maps

$$HH_n(\mathcal{F}_{T,1}) : HH_n(\mathcal{H}(G)^{\mathfrak{s}}) \rightarrow HH_n(\mathcal{O}(X_{\text{nr}}(T)) \otimes \text{End}_{\mathbb{C}}^{\infty}(I_B^G(\text{triv}))) \cong \Omega^n(X_{\text{nr}}(T)),$$

$$HH_n(\mathcal{F}_{G,\text{St}}) : HH_n(\mathcal{H}(G)^{\mathfrak{s}}) \rightarrow HH_n(\text{End}_{\mathbb{C}}^{\infty}(V_{\text{St}})) \cong \Omega^n(\{\text{St}\}),$$

$$HH_n(\mathcal{F}_{G,2}) = HH_n(\pi_+) - HH_n(\pi_-) : HH_n(\mathcal{H}(G)^{\mathfrak{s}}) \rightarrow HH_n(\mathcal{O}(\{\chi_-\})).$$

The sum of the three maps is an isomorphism of $\mathcal{O}(X_{\text{nr}}(T))^{S_2}$ -modules

$$(4.19) \quad HH_n(\mathcal{H}(G)^{\mathfrak{s}}) \rightarrow \Omega^n(X_{\text{nr}}(T))^{S_2} \oplus \Omega^n(\{\text{St}\}) \oplus \Omega^n(\{\chi_-\}).$$

The right hand side of (4.19) is isomorphic to

$$(4.20) \quad HH_n(\mathcal{O}(T) \rtimes S_2) \cong (\Omega^n(X_{\text{nr}}(T)) \oplus \Omega^n(\{\text{triv}\}) \oplus \Omega^n(\{\chi_-\}))^{S_2},$$

and the canonical map from (4.19) to (4.20) is almost linear over $\mathcal{O}(X_{\text{nr}}(T))^{S_2} \cong Z(\text{Rep}(G)^{\mathfrak{s}})$. The only deviation from $\mathcal{O}(X_{\text{nr}}(T))^{S_2}$ -linearity is that the $Z(\text{Rep}(G)^{\mathfrak{s}})$ -character of St does not agree with $\text{triv} \in X_{\text{nr}}(T)/S_2$.

5. HOCHSCHILD HOMOLOGY FOR TOPOLOGICAL ALGEBRAS

We would like to compute the Hochschild homology of topological algebras appearing in the representation theory of p -adic groups, like $C^{\infty}(X) \rtimes \Gamma$ or $\mathcal{S}(G)$. The definition of Hochschild homology in Section 4 can be applied to any algebra, so in particular to $C^{\infty}(X)$ or $C(X)$ for a smooth manifold X . However, that does not give interesting results, because the functor HH_* from Definition 4.1 does not take the topology of an algebra into account. The best way to improve that is by using a topological tensor product.

As is common in noncommutative geometry, we will work mostly with Fréchet algebras. For later use we define precisely which algebras we mean by that.

Definition 5.1. A Fréchet algebra is a $\mathbb{C}$ -algebra A such that:

- A is a Fréchet space,
- the topology on A can be defined by a countable family of seminorms p which are submultiplicative: $p(ab) \leq p(a)p(b)$ for all $a, b \in A$.

The submultiplicativity implies that for any Fréchet algebra A the multiplication map $A \times A \rightarrow A$ is continuous. The class of Fréchet algebras contains all Banach algebras and spaces of smooth functions $C^{\infty}(X)$.

For two Fréchet spaces V and W , we denote their completed projective tensor product by $V \hat{\otimes} W$. This is a completion of $V \otimes W$ and a Fréchet space, with the following universal property: for any Fréchet space Z there is a natural bijection between

- the set of continuous $\mathbb{C}$ -linear maps from $V \hat{\otimes} W$ to Z ,
- the set of continuous $\mathbb{C}$ -bilinear maps from $V \times W$ to Z .

Example 5.2. For two smooth manifolds X and Y , there is an isomorphism of Fréchet spaces

$$C^\infty(X) \hat{\otimes} C^\infty(Y) \cong C^\infty(X \times Y).$$

For a Fréchet algebra A , there is also a notion of the completed projective tensor product of A -modules. Let V be a right Fréchet A -module and let W be a left Fréchet A -module. Then $V \hat{\otimes}_A W$ is a completion of $V \otimes_A W$ which, like $V \hat{\otimes} W$, has a universal property with respect to A -balanced $\mathbb{C}$ -bilinear continuous maps from $V \times W$ to a Fréchet space Z . Concretely, this works out to

$$V \hat{\otimes}_A W = V \hat{\otimes} W / \overline{\text{span}\{va \otimes w - v \otimes aw : v \in V, a \in A, w \in W\}},$$

where the bar means closure.

To define Hochschild homology for Fréchet algebras one needs homological algebra in a topological setting, for which we refer to [Tay]. For Fréchet algebras we will always do that with respect to $\hat{\otimes}$, and we often suppress that from the notations.

Definition 5.3. Let A be a unital Fréchet algebra. For $n \in \mathbb{Z}_{\geq 0}$, Tor_n^A denotes the n -th derived functor of $\hat{\otimes}_A$. The n -th Hochschild homology of A is

$$HH_n(A) = HH_n(A, \hat{\otimes}) = \text{Tor}_n^{A \hat{\otimes} A^{op}}(A, A).$$

For a possibly non-unital Fréchet algebra B we put

$$HH_n(B) = \text{coker}(HH_n(\mathbb{C}) \rightarrow HH_n(B_+)).$$

Each HH_n is a functor from Fréchet spaces to topological vector spaces. These functors share several properties with their purely algebraic counterparts:

- One can compute $HH_*(A)$ as the homology of an explicit differential complex $(A^{\hat{\otimes} m}, d_m)$.
- In degree zero the definition shows that $HH_0(A) = A/\overline{[A, A]}$.
- Additivity holds for unital Fréchet algebras.
- $HH_n(A)$ has the structure of a $Z(A)$ -module.
- Morita invariance holds for unital Fréchet algebras.

Continuity of $HH_n(?, \hat{\otimes})$ is problematic, because a direct limit of Fréchet algebras is often not a Fréchet space.

We point out that Hochschild homology works badly for Banach algebras. Consider a commutative Banach algebra B , for instance $C(Y)$ for a compact Hausdorff space Y . Then $HH_0(B, \hat{\otimes}) = B$ and

$$(5.1) \quad HH_n(B, \hat{\otimes}) = 0 \quad \text{for } n \in \mathbb{Z}_{>0}.$$

In fact (5.1) also holds for large classes of noncommutative Banach algebras [Joh]. Roughly speaking Hochschild homology detects some differentiable structure, and Banach algebras are too complete for that.

5.1. Comparison between algebraic and topological settings.

For a smooth manifold X , we write $\Omega_{sm}^n(X)$ for the space of smooth differential n -forms on X . The following version of the Hochschild–Kostant–Rosenberg theorem (Theorem 4.5) was discovered by Connes, in the case of compact manifolds [Con].

Theorem 5.4. [Tel]

There is a natural isomorphism of Fréchet $C^\infty(X)$ -modules

$$HH_n(C^\infty(X)) \cong \Omega_{sm}^n(X).$$

Let Γ be a finite group acting on X by diffeomorphisms, and let $\natural : \Gamma \times \Gamma \rightarrow \mathbb{C}^\times$ by a 2-cocycle. Then we can form the crossed product $C^\infty(X) \rtimes \Gamma$ and the twisted crossed product $C^\infty(X) \rtimes \mathbb{C}[\Gamma, \natural]$. Like in the algebraic setting, $HH_n(C^\infty(X) \rtimes \Gamma)$ was computed by Brylinski [Bry]. That can be generalized to a topological version of Theorem 4.8:

Theorem 5.5. [Sol11, Proposition 3.12]

There exists an isomorphism of Fréchet $C^\infty(X)^\Gamma$ -modules

$$HH_n(C^\infty(X) \rtimes \mathbb{C}[\Gamma, \natural]) \cong \left(\bigoplus_{\gamma \in \Gamma} \Omega_{sm}^n(X^\gamma) \otimes \natural^\gamma \right)^\Gamma.$$

From on the one hand Theorems 4.5 and 4.8 and on the other hand Theorems 5.7 and 5.4, we see that there is a clear analogy between HH_n in an algebro-geometric setting and HH_n in a differential geometric setting. This can be made precise for larger classes of (noncommutative) algebras.

Let V be a complex affine variety and let X be a real analytic manifold which is contained and Zariski-dense in V . Then $\mathcal{O}(V)$ embeds in $C^\infty(X)$ and $C^\infty(X)$ is a $\mathcal{O}(V)$ -module (usually of infinite rank). Assume that a finite group Γ acts on V by automorphisms of algebraic varieties, and that Γ stabilizes X . Then X/Γ is an orbifold and $C^\infty(X)^\Gamma$ is (by definition) the ring of smooth functions on X/Γ .

Let A be a finite type $\mathcal{O}(V)^\Gamma$ -algebra, as in Definition 4.6. The algebra

$$C^\infty(X)^\Gamma \otimes_{\mathcal{O}(V)^\Gamma} A$$

has finite rank over $C^\infty(X)^\Gamma$, and by [KaSo, Lemma 1.3] it is a Fréchet algebra.

Example 5.6. The crossed product $\mathcal{O}(V) \rtimes \Gamma$ is a finite type $\mathcal{O}(V/\Gamma)$ -algebra and

$$C^\infty(X/\Gamma) \otimes_{\mathcal{O}(V/\Gamma)} \mathcal{O}(V) \rtimes \Gamma = C^\infty(X) \rtimes \Gamma.$$

Theorem 5.7. [KaSo, Theorems C and D]

Let V, X, Γ, A be as above and assume that $T_x(X) \otimes_{\mathbb{R}} \mathbb{C} = T_x(V)$ for all $x \in X$.

- (a) $C^\infty(X)^\Gamma$ is flat over $\mathcal{O}(V)^\Gamma$.
- (b) *There is an isomorphism of Fréchet $C^\infty(X)^\Gamma$ -modules*

$$HH_n(C^\infty(X)^\Gamma \otimes_{\mathcal{O}(V)^\Gamma} A) \cong C^\infty(X)^\Gamma \otimes_{\mathcal{O}(V)^\Gamma} HH_n(A).$$

One can recover Theorems 5.4 and 5.5 from Theorems 4.5 and 4.8 by applying Theorem 5.7, see [KaSo, §4]. In a similar way, Theorem 5.7 will help us to determine $HH_n(\mathcal{S}(G, K)_\mathfrak{d})$ for some $\mathfrak{d} \in \Delta(G)$.

5.2. Hochschild homology of $\mathcal{S}(G)$.

Let G be a reductive p -adic group and let $\mathcal{S}(G)$ be its Harish-Chandra–Schwartz algebra, as in Definition 1.6. We want to describe its Hochschild homology in terms of representation theory, like we did for $HH_*(\mathcal{H}(G))$ in Paragraphs 4.5 and 4.6.

However, since $\mathcal{S}(G)$ is not a Fréchet space, Definition 5.3 will probably produce suboptimal results. The best solution for that technical problem is to consider $\mathcal{S}(G)$ as a bornological algebra [Mey1] and to use Hochschild homology based on the completed bornological tensor product $\hat{\otimes}_b$ [Mey2, Chapter I]. This fits with Definition 5.3, because for Fréchet spaces $\hat{\otimes}_b$ and $\hat{\otimes}$ agree [Mey2, Theorem I.87]. The functor $HH_*(?, \hat{\otimes}_b)$ enjoys continuity properties for strict inductive limits of Fréchet algebras, which we state only in two concrete cases.

Theorem 5.8. [BrPl, Theorem 2] and [Mey2, Theorem I.93]

Let $(K_m)_{m=1}^\infty$ be as in Theorem 1.45 and let $\mathfrak{s} \in \mathfrak{B}(G)$. There are natural isomorphisms of topological vector spaces

$$\begin{aligned} HH_n(\mathcal{S}(G), \hat{\otimes}_b) &= HH_n\left(\lim_{m \rightarrow \infty} \mathcal{S}(G, K_m), \hat{\otimes}_b\right) \cong \lim_{m \rightarrow \infty} HH_n(\mathcal{S}(G, K_m)), \\ HH_n(\mathcal{S}(G)^\mathfrak{s}, \hat{\otimes}_b) &= HH_n\left(\lim_{m \rightarrow \infty} \mathcal{S}(G, K_m)^\mathfrak{s}, \hat{\otimes}_b\right) \cong \lim_{m \rightarrow \infty} HH_n(\mathcal{S}(G, K_m)^\mathfrak{s}). \end{aligned}$$

For each $\mathfrak{s} \in \mathfrak{B}(G)$ we pick $m(\mathfrak{s}) \in \mathbb{Z}_{>0}$ such that $\mathfrak{s} \in \mathfrak{B}(G, K_{m(\mathfrak{s})})$, and we write $K_\mathfrak{s} = K_{m(\mathfrak{s})}$. By Proposition 1.47.a all the spaces $HH_n(\mathcal{S}(G, K_m)^\mathfrak{s})$ with $m \geq m(\mathfrak{s})$ are isomorphic. From Theorem 5.8, Corollary 1.46 and Proposition 1.47.b we obtain

$$\begin{aligned} (5.2) \quad HH_n(\mathcal{S}(G), \hat{\otimes}_b) &\cong \lim_{m \rightarrow \infty} HH_n\left(\bigoplus_{\mathfrak{s} \in \mathfrak{B}(G, K_m)} \mathcal{S}(G, K_m)^\mathfrak{s}\right) \\ &\cong \lim_{m \rightarrow \infty} \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G, K_m)} HH_n(\mathcal{S}(G, K_m)^\mathfrak{s}) \\ &\cong \lim_{m \rightarrow \infty} \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G, K_m)} HH_n(\mathcal{S}(G)^\mathfrak{s}, \hat{\otimes}_b) = \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} HH_n(\mathcal{S}(G)^\mathfrak{s}, \hat{\otimes}_b). \end{aligned}$$

From (5.2) we see that, in order to determine $HH_n(\mathcal{S}(G), \hat{\otimes}_b)$, it suffices to compute $HH_n(\mathcal{S}(G, K_\mathfrak{s})^\mathfrak{s})$ for each $\mathfrak{s} \in \mathfrak{B}(G)$. This brings us back to unital Fréchet algebras.

We recall from Theorem 1.32 and (1.36) that

$$\begin{aligned} \mathcal{S}(G, K_\mathfrak{s})^\mathfrak{s} &= \bigoplus_{\mathfrak{d} \in \Delta(G, \mathfrak{s})} \mathcal{S}(G, K_\mathfrak{s}) \cap \mathcal{S}(G)_\mathfrak{d} \\ &\cong \bigoplus_{[L, \delta]_G \in \Delta(G, \mathfrak{s})} \left(C^\infty(X_{\text{nr}}^u(L)) \otimes \text{End}_{\mathbb{C}}(I_{K_0 \cap P}^{K_0}(V_\delta)^{K_\mathfrak{s}}) \right)^{W_{[L, \delta]}^e}. \end{aligned}$$

Write $\mathfrak{s} = [M, \sigma]_G$ with σ unitary supercuspidal and recall that $\mathfrak{s} \in \Delta(G, \mathfrak{s})$. That brings us in a good position to apply the results from the previous paragraph.

Lemma 5.9. (a) *There is an isomorphism of Fréchet algebras*

$$\mathcal{S}(G, K_\mathfrak{s}) \cap \mathcal{S}(G)_{[M, \sigma]_G} \cong C^\infty(X_{\text{nr}}^u(M))^{W_\mathfrak{s}^e} \otimes_{\mathcal{O}(X_{\text{nr}}(M))^{W_\mathfrak{s}^e}} \mathcal{H}(G, K_\mathfrak{s})^\mathfrak{s}.$$

(b) *There is an isomorphism of Fréchet $C^\infty(X_{\text{nr}}^u(M))^{W_\mathfrak{s}^e}$ -modules*

$$HH_n(\mathcal{S}(G, K_\mathfrak{s}) \cap \mathcal{S}(G)_{[M, \sigma]_G}) \cong C^\infty(X_{\text{nr}}^u(M))^{W_\mathfrak{s}^e} \otimes_{\mathcal{O}(X_{\text{nr}}(M))^{W_\mathfrak{s}^e}} HH_n(\mathcal{H}(G, K_\mathfrak{s})^\mathfrak{s}).$$

Proof. (a) In Theorem 1.42 the underlying intertwining operators do not have singularities on $X_{\text{nr}}^u(M)\sigma$, so the Fourier transform induces an algebra isomorphism

$$\mathcal{H}(G, K_\mathfrak{s})^\mathfrak{s}|_{X_{\text{nr}}^u(M)\sigma} \cong \left(\mathcal{O}(X_{\text{nr}}(M))|_{X_{\text{nr}}^u(M)} \otimes \text{End}_{\mathbb{C}}(I_{K_0 \cap P}^{K_0}(V_\sigma)^{K_\mathfrak{s}}) \right)^{W_\mathfrak{s}^e}.$$

It follows that

$$C^\infty(X_{\text{nr}}^u(M))^{W_\mathfrak{s}^e} \otimes_{\mathcal{O}(X_{\text{nr}}(M))^{W_\mathfrak{s}^e}} \mathcal{H}(G, K_\mathfrak{s})^\mathfrak{s} \cong \left(C^\infty(X_{\text{nr}}^u(M)) \otimes \text{End}_{\mathbb{C}}(I_{K_0 \cap P}^{K_0}(V_\sigma)^{K_\mathfrak{s}}) \right)^{W_\mathfrak{s}^e}.$$

By Theorem 1.32.b, the right hand side equals $\mathcal{S}(G, K_\mathfrak{s}) \cap \mathcal{S}(G)_{[M, \sigma]_G}$.

(b) In view of part (a), this is an instance of Theorem 5.7. $\square$

Unfortunately it remains difficult to determine $HH_n(\mathcal{S}(G, K_\mathfrak{s}) \cap \mathcal{S}(G)_{[M, \sigma]_G})$ in Lemma 5.9.b, because the $\mathcal{O}(X_{\text{nr}}(M))^{W_\mathfrak{s}^e}$ -module structure of $HH_n(\mathcal{H}(G, K_\mathfrak{s})^\mathfrak{s})$ is tricky.

