

INTERTWINING OPERATORS FOR REPRESENTATIONS OF COVERING GROUPS OF REDUCTIVE p -ADIC GROUPS

JANET FLIKKEMA AND MAARTEN SOLLEVELD

ABSTRACT. Let G be a covering group of a reductive p -adic group. We study intertwining operators between parabolically induced representations of G and prove that they satisfy certain adjointness relations. The Harish-Chandra μ -function is defined as a composition of such intertwining operators for opposite parabolic subgroups of G . It can be seen as a complex rational function and we give an explicit formula for it in terms of poles and zeros. The adjointness of the intertwining operators is an important ingredient to prove the formula for the μ -function. Moreover, we need to study the limit of the μ -function at zero and infinity. To locate the poles of μ , we construct a continuous family of Hermitian forms on a family of parabolically induced representations.

CONTENTS

1. Introduction	1
2. Notation	4
3. Adjointness relations for intertwining operators	4
4. Poles and zeros of the Harish-Chandra μ -function	11
4.1. μ as a rational function	12
4.2. An aspect of the Iwasawa decomposition	15
4.3. The limit at infinity	18
4.4. Zeros of the μ -function	22
4.5. Hermitian techniques to find the poles	24
References	32

1. INTRODUCTION

In this paper, we prove a formula for the Harish-Chandra μ -function for covering groups of reductive p -adic groups. By a covering group G of a reductive p -adic group G' , we mean a central extension of G' by a finite abelian group, which is moreover a topological covering. An example of such a group is the metaplectic group, which is a double cover of the symplectic group. Many methods and results from the representation theory of reductive p -adic groups

generalize to covering groups. For example, we have notions of parabolic induction and restriction functors, and there is a Bernstein decomposition, analogous to the reductive case. See [FP22] for an overview.

The Harish-Chandra μ -function is defined using intertwining operators $J_{\bar{P}|P}$ and $J_{P|\bar{P}}$ between parabolically induced representations, for opposite parabolic subgroups P and $\bar{P}$ of G . These intertwining operators are studied in [Wal03]; in fact, they can be considered more generally for parabolic subgroups P, Q which have the same Levi subgroup, but are not necessarily opposite. Such intertwining operators also exist for covering groups, see [Li12]. An important ingredient in our work is an adjointness relation of these intertwining operators, which was stated but not proved in [Wal03]. Section 3 of this paper gives a concrete proof of this adjointness property, using Bruhat-Tits theory. In section 4, we prove a formula for the Harish-Chandra μ function, as a complex rational function. The formula was already given for reductive groups in [Sil80]. Here, we provide an alternative proof of the result, which also works for covering groups.

To be able to more precisely state our results, let us give some more context. Let M be a Levi subgroup of a covering group G and let $P = MU_P, Q = MU_Q$ be parabolic subgroups of G with Levi M . Let π be any irreducible smooth M -representation, and let I_P^G be the functor of normalized parabolic induction. Consider the intertwining operators

$$J_{Q|P}(\pi) : I_P^G(\pi) \rightarrow I_Q^G(\pi)$$

$$f \mapsto \left[g \mapsto \int_{(U_P \cap U_Q) \backslash U_Q} f(ug) du \right],$$

and $J_{P|Q}$ which is defined in a similar way. Let $\bar{\pi}^\vee$ denote the Hermitian dual of π . The M -invariant pairing $\langle \cdot, \cdot \rangle$ between π and $\bar{\pi}^\vee$ gives rise to a G -invariant pairing $\langle \cdot, \cdot \rangle$ between $I_P^G(\pi)$ and $I_P^G(\bar{\pi}^\vee)$. Similarly, we get a pairing $\langle \cdot, \cdot \rangle$ between $I_Q^G(\pi)$ and $I_Q^G(\bar{\pi}^\vee)$. With respect to these pairings, we will prove the following result.

Theorem 1.1. *The intertwining operators $J_{Q|P}(\pi)$ and $J_{P|Q}(\bar{\pi}^\vee)$ are adjoint, in the sense that for $f_1 \in I_P^G(\pi), f_2 \in I_Q^G(\bar{\pi}^\vee)$ we have*

$$\langle J_{Q|P}(\pi)f_1, f_2 \rangle = \langle f_1, J_{P|Q}(\bar{\pi}^\vee)f_2 \rangle.$$

There is a similar result for the contragredient representation $\pi^\vee$ instead of $\bar{\pi}^\vee$. In the special case where π is Hermitian, we obtain the adjointness relation that was already stated in [Wal03] and which we will need to prove the result in Section 4.

Now, let us discuss the Harish-Chandra μ -function. Consider an irreducible supercuspidal representation σ of M . Let $P = MU$ be a parabolic subgroup of G with Levi M , and let $\bar{P} = M\bar{U}$ be the parabolic subgroup of G opposite to P . We have intertwining operators $J_{\bar{P}|P}(\sigma), J_{P|\bar{P}}(\sigma)$ as above; they are integrals over $\bar{U}$ and U , respectively.

The composition

$$j(\sigma) := J_{P|\bar{P}}(\sigma) \circ J_{\bar{P}|P}(\sigma) : I_P^G(\sigma) \rightarrow I_P^G(\sigma)$$

is a scalar, and does not depend on the choice of parabolic subgroup P [Wal03]. The Harish-Chandra μ -function is, up to a positive real scalar, defined to be j^{-1} . This function μ plays a crucial role in the Plancherel formula for reductive p -adic groups [Wal03], generalized to covering groups in [Li12]. Namely, the Plancherel measure for G is a product of μ and some other, much easier terms.

One may view $\mu(\sigma \otimes \chi)$ as a complex rational function in the variable $\chi \in X_{\text{nr}}(M)$, where $X_{\text{nr}}(M)$ is the complex algebraic torus of unramified characters of M . The function μ then decomposes as a product $\mu = \prod_{\alpha} \mu_{\alpha}$, where α runs over the positive roots in the reduced root system corresponding to M in G . Each μ_{α} can be computed in a Levi subgroup M_{α} of G , which has M as a maximal Levi subgroup. Moreover, μ_{α} can be seen as a complex function in a single variable $z = \chi(h_{\alpha}^{\vee}) \in \mathbb{C}^{\times}$, where $h_{\alpha}^{\vee}$ is suitably chosen element of M .

Theorem 1.2. *In the above context, there exists a unitary $\sigma_0 \in \sigma \cdot X_{\text{nr}}(M)$ such that μ_{α} has the form*

$$(1.1) \quad \mu_{\alpha}(\sigma_0 \otimes \chi) = \mu_{\alpha}(\sigma_0, z) = c \cdot \frac{(1-z)(1-z^{-1})}{(1-qz)(1-qz^{-1})} \frac{(1+z)(1+z^{-1})}{(1+q'z)(1+q'z^{-1})},$$

where $c \in \mathbb{R}_{>0}$ and $q, q' \in \mathbb{R}_{\geq 1}$.

Both the zeros and the poles of μ_{α} have clear representation-theoretic significance. If $\sigma \otimes \chi$ is a pole of μ_{α} , then $I_{P \cap M_{\alpha}}^{M_{\alpha}}(\sigma \otimes \chi)$ is reducible (but not decomposable). On the other hand, the set of roots α with $\mu_{\alpha}(\sigma \otimes \chi) = 0$ forms a root system. It plays a role in the Knapp-Stein theory of intertwining operators [Li12; Sil78], which determines the decomposition of $I_P^G(\sigma \otimes \chi)$ in irreducible representations.

Let us give an outline of the proof of the theorem. By setting $\mu_{\alpha}(\sigma, 0) := \lim_{z \rightarrow 0} \mu_{\alpha}(\sigma, z)$ and $\mu_{\alpha}(\sigma, \infty) := \lim_{z \rightarrow \infty} \mu_{\alpha}(\sigma, z)$, we can view μ_{α} as a rational function on the projective curve $\mathbb{P}^1(\mathbb{C})$. Suppose $\mu_{\alpha}(\sigma, z)$ is not constant, then there exists a unitary $\sigma_0 \in \sigma \cdot X_{\text{nr}}(M)$ with $\mu_{\alpha}(\sigma_0) = 0$. The only other possibility for a zero of $\mu_{\alpha}(\sigma_0, z)$ in $\mathbb{C}^{\times}$ is at $z = -1$. Every zero must be of order two, which corresponds to $J_{\bar{P}|P}$ and $J_{P|\bar{P}}$ each having a pole of order one at that point. So $\mu(\sigma_0, z)$ has at most two zeroes in $\mathbb{C}^{\times}$, located at $z = \pm 1$, each of order two. Moreover, we will show that $\mu(\sigma_0, z)$ is not zero at $z = 0$ and $z = \infty$. We will also find two poles for every zero, using techniques concerning Hermitian and unitary representations.

An application of our result is to give a more explicit description of Bernstein blocks in the category of smooth representations of G . It is known that Bernstein blocks can be realized as module categories of endomorphism algebras, see [FP22]. The work in [Sol23] describes the structure of these endomorphism algebras for reductive p -adic groups. With the formula for the μ -function for covering groups, and other results that are already known for covering groups, the results in [Sol23] should generalize.

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2. NOTATION

F : non-archimedean local field
 $G' = \mathcal{G}'(F)$: reductive group over F
 $S' \subset G'$: maximal F -split torus
 $M' \subset G'$: Levi subgroup containing S'
 $P' = M'U_{P'}$: parabolic subgroup of G'
 $\bar{P}' = M'U_{\bar{P}'}$: opposite parabolic subgroup
 $p_G : G \rightarrow G'$: finite central covering
 $M = p_G^{-1}(M')$: Levi subgroup of G
 $P = p_G^{-1}(P'), \bar{P} = p_G^{-1}(\bar{P}')$: parabolic subgroups of G

Since p_G splits over unipotent subgroups, $U_{P'}$ can be identified with a subgroup $U_P \subset P$, and similarly $U_{\bar{P}'} \cong U_{\bar{P}} \subset \bar{P}$. Then $P = MU_P$ and $\bar{P} = MU_{\bar{P}}$.
 $K' = G'_x$: good maximal compact subgroup of G' , associated to a special vertex x in the apartment for S' in the Bruhat–Tits building $\mathcal{B}(G')$
 $K = p_G^{-1}(K') = G_x$: good maximal compact subgroup of G
 π : irreducible smooth M -representation on $\mathbb{C}$ -vector space V_π
 $I_P^G(\pi)$: normalized parabolic induction of π , on the vector space $I_P^G(V_\pi)$
 $(\bar{\pi}^\vee, \overline{V_\pi}^\vee)$: Hermitian dual of (π, V_π)
 $(\pi^\vee, V_\pi^\vee)$: contragredient of (π, V_π)
 $(\sigma, V_\sigma) \in \text{Irr}(M)_{\text{cusp}}$: irreducible supercuspidal representation of M
 $X_{\text{nr}}(M)$: the torus of unramified characters of M
 $\text{Irr}(M)_{[M, \sigma]} = \sigma \cdot X_{\text{nr}}(M)$: the inertial equivalence class of σ for M

3. ADJOINTNESS RELATIONS FOR INTERTWINING OPERATORS

Let G be a covering group of a reductive p -adic group and let M be a Levi subgroup of G . In this section, we prove adjointness relations for the intertwining operators $J_{P|Q}$ and $J_{Q|P}$, where P and Q are parabolic subgroups of G with Levi subgroup M . The adjointness relations will first be proved for opposite parabolic subgroups P and $\bar{P}$, after which they will be generalized to arbitrary parabolic subgroups P and Q with the same Levi M .

Let (π, V_π) be an irreducible smooth representation of M , and denote by $(\bar{\pi}^\vee, \overline{V_\pi}^\vee)$ the Hermitian dual of π . The M -invariant sesquilinear pairing between π and $\bar{\pi}^\vee$ gives rise to a G -invariant sesquilinear pairing between $I_P^G(\pi)$ and $I_P^G(\bar{\pi}^\vee)$, namely

$$(3.1) \quad \langle f, f' \rangle = \int_K \langle f(k), f'(k) \rangle dk \quad f \in I_P^G(V_\pi), f' \in I_P^G(\overline{V_\pi}^\vee).$$

By the Iwasawa decomposition $G = PK$, $\int_K$ in (3.1) is up to a positive scalar factor the same as $\int_{P \backslash G}$. The pairing (3.1) is nondegenerate, so provides an isomorphism $I_P^G(\bar{\pi}^\vee) \cong \overline{I_P^G(\pi)}^\vee$.

There are intertwining operators

$$(3.2) \quad J_{\bar{P}|P}(\pi) : I_P^G(\pi) \rightarrow I_{\bar{P}}^G(\pi), \quad J_{P|\bar{P}}(\bar{\pi}^\vee) : I_{\bar{P}}^G(\bar{\pi}^\vee) \rightarrow I_P^G(\bar{\pi}^\vee),$$

defined by

$$(3.3) \quad (J_{\bar{P}|P}(\pi)f)(g) = \int_{U_{\bar{P}}} f(\bar{u}g) d\bar{u}, \quad (J_{P|\bar{P}}(\bar{\pi}^\vee)f')(g) = \int_{U_P} f'(ug) du.$$

These arise via meromorphic continuation, as in [Wal03, §IV.1].