Example 5.10. Consider $G = SL_2(F)$ and $\mathfrak{s} = [T, \text{triv}]_G$. By Lemma 5.9.b and (4.19) we have

$$\begin{aligned} HH_n(\mathcal{S}(G)_{\mathfrak{s}}, \hat{\otimes}_b) &\cong C^\infty(X_{\text{nr}}^u(T))^{S_2} \otimes_{\mathcal{O}(X_{\text{nr}}(T))^{S_2}} HH_n(\mathcal{H}(G)^{\mathfrak{s}}) \\ &\cong C^\infty(X_{\text{nr}}^u(T))^{S_2} \otimes_{\mathcal{O}(X_{\text{nr}}(T))^{S_2}} (\Omega^n(X_{\text{nr}}(T))^{S_2} \oplus \Omega^n(\{\text{St}\}) \oplus \Omega^n(\{\chi_-\})). \end{aligned}$$

Here $\Omega^n(\{\text{St}\})$ drops out, because its $\mathcal{O}(X_{\text{nr}}(T))$ -character does not lie in $X_{\text{nr}}^u(T)$. The other two summands of $HH_n(\mathcal{H}(G)^{\mathfrak{s}})$ carry the $\mathcal{O}(X_{\text{nr}}(T))$ -action expected from the notations. We find

$$(5.3) \quad HH_n(\mathcal{S}(G)_{\mathfrak{s}}, \hat{\otimes}_b) \cong \Omega_{sm}^n(X_{\text{nr}}^u(T))^{S_2} \oplus \Omega_{sm}^n(\{\chi_-\}).$$

As $\mathcal{S}(G)^{\mathfrak{s}} = \mathcal{S}(G)_{\mathfrak{s}} \oplus \mathcal{S}(G)_{[G, \text{St}]}$ and $\mathcal{S}(G)_{[G, \text{St}]}$ is Morita equivalent to $\mathbb{C}$, (5.3) yields

$$(5.4) \quad HH_n(\mathcal{S}(G)^{\mathfrak{s}}, \hat{\otimes}_b) \cong \Omega_{sm}^n(X_{\text{nr}}^u(T))^{S_2} \oplus \Omega_{sm}^n(\{\chi_-\}) \oplus \Omega^n(\{\text{St}\}).$$

Theorem 5.5 shows that (5.4) is isomorphic to $HH_n(C^\infty(X_{\text{nr}}^u(T)) \rtimes S_2)$.

There is a version of Lemma 5.9.a for $\mathfrak{d} \in \Delta(G, \mathfrak{s})$ which are not represented by a supercuspidal representation [Sol11, Lemma 3.3], but it is more complicated and does not fit in the framework of Theorem 5.7. To handle $HH_n(\mathcal{S}(G, K_{\mathfrak{s}}) \cap \mathcal{S}(G)_{\mathfrak{d}})$ for such $\mathfrak{d}$ we use families of representations, like in Paragraph 4.6. For a parabolic subgroup $P = LU_P \subset G$ and $\tau \in \text{Irr}_{\text{temp}}(L)^{s_L}$ we have a family of tempered representations

$$\mathfrak{F}_{L, \tau}^u = \{I_P^G(\tau \otimes \chi_L) : \chi_L \in X_{\text{nr}}^u(L)\}$$

and a homomorphism of $C^\infty(X_{\text{nr}}^u(M))^{W_{\mathfrak{s}}^e}$ -algebras

$$\mathcal{F}_{L, \tau}^u : \mathcal{S}(G, K_{\mathfrak{s}})^{\mathfrak{s}} \rightarrow C^\infty(X_{\text{nr}}^u(L)) \otimes \text{End}_{\mathbb{C}}(I_{P \cap K_0}^{K_0}(\tau)^{K_{\mathfrak{s}}}).$$

Recall $\zeta^\vee$ from Theorem 3.15. It preserves temperedness and the tempered part of $R(\mathcal{O}(X_{\text{nr}}(M)))$ can be identified with $R(C^\infty(X_{\text{nr}}^u(M)))$, so $\zeta^\vee$ restricts to a group isomorphism

$$(5.5) \quad \zeta_u^\vee : R(\mathcal{S}(G)^{\mathfrak{s}}) \rightarrow R(C^\infty(X_{\text{nr}}^u(M)) \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}]).$$

In particular $\zeta_u^\vee(\mathfrak{F}_{L, \tau}^u) = \mathfrak{F}_{L, \zeta_u^\vee(\tau)}^u$ is a family of $C^\infty(X_{\text{nr}}^u(M)) \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}]$ -modules.

Theorem 5.11. [Sol11, Theorem 3.13]

There exists a unique isomorphism of Fréchet spaces

$$HH_n(\zeta_u^\vee) : HH_n(C^\infty(X_{\text{nr}}^u(M)) \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}]) \rightarrow HH_n(\mathcal{S}(G, K_{\mathfrak{s}})^{\mathfrak{s}})$$

such that

$$HH_n(\mathcal{F}_{L, \tau}^u) \circ HH_n(\zeta_u^\vee) = HH_n(\mathcal{F}_{L, \zeta_u^\vee(\tau)}^u)$$

for all families of tempered representations $\mathfrak{F}_{L, \tau}^u$.

From Theorems 5.5 and 5.11 we obtain an isomorphism of Fréchet spaces

$$(5.6) \quad HH_n(\mathcal{S}(G, K_{\mathfrak{s}})^{\mathfrak{s}}) \cong \left(\bigoplus_{w \in W_{\mathfrak{s}}^e} \Omega_{sm}^n(X_{\text{nr}}^u(M))^w \otimes (\mathfrak{h}_{\mathfrak{s}}^{-1})^w \right)^{W_{\mathfrak{s}}^e}.$$

This isomorphism can also be constructed more directly, with suitable families of tempered representations.

Example 5.12. We give an overview of $HH_n(\mathcal{S}(G)^\mathfrak{s}, \hat{\otimes}_b)$ for all Bernstein blocks of $G = SL_2(F)$. For $\mathfrak{s} = [G, \sigma]_G$ with $\sigma \in \text{Irr}_{\text{cusp}}(G)$, we have

$$HH_*(\mathcal{S}(G, K_\mathfrak{s})^\mathfrak{s}) = HH_0(\mathcal{S}(G, K_\mathfrak{s})^\mathfrak{s}) \cong \mathbb{C}.$$

For $\mathfrak{s} = [T, \chi]_G$ with $\text{ord}(\chi|_{\mathfrak{o}_F^\times}) > 2$ we have

$$HH_n(\mathcal{S}(G, K_\mathfrak{s})^\mathfrak{s}) \cong HH_n(C^\infty(X_{\text{nr}}^u(T))) \cong \Omega_{sm}^n(X_{\text{nr}}^u(T)) \cong \Omega_{sm}^n(S^1).$$

For $\mathfrak{s} = [T, \chi_2]_G$ with $\text{ord}(\chi_2|_{\mathfrak{o}_F^\times}) = 2$, (5.6) shows that

$$HH_n(\mathcal{S}(G, K_\mathfrak{s})^\mathfrak{s}) \cong HH_n(C^\infty(X_{\text{nr}}^u(T)) \rtimes S_2) \cong \Omega_{sm}^n(X_{\text{nr}}^u(T))^{S_2} \oplus \Omega_{sm}^n(\{1, -1\}).$$

Finally, for $\mathfrak{s} = [T, \text{triv}]_G$, we already determined $HH_n(\mathcal{S}(G, K_\mathfrak{s})^\mathfrak{s})$ in Example 5.10.

5.3. Trace Paley–Wiener theorems for $\mathcal{H}(G)$ and $\mathcal{S}(G)$.

The structure of

$$HH_0(\mathcal{H}(G)) = \mathcal{H}(G)/[\mathcal{H}(G), \mathcal{H}(G)]$$

can be described with the trace Paley–Wiener for reductive p -adic groups, which we recall now. The main ingredient is the trace pairing

$$\begin{array}{ccc} HH_0(\mathcal{H}(G)) \times R(G) & \rightarrow & \mathbb{C} \\ (f, \pi) & \mapsto & \text{tr } \pi(f) \end{array}.$$

To see that this pairing is well-defined, pick $K \in \mathcal{CO}(G)$ such that $f \in \mathcal{H}(G, K)$. For $(\pi, V) \in \text{Rep}_{\text{fl}}(G)$, $\text{tr } \pi(f)$ equals $\text{tr } \pi(f)|_{V^K}$. As $\dim_{\mathbb{C}} V^K$ is finite (Theorem 1.12), $\text{tr } \pi(f)|_{V^K} \in \mathbb{C}$ is defined.

We say that a linear form $\lambda \in \text{Hom}_{\mathbb{Z}}(R(G), \mathbb{C})$ is regular if

- λ is supported on finitely many Bernstein blocks of $\text{Rep}(G)$,
- for every parabolic subgroup $P = LU_P$ and every $\pi \in \text{Irr}(L)$, the function

$$X_{\text{nr}}(L) \rightarrow \mathbb{C} : \chi \mapsto \lambda(I_P^G(\pi \otimes \chi)) \quad \text{is regular.}$$

We denote the $\mathbb{C}$ -vector space of such λ by $\text{Hom}_{\mathbb{Z}}(R(G), \mathbb{C})_{\text{reg}}$.

Theorem 5.13. [BDK]

The trace pairing induces an isomorphism of $Z(\text{Rep}(G))$ -modules

$$HH_0(\mathcal{H}(G)) \longrightarrow \text{Hom}_{\mathbb{Z}}(R(G), \mathbb{C})_{\text{reg}}.$$

Next we consider a twisted crossed product $A \rtimes \mathbb{C}[\Gamma, \mathfrak{h}]$ as in (2.6), where Γ is a finite group and A denotes either $C^\infty(X)$ for a smooth manifold or $\mathcal{O}(X)$ for a non-singular affine $\mathbb{C}$ -variety. From Theorems 4.8 and 5.5 we know that there are isomorphisms of A^Γ -modules

$$\begin{aligned} HH_0(C^\infty(X) \rtimes \mathbb{C}[\Gamma, \mathfrak{h}]) &\cong \left(\bigoplus_{\gamma \in \Gamma} C^\infty(X^\gamma) \otimes \mathfrak{h}^\gamma \right)^\Gamma \\ (5.7) \qquad \qquad \qquad &\cong \bigoplus_{\gamma \in \Gamma/\text{conjugacy}} (C^\infty(X^\gamma) \otimes \mathfrak{h}^\gamma)^{Z_\Gamma(\gamma)}, \end{aligned}$$

and similarly with $\mathcal{O}$ instead of C^∞ . The specialisation of $HH_0(A)$ at an arbitrary character $\Gamma x \in X/\Gamma$ of A^Γ can be identified with

$$(5.8) \qquad \mathbb{C}\{T_\gamma : \gamma \in \Gamma_x, \mathfrak{h}^\gamma|_{Z_{\Gamma_x}(\gamma)} = 1, \text{ modulo } \Gamma_x\text{-conjugation}\}.$$

From Theorem 2.4 and Lemma 2.2 one deduces that (5.8) pairs nondegenerately with the Grothendieck group of the category of finite dimensional $A \rtimes \mathbb{C}[\Gamma, \mathfrak{h}]$ -modules with A^Γ -character Γx . This implies that the trace pairing

$$(5.9) \qquad HH_0(A \rtimes \mathbb{C}[\Gamma, \mathfrak{h}]) \times R(A \rtimes \mathbb{C}[\Gamma, \mathfrak{h}]) \longrightarrow \mathbb{C} \quad \text{is nondegenerate.}$$

Reasoning in such ways, one can recover Theorem 5.13 from Theorems 3.15 and 4.13, see [Sol11, Proposition 2.9.b].

As we saw in Paragraph 5.2, there are versions of Theorems 3.15 and 4.13 for $\mathcal{S}(G)$. That makes the existence of a trace Paley–Wiener theorem for $\mathcal{S}(G)$ plausible, and (5.7) suggests that it should involve smooth rather than regular functions. We call a linear form $\lambda \in \text{Hom}_{\mathbb{Z}}(R(\mathcal{S}(G)), \mathbb{C})$ smooth if:

- λ is supported on finitely many Bernstein blocks of $\text{Rep}(G)$,
- for every parabolic subgroup $P = LU_P$ and every $\tau \in \text{Irr}_{\text{temp}}(L)$, the function

$$X_{\text{nr}}(L) \rightarrow \mathbb{C} : \chi \mapsto \lambda(I_P^G(\tau \otimes \chi)) \quad \text{is smooth.}$$

We denote the $\mathbb{C}$ -vector space of such linear forms by $\text{Hom}_{\mathbb{Z}}(R(\mathcal{S}(G)), \mathbb{C})_{\infty}$.

Theorem 5.14. [Sol11, Theorem D.ii and (3.4)]

The trace pairing induces an isomorphism of $Z(\text{Rep}(G))$ -modules

$$HH_0(\mathcal{S}(G), \hat{\otimes}_b) = \mathcal{S}(G) / [\mathcal{S}(G), \mathcal{S}(G)] \longrightarrow \text{Hom}_{\mathbb{Z}}(R(\mathcal{S}(G)), \mathbb{C})_{\infty}.$$

5.4. Periodic cyclic homology.

Periodic cyclic homology of algebras consists of two functors HP_0, HP_1 from algebras to vector spaces. They are periodic in the sense that $HP_{n+2m} = HP_n$ for all $n, m \in \mathbb{Z}$. The functor $HP_* = HP_0 \oplus HP_1$ plays an important role in non-commutative geometry, because it is an analogue of DeRham cohomology (which can be defined for commutative algebras). Its functorial properties are analogous to those of topological K-theory (see Paragraph 6.1), in particular it is Morita invariant [Cun2] and there are six-term exact sequences associated to short exact sequences of algebras [CuQu]. In practice, these make it possible to determine $HP_*(A)$ for many algebras A .

The periodic cyclic homology of a $\mathbb{C}$ -algebra A is defined as the homology of an explicit infinite double complex $CC_{\text{per}}(A)$ with spaces $A^{\otimes n}$ [Lod, §5.1]. A part of $CC_{\text{per}}(A)$ computes $HH_*(A)$, and as a result $HP_*(A)$ and $HH_*(A)$ are closely related. In many cases $HP_*(A)$ can be computed as the homology of $HH_*(A)$ with respect to a new differential B discovered by Connes.

Periodic cyclic homology is also defined for topological algebras, when one fixes a topological tensor product. We focus on Fréchet algebras and $\hat{\otimes}$. Then $HP_*(A, \hat{\otimes})$ is the homology of a double complex $CC_{\text{per}}(A, \hat{\otimes})$ obtained from $CC_{\text{per}}(A)$ by replacing all terms $A^{\otimes n}$ by $A^{\hat{\otimes} n}$.

We can now formulate the Hochschild–Kostant–Rosenberg–Connes theorem. In these cases $HP_*(A)$ is computed from $HH_*(A)$ (which is known from Theorems 4.5 and 5.4) as the homology with respect to usual d for differential forms.

Theorem 5.15. [Lod, §5.1.12] and [Con, Tel]

Let $i \in \{0, 1\}$ and let H_{dR}^ denote de Rham cohomology with complex coefficients.*

(a) *Let V be a nonsingular complex affine variety. There is a natural isomorphism*

$$HP_i(\mathcal{O}(V)) \cong \bigoplus_{m \in \mathbb{Z}_{\geq 0}} H_{dR}^{i+2m}(V).$$

(b) *Let X be a smooth manifold. There is a natural isomorphism*

$$HP_i(C^\infty(X), \hat{\otimes}) \cong \bigoplus_{m \in \mathbb{Z}_{\geq 0}} H_{dR}^{i+2m}(X).$$

Suppose that in the setting of Theorem 5.15, a finite group Γ acts on V and on X , by automorphism in the appropriate category. Then we can combine Theorem 5.15 with Theorems 4.8 and 5.5 to obtain:

Corollary 5.16. *There are isomorphisms*

$$\begin{aligned} HP_*(\mathcal{O}(V) \rtimes \mathbb{C}[\Gamma, \mathfrak{h}]) &\cong (\bigoplus_{\gamma \in \Gamma} H_{dR}^*(V^\gamma) \otimes \mathfrak{h}^\gamma)^\Gamma, \\ HP_*(C^\infty(X) \rtimes \mathbb{C}[\Gamma, \mathfrak{h}], \hat{\otimes}) &\cong (\bigoplus_{\gamma \in \Gamma} H_{dR}^*(X^\gamma) \otimes \mathfrak{h}^\gamma)^\Gamma. \end{aligned}$$

Continuity for HP_* is more subtle than for HH_* , because $CC_{per}(A)$ has infinitely many terms in negative degrees.

Theorem 5.17. [Nis, Proposition 2.2] and [BrPl, Theorem 3]

- (a) *Suppose that $\lim_{i \rightarrow \infty} A_i$ is an inductive limit of algebras and that there exists $N \in \mathbb{N}$ such that $HH_n(A_i) = 0$ for all $n \geq N$ and all i . Then $HP_*(\lim_{i \rightarrow \infty} A_i) \cong \lim_{i \rightarrow \infty} HP_*(A_i)$.*
- (b) *Suppose that $\lim_{i \rightarrow \infty} B_i$ is a strict inductive limit of nuclear Fréchet algebras and that there exists $N \in \mathbb{N}$ such that $HH_n(B_i, \hat{\otimes}) = 0$ for all $n \geq N$ and all i . Then $HP_*(\lim_{i \rightarrow \infty} B_i, \hat{\otimes}_b) \cong \lim_{i \rightarrow \infty} HP_*(B_i, \hat{\otimes})$.*

We return to our main players, $\mathcal{H}(G)$ and $\mathcal{S}(G)$. Theorem 5.17 enables us to apply an argument like in (4.9) and (5.2). Recall that for $\mathfrak{s} \in \mathfrak{B}(G)$ we picked $K_{\mathfrak{s}} \in \mathcal{CO}(G)$ in Paragraph 5.2.

Lemma 5.18. [Sol2, (3.3) and (3.4)]

There are natural isomorphisms

$$\begin{aligned} HP_*(\mathcal{H}(G)) &\cong \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} HP_*(\mathcal{H}(G)^{\mathfrak{s}}) \cong \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} HP_*(\mathcal{H}(G, K_{\mathfrak{s}})^{\mathfrak{s}}), \\ HP_*(\mathcal{S}(G), \hat{\otimes}_b) &\cong \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} HP_*(\mathcal{S}(G)^{\mathfrak{s}}, \hat{\otimes}_b) \cong \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} HP_*(\mathcal{S}(G, K_{\mathfrak{s}})^{\mathfrak{s}}, \hat{\otimes}). \end{aligned}$$

It turns out that the isomorphisms (4.16) and (5.6) (which come from Theorems 4.13 and 5.11) are very suitable to determine $HP_*(\mathcal{H}(G))$ and $HP_*(\mathcal{S}(G), \hat{\otimes}_b)$ in representation theoretic terms. By [Sol11, (4.4) and (4.5)], Connes' differential B on $HH_*(\mathcal{H}(G))$ and on $HH_*(\mathcal{S}(G), \hat{\otimes}_b)$ corresponds to the usual exterior differential d on the differential forms in (4.18) and (5.6). Taking homology with respect to d yields:

Theorem 5.19. [Sol11, Theorem 4.1 and (4.7)–(4.8)]

Let $i \in \{0, 1\}$. There are isomorphisms of vector spaces

$$\begin{aligned} HP_i(\mathcal{H}(G, K_{\mathfrak{s}})^{\mathfrak{s}}) &\cong HP_i(\mathcal{O}(X_{\text{nr}}(L)) \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}]) \\ &\cong \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \left(\bigoplus_{w \in W_{\mathfrak{s}}^e} H_{dR}^{i+2m}(X_{\text{nr}}(L)^w) \otimes (\mathfrak{h}_{\mathfrak{s}}^{-1})^w \right)^{W_{\mathfrak{s}}^e}, \\ HP_i(\mathcal{S}(G, K_{\mathfrak{s}})^{\mathfrak{s}}, \hat{\otimes}) &\cong HP_i(C^\infty(X_{\text{nr}}^u(L)) \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}], \hat{\otimes}) \\ &\cong \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \left(\bigoplus_{w \in W_{\mathfrak{s}}^e} H_{dR}^{i+2m}(X_{\text{nr}}^u(L)^w) \otimes (\mathfrak{h}_{\mathfrak{s}}^{-1})^w \right)^{W_{\mathfrak{s}}^e}. \end{aligned}$$

As $X_{\text{nr}}^u(L)$ is $W_{\mathfrak{s}}^e$ -equivariant deformation retract of $X_{\text{nr}}(L)$, Lemma 5.18 and Theorem 5.19 recover [Sol2, Theorem 3.3] and [BHP, Conjecture 8.9]:

Corollary 5.20. *The inclusions $\mathcal{H}(G, K_{\mathfrak{s}})^{\mathfrak{s}} \rightarrow \mathcal{S}(G, K_{\mathfrak{s}})^{\mathfrak{s}}$ and $\mathcal{H}(G) \rightarrow \mathcal{S}(G)$ induce isomorphisms on periodic cyclic homology.*

Furthermore, by [Sol11, Lemma 4.4] $HP_*(\mathcal{H}(G, K_{\mathfrak{s}})^{\mathfrak{s}})$ can be realized as the subset of $HH_*(\mathcal{H}(G, K_{\mathfrak{s}})^{\mathfrak{s}})$ consisting of differential forms that are locally constant on the various varieties $X_{\text{nr}}(L)^w$, in the picture from (4.18).

Similarly $HP_*(\mathcal{S}(G, K_5)^5, \hat{\otimes})$ has a canonical realization as the set of locally constant differential forms in $HH_*(\mathcal{S}(G, K_5)^5, \hat{\otimes})$, in the picture (5.6).

These are instances of a more general phenomenon in geometric group theory, already observed in [HiNi, Sch]. For any totally disconnected locally compact group G' there is a decomposition

$$(5.10) \quad HH_*(\mathcal{H}(G')) = HH_*(\mathcal{H}(G'))_{\text{cpt}} \oplus HH_*(\mathcal{H}(G'))_{\text{ncpt}}.$$

The subscript cpt means "supported on compact elements": this part of $HH_*(\mathcal{H}(G'))$ only uses chains

$$f_0 \otimes f_1 \otimes \cdots \otimes f_n \in \mathcal{H}(G')^{\otimes(n+1)}$$

such that $f_0(g_0)f_1(g_1)\cdots f_n(g_n)$ with $g_i \in G'$ is zero unless $g_0g_1\cdots g_n$ lies in a compact subgroup of G' . The subscript ncpt stands for "supported on noncompact elements", which means that it comes from chains such that $f_0(g_0)f_1(g_1)\cdots f_n(g_n)$ is zero whenever $g_0g_1\cdots g_n$ lies in a compact subgroup of G' .

Suppose in addition that G' acts properly on an affine building, like any reductive p -adic group does [Tit, §2]. Then the noncompact part of $HH_*(\mathcal{H}(G'))$ disappears in periodic cyclic homology, by [HiNi, Theorem 6.2] or [Sch, Theorem II].

Theorem 5.21. [HiNi, Theorems 1.1 and 4.2] and [Sch, Theorem I]

In the above setting there is an isomorphism

$$HP_i(\mathcal{H}(G')) \cong \bigoplus_{m \in \mathbb{Z}_{\geq 0}} HH_{i+2m}(\mathcal{H}(G'))_{\text{cpt}}.$$

For our group G acting on its Bruhat–Tits building, Theorem 5.19, (4.18) and the aforementioned [Sol9, Lemma 4.4] recover Theorem 5.21 in a representation-theoretic way.

6. TOPOLOGICAL K-THEORY

K-theory started [Ati] as a way to classify vector bundles on a topological space X , up to stable isomorphism. That gives a contravariant functor K^0 from topological spaces to abelian groups, and another functor K^1 is obtained by composing K^0 with the suspension functor for topological spaces. There are also higher functors K^n , but by Bott periodicity these reduce to K^0 or K^1 . The $\mathbb{Z}/2\mathbb{Z}$ -graded functor $K^* = K^0 \oplus K^1$ forms a generalized cohomology theory, so roughly speaking it behaves like singular or Čech cohomology. K-theory for topological spaces can be extended naturally to pairs of spaces $X_1 \supset X_2$.

In noncommutative geometry, topological K-theory is usually defined and studied for C^* -algebras or Banach algebras, see for instance [Bla]. The (covariant) functor K_0 classifies finitely generated projective modules up to stable isomorphism. The functor K_1 can be obtained as K_0 composed with a suspension functor for algebras. Recall that by the Gelfand–Naimark theorem [FGV, Theorem 1.4] every commutative C^* -algebra has the form

$$C_0(Y) = \{f \in C(Y \cup \{\infty\}) : f(\infty) = 0\},$$

where Y is locally compact Hausdorff space with one-point compactification $Y \cup \{\infty\}$. By the Serre–Swan theorem [FGV, §2.3 and Corollary 3.2.1] there is a natural isomorphism

$$(6.1) \quad K_i(C_0(Y)) \cong K^i(Y \cup \{\infty\}, \{\infty\}) \quad i = 0, 1.$$

Most references state this only for $i = 0$, but the definitions of K^1 in terms of K^0 and K_1 in terms of K_0 are analogous, and therefore (6.1) also holds for $i = 1$. Largely due to the results of Gelfand–Naimark and Serre–Swan, K-theory of C^* -algebras is at the heart of noncommutative geometry.

6.1. Versions and properties of K-theory.

Topological K-theory can also be defined for Fréchet algebras (in the sense of Definition 5.1), see [Cun1, Phi2]. This generalizes K-theory for Banach algebras and enjoys the following properties:

- **Additivity.** For any Fréchet algebras A_n ($n \in \mathbb{N}$) and $i \in \{0, 1\}$:

$$K_i\left(\prod_{n=1}^{\infty} A_n\right) \cong \prod_{n=1}^{\infty} K_i(A_n).$$

- **Stability.** $K_i(M_n(A)) \cong K_i(A)$ for any Fréchet algebra A , and the same with $M_n(\mathbb{C})$ replaced by the algebra of compact operators on a separable Hilbert space.
- **Continuity** [Bla, §5.2.4, §5.5.1 and §8.1.5]. If $\lim_{n \rightarrow \infty} B_n$ is an inductive limit of Banach algebras, then

$$K_i\left(\lim_{n \rightarrow \infty} B_n\right) \cong \lim_{n \rightarrow \infty} K_i(B_n).$$

- **Excision.** Let $0 \rightarrow A \rightarrow B \rightarrow C$ be an exact sequence of Fréchet algebras. Then there exists a natural six-term exact sequence

$$(6.2) \quad \begin{array}{ccccc} K_0(A) & \rightarrow & K_0(B) & \rightarrow & K_0(C) \\ & \uparrow & & & \downarrow \\ K_1(C) & \leftarrow & K_1(B) & \leftarrow & K_1(A) \end{array}.$$

- **Homotopy invariance.** If $\phi_0, \phi_1 : A \rightarrow B$ are homotopic morphisms of Fréchet algebras, then $K_i(\phi_0) = K_i(\phi_1)$.

The suspension of a Fréchet algebra A is defined as

$$\Sigma A = \{f \in C(S^1; A) : f(1) = 0\}.$$

There are natural isomorphisms

$$(6.3) \quad K_0(\Sigma A) \cong K_1(A) \quad \text{and} \quad K_1(\Sigma A) \cong K_0(A).$$

The first is either a definition or [Phi2, Theorem 3.14] and the second is a reformulation of Bott periodicity [Phi2, Theorem 5.5]. These isomorphisms are compatible with excision, in the sense that the exact hexagon (6.2) can be rewritten as

$$(6.4) \quad \begin{array}{ccccc} K_0(A) & \rightarrow & K_0(B) & \rightarrow & K_0(C) \\ & \uparrow & & & \downarrow \\ K_0(\Sigma C) & \leftarrow & K_0(\Sigma B) & \leftarrow & K_0(\Sigma A) \end{array}.$$

Consider the split exact sequence of Fréchet algebras

$$(6.5) \quad 0 \rightarrow \Sigma A \rightarrow C(S^1, A) \xrightarrow{\text{ev}_1} A \rightarrow 0,$$

where the splitting sends $a \in A$ to the constant function on S^1 with value a . Topological K-theory respects split exact sequences, so (6.5) and (6.3) induce isomorphisms

$$(6.6) \quad K_i(C(S^1, A)) \cong K_i(\Sigma A) \oplus K_i(A) \cong K_0(A) \oplus K_1(A) \quad i = 0, 1.$$

The isomorphism $K_*(M_n(\mathbb{C})) \cong K_*(A)$ can be regarded as an instance of Morita invariance of topological K-theory. For unital Fréchet algebras, this can be pushed further.

Theorem 6.1. *Let A and B be unital Fréchet algebras, such that $A^\times \subset A$ and $B^\times \subset B$ are open. Suppose that A and B are Morita equivalent. Then the Morita bimodules induce an isomorphism $K_*(A) \cong K_*(B)$.*

Proof. Let P and Q be the bimodules implementing the Morita equivalence, so $P \otimes_B Q \cong A$ and $Q \otimes_A P \cong B$. Since A and B are unital, P and Q are finitely generated and projective. Then $C(S^1, P)$ and $C(S^1, B)$ are finitely generated projective bimodules for $C(S^1, A)$ and $C(S^1, B)$, such that

$$C(S^1, P) \otimes_{C(S^1, B)} C(S^1, Q) \cong C(S^1, A) \text{ and } C(S^1, Q) \otimes_{C(S^1, A)} C(S^1, P) \cong C(S^1, B).$$

This provides a Morita equivalence between $C(S^1, A)$ and $C(S^1, B)$. The assumptions on A and B and the compactness of S^1 imply that $C(S^1, A)^\times \subset C(S^1, A)$ and $C(S^1, B)^\times \subset C(S^1, B)$ are open. According to [Phi2, Theorem 7.7], $K_0(C(S^1, A))$ is naturally isomorphic to the K-group of the monoid of finitely generated projective $C(S^1, A)$ -modules. The same holds for $C(S^1, B)$. Therefore the maps

$$(6.7) \quad K_0(C(S^1, A)) \longleftrightarrow K_0(C(S^1, B))$$

induced by $C(S^1, Q) \otimes_{C(S^1, A)}$ and $C(S^1, P) \otimes_{C(S^1, B)}$ are group isomorphisms. The decomposition

$$K_0(C(S^1, A)) \cong K_0(A) \oplus K_1(A)$$

from (6.5)–(6.6) comes from $C_0(S^1, \{1\}) \rightarrow C(S^1) \rightarrow C(\{1\})$. It works in the same way for $C(S^1, B)$, so (6.7) respects those decompositions and yields isomorphisms

$$K_0(A) \cong K_0(B) \quad \text{and} \quad K_1(A) \cong K_1(B). \quad \square$$

The Serre–Swan theorem admits a generalization to Fréchet algebras, of which we state a simplified version:

Theorem 6.2. [Phi2, Theorem 7.15]

Let Y be a locally compact Hausdorff space. Let A be a commutative Fréchet algebra such that the maximal ideal space of the unitization A_+ is $Y \cup \{\infty\}$, where ∞ corresponds to the canonical projection $A_+ \rightarrow \mathbb{C}$. (For example A could be $C^\infty(Y)$ if Y is a compact smooth manifold.) Then there are natural isomorphisms

$$K_i(A) \cong K_i(C_0(Y)) \cong K^i(Y \cup \{\infty\}, \{\infty\}) \quad i = 0, 1.$$

In Theorem 6.2 the natural map $A \rightarrow C_0(Y)$ induces an isomorphism on K-theory. That phenomenon holds in much larger generality, and it is called the density theorem in K-theory:

Theorem 6.3. [Bos, Théorème A.2.1]

Let $\phi : A \rightarrow B$ be a morphism of Fréchet algebras. Let A_+ be the unitization of A from (4.1), and extend ϕ to $\phi_+ : A_+ \rightarrow B_+$. Assume that:

- $A_+^\times$ (the set of invertible elements in A_+) is open in A_+ ,
- $B_+^\times$ is open in B_+ ,
- $\phi(A)$ is dense in B ,
- whenever $a \in A_+$ and $\phi_+(a)$ is invertible in B_+ , a is invertible in A_+ .

Then $K_(\phi) : K_*(A) \rightarrow K_*(B)$ is an isomorphism.*

Many versions of K-theory admit an extension to an equivariant setting. We focus on equivariance with respect to actions of a finite group Γ . Atiyah [Ati] already

defined the Γ -equivariant K-theory of a Γ -space Y . By design $K_\Gamma^0(Y)$ classifies Γ -equivariant vector bundles on Y , up to stable equivalence, and $K_\Gamma^1(Y)$ is K_Γ^0 of the suspension of X .

Recall that the classical Chern character assigns to a vector bundle over Y a class in the Čech cohomology of Y . It gives rise to a natural transformation

$$(6.8) \quad \text{Ch} : K^*(Y) \rightarrow \check{H}^*(Y; \mathbb{Q}).$$

If one replaces $K^*(Y)$ by $K^*(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$, then (6.8) is often an isomorphism.

Suppose now that X is a compact Hausdorff space with a Γ -action. In this setting Baum and Connes [BaCo] constructed a Γ -equivariant version of (6.8). Write

$$\tilde{X} = \bigsqcup_{\gamma \in \Gamma} \{\gamma\} \times X^\gamma,$$

and let Γ act on $\tilde{X}$ by $\gamma_1 \cdot (\gamma_2, x) = (\gamma_1 \gamma_2 \gamma^{-1}, \gamma_1 x)$.

Theorem 6.4. [BaCo, Theorem 1.19]

There are natural isomorphisms

$$K_\Gamma^*(X) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} (K^*(\tilde{X}) \otimes_{\mathbb{Z}} \mathbb{C})^\Gamma \xrightarrow{\sim} \check{H}^*(\tilde{X}; \mathbb{C})^\Gamma \cong \check{H}^*(\tilde{X}/\Gamma; \mathbb{C}).$$

The map $K_\Gamma^*(X) \rightarrow \check{H}^*(\tilde{X}; \mathbb{C})^\Gamma$ in Theorem 6.4 is called an equivariant Chern character.