We can build two G -invariant sesquilinear pairings $I_P^G(V_\pi) \times I_{\bar{P}}^G(\bar{V}_\pi^\vee) \rightarrow \mathbb{C}$:

$$(3.4) \quad (f_1, f_2) \mapsto \langle J_{\bar{P}|P}(\pi)f_1, f_2 \rangle$$

$$(3.5) \quad (f_1, f_2) \mapsto \langle f_1, J_{P|\bar{P}}(\bar{\pi}^\vee)f_2 \rangle,$$

which are well-defined provided Condition 3.1(i) below.

Condition 3.1. (i) $J_{\bar{P}|P}(\pi)$ and $J_{P|\bar{P}}(\bar{\pi}^\vee)$ are regular, i.e. the meromorphic continuation of (3.3) does not have a pole at π or $\bar{\pi}^\vee$.
 (ii) $I_P^G(\pi)$ and $I_{\bar{P}}^G(\pi)$ are irreducible. This implies that $I_P^G(\bar{\pi}^\vee) \cong \overline{I_P^G(\pi)}^\vee$ and $I_{\bar{P}}^G(\bar{\pi}^\vee) \cong \overline{I_{\bar{P}}^G(\pi)}^\vee$ are also irreducible.

The second condition holds for supercuspidal M -representations in general position by [Ren10, Théorème VI.8.5], and for irreducible M -representations in general position by [Sau97, Théorème 3.2]. (Although these results are written for reductive p -adic groups, the arguments apply also to finite covers like G .)

Lemma 3.2. *Assuming Condition 3.1, the pairings (3.4) and (3.5) differ by a nonzero scalar factor.*

Proof. Any nonzero G -invariant sesquilinear pairing $V \times W \rightarrow \mathbb{C}$ defines a nonzero homomorphism of G -representations $\phi : V \rightarrow \bar{W}^\vee$, and conversely, by the formula

$$(v, w) = \phi(v)(w).$$

If V and W are irreducible G -representations, then ϕ is unique up to scalars from $\mathbb{C}^\times$, by Schur's lemma. The pairings $I_P^G(V_\pi) \times I_{\bar{P}}^G(\bar{V}_\pi^\vee) \rightarrow \mathbb{C}$ in (3.4) and (3.5) are nonzero because (3.1) is nondegenerate and the operators (3.2) are nonzero [Wal03, p. 283]. Hence they differ only by a factor from $\mathbb{C}^\times$. $\square$

We want to show that the pairings (3.4) and (3.5) are equal. This is claimed, but not proven, for reductive p -adic groups in [Wal03, p. 287]. It requires some preparation.

Let $\Phi(G', S')$ be the root system of G' and write it as

$$\Phi(G', S') = \Phi(M', S') \cup \Phi(U_P, S') \cup \Phi(U_{\bar{P}}, S').$$

Here, $\Phi(U_P, S') \subset \Phi(G', S')$ is the set of roots such that U_P is generated by all root subgroups U_α , where $\alpha \in \Phi(U_P, S')$. The subsets $\Phi(M', S')$ and $\Phi(U_{\bar{P}}, S')$ are given similarly.

For each $r \in \mathbb{R}$, the valuated root datum of G provides compact open subgroups $U_{\alpha,r} \subset U_\alpha$. We normalize the valuations on the root subgroups U_α (of G and of G') so that

$$U_{\alpha,0} = U_\alpha \cap K = U_{\alpha,x}.$$

By [KP23, Lemma 7.3.11.(2)] there are well-defined subgroups

$$U_{P,r} = \prod_{\alpha \in \Phi(U_P, S')} U_{\alpha,r}, \quad U_{\bar{P},r} = \prod_{\alpha \in \Phi(U_{\bar{P}}, S')} U_{\alpha,r}.$$

For $r \geq 0$ we also have the subgroups $Z_G(S')_r = p_G^{-1}(Z_{G'}(S')_r)$ of $Z_G(S')$ and the Moy–Prasad groups [KP23, §13.2]

$$G'_{x,r} \subset G' \text{ and } G_{x,r} = p_G^{-1}(G'_{x,r}).$$

Recall that $p_G : G \rightarrow G'$ is a local homeomorphism and that the groups $G'_{x,r}$ with $r \in \mathbb{R}_{\geq 0}$ form a neighborhood basis of e in G' [KP23, Proposition 13.2.5]. Restricting p_G to a neighborhood of e in G on which it is a homeomorphism, we find $r_x > 0$ and a homomorphic splitting $s_G : G'_{x,r_x} \rightarrow G$ of p_G . Then

$$G_{x,r} = s_G(G'_{x,r}) \times \ker(p_G) \quad \forall r \geq r_x.$$

The advantage of this construction is that the subgroups $s_G(G'_{x,r})$ with $r \geq r_x$ form a neighborhood basis of e in G , in contrast with the $G_{x,r}$. As x lies in an apartment of $\mathcal{B}(M')$, we have $M' \cap K' = M'_x$ and $M \cap K = M_x$, and the groups $M'_{x,r}$ and $M_{x,r} = p_G^{-1}(M'_{x,r})$ are defined.

Let $v_1 \in V_\pi, v_2 \in \bar{V}_\pi^\vee$, and pick $r \in \mathbb{R}_{>0}$ so that $s_G(M'_{x,r})$ fixes v_1 and v_2 . We define $f_1 \in I_P^G(V_\pi)$ and $f_2 \in I_{\bar{P}}^G(\bar{V}_\pi^\vee)$ by the conditions

- $f_1(e) = v_1$ and $f_2(e) = v_2$,
- $\text{supp}(f_1) = PU_{\bar{P},r}$ and $\text{supp}(f_2) = \bar{P}U_{P,r}$,
- f_1 is right $U_{\bar{P},r}$ -invariant and f_2 is right $U_{P,r}$ -invariant.

We note that these conditions are not overdetermined because the multiplication maps

$$(3.6) \quad U_P \times M \times U_{\bar{P}} \rightarrow G \quad \text{and} \quad U_{\bar{P}} \times M \times U_P \rightarrow G$$

are injective. That follows from the analogous statements for G' , which can be found for instance in [CGP15, Proposition 2.1.8.(3)].

We want to analyse

$$(3.7) \quad \langle J_{\bar{P}|P}(\pi)f_1, f_2 \rangle = \int_{U_{\bar{P}}} \int_K \langle f_1(\bar{u}k), f_2(k) \rangle dk d\bar{u}.$$

Lemma 3.3. *The only nonzero contributions to (3.7) come from $k \in U_{\bar{P},0}M_xU_{P,r}$ and $\bar{u} \in U_{\bar{P},0}$ such that $\bar{u}k \in U_{\bar{P},r}M_xU_{P,r}$.*

Proof. From the support of f_2 we see that we can only get nonzero contributions to (3.7) when $k \in K \cap \bar{P}U_{P,r}$. Write

$$k = \bar{u}_1 m_1 u_2 \text{ with } \bar{u}_1 \in U_{\bar{P}}, m_1 \in M, u_2 \in U_{P,r}.$$

As $U_{P,r} \subset K$, we have

$$(3.8) \quad \bar{u}_1 m_1 \in K \cap U_{\bar{P}} M = G_x \cap U_{\bar{P}} M = G_x \cap \bar{P}.$$

By [KP23, Proposition 8.3.1] there exists an integral model $\mathcal{G}'_x$ of $\mathcal{G}'$, such that $\mathcal{G}'_x(\mathfrak{o}_F) = G'_x$. Then $\bar{P}' \cap G'_x$, $U_{\bar{P}'} \cap G'_x$ and $M' \cap G'_x$ determine $\mathfrak{o}_F$ -subgroup schemes $\bar{\mathcal{P}}'_x$, $\mathcal{U}'_{\bar{P}',x}$ and $\mathcal{M}'_x$ of $\mathcal{G}'_x$. Since $\bar{\mathcal{P}} = \mathcal{U}_{\bar{P}} \rtimes \mathcal{M}'$ in $\mathcal{G}'$ and $\mathcal{G}'_x$ is an integral model of $\mathcal{G}'$, also

$$\bar{\mathcal{P}}'_x = \mathcal{U}'_{\bar{P}',x} \rtimes \mathcal{M}'_x.$$

Taking $\mathfrak{o}_F$ -rational points, we obtain

$$\bar{P}' \cap G'_x = \bar{\mathcal{P}}'_x(\mathfrak{o}_F) = \mathcal{U}'_{\bar{P}',x}(\mathfrak{o}_F) \rtimes \mathcal{M}'_x(\mathfrak{o}_F) = (U_{\bar{P}} \cap G'_x) \rtimes (M' \cap G'_x) = U_{\bar{P}',0} \rtimes M'_x.$$

Applying p_G^{-1} , we find that

$$\bar{P} \cap G_x = U_{\bar{P},0} \rtimes M_x.$$

Now (3.8) says that $\bar{u}_1 \in U_{\bar{P},0}$ and $m_1 \in M_x$, so

$$\begin{aligned} k &= \bar{u}_1 m_1 u_2 \in U_{\bar{P},0} M_x U_{P,r}, \\ \bar{u} k &= \bar{u} \bar{u}_1 m_1 u_2 \in U_{\bar{P}} M_x U_{P,r}. \end{aligned}$$

The support of f_1 shows that $f_1(\bar{u}k)$ can only be nonzero when $\bar{u}k \in U_P M U_{\bar{P},r}$. We write

$$\bar{u}k = u_3 m_2 \bar{u}_5 \text{ with } u_3 \in U_P, m_2 \in M, \bar{u}_5 \in U_{\bar{P},r}.$$

Then

$$(3.9) \quad \bar{u}k u_2^{-1} = \bar{u} \bar{u}_1 m_1 = u_3 m_2 \bar{u}_5 u_2^{-1} \in U_{\bar{P}} M_x \cap U_P M U_{\bar{P},r} U_{P,r}.$$

The Iwahori decomposition of the Moy–Prasad group $G'_{x,r}$ [KP23, §13.2] entails that the multiplication map

$$(3.10) \quad M_{x,r} \times U_{P,r} \times U_{\bar{P},r} \rightarrow G_{x,r} \text{ is bijective.}$$

This also works with any other ordering of the three factors. Hence

$$\begin{aligned} U_P M U_{\bar{P},r} U_{P,r} &= U_P M M_{x,r} U_{\bar{P},r} U_{P,r} = \\ &= U_P M M_{x,r} U_{P,r} U_{\bar{P},r} = U_P M U_{P,r} U_{\bar{P},r} = U_P M U_{\bar{P},r}. \end{aligned}$$

This enables us to rewrite (3.9) as

$$\bar{u} \bar{u}_1 m_1 = u_6 m_3 \bar{u}_7 \text{ with } u_6 \in U_P, m_3 \in M, \bar{u}_7 \in U_{\bar{P},r}.$$

Next we observe that

$$u_6 m_3 = \bar{u} \bar{u}_1 m_1 \bar{u}_7^{-1} \in U_P M \cap U_{\bar{P}} M_x U_{\bar{P},r} = U_P M \cap M_x U_{\bar{P}}.$$

By (3.6), the last intersection is just M_x . This forces $u_6 = 1, m_3 = m_1 \in M_x$ and

$$\bar{u} \bar{u}_1 = m_1 \bar{u}_7 m_1^{-1} \in m_1 U_{\bar{P},r} m_1^{-1} = U_{\bar{P},r}.$$

As $\bar{u}_1 \in U_{\bar{P},0} \supset U_{\bar{P},r}$, that also forces $\bar{u} \in U_{\bar{P},0}$. $\square$

Assume that the Haar measure on $G_{x,r} = U_{\bar{P},r} M_{x,r} U_{P,r}$ is the product of the Haar measures on the three factors. Now we are ready to evaluate (3.7).

Lemma 3.4. $\langle J_{\bar{P}|P}(\pi) f_1, f_2 \rangle$ equals $\text{vol}(U_{\bar{P},r}) \text{vol}(U_{\bar{P},0}) \text{vol}(M_x) \text{vol}(U_{P,r}) \langle v_1, v_2 \rangle$.

Proof. By Lemma 3.3 we may replace the integral over $k \in K$ in (3.7) by the integral over $\bar{u}_1 m_1 u_2 \in U_{\bar{P},0} \times M_x \times U_{P,r}$. Lemma 3.3 also says that we may replace the integral over $\bar{u} \in U_{\bar{P},0}$ by the integral over $\bar{u}_2 = \bar{u} \bar{u}_1^{-1} \in U_{\bar{P},r}$. Using the properties of f_2 we compute

$$\begin{aligned} \langle J_{\bar{P}|P}(\pi) f_1, f_2 \rangle &= \int_{U_{\bar{P},r}} \int_{U_{\bar{P},0}} \int_{M_x} \int_{U_{P,r}} \langle f_1(\bar{u}_2 m_1 u_2), f_2(\bar{u}_1 m_1 u_2) \rangle du_2 dm_1 d\bar{u}_1 d\bar{u}_2 \\ &= \text{vol}(U_{\bar{P},0}) \int_{U_{\bar{P},r}} \int_{M_x} \int_{U_{P,r}} \langle f_1(\bar{u}_2 m_1 u_2), f_2(m_1) \rangle du_2 dm_1 d\bar{u}_2. \end{aligned}$$

Since M_x is compact and normalizes $U_{\bar{P},r}$, we may exchange the order of $\bar{u}_2$ and m_1 . That gives

$$\begin{aligned} &\text{vol}(U_{\bar{P},0}) \int_{U_{\bar{P},r}} \int_{M_x} \int_{U_{P,r}} \langle f_1(m_1 \bar{u}_2 u_2), f_2(m_1) \rangle du_2 dm_1 d\bar{u}_2 = \\ &\text{vol}(U_{\bar{P},0}) \int_{U_{\bar{P},r}} \int_{M_x} \int_{U_{P,r}} \langle \pi(m_1) f_1(\bar{u}_2 u_2), \bar{\pi}^\vee(m_1) f_2(e) \rangle du_2 dm_1 d\bar{u}_2 = \\ &\text{vol}(U_{\bar{P},0}) \text{vol}(M_x) \int_{U_{\bar{P},r}} \int_{U_{P,r}} \langle f_1(\bar{u}_2 u_2), f_2(e) \rangle du_2 d\bar{u}_2. \end{aligned}$$

Here $\bar{u}_2 u_2 \in s_G(G'_{x,r})$, so by (3.10) it can be written as

$$\bar{u}_2 u_2 = u_3 m_3 \bar{u}_3 \text{ with } u_3 \in U_{P,r}, m_3 \in s_G(M'_{x,r}), \bar{u}_3 \in U_{\bar{P},r}.$$

The construction of f_1 entails that

$$f_1(\bar{u}_2 u_2) = f_1(u_3 m_3 \bar{u}_3) = f_1(m_3) = \pi(m_3) f_1(e) = \pi(m_3) v_1 = v_1.$$

Thus our integral simplifies to

$$\begin{aligned} &\text{vol}(U_{\bar{P},0}) \text{vol}(M_x) \int_{U_{\bar{P},r}} \int_{U_{P,r}} \langle v_1, f_2(e) \rangle du_2 d\bar{u}_2 = \\ &\text{vol}(U_{\bar{P},0}) \text{vol}(M_x) \text{vol}(U_{\bar{P},r}) \text{vol}(U_{P,r}) \langle v_1, v_2 \rangle. \quad \square \end{aligned}$$

Completely analogous to Lemma 3.4, one can show that

$$(3.11) \quad \langle f_1, J_{P|\bar{P}}(\bar{\pi}^\vee) f_2 \rangle = \text{vol}(U_{P,0}) \text{vol}(M_x) \text{vol}(U_{\bar{P},r}) \text{vol}(U_{P,r}) \langle v_1, v_2 \rangle.$$

For a better comparison, we need to normalize the Haar measures on G , M , U_P , $U_{\bar{P}}$. That can be done by fixing the volume of one compact open subgroup in each of these groups:

$$\begin{aligned} \text{vol}(M_x) &= 1, \text{vol}(U_{P,0}) = 1, \text{vol}(U_{\bar{P},0}) = 1, \\ \text{vol}(G_{x,r}) &= \text{vol}(U_{\bar{P},r}) \text{vol}(M_{x,r}) \text{vol}(U_{P,r}). \end{aligned}$$

The last definition says that (locally) the Haar measure on G is the product of the Haar measures on $U_{\bar{P}}$, M and U_P .

We will also formulate a version of the upcoming theorem for the pairing between a representation π and its contragredient $\pi^\vee$. By the same formula as in (3.1), the M -invariant bilinear pairing between π and $\pi^\vee$ yields a G -invariant bilinear pairing between $I_P^G(\pi)$ and $I_{\bar{P}}^G(\pi^\vee)$.

Theorem 3.5. *Let π be any irreducible smooth M -representation.*

(a) *For any $f_1 \in I_P^G(V_\pi), f_2 \in I_{\bar{P}}^G(\bar{V}_\pi^\vee)$:*

$$\langle J_{\bar{P}|P}(\pi)f_1, f_2 \rangle = \langle f_1, J_{P|\bar{P}}(\bar{\pi}^\vee)f_2 \rangle.$$

(b) *For any $f_1 \in I_P^G(V_\pi), f \in I_{\bar{P}}^G(V_\pi^\vee)$:*

$$\langle J_{\bar{P}|P}(\pi)f_1, f \rangle = \langle f_1, J_{P|\bar{P}}(\pi^\vee)f \rangle.$$

Remark. If $J_{\bar{P}|P}(\pi)$ is singular, then the two sides of the equalities in (a) and (b) of Theorem 3.5 may be infinite.

Proof of Theorem 3.5. (a) First we assume that Condition 3.1 holds. By Lemma 3.2, it suffices to find one pair (f_1, f_2) such that $\langle J_{\bar{P}|P}(\pi)f_1, f_2 \rangle$ equals $\langle f_1, J_{P|\bar{P}}(\bar{\pi}^\vee)f_2 \rangle$ and is nonzero. This is achieved by Lemma 3.4 and (3.11) with v_1, v_2 such that $\langle v_1, v_2 \rangle \neq 0$. That proves the theorem assuming Condition 3.1.

Consider a family of irreducible smooth M -representations $\pi_0 \otimes \chi$, where χ runs through the group $X_{\text{nr}}(M)$ of unramified characters of M . By the rationality (over $\mathbb{R}$) of $J_{\bar{P}|P}(\pi_0 \otimes \chi)$ and $J_{P|\bar{P}}(\overline{\pi_0 \otimes \chi}^\vee)$ as functions of $\chi \in X_{\text{nr}}(M)$, Condition 3.1.(i) is fulfilled for $\pi_0 \otimes \chi$ with χ in a nonempty Zariski-open subset of $X_{\text{nr}}(M)$. By [Ren10, Théorème VI.8.5] and [Sau97, Théorème 3.2], Condition 3.1.(ii) also holds for $\pi_0 \otimes \chi$ with χ in some nonempty Zariski-open subset of $X_{\text{nr}}(M)$. In other words, we have already proven the theorem for all π in a nonempty Zariski-open subset of the real variety

$$\{\pi_0 \otimes \chi : \chi \in X_{\text{nr}}(M)\}.$$

Consider a χ_0 for which Condition 3.1 fails, and let $f_3 \in I_P^G(\pi \otimes \chi_0)$ and $f_4 \in I_{\bar{P}}^G(\overline{\pi_0 \otimes \chi_0}^\vee)$. Via the linear bijections

$$I_P^G(V_{\pi_0 \otimes \chi_0}) \cong I_{P \cap K}^K(V_{\pi_0 \otimes \chi_0}) \cong I_{P \cap K}^K(V_{\pi_0 \otimes \chi}) \cong I_P^G(V_{\pi_0 \otimes \chi})$$

we can regard f_3 also as an element of $I_P^G(V_{\pi_0 \otimes \chi})$ for any $\chi \in X_{\text{nr}}(M)$. Similarly $f_4 \in I_{\bar{P}}^G(\overline{\pi_0 \otimes \chi}^\vee)$. Now the expressions

$$\langle J_{\bar{P}|P}(\pi_0 \otimes \chi)f_3, f_4 \rangle \quad \text{and} \quad \langle f_3, J_{P|\bar{P}}(\overline{\pi_0 \otimes \chi}^\vee)f_4 \rangle$$

are rational functions of $\chi \in X_{\text{nr}}(M)$. They coincide on a Zariski-dense subset of $X_{\text{nr}}(M)$, so they are equal.

(b) This can be shown in the same way as part (a), using $\pi^\vee$ instead of $\bar{\pi}^\vee$ in all the previous arguments. $\square$

Theorem 3.5.a says that $J_{\bar{P}|P}(\pi) : I_P^G(\pi) \rightarrow I_{\bar{P}}^G(\pi)$ is the adjoint operator of

$$J_{P|\bar{P}}(\bar{\pi}^\vee) : \overline{I_{\bar{P}}^G(\pi)}^\vee \cong I_{\bar{P}}^G(\bar{\pi}^\vee) \rightarrow I_{\bar{P}}^G(\bar{\pi}^\vee) \cong \overline{I_P^G(\pi)}^\vee.$$

In the special case that π is Hermitian (e.g. unitary), we may replace $\bar{\pi}^\vee$ by the isomorphic representation π . Then Theorem 3.5.a says that

$$(3.12) \quad J_{\bar{P}|P}(\pi) : I_P^G(\pi) \rightarrow I_{\bar{P}}^G(\pi) \text{ is the adjoint of } J_{P|\bar{P}}(\pi) : I_{\bar{P}}^G(\pi) \rightarrow I_P^G(\pi),$$

with respect to the G -invariant Hermitian forms (3.1) on $I_P^G(V_\pi)$ and on $I_{\bar{P}}^G(V_\pi)$.

Theorem 3.5.b is mentioned (for reductive p -adic groups) in [Wal03, p. 283], but unfortunately without a proof. We note that in the versions of Theorem 3.5 in [Wal03], the two parabolic subgroups share the same Levi factor, but they need not be opposite. We want to extend Theorem 3.5 to that generality, for our covering group G .

Let $Z(M')_s$ be the maximal F -split torus in the centre of M' . We say that a root of $Z(M')_s$ is a character of $Z(M')_s$ appearing in the adjoint action on $\text{Lie}(G) = \text{Lie}(G')$, and that a root α is reduced if $q\alpha$ is not a root for $q \in (0, 1)$. Let $Q = MU_Q$ be another parabolic subgroup of G with Levi factor M . As in [Wal03, p. 279], we define the distance between P and Q as

$$d(P, Q) = \text{number of reduced roots of } Z(M')_s \text{ appearing in } \text{Lie}(U_{\bar{P}} \cap U_Q).$$

In this more general context the definition of $J_{Q|P}(\pi)$ becomes

$$(J_{Q|P}(\pi)f)(g) = \int_{(U_P \cap U_Q) \backslash U_Q} f(ug) \, du.$$

Proposition 3.6. *Theorem 3.5 holds also with Q instead of $\bar{P}$, that is, it holds for any two parabolic subgroups with the same Levi factor.*

Proof. We can find a sequence of parabolic subgroups

$$P = P_0, P_1, \dots, P_d = Q$$

such that $d(P_i, P_j) = i - j$ for all $i \geq j$. According to [Wal03, p. 283],

$$J_{Q|P}(\pi) = J_{P_d|P_{d-1}}(\pi) \circ \dots \circ J_{P_2|P_1}(\pi) \circ J_{P_1|P_0}(\pi).$$

Hence it suffices to prove the proposition for two parabolic subgroups of distance 1. In other words, we only need to consider the cases with $d(P, Q) = 1$.

Now $Z(M')_s$ acts on $\text{Lie}(U_{\bar{P}} \cap U_Q)$ by multiples of a unique reduced root α . We put $T'_\alpha = \ker(\alpha)^\circ$, a codimension one subtorus of $Z(M')_s$. Consider the Levi subgroup $M'_\alpha := Z_{G'}(T'_\alpha)$ of G' . It is generated by

$$M' \cup (U_{\bar{P}} \cap U_Q) \cup (U_P \cap U_Q)$$

and contains M' as a maximal Levi subgroup. Then $M_\alpha := p_G^{-1}(M'_\alpha)$ is a Levi subgroup of G with M as maximal Levi subgroup. Since

$$(U_P \cap U_Q) \backslash U_Q \cong U_{\bar{P}} \cap U_Q$$

and $U_{\bar{P}} \cap U_Q \subset M_\alpha$, both $J_{Q|P}(\pi)$ and $J_{Q \cap M_\alpha | P \cap M_\alpha}(\pi)$ are given by integration over $U_{\bar{P}} \cap U_Q$. It follows that $J_{Q|P}(\pi)$ can be obtained from $J_{Q \cap M_\alpha | P \cap M_\alpha}(\pi)$ by applying the functor $I_{P M_\alpha}^G = I_{Q M_\alpha}^G$.

In M_α there are only two parabolic subgroups with Levi factor M , namely $P \cap M_\alpha$ and $Q \cap M_\alpha$, and they are opposite. Thus Theorem 3.5 applies to $J_{Q \cap M_\alpha | P \cap M_\alpha}(\pi)$, for any irreducible smooth M -representation π . We note that the group $K \cap M_\alpha = M_{\alpha, x}$ is the good maximal compact subgroup of M_α

associated to the special vertex x . For $f_1 \in I_P^G(V_\pi)$ and $f_2 \in I_Q^G(\overline{V_\pi}^\vee)$ we compute

$$\begin{aligned}
 \langle J_{Q|P}(\pi)f_1, f_2 \rangle &= \int_K \langle (J_{Q|P}(\pi)f_1)(k), f_2(k) \rangle dk \\
 (3.13) \quad &= \int_{K \cap M_\alpha \backslash K} \int_{K \cap M_\alpha} \langle (J_{Q|P}(\pi)f_1)(k_\alpha k), f_2(k_\alpha k) \rangle dk_\alpha dk \\
 &= \int_{K \cap M_\alpha \backslash K} \int_{K \cap M_\alpha} \langle (J_{Q|P}(\pi)\pi(k)f_1)(k_\alpha), (\overline{\pi}^\vee(k)f_2)(k_\alpha) \rangle dk_\alpha dk.
 \end{aligned}$$

The restriction of $\pi(k)f_1 : G \rightarrow V_\pi$ to M_α can be regarded as an element of $I_{P \cap M_\alpha}^{M_\alpha}(V_\pi)$, and similarly for $\overline{\pi}^\vee(k)f_2$. Then Theorem 3.5.a says that (3.13) equals

$$\begin{aligned}
 &\int_{K \cap M_\alpha \backslash K} \langle J_{Q \cap M_\alpha | P \cap M_\alpha}(\pi)(\pi(k)f_1)|_{M_\alpha}, (\overline{\pi}^\vee(k)f_2)|_{M_\alpha} \rangle dk = \\
 (3.14) \quad &\int_{K \cap M_\alpha \backslash K} \langle (\pi(k)f_1)|_{M_\alpha}, J_{P \cap M_\alpha | Q \cap M_\alpha}(\overline{\pi}^\vee)(\overline{\pi}^\vee(k)f_2)|_{M_\alpha} \rangle dk = \\
 &\int_{K \cap M_\alpha \backslash K} \int_{K \cap M_\alpha} \langle f_1(k_\alpha k), (J_{P \cap M_\alpha | Q \cap M_\alpha}(\overline{\pi}^\vee)f_2)(k_\alpha k) \rangle dk_\alpha dk = \\
 &\int_K \langle f_1(k), (J_{P|Q}(\overline{\pi}^\vee)f_2)(k) \rangle dk = \langle f_1, J_{P|Q}(\overline{\pi}^\vee)f_2 \rangle.
 \end{aligned}$$

The combination of (3.13) and (3.14) shows that Theorem 3.5.a holds for $J_{Q|P}(\pi)$ and $J_{P|Q}(\overline{\pi}^\vee)$. The same argument applies with $\pi^\vee$ instead of $\overline{\pi}^\vee$, and that generalizes Theorem 3.5.b. $\square$

4. POLES AND ZEROS OF THE HARISH-CHANDRA μ -FUNCTION

In this section, we discuss the Harish-Chandra μ -function. By [Wal03, Lemme V.2.1], it decomposes as a product $\mu = \prod_\alpha \mu_\alpha$, where α runs over the positive roots in the reduced root system corresponding to M in G . Each μ_α can be computed in a Levi subgroup M_α of G , which has M as a maximal Levi subgroup.