For the equivariant K-theory K_Γ^* of Γ - C^* -algebras we refer to [Phi1], here we restrict ourselves to a few remarks. The equivariant Serre–Swan theorem [Phi1, Theorem 2.3.1] says that for any locally compact Hausdorff Γ -space Y there is a natural isomorphism

$$(6.9) \quad K_\Gamma^\Gamma(C_0(Y)) \cong K_\Gamma^*(Y \cup \{\infty\}, \{\infty\}).$$

Besides K_Γ^* , there is an equivalent way to introduce Γ -equivariance:

Theorem 6.5. [Jul]

For any Γ - C^ -algebra B , there is a natural isomorphism $K_\Gamma^\Gamma(B) \cong K_*(B \rtimes \Gamma)$.*

In view of Theorem 6.5, it makes sense to define

$$K_\Gamma^\Gamma(A) = K_*(A \rtimes \Gamma) \quad \text{for any Fréchet } \Gamma\text{-algebra.}$$

When X is a compact Hausdorff Γ -space, Theorems 6.4, 6.5 and (6.9) combine to

$$(6.10) \quad K_*(C(X) \rtimes \Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \cong K_\Gamma^*(X) \otimes_{\mathbb{Z}} \mathbb{C} \cong \check{H}^*(\tilde{X}; \mathbb{C})^\Gamma.$$

Let $\mathfrak{h} : \Gamma \times \Gamma \rightarrow \mathbb{C}^\times$ be a 2-cocycle. Recall from (2.3) that $\mathbb{C}[\Gamma, \mathfrak{h}] \cong e_{\mathfrak{h}} \mathbb{C}[\Gamma^*]$ for a central extension $Z^* \rightarrow \Gamma^* \rightarrow \Gamma$, a character $c_{\mathfrak{h}}$ of Z^* and the associated idempotent $e_{\mathfrak{h}} \in \mathbb{C}[Z^*]$. Since $e_{\mathfrak{h}}$ is a central idempotent in $C(X) \rtimes \Gamma^*$, we can write

$$K_*(C(X) \rtimes \mathbb{C}[\Gamma, \mathfrak{h}]) = K_*(e_{\mathfrak{h}}(C(X) \rtimes \Gamma^*)) \cong e_{\mathfrak{h}} K_*(C(X) \rtimes \Gamma^*) \cong e_{\mathfrak{h}} K_\Gamma^*(X).$$

Definition 6.6. The $\mathfrak{h}$ -twisted equivariant K-theory of a locally compact Hausdorff Γ -space Y is

$$K_{\Gamma, \mathfrak{h}}^*(Y) := e_{\mathfrak{h}} K_\Gamma^*(Y \cup \{\infty\}, \{\infty\}) \cong K_*(C_0(Y) \rtimes \mathbb{C}[\Gamma, \mathfrak{h}]).$$

This is the K-theory of Γ^* -equivariant vector bundles over X on which Z^* acts as the character $c_{\mathfrak{h}}$. From (6.10) and [Sol10, proof of Theorem 1.2] we deduce that the Γ^* -equivariant Chern character induces an isomorphism

$$(6.11) \quad K_{\Gamma, \mathfrak{h}}^*(X) \otimes_{\mathbb{Z}} \mathbb{C} \cong e_{\mathfrak{h}} K_*(C(X) \rtimes \Gamma^*) \xrightarrow{\sim} e_{\mathfrak{h}} \check{H}^* \left(\bigsqcup_{\gamma \in \Gamma^*} \{\gamma\} \times X^\gamma \right)^{\Gamma^*} \cong \left(\bigoplus_{\gamma \in \Gamma} \check{H}^*(X^\gamma) \otimes \mathfrak{h}^\gamma \right)^\Gamma.$$

When X is in addition a smooth manifold, (6.10) and (6.11) can be composed with Theorem 6.3 (for $C^\infty(X) \rightarrow C(X)$ and related inclusions) to obtain

$$(6.12) \quad \begin{aligned} K_*^\Gamma(C^\infty(X)) \otimes_{\mathbb{Z}} \mathbb{C} &\cong K_*(C^\infty(X) \rtimes \Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \cong H_{dR}^*(\tilde{X})^\Gamma, \\ K_*(C^\infty(X) \rtimes \mathbb{C}[\Gamma, \mathfrak{h}]) \otimes_{\mathbb{Z}} \mathbb{C} &\cong \left(\bigoplus_{\gamma \in \Gamma} H_{dR}^*(X^\gamma) \otimes \mathfrak{h}^\gamma \right)^\Gamma. \end{aligned}$$

There is a natural transformation

$$(6.13) \quad Ch : K_* \rightarrow HP_*$$

from K-theory for Fréchet algebras to periodic cyclic homology for Fréchet algebras, also called the Chern character [Cun1]. It is constructed as the restriction of a bivariant Chern character, and latter has a universal property which makes it unique [Cun1, Korollar 6.5]. Therefore, for any smooth manifold Y ,

$$Ch : K_*(C^\infty(Y)) \rightarrow HP_*(C^\infty(Y), \hat{\otimes}) \cong H_{dR}^*(Y)$$

agrees with the classical Chern character (6.8). This is also checked in a more concrete, purely algebraic setting in [Lod, Proposition 8.3.9]. That compatibility of Chern characters and Theorems 5.15.b and 6.2 imply:

Theorem 6.7. [Con, Sol1]

Let X be a compact smooth manifold.

- (a) $Ch \otimes \text{id} : K_*(C^\infty(X)) \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow HP_*(C^\infty(X), \hat{\otimes})$ is an isomorphism.
- (b) Let Γ be a finite group acting on X and let $\mathfrak{h}$ be a 2-cocycle of Γ . Then

$$Ch \otimes \text{id} : K_*(C^\infty(X) \rtimes \mathbb{C}[\Gamma, \mathfrak{h}]) \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow HP_*(C^\infty(X) \rtimes \mathbb{C}[\Gamma, \mathfrak{h}], \hat{\otimes})$$

is an isomorphism.

In fact Theorem 6.7 applies in much larger generality. Firstly, twisted crossed products can be replaced by Γ -invariants for certain actions of Γ on matrix algebras over $C^\infty(X)$ [Sol1, Theorem 6]. Secondly, Theorem 6.5 also holds for many noncompact manifolds X . Not for all though, because $\otimes_{\mathbb{Z}} \mathbb{C}$ does not commute with infinite direct products [Sol1, Appendix].

6.2. K-theory of $C_r^*(G)$ and $\mathcal{S}(G)$, modulo torsion.

We survey the relations between the topological K-theory and the periodic cyclic homology of $C_r^*(G)$ and $\mathcal{S}(G)$. Recall from (1.7), Definition 1.6 and Theorem 1.45 that

$$\begin{aligned} C_r^*(G) &= \lim_{K \in \text{co}(G)} C_r^*(G, K) = \lim_{n \rightarrow \infty} (G, K_n), \\ \mathcal{S}(G) &= \lim_{K \in \text{co}(G)} \mathcal{S}(G, K) = \bigcup_{n=1}^{\infty} \mathcal{S}(G, K_n). \end{aligned}$$

The algebra $\mathcal{S}(G)$ is not Fréchet, so its topological K-theory presents new challenges. While K-theories have been constructed for wider classes of topological algebras, it is

unclear whether those functors commute with direct limits. To avoid such problems, we simply define

$$K_*(\mathcal{S}(G)) := \lim_{K \in \mathcal{CO}(G)} K_*(\mathcal{S}(G, K)).$$

Notice that this makes sense because each $\mathcal{S}(G, K)$ is a Fréchet algebra, see Theorem 1.7.a.

Proposition 6.8. *There is a commutative diagram*

$$\begin{array}{ccc} \lim_{K \in \mathcal{CO}(G)} K_*(C_r^*(G, K)) & \rightarrow & K_*(C_r^*(G)) \\ \uparrow & & \uparrow \\ \lim_{K \in \mathcal{CO}(G)} K_*(\mathcal{S}(G, K)) & = & K_*(\mathcal{S}(G)) \end{array}$$

in which all the arrows are natural isomorphisms.

Proof. The continuity of K-theory for Banach algebras tells us that upper line is an isomorphism, where the map is induced by the inclusions $C_r^*(G, K) \rightarrow C_r^*(G)$. By Theorems 1.7 and 6.3

(6.14) the inclusion $\mathcal{S}(G, K) \rightarrow C_r^*(G, K)$ induces an isomorphism on K-theory.

Hence the left column of the diagram is a natural isomorphism. We define the map in the right column of the diagram as the composition of the other maps. Then it is a natural isomorphism, and it is induced by the inclusion $\mathcal{S}(G) \rightarrow C_r^*(G)$. $\square$

Like in (5.2), one deduces from Propositions 1.47 and 6.8 and Theorems 6.2 and 6.3 that there are natural isomorphisms

$$(6.15) \quad \begin{array}{ccc} K_*(C_r^*(G)) & \cong & \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} K_*(C_r^*(G, K_{\mathfrak{s}})^{\mathfrak{s}}) \cong \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} K_*(C_r^*(G)^{\mathfrak{s}}) \\ K_*(\mathcal{S}(G)) & \cong & \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} K_*(\mathcal{S}(G, K_{\mathfrak{s}})^{\mathfrak{s}}) \end{array}.$$

Example 6.9. Consider $G = SL_2(F)$ and $\mathfrak{s} = [T, \text{triv}]_G$. From Example 1.33 and Theorem 1.34 we see that the $C_r^*(G)^{\mathfrak{s}}$ is Morita equivalent to

$$(C(S^1) \otimes M_2(\mathbb{C}))^{S_2} \oplus \mathbb{C}_{St} := A \oplus \mathbb{C}_{St}.$$

As $K_*(\mathbb{C}_{St}) = K_0(\mathbb{C}) = \mathbb{Z}$, we focus on A . By a suitable choice of coordinates, we can achieve that $S_2 = \{1, s_{\alpha}\}$ acts by

$$(s_{\alpha}a)(z) = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} a(z^{-1}) \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \quad a \in C(S^1) \otimes M_2(\mathbb{C}), z \in S^1.$$

This shows that every S_2 -orbit in $S^1 \setminus \{-1\}$ supports a unique irreducible A -representation, while there are precisely two inequivalent irreducible A -representations with $C(S^1)^{S_2}$ -character -1. The upper half circle is a fundamental domain for S_2 acting on S^1 , and it is homeomorphic to $[-1, 1]$ by taking real parts. Evaluation of A at -1 yields a short exact sequence of C^* -algebras

$$0 \rightarrow C_0((-1, 1]) \otimes M_2(\mathbb{C}) \rightarrow A \rightarrow \mathbb{C}^2 \rightarrow 0.$$

The associated six-term exact sequence is

$$\begin{array}{ccccccc} K^0((-1, 1]) = 0 & \rightarrow & K_0(A) & \rightarrow & K_0(\mathbb{C}^2) = \mathbb{Z}^2 \\ \uparrow & & & & \downarrow \\ K_1(\mathbb{C}^2) = 0 & \leftarrow & K_1(A) & \leftarrow & K^1((-1, 1]) = 0 \end{array}.$$

Here $K_*((-1, 1]) = 0$ because the algebra $C_0((-1, 1])$ is homotopy equivalent to 0. We find that $K_*(A) \cong K_0(A) \cong \mathbb{Z}^2$.

With methods like in Example 6.9 one can compute $K_*(C_r^*(G)^\mathfrak{s})$ for many Bernstein blocks $\text{Rep}(G)^\mathfrak{s}$, but the computations quickly become cumbersome.

To analyse $K_*(\mathcal{S}(G, K_\mathfrak{s})^\mathfrak{s})$ and $K_*(C_r^*(G, K_\mathfrak{s})^\mathfrak{s})$ in general, we relate them to the previous section. Theorem 6.7 can be applied to the Fréchet algebras appearing in the Plancherel isomorphism for $\mathcal{S}(G)$ (Theorem 1.32).

Theorem 6.10. [Sol1, Theorem 10 and Corollary 11]

For any $K \in \mathcal{CO}(G)$, the Chern character induces an isomorphism

$$K_*(\mathcal{S}(G, K)) \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow HP_*(\mathcal{S}(G, K), \hat{\otimes}).$$

A combination of the results in this and the previous paragraph yields a description of $K_*(C_r^*(G))$ modulo torsion:

Corollary 6.11. (a) *Fix $\mathfrak{s} = [L, \sigma]_G \in \mathfrak{B}(G)$. There are isomorphisms*

$$\begin{aligned} K_*(C_r^*(G)^\mathfrak{s}) \otimes_{\mathbb{Z}} \mathbb{C} &\cong K_*(C_r^*(G, K_\mathfrak{s})^\mathfrak{s}) \otimes_{\mathbb{Z}} \mathbb{C} \cong K_*(\mathcal{S}(G, K_\mathfrak{s})^\mathfrak{s}) \otimes_{\mathbb{Z}} \mathbb{C} \\ &\cong HP_*(\mathcal{S}(G, K_\mathfrak{s})^\mathfrak{s}, \hat{\otimes}) \cong HP_*(C^\infty(X_{\text{nr}}^u(L)) \rtimes \mathbb{C}[W_\mathfrak{s}^e, \mathfrak{h}_\mathfrak{s}^{-1}], \hat{\otimes}) \\ &\cong K_*(C^\infty(X_{\text{nr}}^u(L)) \rtimes \mathbb{C}[W_\mathfrak{s}^e, \mathfrak{h}_\mathfrak{s}^{-1}]) \otimes_{\mathbb{Z}} \mathbb{C} \\ &\cong K_*(C(X_{\text{nr}}^u(L)) \rtimes \mathbb{C}[W_\mathfrak{s}^e, \mathfrak{h}_\mathfrak{s}^{-1}]) \otimes_{\mathbb{Z}} \mathbb{C} \cong K_{W_\mathfrak{s}^e, \mathfrak{h}_\mathfrak{s}^{-1}}(X_{\text{nr}}^u(L)) \otimes_{\mathbb{Z}} \mathbb{C}. \end{aligned}$$

(b) *There are isomorphisms*

$$K_*(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{C} \cong K_*(\mathcal{S}(G)) \otimes_{\mathbb{Z}} \mathbb{C} \cong \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} K_{W_\mathfrak{s}^e, \mathfrak{h}_\mathfrak{s}^{-1}}(X_{\text{nr}}^u(L)) \otimes_{\mathbb{Z}} \mathbb{C}.$$

Proof. (a) The first isomorphism is (6.15) and the second comes from (6.14). Then we use Theorems 6.10 and 5.19. The fifth isomorphism is an instance of Theorem 6.7, the sixth comes from Theorem 6.3 and the last step is Definition 6.6. $\square$

(b) This follows from Proposition 6.8, (6.15) and part (a). $\square$

Corollary 6.11 gives a description of the group $K_*(C_r^*(G))$ modulo torsion. In some cases that determines $K_*(C_r^*(G))$ up to isomorphism, because it does not have torsion elements:

Theorem 6.12. [Sol7, Theorem 5.3]

Let $\mathcal{G}$ be an inner form of GL_n , or a symplectic group, or a special orthogonal group (not necessarily F -split). Then $K_(C_r^*(G))$ is a free abelian group.*

6.3. Relation with the Baum–Connes conjecture.

K-theory of group- C^* -algebras figures prominently in the Baum–Connes conjecture [BCH, Val]. For any Hausdorff space Y with a proper G -action, Kasparov’s equivariant KK-theory provides a notion of the G -equivariant K-homology $K_*^G(Y)$. There is an assembly map

$$\mu_{G,Y} : K_*^G(Y) \rightarrow K_*(C_r^*(G)),$$

which can be defined in several (analytic) ways [Val, §6]. The Baum–Connes conjecture asserts that $\mu_{G,Y}$ is an isomorphism when Y is a classifying space for proper G -actions, as in [BCH, §1].

For our reductive p -adic group G , we can take as Y the (extended) Bruhat–Tits building $\mathcal{BG}$, see [BCH, §6]. In this case the Baum–Connes conjecture is known to hold, a celebrated result of V. Lafforgue [Laf].

There is a more algebraic version of equivariant K-homology, called (equivariant) chamber homology, see [HiNi, §3] and [BHP, §2]. It is available for any totally

disconnected locally compact group G' acting on a polysimplicial complex X' , and can be defined as follows. Let X'_n be the set of n -dimensional polysimplices of X' . For every $\sigma \in X'_n$, the stabilizer group G'_σ is compact, and we form its (complex-valued) Hecke algebra $\mathcal{H}(G'_\sigma)$. The usual boundary operator for polysimplices induces a boundary operator

$$\partial_* \quad \text{on} \quad \bigoplus_{n \geq 0} \bigoplus_{\sigma \in X'_n} \mathcal{H}(G'_\sigma),$$

which decreases the degrees by one. Taking G -coinvariants, we form the differential complex with terms $C_n = \left(\bigoplus_{\sigma \in X'_n} \mathcal{H}(G'_\sigma) \right)_{G'}$. The homology of (C_*, ∂_*) is denoted

$$CH_*^{G'}(X') = \bigoplus_{n \geq 0} CH_n^{G'}(X').$$

This is similar to $HH_*(\mathcal{H}(G'))$, in the sense that it looks somewhat like a resolution which could be used in the definition of HH_n as a derived functor. Higson and Nistor made that precise:

Theorem 6.13. [HiNi, Theorem 4.2]

Assume that, for every $K' \in \mathcal{CO}(G')$, the fixed point set of K' in X' is nonempty and contractible. Then there are natural isomorphisms

$$CH_n^{G'}(X') \cong HH_n(\mathcal{H}(G'))_{\text{cpt}},$$

where *cpt* means "supported on compact elements" as in Theorem 5.21.

The definition of $CH_*^{G'}(X')$ can be reformulated with $\bigoplus_{\sigma \in X'_n/G'} R(G'_\sigma) \otimes_{\mathbb{Z}} \mathbb{C}$ instead of C_n , see [BHP, p. 215]. In that way one can regard equivariant chamber homology as an algebraic combination of ordinary homology and virtual representations of compact subgroups. This is reminiscent of how the algebraic K-theory of $\mathcal{H}(G)$ was computed in [BaLü].

Similar to (6.13), Voigt constructed an equivariant Chern character

$$\text{Ch}^{G'} : K_i^{G'} \rightarrow \bigoplus_{m \geq 0} CH_{i+2m}^{G'}.$$

Theorem 6.14. [Voi, §6]

Suppose that X' is a finite dimensional locally finite polysimplicial complex, and that X'/G' is compact. Then

$$\text{Ch}^{G'} \otimes \text{id} : K_i^{G'}(X') \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow \bigoplus_{m \geq 0} CH_{i+2m}^{G'}(X') \quad \text{is an isomorphism.}$$

For G acting on $\mathcal{BG}$, the conditions in Theorem 6.13 are fulfilled by the CAT(0)-property of $\mathcal{BG}$ and the Bruhat–Tits fixed point theorem [Tit, §2.3]. The conditions on X' in Theorem 6.14 hold for $\mathcal{BG}$ by [Tit, §2.2 and §2.5].