Thus from now on, we assume that M is a maximal Levi subgroup of G , which leads to $[N_G(M) : M] \leq 2$. Let $P = MU_P$ be a parabolic subgroup with Levi M and let $\bar{P} = MU_{\bar{P}}$ be its opposite parabolic subgroup. We consider the intertwining operators

$$J_{\bar{P}|P}(\sigma \otimes \chi) : I_P^G(\sigma \otimes \chi) \rightarrow I_{\bar{P}}^G(\sigma \otimes \chi), \quad J_{P|\bar{P}}(\sigma \otimes \chi) : I_{\bar{P}}^G(\sigma \otimes \chi) \rightarrow I_P^G(\sigma \otimes \chi),$$

for $\sigma \in \text{Irr}_{\text{cusp}}(M)$ and $\chi \in X_{\text{nr}}(M)$. They are defined by

$$(J_{\bar{P}|P}(\sigma \otimes \chi)f)(g) = \int_{U_{\bar{P}}} f(\bar{u}g) d\bar{u}, \quad (J_{P|\bar{P}}(\sigma \otimes \chi)f')(g) = \int_{U_P} f'(ug) du.$$

Their composition

$$j(\sigma \otimes \chi) := J_{P|\bar{P}}(\sigma \otimes \chi) \circ J_{\bar{P}|P}(\sigma \otimes \chi)$$

is a rational function in the variable $\chi \in X_{\text{nr}}(M)$, see [Wal03, §IV.3]. Up to a positive real scalar, the Harish-Chandra μ -function is $j(\sigma \otimes \chi)^{-1}$ [Wal03, §V.2], and we want to analyze its poles and zeros.

4.1. μ as a rational function. Let us explain how to view $\mu(\sigma \otimes \chi)$ as a rational function on $\mathbb{C}^\times$. We refer to [Sol22] (and the correction in [Sol23]). Let

$$X_{\text{nr}}(M, \sigma) := \{\chi \in X_{\text{nr}}(M) : \sigma \otimes \chi \cong \sigma\}.$$

Note that $X_{\text{nr}}(M, \tau) = X_{\text{nr}}(M, \sigma)$ for any $\tau \in \text{Irr}(M)_{[M, \sigma]}$. There are subgroups of M ,

$$M^1 = \cap_{\chi \in X_{\text{nr}}(M)} \ker(\chi), \quad M_\sigma^2 = \cap_{\chi \in X_{\text{nr}}(M, \sigma)} \ker(\chi).$$

The subgroup $M \subset G$ is the inverse image of a maximal Levi subgroup M' of G' . Let $Z(M')_s$ be the maximal split torus in the center of M' . Then there is only one positive reduced root α coming from the adjoint representation of $Z(M')_s$ on the Lie algebra of G' . Let $h_\alpha^\vee$ be the unique generator of $(M_\sigma^2 \cap G^1)/M^1 \cong \mathbb{Z}$ such that $|\alpha(h_\alpha^\vee)|_F > 1$. Then for $\chi \in X_{\text{nr}}(M)$, we let the coordinate z be given by $z = \chi(h_\alpha^\vee) \in \mathbb{C}^\times$. The function $\mu(\sigma \otimes \chi) = \mu(\sigma, z)$ only depends on this variable, see [Wal03, p. 283].

Suppose $\tau = \sigma \otimes \chi_\tau \in \text{Irr}(M)_{[M, \sigma]}$ is a different choice of representative in the inertial equivalence class of σ for M . Then for $\chi \in X_{\text{nr}}(M)$, $\mu(\tau \otimes \chi) = \mu(\sigma \otimes \chi_\tau \chi)$. Hence

$$(4.1) \quad \mu(\tau, z) = \mu(\sigma, q_\tau z), \quad \text{where } q_\tau = \chi_\tau(h_\alpha^\vee) \in \mathbb{C}^\times.$$

For G' , it was shown in [Sil80, Theorem 1.6] that there exists a unitary $\sigma_0 \in \text{Irr}(M')_{[M', \sigma]}$ such that as a function in $z = \chi(h_\alpha^\vee)$, $\mu = \mu_\alpha$ has the form

$$(4.2) \quad \mu(\sigma_0 \otimes \chi) = \mu(\sigma_0, z) = c_\mu \cdot \frac{(1-z)(1-z^{-1})}{(1-qz)(1-qz^{-1})} \frac{(1+z)(1+z^{-1})}{(1+q'z)(1+q'z^{-1})},$$

where $c_\mu > 0$ and $q \geq 1$. The possibility that μ is independent of χ is included in this formula; this happens when $q = q' = 1$.

Our goal is to give an alternative proof of this result, which generalizes to covering groups. The idea is as follows. Since $\mu(\sigma \otimes \chi) = \mu(\sigma, z)$ is a rational function on $\mathbb{C}^\times$, we may also view it as a rational function on the projective line $\mathbb{P}^1(\mathbb{C})$, by considering the limits $\mu(\sigma, \infty) := \lim_{z \rightarrow \infty} \mu(\sigma, z)$ and $\mu(\sigma, 0) := \lim_{z \rightarrow 0} \mu(\sigma, z)$. If $\mu(\sigma, z)$ is not constant, then there exists a unitary $\sigma_0 \in \text{Irr}(M)_{[M, \sigma]}$ with $\mu(\sigma_0) = 0$. The only other possibility for a zero of $\mu(\sigma_0, z)$ in $\mathbb{C}^\times$ will be at $z = -1$ and all zeros are of multiplicity 2. Using techniques concerning Hermitian and unitary representations (see [Ren10, §IV] for definitions), we will then locate two poles in $\mathbb{R}^\times$ for every zero of $\mu(\sigma_0, z)$. Moreover, we will show that $\mu(\sigma_0, z)$ can not be zero at $z = \infty$ and $z = 0$. Combining all these results, we obtain the formula as in (4.2).

The result below is similar to the reductive case and will help us to give a description of the supercuspidal representations of M .

Lemma 4.1. *Let $p_G : G \rightarrow G'$ be a finite central covering. Let $G^1 = p_G^{-1}(G'^1)$, where G'^1 is the subgroup of G' generated by all compact subgroups. Then $G^1 \cdot Z(G)$ has finite index in G .*

Proof. Let $g \in p_G^{-1}(Z(G'))$. Consider the commutator map

$$[g, \cdot] : G \rightarrow \ker(p_G) \subset Z(G), \quad x \mapsto gxg^{-1}x^{-1}.$$

Let $k = |\ker(p_G)|$. Then since $gxg^{-1} = [g, x]x$, we obtain

$$g^k x g^{-k} x^{-1} = g^{k-1} [g, x] x g^{1-k} x^{-1} = \dots = [g, x]^k x x^{-1} = [g, x]^k = 1.$$

Therefore, $g^k \in Z(G)$ for all $g \in p_G^{-1}(Z(G'))$. Recall that G'/G'^1 is a finitely generated free abelian group and that $Z(G')/(Z(G') \cap G'^1)$ is a subgroup of finite index in G'/G'^1 . Hence the group of k -th powers $Z(G')^{(k)}/(Z(G')^{(k)} \cap G'^1)$ also has finite index in G'/G'^1 . Applying p_G^{-1} , it follows that $G^1 \cdot Z(G)$ has finite index in G . $\square$

Let $\text{Irr}(M)_{\text{cusp}}^{\text{unit}}$ be the set of unitary supercuspidal representations of M .

Lemma 4.2. *There is a bijection*

$$\text{Irr}(M)_{\text{cusp}}^{\text{unit}} \times \text{Hom}(M, \mathbb{R}_{>0}) \rightarrow \text{Irr}(M)_{\text{cusp}}.$$

Proof. First, let us show that there is a bijection

$$\text{Hom}(Z(M), \mathbb{R}_{>0}) \xrightarrow{\sim} \text{Hom}(M, \mathbb{R}_{>0}).$$

For this, we need to show that every $\chi \in \text{Hom}(Z(M), \mathbb{R}_{>0})$ extends uniquely to a character in $\text{Hom}(M, \mathbb{R}_{>0})$. Since $Z(M)$ is abelian, it has a maximal compact subgroup $Z(M)_{\text{cpt}}$. Note that χ factors through $Z(M)/Z(M)_{\text{cpt}}$, because the image of $Z(M)_{\text{cpt}}$ must be a compact subgroup of $\mathbb{R}_{>0}$, i.e. $\{1\}$. By Lemma 4.1, $M^1 Z(M)$ has finite index in M . Hence,

$$Z(M)/Z(M)_{\text{cpt}} \hookrightarrow M/M^1 \cong \mathbb{Z}^r,$$

is a subgroup of finite index. So $Z(M)/Z(M)_{\text{cpt}} \cong \mathbb{Z}^r$ is a free abelian subgroup of the same rank. So for $x \in M/M^1$, let n be a positive integer such that $n \cdot x \in Z(M)/Z(M)_{\text{cpt}}$. Define $\chi(x) := \chi(n \cdot x)^{1/n}$, which is well-defined because $\mathbb{R}_{>0}$ has unique n -th roots and it does not depend on the choice of n . It is a group homomorphism, since

$$\begin{aligned} \chi(n \cdot x)^{\frac{1}{n}} \chi(m \cdot y)^{\frac{1}{m}} &= \chi(nm \cdot x)^{\frac{1}{nm}} \chi(nm \cdot y)^{\frac{1}{nm}} = (\chi(nm \cdot x) \chi(nm \cdot y))^{\frac{1}{nm}} \\ &= \chi((nm \cdot x)(nm \cdot y))^{\frac{1}{nm}} = \chi(nm \cdot xy)^{\frac{1}{nm}}. \end{aligned}$$

This describes how $\chi : Z(M)/Z(M)_{\text{cpt}} \rightarrow \mathbb{R}_{>0}$ extends uniquely to a character $\chi : M/M^1 \rightarrow \mathbb{R}_{>0}$. Now we can explicitly describe the bijection. It is given by

$$\text{Irr}(M)_{\text{cusp}}^{\text{unit}} \times \text{Hom}(M, \mathbb{R}_{>0}) \rightarrow \text{Irr}(M)_{\text{cusp}}, \quad (\pi, \chi) \mapsto \pi \otimes \chi,$$

and its inverse is given by $\sigma \mapsto (\sigma \otimes |\text{cc}(\sigma)|^{-1}, |\text{cc}(\sigma)|)$, where $\text{cc}(\sigma)$ is the central character of σ , and we extend $|\text{cc}(\sigma)|$ to a character of M as above. By [Ren10, p. 116], [FP22, Theorem 6.2], a smooth irreducible representation π of M is supercuspidal and unitary if and only if it is supercuspidal and its central

character $\text{cc}(\pi)$ is unitary. That explains why the representation $\sigma \otimes |\text{cc}(\sigma)|^{-1}$ is unitary. $\square$

By Lemma 4.2, there exists a unitary $\tilde{\sigma} \in \text{Irr}(M)_{[M, \sigma]}$. Knowing $\mu(\tilde{\sigma}, z)$ is equivalent to knowing $\mu(\sigma, z)$, by (4.1).

Suppose that σ is unitary, then by [Ren10, §IV.2.3], the parabolically induced representation $I_P^G(\sigma)$ is unitary as well. We can use the adjointness relation from Section 3 to show that μ must be real and non-negative on the unit circle.