The various homology theories associated to G , its group algebras and $\mathcal{BG}$ can be combined in the following diagram, which is an extended version of [BHP, Proposition 9.4]:

$$(6.16) \quad \begin{array}{ccccc} K_i^G(\mathcal{BG}) & \xrightarrow{\mu_{G, \mathcal{BG}}} & K_i(C_r^*(G)) & \longleftarrow & K_i(\mathcal{S}(G)) \\ \downarrow \text{Ch}^G & & & & \downarrow \text{Ch} \\ \bigoplus_{m \geq 0} CH_{i+2m}^G(\mathcal{BG}) & \xrightarrow{\sim} & \bigoplus_{m \geq 0} HH_{i+2m}(\mathcal{H}(G))_{\text{cpt}} \cong HP_i(\mathcal{H}(G)) & \longrightarrow & HP_i(\mathcal{S}(G), \hat{\otimes}_b) \end{array}$$

Here the Chern character $\text{Ch} : K_*(\mathcal{S}(G)) \rightarrow HP_*(\mathcal{S}(G), \hat{\otimes}_b)$ is defined as the direct limit over $K \in \mathbf{CO}(G)$ of $\text{Ch} : K_*(\mathcal{S}(G, K)) \rightarrow HP_*(\mathcal{S}(G, K), \hat{\otimes})$, which makes sense by Theorem 5.17 and our definition of $K_*(\mathcal{S}(G))$.

Theorem 6.15. (a) *All the horizontal maps in the diagram (6.16) are natural isomorphisms.*
 (b) *The vertical maps in (6.16) become isomorphisms if we apply $\otimes_{\mathbb{Z}} \mathbb{C}$ to the K -groups.*
 (c) *If we extend (6.16) with inverses of all the arrows that are isomorphisms, then the diagram commutes.*

Proof. (a) For the upper line we refer to [Laf] and Proposition 6.8. The first isomorphism on the lower line is Theorem 6.13, the second is Theorem 5.21 and the third is Corollary 5.20.

(b) This follows from Theorems 6.14 and 6.10.

(c) See [Sol2, Theorem 3.7 and Lemma 3.9]. $\square$

All the terms in (6.16) that do not involve $\mathcal{B}G$ admit a natural Bernstein decomposition, see Lemma 5.18 and (6.15). We computed the summands associated to an arbitrary $\mathfrak{s} \in \mathfrak{B}(G)$ explicitly in Theorem 5.19 and Corollary 6.11. Thus the isomorphisms $\mu_{G, \mathcal{B}G}$ and Theorem 6.13 provide natural Bernstein decompositions of $K_*^G(\mathcal{B}G)$ and $CH_*^G(\mathcal{B}G)$.

Unfortunately it remains unclear how to describe these decompositions in terms of the action of G on its Bruhat–Tits building $\mathcal{B}G$. This reflects the difficulty of recovering the Bernstein decomposition via restrictions of G -representations to compact open subgroups of G .

6.4. A progenerator of $\text{Mod}(C_r^*(G)^{\mathfrak{s}})$.

These and the next paragraphs contain some new material, which aims to compute $K_*(C_r^*(G))$ including torsion elements. Fix $\mathfrak{s} = [L, \sigma]_G \in \mathfrak{B}(G)$, with a unitary $\sigma \in \text{Irr}_{\text{cusp}}(L)$. Recall from Proposition 3.4 that $\text{Rep}(G)^{\mathfrak{s}}$ has a canonical progenerator

$$\Pi_{\mathfrak{s}} = I_P^G(\text{ind}_{L_1}^L(\sigma)).$$

For the remainder of the section, we pick $K \in \mathbf{CO}(G)$ such that

$$(6.17) \quad \mathcal{H}(G)^{\mathfrak{s}} \quad \text{and} \quad \mathcal{H}(G, K)^{\mathfrak{s}} \text{ are Morita equivalent,}$$

for instance $K_{\mathfrak{s}}$ from (5.2). Then $\langle K \rangle \Pi_{\mathfrak{s}}$ is a progenerator of $\text{Mod}(\mathcal{H}(G, K)^{\mathfrak{s}})$. Furthermore Proposition 1.47 shows that

$$(6.18) \quad C_r^*(G)^{\mathfrak{s}} \quad \text{and} \quad C_r^*(G, K)^{\mathfrak{s}} \text{ are Morita equivalent.}$$

We define

$$\begin{aligned} i_{\mathfrak{s}}^c &:= C_r(G)^{\mathfrak{s}} \otimes_{\mathcal{H}(G)^{\mathfrak{s}}} \Pi_{\mathfrak{s}}, \\ i_{\mathfrak{s}, K}^c &:= C_r(G, K)^{\mathfrak{s}} \otimes_{\mathcal{H}(G, K)^{\mathfrak{s}}} \langle K \rangle \Pi_{\mathfrak{s}}. \end{aligned}$$

Lemma 6.16. (a) *The $C_r^*(G, K)^{\mathfrak{s}}$ -module $\Pi_{\mathfrak{s}, K}^c$ is a progenerator of $\text{Mod}(C_r^*(G, K)^{\mathfrak{s}})$.*
 (b) *The $C_r^*(G)^{\mathfrak{s}}$ -module $\Pi_{\mathfrak{s}}^c$ is a progenerator of $\text{Mod}(C_r^*(G)^{\mathfrak{s}})$ and $\Pi_{\mathfrak{s}, K}^c = \langle K \rangle \Pi_{\mathfrak{s}}^c$.*

Proof. (a) The algebra $\mathcal{H}(G, K)^{\mathfrak{s}}$ is unital and its module $\langle K \rangle \Pi_{\mathfrak{s}}$ is finitely generated projective. So it is a direct summand of a free module of finite rank, say $\mathcal{H}(G, K)^n$. Then $\Pi_{\mathfrak{s}, K}^c$ is a direct summand of $(C_r^*(G, K)^{\mathfrak{s}})^n$, so it is finitely generated and projective. For any nonzero $M \in \text{Mod}(C_r^*(G, K)^{\mathfrak{s}})$ we have

$$\text{Hom}_{C_r^*(G, K)^{\mathfrak{s}}}(\Pi_{\mathfrak{s}, K}^c, M) \cong \text{Hom}_{\mathcal{H}(G, K)^{\mathfrak{s}}}(\langle K \rangle \Pi_{\mathfrak{s}}, M) \neq 0.$$

Therefore $\Pi_{\mathfrak{s},K}^c$ generates $\text{Mod}(C_r^*(G, K)^{\mathfrak{s}})$.

(b) By part (a) and (6.18), $C_r^*(G)^{\mathfrak{s}}\langle K \rangle \otimes_{C_r^*(G,K)^{\mathfrak{s}}} \Pi_{\mathfrak{s},K}^c$ is a progenerator of $\text{Mod}(C_r^*(G)^{\mathfrak{s}})$. It can be rewritten as

$$(6.19) \quad C_r^*(G)^{\mathfrak{s}}\langle K \rangle \otimes_{C_r^*(G,K)^{\mathfrak{s}}} C_r^*(G, K)^{\mathfrak{s}} \otimes_{\mathcal{H}(G,K)^{\mathfrak{s}}} \langle K \rangle \mathcal{H}(G)^{\mathfrak{s}} \otimes_{\mathcal{H}(G)^{\mathfrak{s}}} \Pi_{\mathfrak{s}} = \\ C_r^*(G)^{\mathfrak{s}}\langle K \rangle \otimes_{\mathcal{H}(G,K)^{\mathfrak{s}}} \langle K \rangle \mathcal{H}(G)^{\mathfrak{s}} \otimes_{\mathcal{H}(G)^{\mathfrak{s}}} \Pi_{\mathfrak{s}}.$$

By (6.17) this reduces to $C_r(G)^{\mathfrak{s}} \otimes_{\mathcal{H}(G)^{\mathfrak{s}}} \Pi_{\mathfrak{s}} = \Pi_{\mathfrak{s}}^c$, which is therefore also a progenerator. It follows that

$$\langle K \rangle \Pi_{\mathfrak{s}}^c = \langle K \rangle C_r^*(G)^{\mathfrak{s}}\langle K \rangle \otimes_{C_r^*(G,K)^{\mathfrak{s}}} \Pi_{\mathfrak{s},K}^c = \Pi_{\mathfrak{s},K}^c. \quad \square$$

It is easier to work with $\Pi_{\mathfrak{s},K}^c$ than with $\Pi_{\mathfrak{s}}^c$ because the algebra $C_r^*(G, K)^{\mathfrak{s}}$ is unital and its irreducible modules have finite dimension. The upcoming results have versions for $\Pi_{\mathfrak{s}}^c$, which can be derived with arguments like in the proof of Lemma 6.16.b. By Lemma 6.16.a there exists $n_{\mathfrak{s}}^c \in \mathbb{N}$ and an idempotent $e_{\mathfrak{s}}^c \in M_{n_{\mathfrak{s}}^c}(C_r^*(G, K)^{\mathfrak{s}})$ such that

$$\Pi_{\mathfrak{s},K}^c \cong (C_r^*(G, K)^{\mathfrak{s}})^{n_{\mathfrak{s}}^c} e_{\mathfrak{s}}^c.$$

Then $\text{End}_{C_r^*(G,K)^{\mathfrak{s}}}(\Pi_{\mathfrak{s},K}^c)^{op}$, which by definition acts from the right on $\Pi_{\mathfrak{s},K}^c$, is isomorphic to $e_{\mathfrak{s}}^c M_{n_{\mathfrak{s}}^c}(C_r^*(G, K)^{\mathfrak{s}}) e_{\mathfrak{s}}^c$. In particular $\text{End}_{C_r^*(G,K)^{\mathfrak{s}}}(\Pi_{\mathfrak{s},K}^c)^{op}$ is isomorphic to a corner in the Banach algebra $M_{n_{\mathfrak{s}}^c}(C_r^*(G, K)^{\mathfrak{s}})$, so

$$(6.20) \quad \text{End}_{C_r^*(G,K)^{\mathfrak{s}}}(\Pi_{\mathfrak{s},K}^c)^{op} \quad \text{and} \quad \text{End}_{C_r^*(G,K)^{\mathfrak{s}}}(\Pi_{\mathfrak{s},K}^c) \text{ are Banach algebras.}$$

As an instance of Proposition 3.1, we find that

$$(6.21) \quad C_r^*(G, K)^{\mathfrak{s}} \quad \text{and} \quad \text{End}_{C_r^*(G,K)^{\mathfrak{s}}}(\Pi_{\mathfrak{s},K}^c)^{op} \text{ are Morita equivalent.}$$

This works both as abstract rings and as Banach algebras, with the Morita bimodules

$$\Pi_{\mathfrak{s},K}^c \quad \text{and} \quad e_{\mathfrak{s}}^c (C_r^*(G, K)^{\mathfrak{s}})^{n_{\mathfrak{s}}^c} \cong \text{Hom}_{C_r^*(G,K)^{\mathfrak{s}}}(\Pi_{\mathfrak{s},K}^c, C_r^*(G, K)^{\mathfrak{s}}).$$

Recall the subset $\Delta(G, \mathfrak{s}) \subset \Delta(G)$ from (1.34).

Lemma 6.17. *There exist positive integers $n_{\mathfrak{d}}$ for $\mathfrak{d} = [M, \delta] \in \Delta(G, \mathfrak{s})$, such that*

$$\Pi_{\mathfrak{s},K}^c \cong \bigoplus_{[M,\delta] \in \Delta(G,\mathfrak{s})} \langle K \rangle I_{MU_Q}^Q(\delta \otimes C(X_{\text{nr}}^u(M)))^{n_{[M,\delta]}} \quad \text{as } C_r^*(G, K)^{\mathfrak{s}}\text{-modules.}$$

Proof. The $\mathcal{H}(G, K)^{\mathfrak{s}}$ -module $\langle K \rangle \Pi_{\mathfrak{s}}$ is a direct integral (in an algebraic sense) of the representations $I_P^G(\sigma \otimes \chi)$ with $\chi \in X_{\text{nr}}(L)$. Tensoring with $C_r(G, K)^{\mathfrak{s}}$ kills all nontempered irreducible subquotients. By Theorem 1.27 every irreducible tempered subquotient of $\Pi_{\mathfrak{s}}$ arises as a direct summand of $I_Q^G(\delta)$, where $Q \subset G$ is a parabolic subgroup containing P , M is a Levi factor of Q containing L and $\delta \in \text{Irr}(M)$ is square-integrable modulo centre. Then δ is a subquotient of $I_{M \cap P}^M(\sigma \otimes \chi)$ for some $\chi \in X_{\text{nr}}(L)$. Sometimes the copy of $I_Q^G(\delta)$ in $I_P^G(\sigma \otimes \chi)$ is annihilated by applying $C_r(G, K)^{\mathfrak{s}} \otimes_{\mathcal{H}(G,K)^{\mathfrak{s}}}$, because it belongs to a subrepresentation of $I_P^G(\sigma \otimes \chi)$ generated by elements from nontempered representations.

For each $\chi_M \in X_{\text{nr}}^u(M)$ the discrete series representation

$$(6.22) \quad \delta \otimes \chi_M \quad \text{is a subquotient of} \quad I_{M \cap P}^M(\sigma \otimes \chi_M|_{L\chi}).$$

Here $(L, \sigma \otimes \chi)$ represents the cuspidal support of δ and, while χ is not unique, there are only finitely many possibilities because $\text{Sc}(\delta)$ has only finitely many representatives in $\text{Irr}(L)$. It follows that $\Pi_{\mathfrak{s},K}^c$ is a direct integral of the representations

$I_Q^G(\delta \otimes \chi_M)$ with $[M, \delta] \in \Delta(G, \mathfrak{s})$, and the number of times such a representations appears as a subquotient of $\Pi_{\mathfrak{s}, K}^c$ depends only on $[M, \delta]$. Each $n_{\mathfrak{d}}$ is nonzero because $\Pi_{\mathfrak{s}, K}^c$ is a progenerator (Lemma 6.16.a).

We know the structure of $C_r^*(G, K)^{\mathfrak{s}}$ from Theorem 1.34 and (1.34). More concretely, by (1.27) $C_r^*(G, K)^{\mathfrak{s}}$ has direct summands

$$C_r^*(G, K)_{[M, \delta]} = \left(C(X_{\text{nr}}^u(M)) \otimes \text{End}_{\mathbb{C}}(\text{ind}_{K_0 \cap Q}^{K_0}(V_{\delta})^K) \right)^{W_{[M, \delta]}}.$$

The projectivity of $\Pi_{\mathfrak{s}, K}^c$ as $C_r^*(G, K)^{\mathfrak{s}}$ -module (Lemma 6.16.a) implies that the direct integral of the representations $I_Q^G(\delta \otimes \chi_M)$ from (6.22) occurs in the form

$$\langle K \rangle I_Q^G(C(X_{\text{nr}}^u(M)) \otimes \delta). \quad \square$$

6.5. Construction of an action on the progenerator.

We want to define an action of $C(X_{\text{nr}}^u(L)) \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{d}_{\mathfrak{s}}]$ on $\Pi_{\mathfrak{s}, K}^c$ by $C_r^*(G, K)^{\mathfrak{s}}$ -intertwiners, which for finite length tempered representations recovers Theorems 3.14 and 3.15. Unfortunately this will be a rather technical affair, because we have to involve a lot of arguments from [Sol9]. The easy part is to let $C(X_{\text{nr}}^u(L))$ act canonically on $\Pi_{\mathfrak{s}}^c$ and $\Pi_{\mathfrak{s}, K}^c$, as follows.

The set of tempered representations in $X_{\text{nr}}(L)\sigma$ is $X_{\text{nr}}^u(L)\sigma$, and by Lemma 1.37 every $\sigma \otimes \chi$ can be written uniquely as $\sigma \otimes \chi |\chi|^{-1} \otimes |\chi|$, where $\sigma \otimes \chi |\chi|^{-1}$ is unitary and $\chi \in X_{\text{nr}}^+(L)$. We let $f \in C(X_{\text{nr}}^u(L))$ act on $I_P^G(\sigma \otimes \chi)$ as multiplication by $f(\chi |\chi|^{-1})$. This can be restricted to tempered subquotients and in particular to any $I_Q^G(\delta \otimes \chi_M)$ occurring in $I_P^G(\sigma \otimes \chi)$. When we vary χ_M in $X_{\text{nr}}^u(M)$, we see that this integrates to an action of f on $I_Q^G(C(X_{\text{nr}}^u(M)) \otimes \delta)$, namely

$$(6.23) \quad I_Q^G \text{ applied to multiplication by } [\chi_M \mapsto f(\chi_M |L\chi| |\chi|^{-1})] \text{ on } C(X_{\text{nr}}^u(M)) \otimes \delta.$$

Via Lemma 6.16 this gives rise to a canonical action of $C(X_{\text{nr}}^u(L))$ on $\Pi_{\mathfrak{s}}^c$ by G -intertwiners, which restricts to an action of $C(X_{\text{nr}}^u(L))$ on $\Pi_{\mathfrak{s}, K}^c$ by $C_r^*(G, K)^{\mathfrak{s}}$ -intertwiners.

Recall from Theorem 3.8 or [Sol9] that

$$(6.24) \quad \text{End}_G(\Pi_{\mathfrak{s}}) \otimes_{\mathcal{O}(X_{\text{nr}}(L))} \mathbb{C}(X_{\text{nr}}(L)) \cong \mathbb{C}(X_{\text{nr}}(L)) \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{d}_{\mathfrak{s}}].$$

Here $W_{\mathfrak{s}}^e$ appears as a set of intertwining operators $\mathcal{T}_w$ that multiply as in $\mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{d}_{\mathfrak{s}}]$. More precisely, the specialization

$$\mathcal{T}_w : I_P^G(\sigma \otimes \chi) \rightarrow I_P^G(\sigma \otimes w(\chi))$$

is rational as a function of $\chi \in X_{\text{nr}}(L)$. On the other hand, from Theorem 3.13 or the underlying [Sol9, Proposition 7.3] we obtain

$$\mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{d}_{\mathfrak{s}}] \subset \mathbb{H}(\mathfrak{t}, W_{\mathfrak{s}, \sigma \otimes \chi}, k_{\sigma \otimes \chi}, \mathfrak{d}_{\sigma \otimes \chi}) \subset \text{End}_G(\Pi_{\mathfrak{s}})_{U_{\sigma \otimes \chi}}^{an}.$$

Here $U_{\sigma \otimes \chi}$ is a neighborhood of $X_{\text{nr}}^+(L)\chi$ in $X_{\text{nr}}(L)$, a superscript an stands for complex analytic functions and $\text{End}_G(\Pi_{\mathfrak{s}})_{U_{\sigma \otimes \chi}}^{an}$ consists of endomorphisms of $\Pi_{\mathfrak{s}} \otimes_{\mathcal{O}(X_{\text{nr}}(L))} C^{an}(U_{\sigma \otimes \chi})$. To construct an action of $\mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{d}_{\mathfrak{s}}]$ on $\Pi_{\mathfrak{s}}^c$ and on $\Pi_{\mathfrak{s}, K}^c$, we will need to combine both pictures of $\mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{d}_{\mathfrak{s}}]$.