Lemma 4.3. *Let σ be unitary. Then the function $\mu(\sigma, z)$ has all its values in $\mathbb{R}_{\geq 0}$ for z on the unit circle $S^1 = \{e^{i\varphi} \mid \varphi \in \mathbb{R}\} \subset \mathbb{C}^\times$.*

Proof. We claim that for $z = e^{i\varphi} \in S^1$, there always exists a unitary character $\chi_z \in X_{\text{nr}}(M)$ such that $\chi_z(h_\alpha^\vee) = z$. Indeed, recall that $X_{\text{nr}}(M) = \text{Hom}(M/M^1, \mathbb{C}^\times)$. Moreover, M/M^1 is a free abelian group of finite rank, and $h_\alpha^\vee$ is a generator of $M_\sigma^2 \cap G^1/M^1$, which is a free rank 1 subgroup of M/M^1 . Therefore, we can choose a basis of M/M^1 , such that $h_\alpha^\vee$ is an integer multiple of one of these basis elements. So write $h_\alpha^\vee = n \cdot b$ with $n \in \mathbb{Z}$ and b a basis element of M/M^1 . Then we can define $\chi_z(b) = e^{i\varphi/n}$, and let χ_z be trivial on the other basis elements of M/M^1 . This gives us a unitary character with

$$\chi_z(h_\alpha^\vee) = \chi_z(n \cdot b) = \chi_z(b)^n = (z^{1/n})^n = z.$$

Since we chose σ to be unitary, $\sigma \otimes \chi_z$ is unitary as well, and the adjunction formula from Theorem 3.5.a applies with $\overline{\sigma \otimes \chi_z}^\vee$ replaced by $\sigma \otimes \chi_z$. Since $\sigma \otimes \chi_z$ is unitary, we may assume that the induced pairing on $I_P^G(\sigma \otimes \chi_z)$ (see (3.1)) is positive definite. So for $f \in I_P^G(\sigma \otimes \chi_z)$, we have

$$(4.3) \quad \begin{aligned} j(\sigma \otimes \chi_z) \langle f, f \rangle &= \langle f, J_{P|\bar{P}}(\sigma \otimes \chi_z) \circ J_{\bar{P}|P}(\sigma \otimes \chi_z) f \rangle \\ &= \langle J_{\bar{P}|P}(\sigma \otimes \chi_z) f, J_{\bar{P}|P}(\sigma \otimes \chi_z) f \rangle \in \mathbb{R}_{\geq 0} \cup \{\infty\}. \end{aligned}$$

Since $J_{P|\bar{P}}(\sigma \otimes \chi_z)$ is nonzero, (4.3) shows that $j(\sigma \otimes \chi_z) \neq 0$. Thus $z \mapsto j(\sigma \otimes \chi_z)$ is a continuous map from S^1 to $\mathbb{R}_{>0} \cup \{\infty\}$. We obtain that $\mu(\sigma, z) = \mu(\sigma \otimes \chi_z) = j(\sigma \otimes \chi_z)^{-1} \in \mathbb{R}_{\geq 0}$. $\square$

Using the result above, we can now show that for every zero (resp. pole) of μ , there must be another zero (resp. pole) of μ , of the same multiplicity.

Lemma 4.4. *Let σ be unitary. Then the function $\mu(\sigma, z)$ has the form*

$$\mu(\sigma, z) = c_\mu \prod_i (z - \lambda_i)^{n_i} (z^{-1} - \overline{\lambda_i})^{n_i},$$

where $c_\mu \in \mathbb{R}_{>0}$, $\lambda_i \in \mathbb{C}$ and $n_i \in \mathbb{Z}$.

Proof. By Lemma 4.3, we have that $\mu(\sigma, z) \in \mathbb{R}_{\geq 0}$ for all z on the unit circle. Since $\mu(\sigma, z)$ is a nonzero complex rational function in the variable z , we can write

$$\mu(\sigma, z) = c_\mu \prod_i (z - \lambda_i)^{n_i},$$

where $c_\mu \in \mathbb{C}^\times$, $\lambda_i \in \mathbb{C}$, $n_i \in \mathbb{Z}$. For every negative power $n_i \in \mathbb{Z}_{<0}$ appearing in the factorization above, we may multiply $\mu(\sigma, z)$ by $(z - \lambda_i)^{-n_i}(z^{-1} - \overline{\lambda_i})^{-n_i}$, to obtain a function of the form

$$\tilde{\mu}(\sigma, z) = c_\mu \prod_i (z - \lambda_i)^{n_i} \prod_j (z^{-1} - \gamma_j)^{\ell_j},$$

where $\lambda_i, \gamma_j \in \mathbb{C}$ and $n_i, \ell_j \in \mathbb{Z}_{>0}$. Note that $\tilde{\mu}(\sigma, z)$ is again real and non-negative on the unit circle, and it suffices to show the statement of the lemma for $\tilde{\mu}(\sigma, z)$. For $\varphi \in \mathbb{R}$, we define $f(\varphi) := \tilde{\mu}(\sigma, e^{i\varphi})$. Then we have $f(\varphi) = \overline{f(\varphi)}$ for all $\varphi \in \mathbb{R}$, i.e.

$$c_\mu \prod_i (e^{i\varphi} - \lambda_i)^{n_i} \prod_j (e^{-i\varphi} - \gamma_j)^{\ell_j} = \overline{c_\mu} \prod_i (e^{-i\varphi} - \overline{\lambda_i})^{n_i} \prod_j (e^{i\varphi} - \overline{\gamma_j})^{\ell_j}.$$

Both sides in the equation are trigonometric polynomials, which have a unique factorization. So the set of $\overline{\lambda_i}$'s, where each $\overline{\lambda_i}$ appears n_i times, equals the set of γ_j 's, where each γ_j appears ℓ_j times. It follows that

$$\tilde{\mu}(\sigma, z) = c_\mu \prod_i (z - \lambda_i)^{n_i} (z^{-1} - \overline{\lambda_i})^{n_i}.$$

Since μ is real and non-negative on the unit circle and not identically zero, we see that $c_\mu \in \mathbb{R}_{>0}$, which completes the proof. $\square$

4.2. An aspect of the Iwasawa decomposition. In this subsection, we discuss an aspect of the Iwasawa decomposition. This will be necessary to study the limit of the intertwining operator $J_{\bar{P}|P}(\sigma \otimes \chi)$ and the function $\mu(\sigma, z)$ at $z = \infty$. We use the notations from Section 3. Recall the Iwasawa decomposition $G' = P'K' = K'P'$ from [KP23, Theorem 5.3.4]. It lifts to the Iwasawa decomposition

$$(4.4) \quad G = PK = KP.$$

As $P = U_P \rtimes M$, this means that for each $g \in G$ there exist $k_P(g) \in K$, $m_P(g) \in M$ and $u_P(g) \in U_P$ such that $g = u_P(g)m_P(g)k_P(g)$. The decomposition of g along (4.4) is unique up to $P \cap K$. Hence $k_P(g)$, $m_P(g)$ and $u_P(g)$ are unique up to, respectively, $P \cap K$, $M \cap K$ and $U_P \cap K$.

For any subgroup H of G or G we will write H_x for the stabilizer of $x \in \mathcal{B}(G')$ in H . In particular $K' = G'_x$ and $K = G_x$. We choose a positive system $\Phi(G, S)^+$ in $\Phi(G, S)$, such that $\Phi(U_P, S) \subset \Phi(G, S)^+$. Let U^+ (respectively U^-) be the subgroup of G generated by the root subgroups U_β with β in $\Phi(G, S)^+$ (respectively in $\Phi(G, S)^- = -\Phi(G, S)^+$).

We note that the normalizer $N_G(S)$ equals $p_G^{-1}(N_{G'}(S'))$.

Theorem 4.5. *The product map*

$$\prod_{\beta \in \Phi(G, S)^\pm} U_{\beta, x} \longrightarrow U_x^\pm = U^\pm \cap G_x$$

is a homeomorphism. Further, G_x can be obtained as

$$G_x = U_x^- U_x^+ N_G(S)_x = U_x^+ U_x^- N_G(S)_x.$$

Proof. For G' , the theorem is the combination of [BT72, Proposition 6.4.9] and the proof of [KP23, Lemma 7.7.3]. The theorem for G follows from that for G' by the covering $p_G : G \rightarrow G'$ and the canonical lifting of each root subgroup of G' to G . $\square$

We will also work with the Lie algebra $\text{Lie}(G')$, which can be defined either as the F -points of $\text{Lie}(G')$ or as the tangent space at e of G considered as manifold over the local field F . Since $p_G : G \rightarrow G'$ is a finite covering, we can identify $\text{Lie}(G')$ (defined using F -manifolds) with $\mathfrak{g} = \text{Lie}(G)$. Similarly $\text{Lie}(G'_x)$ can be viewed either as the $\mathfrak{o}_F$ -points of the Lie algebra of the $\mathfrak{o}_F$ -scheme G'_x , or as the Lie algebra of G'_x in the sense of manifolds over $\mathfrak{o}_F$. Then $\mathfrak{g}_x = \text{Lie}(G_x)$ can be identified with $\text{Lie}(G'_x)$, and it is an $\mathfrak{o}_F$ -lattice in $\mathfrak{g}$. We write $\mathfrak{u}_\beta = \text{Lie}(U_\beta)$ and

$$\mathfrak{z} = \text{Lie}(Z_{G'}(S')) = \text{Lie}(N_{G'}(S')) = \text{Lie}(N_G(S)).$$

By $\text{Lie}(H)_x$ we mean always $\text{Lie}(H_x)$. With these notations, Theorem 4.5 implies that

(4.5)

$$\begin{aligned} \mathfrak{g}_x &= \bigoplus_{\beta \in \Phi(G,S)^-} \mathfrak{u}_{\beta,x} \oplus \bigoplus_{\beta \in \Phi(G,S)^+} \mathfrak{u}_{\beta,x} \oplus \mathfrak{z}_x \\ &= \bigoplus_{\beta \in \Phi(U_{\bar{P}},S)} \mathfrak{u}_{\beta,x} \oplus \left(\bigoplus_{\beta \in \Phi(M,S)^-} \mathfrak{u}_{\beta,x} \oplus \bigoplus_{\beta \in \Phi(M,S)^+} \mathfrak{u}_{\beta,x} \oplus \mathfrak{z}_x \right) \oplus \bigoplus_{\beta \in \Phi(U_P,S)} \mathfrak{u}_{\beta,x} \\ &= \text{Lie}(U_{\bar{P},x}) \oplus \text{Lie}(M_x) \oplus \text{Lie}(U_{P,x}). \end{aligned}$$

Proposition 4.6. *Let $\bar{u} \in U_{\bar{P}}$ and write it as $\bar{u} = umk$ with $u = u_P(\bar{u}) \in U_P$, $m = m_P(\bar{u}) \in M$ and $k = k_P(\bar{u}) \in G_x$. Then $\delta_P(m_P(\bar{u})) \leq 1$.*

Proof. Since $k \in G_x$, we have

$$\bar{u}G_x\bar{u}^{-1} = G_{\bar{u}x} = G_{umkx} = G_{umx} = uG_{mx}u^{-1}.$$

By (4.5), the Lie algebra of this group is

$$\begin{aligned} \mathfrak{g}_{\bar{u}x} &= \text{Ad}(u)\mathfrak{g}_{mx} = \text{Ad}(u)(\text{Lie}(U_{\bar{P},mx}) \oplus \text{Lie}(M_{mx}) \oplus \text{Lie}(U_{P,mx})) \\ (4.6) \quad &= \text{Ad}(u)\text{Lie}(U_{\bar{P},mx}) \oplus \text{Ad}(u)\text{Lie}(P_{mx}). \end{aligned}$$

The decomposition $\mathfrak{g} = \text{Lie}(U_{\bar{P}}) \oplus \mathfrak{p}$ gives a projection $\text{pr}_{\mathfrak{g}/\mathfrak{p}} : \mathfrak{g} \rightarrow \text{Lie}(U_{\bar{P}})$ with kernel $\mathfrak{p} = \text{Lie}(P)$. Clearly $\text{pr}_{\mathfrak{g}/\mathfrak{p}}(\mathfrak{g}_{\bar{u}x})$ contains $\text{Lie}(U_{\bar{P},\bar{u}x}) = \text{Ad}(\bar{u})\text{Lie}(U_{\bar{P},x})$. From (4.6) we obtain, using $\text{Ad}(u)\text{Lie}(P_{mx}) \subset \mathfrak{p}$:

$$(4.7) \quad \text{Ad}(\bar{u})\text{Lie}(U_{\bar{P},x}) \subset \text{pr}_{\mathfrak{g}/\mathfrak{p}}(\mathfrak{g}_{\bar{u}x}) = \text{pr}_{\mathfrak{g}/\mathfrak{p}}(\text{Ad}(u)\text{Lie}(U_{\bar{P},mx})).$$

The Haar measures on the groups $G, M, U_P, P, U_{\bar{P}}, \bar{P}$ induce measures on their Lie algebras. By the unimodularity of the unipotent group $U_{\bar{P}}$, the lattice $\text{Ad}(\bar{u})\text{Lie}(U_{\bar{P},x})$ in $\text{Lie}(U_{\bar{P}})$ has the same volume as $\text{Lie}(U_{\bar{P},x})$.

Since u is unipotent, so is the F -linear operator $\text{Ad}(u) : \mathfrak{g} \rightarrow \mathfrak{g}$. Moreover $\text{Ad}(u)$ stabilizes $\mathfrak{p}$, so there exist a basis of $\mathfrak{p}$ and a basis of $\text{Lie}(U_{\bar{P}})$ such that

the matrix of $\text{Ad}(u)$ with respect to the combined basis of $\mathfrak{g}$ is unipotent and upper triangular. It follows that the matrix of

$$(4.8) \quad \text{pr}_{\mathfrak{g}/\mathfrak{p}} \circ \text{Ad}(u) : \text{Lie}(U_{\bar{P}}) \rightarrow \text{Lie}(U_{\bar{P}})$$

with respect to the chosen basis is also unipotent and upper triangular. Therefore (4.8) preserves volumes. Now we can interpret (4.7) in terms of volumes:

$$\text{vol}(\text{Lie}(U_{\bar{P},x})) \leq \text{vol}(\text{Lie}(U_{\bar{P},mx})).$$

This can be rewritten as

$$(4.9) \quad 1 \leq \frac{\text{vol}(\text{Lie}(U_{\bar{P},mx}))}{\text{vol}(\text{Lie}(U_{\bar{P},x}))} = \frac{\text{vol}(\text{Ad}(m)\text{Lie}(U_{\bar{P},x}))}{\text{vol}(\text{Lie}(U_{\bar{P},x}))} \\ = \left| \det(\text{Ad}(m) : \text{Lie}(U_{\bar{P}}) \rightarrow \text{Lie}(U_{\bar{P}})) \right|_F.$$

The modular function $\delta_{\bar{P}}$ of $\bar{P}$ equals $\delta_{\bar{P}'} \circ p_G$, because $\ker p_G$ is finite. By [Sil80, Lemma 1.2.1.1], we have

$$(4.10) \quad \delta_{\bar{P}'}(m') = \left| \det(\text{Ad}(m') : \text{Lie}(U_{\bar{P}}) \rightarrow \text{Lie}(U_{\bar{P}})) \right|_F \quad \text{for all } m' \in M'.$$

The adjoint representation of $\bar{P}$ factors through $p_G : \bar{P} \rightarrow \bar{P}'$, so the inequality (4.9) says that $\delta_{\bar{P}}(m) \geq 1$. Finally, we use that $\delta_{\bar{P}} = \delta_{\bar{P}'}^{-1}$ [Ren10, Lemme V.5.4]. $\square$

One may reformulate the conclusion of Proposition 4.6 as: conjugation by $m_P(\bar{u})$ contracts U_P and inflates $U_{\bar{P}}$. Thus the image of $\bar{U}$ in M under the Iwasawa decomposition (well-defined up to $M \cap G_x$) sits inside M in asymmetric way, like a semigroup but certainly not like a group.

Lemma 4.7. *In the setting of Proposition 4.6, suppose that P is a maximal proper parabolic subgroup of G and that $\delta_P(m_P(\bar{u})) = 1$. Then $m_P(\bar{u}) \in M^1$.*

Proof. By construction $M^1 = p_G^{-1}(M^1)$. As the Iwasawa decomposition of G is lifted from that for G' , it suffices to prove the lemma with G' instead of G . To use that simplification without many extra primes, we assume in the remainder of the proof that $G = G'$.

The derived group $G_{\text{der}} = \mathcal{G}_{\text{der}}(F)$ also has an Iwasawa decomposition, namely

$$(4.11) \quad G_{\text{der}} = U_P(M \cap G_{\text{der}})G_{\text{der},x}.$$

This is a restriction of the Iwasawa decomposition $G = U_P M G_x$. As $\bar{u} \in G_{\text{der}}$, we can take $k_P(\bar{u})$, $m_P(\bar{u})$ and $u_P(\bar{u})$ in G_{der} as well, using (4.11).

By assumption $P \cap G_{\text{der}} = (M \cap G_{\text{der}}) \ltimes U_P$ is a maximal proper parabolic subgroup of the semisimple group G_{der} . Then the algebraic group $Z(\mathcal{M} \cap \mathcal{G}_{\text{der}})^\circ$ is a onedimensional F -split torus. It follows that $(M \cap G_{\text{der}})/(M \cap G_{\text{der}})^1$ is a free abelian group of rank one, so isomorphic to $\mathbb{Z}$.

On the right hand side of (4.10) we can replace $\bar{P}$ by P or by $P \cap G_{\text{der}}$, both give the same expression. Hence the modular functions δ_P and $\delta_{P \cap G_{\text{der}}}$ coincide on $M \cap G_{\text{der}}$. As $\delta_{P \cap G_{\text{der}}}$ is trivial on $(M \cap G_{\text{der}})^1$, it defines a character

$$(4.12) \quad \mathbb{Z} \cong (M \cap G_{\text{der}})/(M \cap G_{\text{der}})^1 \rightarrow \mathbb{R}_{>0}^\times.$$

The restrictions of elements of $\Phi(U_P, S)$ to $Z(M \cap G_{\text{der}})$ are positive powers of a single nontrivial character α . Then (4.10) shows that $\delta_{P \cap G_{\text{der}}}|_{Z(M \cap G_{\text{der}})}$ is a positive power of $|\alpha|$. In particular $\delta_{P \cap G_{\text{der}}}$ is nontrivial, so regarded as a character of $\mathbb{Z}$ via (4.12) it is injective. The assumptions of the lemma and Proposition 4.6 entail that

$$\delta_{P \cap G_{\text{der}}}(m_P(\bar{u})) = \delta_P(m_P(\bar{u})) = 1.$$

Therefore $m_P(\bar{u})$ lies in the kernel of (4.12), which means that

$$m_P(\bar{u}) \in (M \cap G_{\text{der}})^1 \subset M^1.$$

That proves the lemma for one choice of $m_P(\bar{u})$ in the Iwasawa decomposition of G . But $m_P(\bar{u})$ is unique up to the compact group $M \cap G_x$, so $m_P(\bar{u}) \in M^1$ for any other choice of $m_P(\bar{u})$ as well. $\square$

4.3. The limit at infinity. In this subsection, we study the limit of the intertwining operator $J_{\bar{P}|P}(\sigma \otimes \chi)$ and the function $\mu(\sigma, z)$ when $z = \chi(h_\alpha^\vee)$ goes to infinity.

By (4.10), the modulus character δ_P only takes values in the integer powers of q . By Proposition 4.6, $\delta_P(m_P(\bar{u})) \leq 1$ for all $\bar{u} \in U_{\bar{P}}$. Hence, $U_{\bar{P}}$ is the disjoint union of the subsets

$$U_{\bar{P}}(n) := \{\bar{u} \in U_{\bar{P}} : \delta_P(m_P(\bar{u})) = q^{-n}\},$$

where n runs over all non-negative integers. The set

$$U_{\bar{P}}(0) = \{\bar{u} \in U_{\bar{P}} : \delta_P(m_P(\bar{u})) = 1\}$$

is compact by [Wal03, Lemme II.3.4] and $m_P(U_{\bar{P}}(0)) \subset M^1$ by Lemma 4.7.

Lemma 4.8. *There exist $C > 0$, $R > 0$ and $d \in \mathbb{N}$ such that for every $n \geq 1$,*

$$\text{vol}(U_{\bar{P}}(n)) \leq Cq^{nR}n^d.$$

Proof. In [Wal03, Lemme II.4.1], this statement (with $R = 1/2$) is proven for $\delta_0 := \delta_{P_0}$, where P_0 is a minimal parabolic subgroup of G . So let $P_0 = M_0U_P$ with $M_0 \subset M$ a minimal Levi subgroup. Consider the Iwasawa decomposition $G = P_0K$, and for every $\bar{u} \in U_{\bar{P}}$, write

$$\bar{u} = u_0(\bar{u})m_0(\bar{u})k_0(\bar{u}),$$

where $u_0(\bar{u}) \in U_P$, $m_0(\bar{u}) \in M_0$ and $k_0(\bar{u}) \in K$. Define, for all $n \geq 1$, the subsets

$$U_{\bar{P},0}(n) := \{\bar{u} \in U_{\bar{P}} : \delta_0(m_0(\bar{u})) = q^{-n}\}.$$

Then, by [Wal03, Lemme II.4.1], there exist $C_0 > 0$ and $d_0 \in \mathbb{N}$ such that for all $n \geq 1$,

$$\text{vol}(U_{\bar{P},0}(n)) \leq C_0q^{n/2}n^{d_0}.$$

By [Wal03, Lemme II.3.4] and Proposition 4.6, there exist $C_1 > 0$, $r > 0$, such that for all $\bar{u} \in U_{\bar{P}}$,

$$C_1\delta_P(m_P(\bar{u}))^r \leq \delta_0(m_0(\bar{u})) \leq 1.$$

Suppose $\bar{u} \in U_{\bar{P}}(n)$, i.e. $\delta_P(m_P(\bar{u})) = q^{-n}$. Then by the inequalities above, we get

$$C_1 q^{-nr} \leq \delta_0(m_0(\bar{u})) \leq 1.$$

Write $C_1 = q^{-b}$ for some $b \in \mathbb{R}$. Let $N \in \mathbb{N}$ such that $N \geq r + \max(b, 0)$. Then for all $n \geq 1$,

$$q^{-nN} \leq q^{-nr-b} = C_1 q^{-nr} \leq \delta_0(m_0(\bar{u})) \leq 1.$$

Hence, $\bar{u} \in U_{\bar{P}}(n)$ can only be contained in one of the sets

$$U_{\bar{P},0}(nN), U_{\bar{P},0}(nN-1), \dots, U_{\bar{P},0}(0).$$

Moreover, each of the sets above has volume bounded above by $C_0 q^{nN/2} (nN)^{d_0}$. Therefore

$$\text{vol}(U_{\bar{P}}(n)) \leq (nN+1)C_0 q^{nN/2} (nN)^{d_0} \leq n(N+1)C_0 q^{nN/2} (nN)^{d_0} = C q^{nR} n^d,$$

where $C = (N+1)C_0 N^{d_0}$, $R = N/2$ and $d = d_0 + 1$. $\square$

For $\chi \in X_{\text{nr}}(M)$, we define the variable $r(\chi) \in \mathbb{R}$ determined by

$$|\chi|_{M \cap G_{\text{der}}} = \delta_P^{r(\chi)}|_{M \cap G_{\text{der}}}.$$

When this variable goes to infinity, the operator $J_{\bar{P}|P}(\sigma \otimes \chi)$ converges to an operator which is independent of χ , as we will show below.

Proposition 4.9. *For $f \in I_{P \cap K}^K(\sigma)$, $g \in K$,*

$$\lim_{r(\chi) \rightarrow \infty} J_{\bar{P}|P}(\sigma \otimes \chi)f(g) = \int_{U_{\bar{P}}(0)} f(\bar{u}g) d\bar{u},$$

and this integral is independent of χ . In particular, $\lim_{r(\chi) \rightarrow \infty} J_{\bar{P}|P}(\sigma \otimes \chi)$ exists.

Proof. For $\chi \in X_{\text{nr}}(M)$, $f \in I_P^G(\sigma \otimes \chi) \cong I_{P \cap K}^K(\sigma)$ and $g \in K$, we have that

$$J_{\bar{P}|P}(\sigma \otimes \chi)f(g) = \int_{U_{\bar{P}}} f(\bar{u}g) d\bar{u} = \int_{U_{\bar{P}}(0)} f(\bar{u}g) d\bar{u} + \int_{\cup_{n \geq 1} U_{\bar{P}}(n)} f(\bar{u}g) d\bar{u}.$$

Recall that for $\bar{u} \in U_{\bar{P}}$, using the Iwasawa decomposition $G = PK$ we write

$$\bar{u} = u_P(\bar{u})m_P(\bar{u})k_P(\bar{u}),$$

where $u_P(\bar{u}) \in U_P$, $m_P(\bar{u}) \in M$ and $k_P(\bar{u}) \in K$. Then

$$f(\bar{u}g) = \chi \delta_P^{1/2}(m_P(\bar{u})) \sigma(m_P(\bar{u})) f(k_P(\bar{u})g).$$

For $\bar{u} \in U_{\bar{P}}(0)$, we have that $\delta_P(m_P(\bar{u})) = 1$. Then by Lemma 4.7, $m_P(\bar{u}) \in M^1$, so also $\chi(m_P(\bar{u})) = 1$. Hence

$$\int_{U_{\bar{P}}(0)} f(\bar{u}g) d\bar{u} = \int_{U_{\bar{P}}(0)} \sigma(m_P(\bar{u})) f(k_P(\bar{u})g) d\bar{u},$$

which does not depend on χ , since $k_P(\bar{u})g \in K$ and $f \in I_{P \cap K}^K(\sigma)$.