Example 6.18. Consider $G = SL_2(F)$ and $\mathfrak{s} = [T, \text{triv}]$. Lemma 6.17 works out to

$$\Pi_{\mathfrak{s}, K}^c = \langle K \rangle I_B^G(C(X_{\text{nr}}^u(T))) \oplus \langle K \rangle \text{St}_G.$$

By [Sol9, (5.20)] the operator $\mathcal{T}_{s_\alpha}$ has only one singularity, at $\chi(h_\alpha^\vee) = q_F$, or equivalently for $\chi = \delta_B^{1/2}$. In particular $\mathcal{T}_{s_\alpha}$ is regular on $X_{\text{nr}}^u(T)$, which means that the action of $\mathcal{T}_{s_\alpha}$ on

$$\Pi_{\mathfrak{s}} \otimes_{\mathcal{O}(X_{\text{nr}}(T))} \mathbb{C}(X_{\text{nr}}(T))$$

from (6.24) extends naturally to an action on

$$\Pi_{\mathfrak{s}} \otimes_{\mathcal{O}(X_{\text{nr}}(T))} C(X_{\text{nr}}^u(T)) = I_B^G(C(X_{\text{nr}}^u(T)))$$

and on $\langle K \rangle I_B^G(C(X_{\text{nr}}^u(T)))$. Let us check that the specialization of this action at any $\chi \in X_{\text{nr}}^u(T)$ recovers Theorem 3.14.

When $s_\alpha(\chi) \neq \chi$, that follows from [Sol9, Lemma 8.2]. When $s_\alpha(\chi) = \chi$, we have $\chi = \chi_-$ or $\chi = \text{triv}$. For χ_- , the relevant Hecke algebra is $\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W_{\mathfrak{s}, \chi_-}]$ with $W_{\mathfrak{s}, \chi_-} = S_2$. Via [Sol9, Lemma 7.1], $\mathcal{T}_{s_\alpha}$ specialized at χ_- is identified with a scalar multiple of N_{s_α} . The same identification is used in the constructions for Theorem 3.14. For triv_T the relevant Hecke algebra is $\mathbb{H}(\mathfrak{t}, S_2, \log(q_F))$, and [Sol9, Lemma 7.1] identifies the specialization of $\mathcal{T}_{s_\alpha}$ at triv_T with

$$\tau_{s_\alpha} = -1 + (N_{s_\alpha} + 1) \frac{h_\alpha^\vee}{\log(q_F) + h_\alpha^\vee}$$

from Theorem 2.10. Now the group $W_{\mathfrak{s}}$ has two incarnations $\{1, N_{s_\alpha}\}$ and $\{1, \tau_{s_\alpha}\}$, both of which act on the specialization of $\Pi_{\mathfrak{s}}$ at $\chi = \text{triv}_T$. By direct calculations one checks that the two actions of $W_{\mathfrak{s}}$ on $I_B^G(\text{triv}_T)$ are both equivalent to the regular representation.

We move on to the Steinberg representation. There are short exact sequences

$$(6.25) \quad \begin{array}{ccccccc} 0 & \rightarrow & \text{St}_G & \rightarrow & I_B^G(\delta_B^{1/2}) & \rightarrow & \text{triv}_G \rightarrow 0, \\ 0 & \rightarrow & \text{triv}_G & \rightarrow & I_B^G(\delta_B^{-1/2}) & \rightarrow & \text{St}_G \rightarrow 0. \end{array}$$

The standard intertwining operator

$$J(s_\alpha, B, \text{triv}_T, \delta_B^{1/2}) : I_B^G(\delta_B^{1/2}) \rightarrow I_B^G(\delta_B^{-1/2})$$

annihilates St_G and sends $\text{triv}_G \cong I_B^G(\delta_B^{1/2})/\text{St}_G$ bijectively to $\text{triv}_G \subset I_B^G(\delta_B^{-1/2})$. Similarly

$$J(s_\alpha, B, \text{triv}_T, \delta_B^{-1/2}) : I_B^G(\delta_B^{-1/2}) \rightarrow I_B^G(\delta_B^{1/2})$$

induces an isomorphism $I_B^G(\delta_B^{-1/2})/\text{triv}_G \rightarrow \text{St}_G$. The specialization of $\mathcal{T}_{s_\alpha}$ at $I_B^G(\delta_B^{1/2})$ equals $J(s_\alpha, B, \text{triv}_T, \delta_B^{-1/2})$ times a function of $\chi \in X_{\text{nr}}(L)$ which has a pole at $\delta_B^{1/2}$. The specialization of $\mathcal{T}_{s_\alpha}$ at $I_B^G(\delta_B^{-1/2})$ equals $J(s_\alpha, B, \text{triv}_T, \delta_B^{1/2})$ times a nonzero scalar. In both cases $\mathcal{T}_{s_\alpha}$ induces a "singular map" from St_G to St_G , 0 in one direction and ∞ in the other direction. When we apply $C_r^*(G, K)^{\mathfrak{s}} \otimes_{\mathcal{H}(G, K)^{\mathfrak{s}}}$ to the K -invariant vectors in (6.25), $\langle K \rangle I_B^G(\delta_B^{-1/2})$ becomes $\langle K \rangle \text{St}_G$ because triv_G is killed. By this tensoring the entire representation $I_B^G(\delta_B^{1/2})$ is annihilated, because it is generated by vectors from a copy of the nontempered representation triv_G . Therefore (6.25) gives rise to only one copy of $\langle K \rangle \text{St}_G$ in $\Pi_{\mathfrak{s}, K}^c$. The $\text{End}_G(\Pi_{\mathfrak{s}})$ -module

$$\text{Hom}_G(\Pi_{\mathfrak{s}}, \text{St}_G) \cong \text{Hom}_G(I_B^G(\delta_B^{-1/2}), \text{St}_G) \cong \mathbb{C}$$

admits an action of the graded Hecke algebra $\mathbb{H}(\mathfrak{t}, S_2, \log(q_F))$. The element N_{s_α} of that algebra acts as -1 on $\text{Hom}_G(\Pi_{\mathfrak{s}}, \text{St}_G)$, because it corresponds to the Steinberg

representation $\text{St}_{\mathbb{H}}$ of $\mathbb{H}(\mathfrak{t}, S_2, \log(q_F))$. In [Sol9, (7.7) and Proposition 7.3],

$$(6.26) \quad \mathcal{T}_{s_\alpha} \text{ is mapped to } \tau_{s_\alpha}.$$

There is still a singularity involved, because $\text{St}_{\mathbb{H}}(h_\alpha^\vee) = -\log(q_F)$, see Example 2.14. But in the end we want to relate to Theorems 3.14 and 3.15, which run via Theorem 2.24. In Theorem 2.24, ζ_0 sends tempered $\mathbb{H}(\mathfrak{t}, S_2, \log(q_F))$ -modules to tempered $\mathbb{H}(\mathfrak{t}, S_2, 0)$ -modules. This means that we actually have to map $\mathcal{T}_{s_\alpha}$ to an element of $\mathbb{H}(\mathfrak{t}, S_2, 0)$, namely

$$(6.27) \quad -1 + (N_{s_\alpha} + 1) \frac{h_\alpha^\vee}{0 + h_\alpha^\vee} = N_{s_\alpha}.$$

In this way the singularity disappears. Since N_{s_α} acts as -1 on $\text{St}_{\mathbb{H}}$ and on $\zeta_0(\text{St}_{\mathbb{H}}) = \mathbb{C}_0 \otimes \text{sign}$, $\mathcal{T}_{s_\alpha}$ acts as -1 on $\text{Hom}_G(\Pi_{\mathfrak{s}}, \text{St}_G)$ in Theorem 3.14.a. This forces us to define that $\mathcal{T}_{s_\alpha}$ acts as -1 on St_G and on

$$\langle K \rangle \text{St}_G \cong C_r^*(G, K)^s \otimes_{\mathcal{H}(G, K)^s} \langle K \rangle I_B^G(\delta_B^{-1/2}).$$

Summarizing: we constructed a group action of $\{1, \mathcal{T}_{s_\alpha}\}$ on $\Pi_{\mathfrak{s}}^c$ and on $\Pi_{\mathfrak{s}, K}^c$, such that the induced action on

$$\text{Hom}_G(\Pi_{\mathfrak{s}}^c, I_B^G(\chi)) \cong \text{Hom}_{C_r^*(G, K)^s}(\Pi_{\mathfrak{s}, K}^c, \langle K \rangle I_B^G(\chi))$$

is equivalent to the action of $\{1, N_{s_\alpha}\}$ on that space (as in Theorem 3.14).

Example 6.18 and [Lus, §8] guide us to the desired action on $\Pi_{\mathfrak{s}, K}^c$.

Theorem 6.19. *There exists an action of $\mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{d}_{\mathfrak{s}}]$ on $\Pi_{\mathfrak{s}, K}^c$ by $C_r^*(G, K)^s$ -intertwiners, such that:*

- (a) *It combines with (6.23) to an action of $C(X_{\text{nr}}^u(L)) \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{d}_{\mathfrak{s}}]$ on $\Pi_{\mathfrak{s}, K}^c$.*
- (b) *Part (a) provides a homomorphism of Banach algebras*

$$\phi_{\mathfrak{s}, K} : C(X_{\text{nr}}^u(L)) \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{d}_{\mathfrak{s}}] \rightarrow \text{End}_{C_r^*(G, K)^s}(\Pi_{\mathfrak{s}, K}^c).$$

Proof. We start by defining the action locally. Consider $\sigma' := \sigma \otimes \chi'$ for some $\chi' \in X_{\text{nr}}^u(L)$. Let $U_{\chi'}$ be a small neighborhood of $X_{\text{nr}}^+(L)\chi'$ in $X_{\text{nr}}(L)$, as in [Sol9, §7] and Theorem 3.14. We use the analytic localization $\Pi_{\mathfrak{s}} \otimes_{\mathcal{O}(X_{\text{nr}}(L))} C^{an}(U_{\chi'})$ of $\Pi_{\mathfrak{s}}$ at $U_{\chi'}$, like in (3.15)–(3.16). This can also be done with $\Pi_{\mathfrak{s}, K}^c$ instead of $\Pi_{\mathfrak{s}}$.

By (3.14), any element of $W_{\mathfrak{s}}^e$ can be written uniquely as γw , where γ sends $\Phi(G, Z^\circ(L))_{\sigma'}$ to $\Phi(G, Z^\circ(L))_{\gamma\sigma'}$ and $w \in W(\Phi(G, Z^\circ(L))_{\sigma'})$. Then $\mathcal{T}_\gamma$ has no singularities on $U_{\chi'}$ [Sol9, (6.7)]. We define the action of γ to be that of $\mathcal{T}_\gamma$, at least locally over $U_{\chi'}$. That intertwines the G -action and is consistent with [Sol9, §8], although the latter is not explicit in [Sol9].

The Weyl group $W(\Phi(G, Z^\circ(L))_{\sigma'})$ is generated by simple reflections s_α , so we look at one of those. By [Sol9, (5.20)], the only poles of $\mathcal{T}_{s_\alpha}$ on $U_{\chi'}$ are at $\chi(h_\alpha^\vee) = q_{\sigma', \alpha}$ (for $\mathcal{T}_{s_\alpha}$ considered from the left) and at $\chi(h_\alpha^\vee) = q_{\sigma', \alpha}^{-1}$ (considered from the right). By [Sol9, (6.11)], there is a group isomorphism $W_{\mathfrak{s}, \sigma'} \cong (W_{\mathfrak{s}}^e)_{\chi'}$. Next [Sol9, Lemma 7.2]

$$(6.28) \quad \text{identifies the specialization of } \mathcal{T}_{s_\alpha} \text{ at } U_{\chi'} \text{ with } f_{s_\alpha}^{\chi'} \mathcal{T}_{s_\alpha}^{\chi'},$$

where $f_{s_\alpha}^{\chi'} \in \mathbb{C}(X_{\text{nr}}(L))$ without zeros or poles on $X_{\text{nr}}^u(L)$, and $\mathcal{T}_{s_\alpha}^{\chi'}$ is a version of $\mathcal{T}_{s_\alpha}$ defined with respect to $\sigma \otimes \chi'$ as basepoint. The action of $\mathcal{T}_{s_\alpha}$ specialized at any point of $\chi_1 X_{\text{nr}}^+(L)$ with $\chi_1 \in X_{\text{nr}}^u(L) \cap U_{\chi'}$ will be defined as that of $f_{s_\alpha}^{\chi'} \mathcal{T}_{s_\alpha}^{\chi'}$.

As $f_{s_\alpha}^{\chi'}$ restricts to a continuous function on $X_{\text{nr}}^u(L)$ and we already defined how $C(X_{\text{nr}}^u(L))$ acts on $\Pi_{s,K}^c$, the task at hand is to define how $\mathcal{T}_{s_\alpha}^{\chi'}$ acts. In [Sol9, Proposition 7.3], $\mathcal{T}_{s_\alpha}^{\chi'}$ is mapped to

$$(6.29) \quad \tau_{s_\alpha} = -1 + (N_{s_\alpha} + 1) \frac{h_\alpha^\vee}{\log(q_{\sigma',\alpha}) + h_\alpha^\vee} \in \mathbb{H}(\mathfrak{t}, W_{s,\sigma'}, k_{\sigma'}, \mathfrak{d}_{\sigma'}) \otimes_{\mathcal{O}(\mathfrak{t})^{W_{s,\sigma'}}} \mathbb{C}(\mathfrak{t})^{W_{s,\sigma'}}$$

from (2.18). Recall from Theorem 2.10.a that both

$$(6.30) \quad \{\tau_w : w \in W_{s,\sigma'}\} \quad \text{and} \quad \{N_w \tau_r : w \in W(\Phi(G, Z^\circ(L))_{\sigma'}), r \in \Gamma_{\sigma'}\}$$

satisfy the multiplication rules for the standard basis elements of $\mathbb{C}[W_s^e, \mathfrak{d}_s]$.

Consider $Q = MU_Q \subset P$ and $\delta \in \text{Irr}_{\text{disc}}(M)$ which arises as a subquotient of $I_{L(M \cap P)}^M(\sigma \otimes \chi)$ for some $\chi \in U_{\chi'}$, and such that $\langle K \rangle I_Q^G(\delta)$ is a quotient of $\Pi_{s,K}^c$. If $\chi(h_\alpha^\vee) \notin \{q_{\sigma',\alpha}, q_{\sigma',\alpha}^{-1}\}$, then we let $\mathcal{T}_{s_\alpha}^{\chi'}$ acts on this copy of $\langle K \rangle I_Q^G(\delta)$ as τ_{s_α} (which is regular there). If $\chi(h_\alpha^\vee) \in \{q_{\sigma',\alpha}, q_{\sigma',\alpha}^{-1}\}$, then τ_{s_α} has a singularity at $\sigma \otimes \chi$, and we let $\mathcal{T}_{s_\alpha}^{\chi'}$ act as N_{s_α} instead. Notice that in both cases we act by a G -intertwiner.

The dichotomy $\tau_{s_\alpha}/N_{s_\alpha}$ is analogous to [Lus, §8.8]. The arguments from [Lus, §8] show that the above leads to an action of the group

$$(6.31) \quad \{\mathcal{T}_w^{\chi'} : w \in W(\Phi(G, Z^\circ(L))_{\sigma'})\}$$

on the sum of the copies of $I_Q^G(\delta)$ that arise in the above way. The main idea in [Lus, §8] is a more precise version of (6.30), with τ_{s_α} or N_{s_α} depending on one $W_{s,\sigma'}$ -orbit in $U_{\chi'}$. Next we can vary δ , which yields an action of (6.31) on the localization of $\Pi_{s,K}^c$ at $U_{\chi'}$.