Now, let us analyze the integral

$$\int_{\cup_{n \geq 1} U_{\bar{P}}(n)} f(\bar{u}g) d\bar{u},$$

in particular we want to show that this integral converges to zero in the limit $r(\chi) \rightarrow \infty$. For this, we follow the proof of [Wal03, Théorème IV.1.1]. This proof shows that we can reduce the problem to showing that the integral

$$\int_{\cup_{n \geq 1} U_{\bar{P}}(n)} |\chi \delta_P^{r+1/2}(m_P(\bar{u}))| d\bar{u} = \int_{\cup_{n \geq 1} U_{\bar{P}}(n)} \delta_P^{r(\chi)+r+1/2}(m_P(\bar{u})) d\bar{u},$$

where $r \in \mathbb{R}$ is some fixed number, converges to zero as $r(\chi) \rightarrow \infty$. By Lemma 4.8, there exist $C > 0$, $R > 0$ and $d \in \mathbb{N}$ such that for each $n \geq 1$,

$$\text{vol}(U_{\bar{P}}(n)) \leq C q^{nR} n^d.$$

Hence, we obtain that

$$\begin{aligned} \int_{\cup_{n \geq 1} U_{\bar{P}}(n)} \delta_P^{r(\chi)+r+1/2}(m_P(\bar{u})) d\bar{u} &\leq \sum_{n \geq 1} q^{-n(r(\chi)+r+1/2)} C q^{nR} n^d \\ &= C \sum_{n \geq 1} q^{-n(r(\chi)+r')} n^d, \end{aligned}$$

where $r' \in \mathbb{R}$ is independent of n . This sum converges for $r(\chi)$ large enough, and it converges to zero when $r(\chi) \rightarrow \infty$. $\square$

Recall that for every χ , we may view $J_{\bar{P}|P}(\sigma \otimes \chi)$ as a linear operator

$$J_{\bar{P}|P}(\sigma \otimes \chi) : I_{P \cap K}^K(V_\sigma) \rightarrow I_{\bar{P} \cap K}^K(V_\sigma),$$

i.e. these are all linear operators on the same vector space, with the same range. If $z = \chi(h_\alpha^\vee) \rightarrow \infty$, then $r(\chi) \rightarrow \infty$. So by Proposition 4.9, $\lim_{z \rightarrow \infty} J_{\bar{P}|P}(\sigma \otimes \chi)$ exists and is given by

$$\lim_{z \rightarrow \infty} J_{\bar{P}|P}(\sigma \otimes \chi) = \int_{U_{\bar{P}}(0)} \lambda(\bar{u})^{-1} d\bar{u} : I_{P \cap K}^K(V_\sigma) \rightarrow I_{\bar{P} \cap K}^K(V_\sigma).$$

One can also directly prove that the image of this operator lies in $I_{\bar{P} \cap K}^K(V_\sigma)$, as we will do below.

Lemma 4.10. *Let $f \in I_{P \cap K}^K(V_\sigma)$ and let $\bar{f} := \int_{U_{\bar{P}}(0)} \lambda(\bar{u})^{-1} f d\bar{u}$. Then $\bar{f} \in I_{\bar{P} \cap K}^K(V_\sigma)$.*

Proof. Let $f \in I_{P \cap K}^K(V_\sigma)$, so $f : K \rightarrow V_\sigma$ is a function right invariant under some compact open subgroup and $f(mug) = \sigma(m)f(g)$ for all $g \in K$, $m \in M \cap K$, $u \in U_P \cap K$. We want to show that for $m \in M \cap K$, $\bar{u} \in U_{\bar{P}} \cap K$, $g \in K$, we have $\bar{f}(m\bar{u}g) = \sigma(m)\bar{f}(g)$. First, let us show that $U_{\bar{P}} \cap K \subset U_{\bar{P}}(0)$. For $\bar{u} \in U_{\bar{P}} \cap K$, write

$$\bar{u} = u_P(\bar{u})m_P(\bar{u})k_P(\bar{u})$$

using the Iwasawa decomposition. We may choose $k_P(\bar{u}) = \bar{u} \in K$. Then $m_P(\bar{u}) = 1$, hence $\delta_P(m_P(\bar{u})) = 1$ so $\bar{u} \in U_{\bar{P}}(0)$.

Now let $m \in M \cap K$, $\bar{u} \in U_{\bar{P}} \cap K$, $g \in K$. Then

$$\begin{aligned}\bar{f}(m\bar{u}g) &= \int_{U_{\bar{P}}(0)} f(\bar{u}_1 m \bar{u} g) d\bar{u}_1 = \int_{U_{\bar{P}}(0)} f(mm^{-1}\bar{u}_1 m \bar{u} g) d\bar{u}_1 \\ &= \sigma(m) \int_{U_{\bar{P}}(0)} f(m^{-1}\bar{u}_1 m \bar{u} g) d\bar{u}_1.\end{aligned}$$

We have that

$$m^{-1}\bar{u}_1 m = (m^{-1}u_P(\bar{u}_1)m)(m^{-1}m_P(\bar{u}_1)m)(m^{-1}k_P(\bar{u}_1)m),$$

therefore

$$\delta_P(m_P(m^{-1}\bar{u}_1 m)) = \delta_P(m^{-1}m_P(\bar{u}_1)m) = 1,$$

since $m \in K$ and $\bar{u}_1 \in U_{\bar{P}}(0)$. So $m^{-1}\bar{u}_1 m \in U_{\bar{P}}(0)$ and this gives

$$\text{Ad}(m^{-1}) : U_{\bar{P}}(0) \rightarrow U_{\bar{P}}(0),$$

which preserves measures since $m \in K \cap M$, so $\delta_P(m) = 1$. Thus

$$\bar{f}(m\bar{u}g) = \sigma(m) \int_{U_{\bar{P}}(0)} f(m^{-1}\bar{u}_1 m \bar{u} g) d\bar{u}_1 = \sigma(m) \int_{U_{\bar{P}}(0)} f(\bar{u}_1 \bar{u} g) d\bar{u}_1.$$

We can write

$$\bar{u}_1 \bar{u} = u_P(\bar{u}_1)m_P(\bar{u}_1)(k_P(\bar{u}_1)\bar{u}),$$

so $\bar{u}_1 \bar{u} \in U_{\bar{P}}(0)$, and we may change the variable $\bar{u}_1 \bar{u}$ to $\bar{u}_2 \in U_{\bar{P}}(0)$, by right invariance of the Haar measure on $U_{\bar{P}}$. So we obtain

$$\bar{f}(m\bar{u}g) = \sigma(m) \int_{U_{\bar{P}}(0)} f(\bar{u}_2 g) d\bar{u}_2 = \sigma(m) \bar{f}(g),$$

hence $\bar{f} \in I_{\bar{P} \cap K}^K(V_\sigma)$. □

Since

$$J_{P|\bar{P}}(\sigma \otimes \chi) \circ J_{\bar{P}|P}(\sigma \otimes \chi) = j(\sigma \otimes \chi) = \mu(\sigma \otimes \chi)^{-1}$$

is a rational function in the variable $z = \chi(h_\alpha^\vee)$, it has only finitely many poles and zeros. So for $z = \chi(h_\alpha^\vee)$ large enough, $J_{\bar{P}|P}(\sigma \otimes \chi)$ is invertible. We can use this to show that $\lim_{z \rightarrow \infty} J_{\bar{P}|P}(\sigma \otimes \chi)$ is invertible.

Lemma 4.11. *The operator*

$$A := \int_{U_{\bar{P}}(0)} \lambda(\bar{u})^{-1} d\bar{u} : I_{\bar{P} \cap K}^K(V_\sigma) \rightarrow I_{\bar{P} \cap K}^K(V_\sigma)$$

is invertible.

Proof. It suffices to show that for every compact open subgroup C of K , the operator

$$A^C := \int_{U_{\bar{P}}(0)} \lambda(\bar{u})^{-1} d\bar{u} : I_{\bar{P} \cap K}^K(V_\sigma)^C \rightarrow I_{\bar{P} \cap K}^K(V_\sigma)^C,$$

which is a map of finite-dimensional vector spaces, is bijective. Let $z = \chi(h_\alpha^\vee)$ be large enough such that $J_{\bar{P}|P}(\sigma \otimes \chi)$ is invertible. Let $f \in I_{P \cap K}^K(V_\sigma)^C$, but view it as an element of $I_P^G(\sigma \otimes \chi)$. Let $g \in K$. Then

$$\int_{U_{\bar{P}}} \left[\int_{U_{\bar{P}}(0)} f(\bar{u}_1 \bar{u}_2 g) d\bar{u}_1 \right] d\bar{u}_2 = \text{vol}(U_{\bar{P}}(0)) \int_{U_{\bar{P}}} f(\bar{u}g) d\bar{u},$$

hence

$$\int_{U_{\bar{P}}} \lambda(\bar{u})^{-1} d\bar{u} \circ \int_{U_{\bar{P}}(0)} \lambda(\bar{u})^{-1} d\bar{u} = \text{vol}(U_{\bar{P}}(0)) J_{\bar{P}|P}(\sigma \otimes \chi).$$

Since $J_{\bar{P}|P}(\sigma \otimes \chi)$ is invertible, $\int_{U_{\bar{P}}(0)} \lambda(\bar{u})^{-1} d\bar{u}$ is injective, hence A^C is bijective since it is a map of finite dimensional vector spaces. We can do this for every C , so A is invertible. $\square$

We can use the results above to show that $\mu(\sigma, z) = \mu(\sigma \otimes \chi) = j(\sigma \otimes \chi)^{-1}$ is not zero at $z = \infty$ and $z = 0$.

Proposition 4.12. *The function $\mu(\sigma, z)$ is not zero at $z = \infty$ and $z = 0$, i.e.*

$$\lim_{z \rightarrow \infty} j(\sigma \otimes \chi) \neq \infty \text{ and } \lim_{z \rightarrow 0} j(\sigma \otimes \chi) \neq \infty.$$

Proof. We will first show that $\lim_{z \rightarrow \infty} j(\sigma \otimes \chi) \neq \infty$. Let $f_\chi \in I_P^G(\sigma \otimes \chi)$ be determined by $\text{supp}(f_\chi) = \bar{P}U_{P,r}$, $f_\chi(1) = v$, and f_χ is right $U_{P,r}$ -invariant. Then

$$J_{P|\bar{P}}(\sigma \otimes \chi) f_\chi(1) = \int_{U_P} f_\chi(u) du = \int_{U_{P,r}} f_\chi(u) du = \text{vol}(U_{P,r}) v,$$

which is independent of χ . Let $z = \chi(h_\alpha^\vee)$ be large enough such that $J_{\bar{P}|P}(\sigma \otimes \chi)$ is invertible, and write $g_\chi = J_{\bar{P}|P}(\sigma \otimes \chi)^{-1} f_\chi \in I_P^G(\sigma \otimes \chi)$. Then

$$j(\sigma \otimes \chi) g_\chi(1) = J_{P|\bar{P}}(\sigma \otimes \chi) \circ J_{\bar{P}|P}(\sigma \otimes \chi) g_\chi(1) = J_{P|\bar{P}}(\sigma \otimes \chi) f_\chi(1) = \text{vol}(U_{P,r}) v.$$

By Proposition 4.9 and Lemma 4.11, $\lim_{z \rightarrow \infty} J_{\bar{P}|P}(\sigma \otimes \chi)$ exists and is invertible. Hence, $\lim_{z \rightarrow \infty} g_\chi = g \in I_{P \cap K}^K(V_\sigma)$ exists. Hence,

$$\lim_{z \rightarrow \infty} j(\sigma \otimes \chi) g_\chi(1) = \lim_{z \rightarrow \infty} j(\sigma \otimes \chi) g(1) = \text{vol}(U_{P,r}) v,$$

so $\lim_{z \rightarrow \infty} j(\sigma \otimes \chi) \neq \infty$.

So far, we showed that $\mu(\sigma, \infty) \neq 0$. By Lemma 4.2, $\sigma = \tilde{\sigma} \otimes \tilde{\chi}$ for some unitary $\tilde{\sigma} \in \text{Irr}(M)_{[M, \sigma]}$ and $\tilde{\chi} \in \text{Hom}(M, \mathbb{R}_{>0})$. By (4.1), $\mu(\sigma, 0) = \mu(\tilde{\sigma}, 0)$ and $\mu(\sigma, \infty) = \mu(\tilde{\sigma}, \infty)$. By Lemma 4.4, $\mu(\tilde{\sigma}, \infty) \neq 0$ implies that $\mu(\tilde{\sigma}, 0) \neq 0$, i.e. $\mu(\sigma, 0) \neq 0$. $\square$

4.4. Zeros of the μ -function. In this subsection, we analyze the zeros of the μ -function. Recall that since M is maximal, we have $[N_G(M) : M] \leq 2$, so we will distinguish between two cases. In the case where $N_G(M)/M$ is trivial, it turns out that μ is constant. In the case where $N_G(M)/M$ is not trivial, we will prove that μ has at most two zeros in $\mathbb{C}^\times$.

Lemma 4.13. *If $\mu(\sigma, z)$ has no zeros on $\mathbb{C}^\times$, then $\mu(\sigma, z)$ is constant.*

Proof. Take a unitary $\tilde{\sigma} \in \text{Irr}(M)_{[M, \sigma]}$. Recall from Lemma 4.4 that we can write

$$\mu(\tilde{\sigma}, z) = c_\mu \prod_i (z - \lambda_i)^{n_i} (z^{-1} - \bar{\lambda}_i)^{n_i}.$$

Suppose $\mu(\tilde{\sigma}, z)$ has no zeros in $\mathbb{C}^\times$, then there are no factors with n_i positive. So there can only be factors with n_i negative. However, by Proposition 4.12, $\mu(\tilde{\sigma}, \infty) \neq 0$, so $\mu(\tilde{\sigma}, z)$ is constant, hence $\mu(\sigma, z)$ is also constant by (4.1). $\square$

Lemma 4.14. *If $N_G(M)/M$ is trivial, then $\mu(\sigma, z)$ is constant.*

Proof. First let us show that if $N_G(M)/M$ is trivial, then the parabolic subgroups P and $\bar{P}$ are not conjugate. Suppose that $N_G(M) = M$ and that P and $\bar{P}$ are conjugate, so there exists $g \in G$ such that $gPg^{-1} = \bar{P}$. Write $P = MU_P$, $\bar{P} = MU_{\bar{P}}$ then

$$gMU_Pg^{-1} = gMg^{-1}gU_Pg^{-1} = MU_{\bar{P}}.$$

A Levi factor of $\bar{P}$ is unique up to conjugation by elements of $U_{\bar{P}}$, see [Spr09, Proposition 16.1.1]. Thus $gMg^{-1} = \bar{u}M\bar{u}^{-1}$ for some $\bar{u} \in U_{\bar{P}}$. So $g^{-1}\bar{u} \in N_G(M) = M$, i.e. $g = \bar{u}m$ for some $m \in M$. But then $g \in \bar{P}$ and $P = g^{-1}\bar{P}g = \bar{P}$, a contradiction since P and $\bar{P}$ are opposite parabolic subgroups.