From Theorem 3.8 and the multiplication rules in $\mathbb{C}[W_s^e, \mathfrak{d}_s]$ we see that $\mathcal{T}_\gamma \mathcal{T}_{s_\alpha} \mathcal{T}_\gamma^{-1}$ equals $\mathcal{T}_{\gamma s_\alpha \gamma^{-1}}$ for all simple roots $\alpha \in \Phi(G, Z^\circ(L))_{\sigma'}$, and hence

$$\mathcal{T}_\gamma \mathcal{T}_w \mathcal{T}_\gamma^{-1} = \mathcal{T}_{\gamma w \gamma^{-1}} \quad \text{for all } w \in W(\Phi(G, Z^\circ(L))_{\sigma'}).$$

Then (6.28) says that

$$(6.32) \quad \mathcal{T}_\gamma f_{s_\alpha}^{\chi'} \mathcal{T}_{s_\alpha}^{\chi'} \mathcal{T}_\gamma^{-1} = (f_{s_\alpha}^{\chi'} \circ \gamma^{-1}) \mathcal{T}_\gamma \mathcal{T}_w^{\chi'} \mathcal{T}_\gamma^{-1} = f_{\gamma s_\alpha \gamma^{-1}}^{\gamma(\chi')} \mathcal{T}_{\gamma s_\alpha \gamma^{-1}}^{\gamma(\chi')}.$$

The relations between $\mathcal{T}_{s_\alpha}^{\chi'}, \tau_{s_\alpha}^{\chi'}$ and $N_{s_\alpha} = N_{s_\alpha}^{\chi'}$ look the same after replacing χ' by $\gamma(\chi')$, so (6.32) implies that

$$\mathcal{T}_\gamma f_{s_\alpha}^{\chi'} \mathcal{T}_{s_\alpha}^{\chi'} \mathcal{T}_\gamma^{-1} = f_{\gamma s_\alpha \gamma^{-1}}^{\gamma(\chi')} \tau_{\gamma s_\alpha \gamma^{-1}}^{\gamma(\chi')} \quad \text{and} \quad \mathcal{T}_\gamma f_{s_\alpha}^{\chi'} N_{s_\alpha}^{\chi'} \mathcal{T}_\gamma^{-1} = f_{\gamma s_\alpha \gamma^{-1}}^{\gamma(\chi')} N_{\gamma s_\alpha \gamma^{-1}}^{\gamma(\chi')}.$$

It follows that, as operators on a localization of $\Pi_{s,K}^c$ at $U_{\gamma(\chi')}$:

$$\mathcal{T}_\gamma \circ \mathcal{T}_{s_\alpha} \circ \mathcal{T}_\gamma^{-1} = \mathcal{T}_{\gamma s_\alpha \gamma^{-1}}.$$

Combining instances of this relation leads to

$$\mathcal{T}_\gamma \circ \mathcal{T}_w \circ \mathcal{T}_\gamma^{-1} = \mathcal{T}_{\gamma w \gamma^{-1}} \quad \text{and} \quad \mathcal{T}_\gamma \circ \mathcal{T}_w \circ \mathcal{T}_\gamma^{-1} = \mathcal{T}_{\gamma w \gamma^{-1}} \circ \mathcal{T}_w$$

as operators on localized versions of $\Pi_{s,K}^c$. Consider now w_2, γ_2 like w, γ , only with respect to $\gamma(\chi')$ instead of χ . We compute

$$\begin{aligned} \mathcal{T}_{\gamma_2} \circ \mathcal{T}_{w_2} \circ \mathcal{T}_\gamma \circ \mathcal{T}_w &= \mathcal{T}_{\gamma_2} \circ \mathcal{T}_\gamma \circ \mathcal{T}_{\gamma^{-1} w_2 \gamma} \circ \mathcal{T}_w = \mathfrak{d}_s(\gamma_2, \gamma) \mathcal{T}_{\gamma_2 \gamma} \circ \mathcal{T}_{\gamma^{-1} w_2 \gamma w} \\ &= \mathfrak{d}(\gamma_2, \gamma) \mathcal{T}_{\gamma_2 w_2 \gamma w} = \mathfrak{d}(\gamma_2 w_2, \gamma w) \mathcal{T}_{\gamma_2 w_2 \gamma w} \end{aligned}$$

as operators from

$$\Pi_{s,K}^c \otimes_{\mathcal{O}(X_{\text{nr}}(L))} C^{an}(U_{\chi'}) \quad \text{to} \quad \Pi_{s,K}^c \otimes_{\mathcal{O}(X_{\text{nr}}(L))} C^{an}(\gamma_2 w_2 \gamma w U_{\chi'}).$$

This shows that our actions defined locally near $X_{\text{nr}}^+(L)\chi'$ extend to an action of $\mathbb{C}[W_s^e, \mathfrak{h}_s]$ defined locally near $W_s^e X_{\text{nr}}^+(L)\chi'$.

Next we check that the above constructions do not depend on the choice of the neighborhoods $U_{\chi'}$ (as long as they have the shape from [Sol9, §7]). Suppose that we have another set $U_{\chi''}$ like $U_{\chi'}$, and that $U_{\chi''} \cap U_{\chi'}$ is nonempty. We need to verify that our locally defined actions coincide on

$$\Pi_{s,K}^c \otimes_{\mathcal{O}(X_{\text{nr}}(L))} C^{an}(U_{\chi'} \cap U_{\chi''}).$$

We may pick $\chi_2 \in U_{\chi'} \cap U_{\chi''}$ with a neighborhood U_{χ_2} of $X_{\text{nr}}^+(L)\chi_2$ contained in $U_{\chi'} \cap U_{\chi''}$. If we can compare the actions relative to $U_{\chi'}$ and U_{χ_2} and the actions relative to $U_{\chi''}$ and U_{χ_2} , then a comparison between $U_{\chi'}$ and $U_{\chi''}$ follows. Therefore it suffices to consider the case where $U_{\chi''} \subset U_{\chi'}$.

The conditions on $U_{\chi''}$ from [Sol9, §7] ensure that

$$\Phi(G, Z^\circ(L))_{\sigma'} \supset \Phi(G, Z^\circ(L))_{\sigma'}.$$

We consider w, γ, s_α as above. For γ the actions with respect to $U_{\chi'}$ and $U_{\chi''}$ are defined in exactly the same way, so they agree where they are both defined.

Suppose that the action of $\mathcal{T}_{s_\alpha}$ (from $U_{\chi'}$) is defined via that of τ_{s_α} in (6.29). Then $\mathcal{T}_{s_\alpha}$ acts on $I_Q^G(\delta)$ in the same way as $\mathcal{T}_{s_\alpha}$ on $\Pi_s \otimes_{\mathcal{O}(X_{\text{nr}}(L))} \mathbb{C}(X_{\text{nr}}(L))$. In that case the action with respect to $U_{\chi''}$ is defined just like that.

Suppose now that $\mathcal{T}_{s_\alpha}$ acts on $I_Q^G(\delta)$ via N_{s_α} in (6.29). Then $\chi(h_\alpha^\vee) \in \{q_{\sigma',\alpha}, q_{\sigma',\alpha}^{-1}\}$ with $\chi \in U_{\chi'} \cap U_{\chi''}$. In that case (6.29) holds also with respect to $\sigma \otimes \chi''$ instead of σ' . Further $f_{s_\alpha}^{\chi''} N_{s_\alpha}^{\chi''} = f_{s_\alpha}^{\chi'} N_{s_\alpha}^{\chi'}$ because

$$f_{s_\alpha}^{\chi''} \tau_{s_\alpha}^{\chi''} = f_{s_\alpha}^{\chi'} \tau_{s_\alpha}^{\chi'} \text{ in } \mathbb{H}(\mathfrak{t}, \langle s_\alpha \rangle, k_{\sigma'}, \mathfrak{h}_{\sigma'}) \otimes_{\mathcal{O}(\mathfrak{t})} \mathbb{C}(\mathfrak{t}).$$

The action of $\mathcal{T}_{s_\alpha}$ on $I_Q^G(\delta)$ with respect to $U_{\chi''}$ is defined to be that of $f_{s_\alpha}^{\chi''} N_{s_\alpha}^{\chi''} = f_{s_\alpha}^{\chi'} N_{s_\alpha}^{\chi'}$. This concludes the definition of an action of $\mathbb{C}[W_s^e, \mathfrak{h}_s]$ on $\Pi_{s,K}^c$.

(a) By Theorem 2.10.b and the multiplication rules in $\mathbb{C}(X_{\text{nr}}(L)) \rtimes \mathbb{C}[W_s^e, \mathfrak{h}_s]$, we have, as operators on $\Pi_{s,K}^c$:

$$(6.33) \quad \mathcal{T}_w \circ f = (f \circ w^{-1}) \circ \mathcal{T}_w \quad \text{for all } w \in W_s^e \text{ and } f \in \mathcal{O}(X_{\text{nr}}(L)).$$

The action of $\mathcal{T}_w$ is defined via its action on the specializations of $\Pi_{s,K}^c$ at the various $\chi \in X_{\text{nr}}(L)$, so (6.33) extends to $f \in C(X_{\text{nr}}(L))$. Then it holds also for $f \in C(X_{\text{nr}}^u(L))$, regarded as $X_{\text{nr}}^+(L)$ -invariant function on $X_{\text{nr}}(L)$. Therefore the actions of $\mathbb{C}[W_s^e, \mathfrak{h}_s]$ and $C(X_{\text{nr}}^u(L))$ on $\Pi_{s,K}^c$ combine in the required way.

(b) Part (a) provides an algebra homomorphism $\phi_{s,K}$, and from (6.20) we know that both source and target are Banach algebras. It remains to check the continuity of $\phi_{s,K}$. On the finite dimensional algebra $\mathbb{C}[W_s^e, \mathfrak{h}_s]$ continuity is automatic, so we may focus on $C(X_{\text{nr}}^u(L))$. From (6.23) we see that its action on $\Pi_{s,K}^c$ is a direct sum of actions of the following kind. First a projection $C(X_{\text{nr}}^u(L)) \rightarrow C(X)$ for some closed subspace $X \subset X_{\text{nr}}^u(L)$, then multiplication by $f|_X$ on some vector bundle over X . In terms of the isomorphism

$$\text{End}_{C_r^*(G,K)^s}(\Pi_{s,K}^c)^{op} \cong e_s^c M_{n_s^c}(C_r^*(G,K)^s) e_s^c$$

and the description of $C_r^*(G,K)^s$ in Theorem 1.34, this means that $\phi_{s,K}|_{C(X_{\text{nr}}^u(L))}$ is a finite direct sum of homomorphisms of the form $C(X_{\text{nr}}(L)) \rightarrow C(X)$. Those are continuous, so $\phi_{s,K}$ is continuous. $\square$

Theorem 6.19 gives rise to a version of Theorem 3.15 for $C_r^*(G, K)^\mathfrak{s}$.

Lemma 6.20. *Let $\pi \in \text{Irr}_{\text{temp}}(G) \cap \text{Rep}_{\text{fl}}(G)_{X_{\text{nr}}^+(L)W_{\mathfrak{s}}(\sigma \otimes \chi)}^\mathfrak{s}$ with $\chi \in X_{\text{nr}}^u(L)$, and recall from Theorem 3.15 that*

$$\zeta^\vee(\pi) = \text{ind}_{\mathcal{O}(X_{\text{nr}}(L)) \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}]}^{\mathcal{O}(X_{\text{nr}}(L)) \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}]} (\mathbb{C}_\chi \otimes \pi_\chi).$$

Then, as left modules over $(C(X_{\text{nr}}^u(L)) \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}])^{op} \cong C(X_{\text{nr}}^u(L)) \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}]$ via Theorem 6.19:

$$\text{Hom}_{C_r^*(G, K)^\mathfrak{s}}(\Pi_{\mathfrak{s}, K}^c, \langle K \rangle \pi) \cong \text{ind}_{C(X_{\text{nr}}^u(L)) \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}]}^{C(X_{\text{nr}}^u(L)) \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}]} (\mathbb{C}_\chi \otimes \pi_\chi).$$

Proof. Both $\zeta^\vee(\pi)$ and $\text{Hom}_{C_r^*(G, K)^\mathfrak{s}}(\Pi_{\mathfrak{s}, K}^c, \langle K \rangle \pi)$ are determined by their specializations at $\chi \in X_{\text{nr}}^u(L)$, and for both the operation to go from there to the full module is $\text{ind}_{\mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}]}^{\mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}]}$. Therefore it suffices show that the specialization of

$\text{Hom}_{C_r^*(G, K)^\mathfrak{s}}(\Pi_{\mathfrak{s}, K}^c, \langle K \rangle \pi)$ at U_χ is isomorphic to $\mathbb{C}_\chi \otimes \pi_\chi$, as left module of $C(X_{\text{nr}}^u(L)) \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}]$. By definition $C(X_{\text{nr}}^u(L))$ acts on

$$\text{Hom}_{C_r^*(G, K)^\mathfrak{s}}(\Pi_{\mathfrak{s}, K}^c, \langle K \rangle \pi) \otimes_{\mathcal{O}(X_{\text{nr}}(L))} C^{an}(U_\chi)$$

by evaluation at $X_{\text{nr}}^+(L)\chi$, or equivalently evaluation at χ . Thus we only need to show that $\text{Hom}_{C_r^*(G, K)^\mathfrak{s}}(\Pi_{\mathfrak{s}, K}^c, \langle K \rangle \pi)$ is isomorphic to π_χ as right $\mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}]$ -module, where $\text{Hom}_G(\Pi_{\mathfrak{s}}, \pi)$ is isomorphic to π_χ as right module for

$$\mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}] \subset \mathbb{H}(\mathfrak{t}, W_{\mathfrak{s}, \sigma \otimes \chi}, k_{\sigma \otimes \chi}, \mathfrak{h}_{\sigma \otimes \chi}).$$

In the way we set it up, $\text{Hom}_{C_r^*(G, K)^\mathfrak{s}}(\Pi_{\mathfrak{s}, K}^c, \langle K \rangle \pi)$ is acted upon by a copy of $\mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}]$ which is generated by three kinds of elements:

- (i) $\mathcal{T}_\gamma$ for $\gamma \in \Gamma_{\sigma \otimes \chi}$,
- (ii) $\mathcal{T}_{s_\alpha}^\chi$ for simple roots $\alpha \in \Phi(G, Z^\circ(L))_{\sigma \otimes \chi}$ such that $\chi(h_\alpha^\vee) \notin \{q_{\sigma \otimes \chi, \alpha}, q_{\sigma \otimes \chi, \alpha}^{-1}\}$,
- (iii) $N_{s_\alpha}^\chi$ for simple roots $\alpha \in \Phi(G, Z^\circ(L))_{\sigma \otimes \chi}$ such that $\chi(h_\alpha^\vee) \in \{q_{\sigma \otimes \chi, \alpha}, q_{\sigma \otimes \chi, \alpha}^{-1}\}$.

We can regard $\text{Hom}_{C_r^*(G, K)^\mathfrak{s}}(\Pi_{\mathfrak{s}, K}^c, \langle K \rangle \pi) \cong \text{Hom}_G(\Pi_{\mathfrak{s}}, \pi)$ as a module over

$$\mathbb{H}(\mathfrak{t}, W_{\mathfrak{s}, \sigma \otimes \chi}, k_{\sigma \otimes \chi}, \mathfrak{h}_{\sigma \otimes \chi}) \otimes_{\mathcal{O}(\mathfrak{t}/W_{\mathfrak{s}, \sigma \otimes \chi})} C^{an}(U_\chi)^{W_{\mathfrak{s}, \sigma \otimes \chi}},$$

as in (3.16). In that algebra we continuously scale the parameters $k_{\sigma \otimes \chi, \alpha}$ to 0 (or equivalently deform $q_{\sigma \otimes \chi, \alpha}$ to 1), like in (2.15) and Paragraph 2.4. Such continuous deformations do not change representations of the finite dimensional semisimple algebra $\mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}]$. In the limit case $k_{\sigma \otimes \chi, \alpha} = 0, q_{\sigma \otimes \chi, \alpha} = 1$ the difference between $\mathcal{T}_{s_\alpha}^\chi$ and $N_{s_\alpha}^\chi$ disappears, see (6.27). Hence $\text{Hom}_{C_r^*(G, K)^\mathfrak{s}}(\Pi_{\mathfrak{s}, K}^c, \langle K \rangle \pi)$ as a representation of $\mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}]$ as generated by elements of the above kinds (i), (ii) and (iii) is equivalent to $\text{Hom}_G(\Pi_{\mathfrak{s}}, \pi)$ as representation of $\mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}]$ as generated by the $\mathcal{T}_\gamma$ with $\gamma \in \Gamma_{\sigma \otimes \chi}$ and the $N_{s_\alpha}^\chi$ for simple roots $\alpha \in \Phi(G, Z^\circ(L))_{\sigma \otimes \chi}$. The latter representation recovers exactly how π_χ is constructed in [Sol9, §7]. $\square$

6.6. $K_*(C_r^*(G)^\mathfrak{s})$ via the progenerator.

The goal of this paragraph is to compute $K_*(C_r^*(G)^\mathfrak{s})$ in representation theoretic terms. The improvement on Paragraph 6.2 is that we do not need to tensor with $\mathbb{C}$ over $\mathbb{Z}$, so that we can also detect torsion elements in $K_*(C_r^*(G)^\mathfrak{s})$. Our main tools are the progenerator from Paragraph 6.4 and the below general technique to compute K-groups.

Let X be a compact Hausdorff space which has the structure of a finite simplicial complex. Let A and B be $C(X)$ -Banach algebras, by which we mean that $C(X)$ acts on A via a Banach algebra morphism from $C(X)$ to the centre of the multiplier algebra of A (and similarly for B).

Lemma 6.21. [Sol5, Lemma 5.1.3]

Suppose that $\phi : A \rightarrow B$ is a homomorphism of $C(X)$ -Banach algebras, such that for each simplex σ of X the specialization

$$\phi_{\sigma, \partial\sigma} : C_0(X, \partial\sigma)A / C_0(X, \sigma)A \rightarrow C_0(X, \partial\sigma)B / C_0(X, \sigma)B$$

induces an isomorphism on K -theory. Then $K_*(\phi) : K_*(A) \rightarrow K_*(B)$ is an isomorphism.

Recall that $K \in \mathbf{CO}(G)$ has been chosen in (6.17). In this paragraph we abbreviate

$$A = C(X_{\text{nr}}^u(L)) \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}], \quad B = \text{End}_{C_r^*(G, K)^{\mathfrak{s}}}(\Pi_{\mathfrak{s}, K}^c)^{op}, \quad X = X_{\text{nr}}^u(L) / W_{\mathfrak{s}}^e.$$

Clearly A is a $C(X)$ -Banach algebra, and via $\phi_{\mathfrak{s}, K} : A \rightarrow B$ we make B into a $C(X)$ -Banach algebra. We may regard $C(X)$ as the algebra of $X_{\text{nr}}^+(L)$ -invariant continuous functions on $X_{\text{nr}}(L) / W_{\mathfrak{s}}^e \cong X_{\text{nr}}(L) \sigma / W_{\mathfrak{s}}^e$. The latter algebra acts naturally on the family of representations $I_P^G(\sigma \otimes \chi)$ with $\chi \in X_{\text{nr}}(L)$, namely f acts as multiplication by

$$(6.34) \quad f(\sigma \otimes \chi) = f(\sigma \otimes \chi |\chi|^{-1}) = f(\chi |\chi|^{-1}).$$

From Theorem 3.7 we see that $C(X)$ almost acts by elements of the Bernstein centre of $\text{Rep}(G)^{\mathfrak{s}}$, but not entirely because it consists of continuous rather than regular (algebraic) functions. Taking one step back, we can say that

$$(6.35) \quad C(X) \text{ acts on finite length } G\text{-representations via } Z(\text{Rep}_{\text{fl}}(G)^{\mathfrak{s}}).$$

From the Fourier transform of $C_r^*(G, K)^{\mathfrak{s}}$ (Theorem 1.34 and (1.34)), we see that the same recipe yields an action of $C(X)$ on $C_r^*(G, K)^{\mathfrak{s}}$. Moreover (6.35) implies that $C(X)$ acts via a map to $Z(C_r^*(G, K)^{\mathfrak{s}})$, which makes $C_r^*(G, K)^{\mathfrak{s}}$ into a $C(X)$ -Banach algebra. Comparing (6.23) and (6.34), we deduce that the Morita equivalence (6.21) is $C(X)$ -linear.

It is known from [Ill] that the manifold $X_{\text{nr}}^u(L)$ admits a smooth $W_{\mathfrak{s}}^e$ -equivariant triangulation. This means that $X_{\text{nr}}^u(L)$ can be made into a finite simplicial complex, such that all points in the interior of one simplex have the same stabilizer in $W_{\mathfrak{s}}^e$. Then we can divide by the action of $W_{\mathfrak{s}}^e$, which produces a convenient triangulation of $X = X_{\text{nr}}(L)^u / W_{\mathfrak{s}}^e$. Let τ' be a simplex in $X_{\text{nr}}^u(L)$ (from the chosen triangulation) and write $\tau = W_{\mathfrak{s}}^e \tau' / W_{\mathfrak{s}}^e$, which is a simplex in X .

Lemma 6.22. The C^* -algebra $C_0(X, \partial\tau)A / C_0(X, \tau)A$ is Morita equivalent to $C_0(\tau, \partial\tau) \otimes \mathbb{C}[W_{\mathfrak{s}, \tau'}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}]$, and there is an isomorphism

$$K_*(C_0(X, \partial\tau)A / C_0(X, \tau)A) \cong K_*(C_0(\tau, \partial\tau)) \otimes_{\mathbb{Z}} R(\mathbb{C}[W_{\mathfrak{s}, \tau'}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}]).$$

Proof. We can write

$$\begin{aligned} C_0(X, \partial\tau)A / C_0(X, \tau)A &\cong C_0(W_{\mathfrak{s}}^e \tau', W_{\mathfrak{s}}^e \partial\tau') \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}] \\ &= \left(\bigoplus_{w \in W_{\mathfrak{s}}^e / W_{\mathfrak{s}, \tau'}^e} C_0(w\tau', w\partial\tau') \right) \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}]. \end{aligned}$$

This is Morita equivalent to $C_0(\tau', \partial\tau') \rtimes \mathbb{C}[W_{\mathfrak{s}, \tau'}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}]$ via the bimodules

$$1_{\tau'} C_0(W_{\mathfrak{s}}^e \tau', W_{\mathfrak{s}}^e \partial\tau') \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}] \quad \text{and} \quad C_0(W_{\mathfrak{s}}^e \tau', W_{\mathfrak{s}}^e \partial\tau') \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}] 1_{\tau'}.$$

Since all points of $\tau' \setminus \partial\tau'$ have the same stabilizer in W_s^e , that algebra equals

$$(6.36) \quad C_0(\tau', \partial\tau') \otimes \mathbb{C}[W_{s,\tau'}^e, \mathfrak{h}_s^{-1}] \cong C_0(\tau, \partial\tau) \otimes \mathbb{C}[W_{s,\tau'}^e, \mathfrak{h}_s^{-1}].$$

The K-theory of (6.36) is easy to compute, because $\mathbb{C}[W_{s,\tau'}^e, \mathfrak{h}_s^{-1}]$ is a direct sum of matrix algebras $M_{n_i}(\mathbb{C})$. Every such summand contributes a copy of $K_*(C_0(\tau, \partial\tau))$ to the K-theory of (6.36). On the other hand $\oplus_i K_0(M_{n_i}(\mathbb{C})) = \oplus_i \mathbb{Z}$ is naturally isomorphic to the representation ring of $\mathbb{C}[W_{s,\tau'}^e, \mathfrak{h}_s^{-1}]$. We deduce that

$$\begin{aligned} K_*(C_0(X, \partial\tau)A/C_0(X, \tau)A) &\cong K_*(C_0(\tau, \partial\tau) \otimes \mathbb{C}[W_{s,\tau'}^e, \mathfrak{h}_s^{-1}]) \\ &\cong K_*(C_0(\tau, \partial\tau)) \otimes_{\mathbb{Z}} R(\mathbb{C}[W_{s,\tau'}^e, \mathfrak{h}_s^{-1}]). \quad \square \end{aligned}$$

The analogue of Lemma 6.22 for $C_r^*(G, K)^s$ is more involved.

Proposition 6.23. (a) *The Banach algebra*

$$C_0(X, \partial\tau)C_r^*(G, K)^s/C_0(X, \tau)C_r^*(G, K)^s$$

is homotopy equivalent to the tensor product of $C_0(\tau, \partial\tau)$ and a finite dimensional semisimple algebra.

(b) *Its K-theory is*

$$K_*(C_0(\tau, \partial\tau)) \otimes R(\text{Mod}_{\mathfrak{H}}(C_r^*(G, K)^s)_{X_{\text{nr}}^+(L)W_s(\sigma \otimes \chi)}).$$

(c) *The same holds for the Banach algebra $C_0(X, \partial\tau)B/C_0(X, \tau)B$.*

Proof. (a) From Theorem 1.34 we see that

$$C_0(X, \partial\tau)C_r^*(G, K)^s/C_0(X, \tau)C_r^*(G, K)^s$$

is a direct sum of algebras of the form

$$(6.37) \quad (C_0(X_{\text{nr}}^u(M)\text{cc}(\delta) \cap X_{\text{nr}}^+(L)\tilde{\tau}, X_{\text{nr}}^u(M)\text{cc}(\delta) \cap X_{\text{nr}}^+(L)\partial\tilde{\tau}) \otimes \text{End}_{\mathbb{C}}(I_Q^G(V_\delta)^K))^{W_{[M, \delta]}},$$

where $\tilde{\tau} = W_s^e\tau' \subset X_{\text{nr}}^u(L)$, $X_{\text{nr}}^u(M)$ is identified with its restriction to L and $\text{cc}(\delta) \in X_{\text{nr}}(L)$ is chosen such that δ is a quotient of $I_{L(M \cap P)}^M(\sigma \otimes \text{cc}(\delta))$. The set of summands (6.37) whose specialization at a point $W_s^e\chi \in \tau$ is nonzero depends only on which discrete series representations appear in the parabolic inductions (for the various Levi subgroups of G containing L) of $\sigma \otimes \chi'$ with $\chi' \in X_{\text{nr}}^+(L)\chi$.

We claim that, when χ varies continuously over $X_{\text{nr}}^u(L)$, this set of discrete series can only jump when $W_{s,\chi}^e$ changes. Clearly this is an issue that can be studied locally on $X_{\text{nr}}^u(L)$, and that can be done like in Theorem 3.13. In [Sol9, §7], Theorem 3.13 is in fact proven in larger generality, not for $\text{Rep}_{\mathfrak{H}}(G)_{X_{\text{nr}}^+(L)W_s(\sigma \otimes \chi)}^s$ but for $\text{Rep}_{\mathfrak{H}}(G)_{W_s U_{\sigma \otimes \chi}}^s$ where $U_{\sigma \otimes \chi}$ is a small neighborhood of $X_{\text{nr}}^+(L)(\sigma \otimes \chi)$. This reduces our claim to the analogous claim for

$$\mathbb{H}(\mathfrak{t}, W_{s,\sigma \otimes \chi}, k_{\sigma \otimes \chi}, \mathfrak{h}_{\sigma \otimes \chi}) - \text{Mod}_{\mathfrak{H}, U_{\mathbb{R}}},$$

where $U_{\mathbb{R}}$ is a small neighborhood of $\mathfrak{t}_{\mathbb{R}}$ in $\mathfrak{t}$, such that $U_{\mathbb{R}}/\mathfrak{t}_{\mathbb{R}}$ is a $W_{s,\sigma \otimes \chi}$ -invariant ball around 0 in $i\mathfrak{t}_{\mathbb{R}}$. Now we have to consider discrete series representations in the parabolic inductions of $\mathbb{C}_{\lambda+\nu} \in \text{Irr}(\mathcal{O}(\mathfrak{t}))$, where $\nu \in \mathfrak{t}_{\mathbb{R}}$ and λ varies in $U_{\mathbb{R}} \cap i\mathfrak{t}_{\mathbb{R}}$. This version of our claim is proven in Lemma 2.27.

One nice feature of the chosen triangulation of $X_{\text{nr}}^u(L)$ is that all points of $\tau' \setminus \partial\tau'$ have the same stabilizer in W_s^e . Together with the above claim, that implies that every summand (6.37) has a nonzero specialization at every point of $\tau \setminus \partial\tau \subset X_{\text{nr}}^u(L)/W_{s^\vee}$.

For a given summand (6.37), we can choose $w_\delta \in W_{\mathfrak{s}^e}$ such that $w_\delta \tau'$ is contained in $X_{\text{nr}}^u(M)X_{\text{nr}}^+(L)\text{cc}(\delta)$. Then (6.37) reduces to

$$(6.38) \quad (C_0(w_\delta \tau', w_\delta \partial \tau') \otimes \text{End}_{\mathbb{C}}(I_Q^G(V_\delta)^K))^{W_{[M, \delta], \tau'}}.$$

The group $W_{[M, \delta]}$ is a subquotient of $W_{\mathfrak{s}^e}$ (Lemma 1.44), so by the equivariance of the triangulation of $X_{\text{nr}}^u(L)$ every element of $W_{[M, \delta], \tau'}$ acts trivially on τ' . Furthermore $w_\delta \tau', w_\delta \partial \tau'$ is $W_{[M, \delta], \tau'}$ -equivariantly contractible, so can apply [Sol1, Lemma 7]. It says that, if we replace $C_0(w_\delta \tau', w_\delta \partial \tau')$ by $C^\infty(w \tau')$ in (6.38), then the algebra becomes homotopy equivalent to its specialization at any point of $w_\delta \tau' \setminus w_\delta \partial \tau'$. For (6.38) itself the proof of [Sol1, Lemma 7] can be followed up to [Sol1, (20)], and then it says that (6.38) is homotopy equivalent to

$$(6.39) \quad C_0(w_\delta \tau', w_\delta \partial \tau') \otimes \text{End}_{\mathbb{C}}(I_Q^G(V_\delta)^K)^{W_{[M, \delta], \chi}} \quad \text{for any } \chi \in w_\delta \tau' \setminus w_\delta \partial \tau'.$$

We note that $\text{End}_{\mathbb{C}}(I_Q^G(V_\delta)^K)^{W_{[M, \delta], \chi}}$ is a finite dimensional semisimple algebra. Now we consider the direct sum over all summands (6.37) of

$$C_0(X, \partial \tau) C_r^*(G, K)^{\mathfrak{s}} / C_0(X, \tau) C_r^*(G, K)^{\mathfrak{s}}.$$

We obtain a tensor product of

$$C_0(\tau, \partial \tau) \cong C_0(w_\delta \tau', w_\delta \partial \tau')$$

with a finite dimensional semisimple algebra.

(b) The algebra (6.39) is of the same kind as (6.36). The arguments in the proof of Lemma 6.22 show that

$$(6.40) \quad K_*(6.37) \cong K_*(6.39) \cong K_*(C_0(w_\delta \tau', w_\delta \partial \tau')) \otimes_{\mathbb{Z}} R\left(\text{End}_{\mathbb{C}}(I_Q^G(V_\delta)^K)^{W_{[M, \delta], \chi}}\right).$$

Identifying χ with $W_{\mathfrak{s}^e} \chi \in \tau \setminus \partial \tau$, we can take the direct sum of the groups $R(\text{End}_{\mathbb{C}}(I_Q^G(V_\delta)^K)^{W_{[M, \delta], \chi}})$, over all involved instances of (6.37). By Theorem 1.34 that gives precisely

$$R(\text{Mod}_{\mathfrak{h}}(C_r^*(G, K)^{\mathfrak{s}})_{X_{\text{nr}}^+(L)W_{\mathfrak{s}}(\sigma \otimes \chi)}).$$

That and (6.40) provide the required description of

$$K_*(C_0(X, \partial \tau) C_r^*(G, K)^{\mathfrak{s}} / C_0(X, \tau) C_r^*(G, K)^{\mathfrak{s}}).$$

(c) From part (a) and the $C(X)$ -linear Morita equivalence between $C_r^*(G, K)^{\mathfrak{s}}$ and $B \cong e_{\mathfrak{s}}^c M_{n_{\mathfrak{s}}}^*(C_r^*(G, K)^{\mathfrak{s}}) e_{\mathfrak{s}}^c$, we deduce that (a) holds for $C_0(X, \partial \tau) B / C_0(X, \tau) B$. The involved finite dimensional semisimple algebra may not be the same as before, but it is Morita equivalent to that from part (a).

The calculation of $K_*(C_0(X, \partial \tau) B / C_0(X, \tau) B)$ is the same as in part (b), and also follows from Morita equivalence. $\square$

The following result shows that the K-groups in Lemma 6.22 and Proposition 6.23 are isomorphic.

Lemma 6.24. (a) *The map*

$$R(\mathbb{C}[W_{\mathfrak{s}, \chi}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}]) \rightarrow R(\text{Mod}_{\mathfrak{h}}(C_r^*(G, K)^{\mathfrak{s}})_{X_{\text{nr}}^+(L)W_{\mathfrak{s}}(\sigma \otimes \chi)})$$

induced by $\otimes_{C(X_{\text{nr}}^u(L)) \times \mathbb{C}[W_{\mathfrak{s}, \chi}^e, \mathfrak{h}_{\mathfrak{s}}]} \Pi_{\mathfrak{s}, K}^c$ is a group isomorphism.

(b) *Let $\phi_{\mathfrak{s}, K, \tau, \partial \tau} : C_0(X, \partial \tau) A / C_0(X, \tau) A \rightarrow C_0(X, \partial \tau) B / C_0(X, \tau) B$ be the homomorphism of Banach algebras induced by $\phi_{\mathfrak{s}, K}$. Then $K_*(\phi_{\mathfrak{s}, K, \tau, \partial \tau})$ is an isomorphism.*

Proof. (a) For

$$\pi \in \text{Irr}_{\text{temp}}(G)_{X_{\text{nr}}^+(L)W_{\mathfrak{s}}(\sigma \otimes \chi)}^{\mathfrak{s}} \cong \text{Irr}(C_r^*(G, K)^{\mathfrak{s}})_{X_{\text{nr}}^+(L)W_{\mathfrak{s}}(\sigma \otimes \chi)},$$

Theorem 6.19 shows that $\zeta^{\vee}(\pi)$ is equivalent to $\text{Hom}_{C_r^*(G, K)^{\mathfrak{s}}}(\Pi_{\mathfrak{s}, K}^c, \langle K \rangle \pi)$, in the sense that both have the same central character $W_{\mathfrak{s}}^e \chi$ and the same $\mathbb{C}[W_{\mathfrak{s}, \chi}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}]$ -representation π_{χ} . In combination with Theorem 3.14.a we find that

$$(6.41) \quad \phi_{\mathfrak{s}, K}^* \circ \text{Hom}_{C_r^*(G, K)^{\mathfrak{s}}}(\Pi_{\mathfrak{s}, K}^c, ?) : \\ R(\text{Mod}_{\mathfrak{H}}(C_r^*(G, K)^{\mathfrak{s}})_{X_{\text{nr}}^+(L)W_{\mathfrak{s}}(\sigma \otimes \chi)}) \rightarrow R(\mathbb{C}[W_{\mathfrak{s}, \chi}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}]) \cong R(\text{Mod}(A)_{W_{\mathfrak{s}}^e \chi})$$

is a group isomorphism. The inverse of this map is $\otimes_{C(X_{\text{nr}}^u(L)) \rtimes \mathbb{C}[W_{\mathfrak{s}, \chi}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}]} \Pi_{\mathfrak{s}, K}^c$.

(b) From Lemma 6.22, Proposition 6.23 and (6.41) we see that $K_*(\phi_{\mathfrak{s}, K, \tau, \partial \tau})$ is the tensor product of the identity on $K_*(C_0(\tau, \partial \tau))$ and the isomorphism from part (a). $\square$

Our preparations to apply Lemma 6.21 are complete.

Theorem 6.25. *Let $\mathfrak{s} \in \mathfrak{B}(G)$ and choose $K \in \mathcal{CD}(G)$ as in (6.17). There are isomorphisms of $\mathbb{Z}/2\mathbb{Z}$ -graded groups*

$$K_*(C(X_{\text{nr}}^u(L)) \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}]) \xrightarrow{K_*(\phi_{\mathfrak{s}, K})} K_*(\text{End}_{C_r^*(G, K)^{\mathfrak{s}}}(\Pi_{\mathfrak{s}, K}^c)^{op}) \cong K_*(C_r^*(G, K)^{\mathfrak{s}}).$$

Proof. Lemma 6.24 shows that the conditions of Lemma 6.21 are fulfilled. Then Lemma 6.21 tells us that $K_*(\phi_{\mathfrak{s}, K})$ is an isomorphism. The second isomorphism follows from the Morita equivalence (6.21) and Theorem 6.1. $\square$

Recall from Definition 6.6 that we interpreted $K_*(C(X_{\text{nr}}^u(L)) \rtimes \mathbb{C}[W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}])$ as the K-theory of $\mathfrak{h}_{\mathfrak{s}}^{-1}$ -twisted $W_{\mathfrak{s}}^e$ -equivariant vector bundles on $X_{\text{nr}}^u(L)$. From (6.15), Theorem 6.25 and Definition 6.6 we conclude:

Corollary 6.26. *There are isomorphisms*

$$K_*(C_r^*(G)) \cong \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} K_*(C_r^*(G, K_{\mathfrak{s}})^{\mathfrak{s}}) \cong \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} K_{W_{\mathfrak{s}}^e, \mathfrak{h}_{\mathfrak{s}}^{-1}}^*(X_{\text{nr}}^u(L)).$$

We note that Theorem 6.25 and Corollary 6.26 prove [ABPS2, Conjecture 5], a version of the ABPS conjecture in topological K-theory.

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