Now suppose that $\mu(\sigma \otimes \chi) = 0$ for some χ . This corresponds to $J_{\bar{P}|P}$ and/or $J_{P|\bar{P}}$ being singular at $\sigma \otimes \chi$. However, this can not happen if P and $\bar{P}$ are not conjugate, see [Wal03, Corollaire IV.1.2]. Therefore, $\mu(\sigma, z)$ has no zeros in $\mathbb{C}^\times$, and by Lemma 4.13, it is constant. $\square$

So from now on, let us assume that $[N_G(M) : M] = 2$ and that μ has a zero, otherwise there is nothing left to prove. Let s_α be the unique nontrivial element of $N_G(M)/M$, with representative $w \in N_G(M)$. Then s_α acts on $\text{Irr}(M)$ via $(s_\alpha \cdot \pi)(m) = \pi(w^{-1}mw)$.

Proposition 4.15. *Suppose that $\mu(\sigma, z)$ is not constant. Then there exists a unitary $\sigma_0 \in \text{Irr}(M)_{[M, \sigma]}$ such that the following statements hold:*

- (i) $\mu(\sigma_0) = 0$;
- (ii) $\mu(\sigma_0, z) = 0$ implies that $z = 1$ or $z = -1$;
- (iii) if $\mu(\sigma_0, z) = 0$, the zero at z is of order 2.

Proof. Since $\mu(\sigma, z)$ is not constant, $\mu(\tau) = 0$ for some $\tau \in \text{Irr}(M)_{[M, \sigma]}$, by Lemma 4.13. Moreover, by Lemma 4.14, $N_G(M)/M$ has a unique nontrivial element s_α . Since $\mu(\tau) = 0$, it follows that $s_\alpha\tau \cong \tau$, see [Wal03, p. 283]. By Lemma 4.2, there is a bijection

$$\text{Irr}(M)_{[M, \sigma]}^{\text{unit}} \times \text{Hom}(M, \mathbb{R}_{>0}) \rightarrow \text{Irr}(M)_{[M, \sigma]} : (\pi, \chi) \mapsto \pi \otimes \chi.$$

This bijection is s_α -equivariant by construction,

$$s_\alpha \cdot (\pi, \chi) = (s_\alpha\pi, s_\alpha\chi) \mapsto s_\alpha\pi \otimes s_\alpha\chi = s_\alpha \cdot (\pi \otimes \chi).$$

So $\tau = \sigma_0 \otimes \chi_0$ for a unitary representation σ_0 and $\chi_0 \in \text{Hom}(M, \mathbb{R}_{>0})$, with $s_\alpha\sigma_0 \cong \sigma_0$ and $s_\alpha\chi_0 \cong \chi_0$.

We will first prove (ii) and use that result to prove (i). For $\mu(\sigma_0 \otimes \chi) = \mu(\sigma_0, z)$ to be zero, we must have $s_\alpha(\sigma_0 \otimes \chi) \cong \sigma_0 \otimes \chi$ [Wal03, p. 283]. We

have $s_\alpha \sigma_0 \cong \sigma_0$, so $s_\alpha(\sigma_0 \otimes \chi) \cong \sigma_0 \otimes s_\alpha \chi$. Therefore, $\mu(\sigma_0 \otimes \chi) = 0$ implies that $\sigma_0 \cong \sigma_0 \otimes \chi(s_\alpha \chi)^{-1}$, so $\chi(s_\alpha \chi)^{-1}(h_\alpha^\vee) = 1$. Write $\chi(h_\alpha^\vee) = z$, then $(s_\alpha \chi)(h_\alpha^\vee) = z^{-1}$, so $\chi(s_\alpha \chi)^{-1}(h_\alpha^\vee) = z^2$. Hence, $\mu(\sigma_0 \otimes \chi) = \mu(\sigma_0, z) = 0$ implies that $z^2 = 1$, hence $z = \pm 1$. This proves (ii).

Recall that $\mu(\tau) = \mu(\sigma_0 \otimes \chi_0) = 0$, where $\chi_0 \in \text{Hom}(M, \mathbb{R}_{>0})$. By the above, we obtain that $\chi_0(h_\alpha^\vee) = 1$, hence $\mu(\sigma_0) = 0$. This proves (i).

It remains to prove (iii). Suppose $\mu(\sigma_0, z) = 0$, then $z = \pm 1$ by (ii). In particular $z \in S^1$, so by the proof of Lemma 4.3, there exists a unitary character $\chi_z \in X_{\text{nr}}(M)$ such that $\chi_z(h_\alpha^\vee) = z$. Then $\sigma_0 \otimes \chi_z$ is unitary with $\mu(\sigma_0 \otimes \chi_z) = 0$. This means that at least one of $J_{\bar{P}|P}(\sigma_0 \otimes \chi_z)$ and $J_{P|\bar{P}}(\sigma_0 \otimes \chi_z)$ is singular. Since $\sigma_0 \otimes \chi_z$ is unitary (in particular Hermitian), we have the adjointness relation from Theorem 3.5.a with $\overline{\sigma_0 \otimes \chi_z}^\vee$ replaced by $\sigma_0 \otimes \chi_z$. So

$$\langle J_{\bar{P}|P}(\sigma_0 \otimes \chi_z) f_1, f_2 \rangle = \langle f_1, J_{P|\bar{P}}(\sigma_0 \otimes \chi_z) f_2 \rangle$$

for all $f_1 \in I_P^G(\sigma_0 \otimes \chi_z)$, $f_2 \in I_{\bar{P}}^G(\sigma_0 \otimes \chi_z)$. Thus, singularity of one of the two intertwining operators $J_{\bar{P}|P}$ and $J_{P|\bar{P}}$ at $\sigma_0 \otimes \chi_z$ automatically implies singularity of the other one at $\sigma_0 \otimes \chi_z$. By [Wal03, Corollaire IV.1.2], both singularities have order 1. Since

$$\mu(\sigma_0 \otimes \chi)^{-1} = j(\sigma_0 \otimes \chi) = J_{P|\bar{P}}(\sigma_0 \otimes \chi) \otimes J_{\bar{P}|P}(\sigma_0 \otimes \chi),$$

we obtain that the zero of μ at $\sigma_0 \otimes \chi_z$ has order 2. This proves (iii). $\square$

4.5. Hermitian techniques to find the poles. In this subsection, we locate the poles of $\mu(\sigma, z)$ using techniques with Hermitian and unitary representations. We will prove the main result of this paper, which is a formula for $\mu(\sigma, z)$ in terms of poles and zeros. Recall that in Lemma 4.4, we have already written the function $\mu(\sigma, z)$ in a form that is closer to the desired result. We also showed in Lemma 4.13 that $\mu(\sigma, z)$ is constant if it has no zeros, which is satisfied when $[N_G(M) : M] = 1$, see Lemma 4.14.

Therefore, from now on, we assume that $[N_G(M) : M] = 2$ and that $\mu(\sigma, z)$ is not constant. So $N_G(M)/M$ has a unique nontrivial element s_α . For simplicity, we assume in this section that $\sigma = \sigma_0$, where σ_0 is as in Proposition 4.15. So σ is unitary with $s_\alpha \sigma \cong \sigma$ and $\mu(\sigma) = 0$. There is possibly another zero of $\mu(\sigma, z)$ at $z = -1$ and each zero is of order 2. For each zero of μ , we will find two poles in $\mathbb{R}^\times$. To do this, it suffices to find one pole for each zero, because by Lemma 4.4, one obtains the other pole for free. We already showed in Proposition 4.12 that $\mu(\sigma, z)$ is not zero at $z = \infty$ and $z = 0$, so we can conclude that $\mu(\sigma, z)$ does not have any more poles in $\mathbb{C}^\times$.

We will use arguments concerning Hermitian and unitary representations to locate the poles. See [Ren10, §IV] for definitions and useful results about Hermitian and unitary representations. Let $\chi \in \text{Hom}(M, \mathbb{R}_{>0}) \subset X_{\text{nr}}(M)$, then $\bar{\chi} = \chi$ and $\bar{\chi}^\vee = \chi^\vee = \chi^{-1}$. By unitarity of σ , we have $\overline{\sigma \otimes \chi}^\vee \cong \sigma \otimes \chi^{-1}$. So $\sigma \otimes \chi$ is only Hermitian if $\sigma \otimes \chi \cong \sigma \otimes \chi^{-1}$, or equivalently if $\chi^2 \in X_{\text{nr}}(M, \sigma)$. We will show that the induced representation $I_P^G(\sigma \otimes \chi)$ is Hermitian whenever $\chi \in \text{Hom}(M/Z(G), \mathbb{R}_{>0})$ and $\mu(\sigma \otimes \chi) \neq \infty$. This will allow us to construct a

continuous family of Hermitian forms on a family of Hermitian representations, which will help us to show that somewhere in this family, a pole of $\mu(\sigma \otimes \chi)$ must occur.

Lemma 4.16. *Let L be a Levi subgroup of G and $\tau \in \text{Irr}_{\text{cusp}}(L)$, $w \in N_G(L)$. Let $Q = LV$ be a parabolic subgroup of G . Then $I_Q^G(\tau)$ and $I_Q^G(w \cdot \tau)$ have the same irreducible subquotients with the same multiplicities. In particular, if $I_Q^G(\tau)$ is irreducible, then $I_Q^G(\tau) \cong I_Q^G(w \cdot \tau)$.*

Proof. This follows from [Ren10, Théorème VI.5.4]. $\square$

Lemma 4.17. *Suppose that $s_\alpha(\chi) = \chi^{-1}$ and that $I_P^G(\sigma \otimes \chi)$ is irreducible. Then $I_P^G(\sigma \otimes \chi)$ is Hermitian.*

Proof. Since $s_\alpha(\sigma \otimes \chi) \cong \sigma \otimes \chi^{-1}$, we have by Lemma 4.16 that $I_P^G(\sigma \otimes \chi) \cong I_P^G(\sigma \otimes \chi^{-1})$. Taking the complex conjugate representation commutes with parabolic induction. By [Ren10, §VI.1.2], taking the contragredient representation also commutes with parabolic induction. So we get

$$I_P^G(\sigma \otimes \chi) \cong I_P^G(\sigma \otimes \chi^{-1}) \cong I_P^G(\overline{\sigma \otimes \chi}^\vee) \cong \overline{I_P^G(\sigma \otimes \chi)}^\vee,$$

where the middle isomorphism follows from unitarity of σ , so $\overline{\sigma \otimes \chi}^\vee \cong \sigma \otimes \chi^{-1}$. This completes the proof that $I_P^G(\sigma \otimes \chi)$ is Hermitian. $\square$

The condition $s_\alpha \chi = \chi^{-1}$ from Lemma 4.17 is satisfied by unramified characters $\chi \in \text{Hom}(M/Z(G), \mathbb{R}_{>0})$, as shown below.

Lemma 4.18. *If $\chi \in \text{Hom}(M/Z(G), \mathbb{R}_{>0})$, then $s_\alpha \chi = \chi^{-1}$.*

Proof. The unramified character χ factors through M/M^1 , so

$$\chi : M/M^1 Z(G) \rightarrow \mathbb{R}_{>0}.$$

Because M is a maximal Levi subgroup, we have

$$M/M^1 \cong \mathbb{Z}^r \text{ and } Z(G)/Z(G)_{\text{cpt}} \cong \mathbb{Z}^{r-1}.$$

This is because the rank r of M/M^1 equals the dimension of the maximal split torus in M , which is one higher than the dimension of the maximal split torus in G , i.e. the rank of $Z(G)/Z(G)_{\text{cpt}}$, which is the same as the rank $r - 1$ of G/G^1 . Thus, we obtain the quotient

$$M/M^1 Z(G) = (M/M^1)/(Z(G)/Z(G)_{\text{cpt}}) \cong \mathbb{Z} \times H,$$

with H a finite group. So we can consider $\chi : \mathbb{Z} \times H \rightarrow \mathbb{R}_{>0}$, which must be trivial on H since it is a finite group. So $\chi : \mathbb{Z} \rightarrow \mathbb{R}_{>0}$ is of the form $n \mapsto x^n$ where $x \in \mathbb{R}_{>0}$. Now s_α acts on $\mathbb{Z}$ as a nontrivial group homomorphism and an involution, so the only possibility is multiplication by -1 . This induces the action on χ , which we conclude is given by $s_\alpha \chi = \chi^{-1}$. $\square$

Under the assumptions of Lemma 4.17, we will now construct a Hermitian form on $I_P^G(\sigma \otimes \chi)$, using the intertwining operator $J_{\bar{P}|P}(\sigma \otimes \chi)$. Write $\lambda(s_\alpha)$ for

translation by s_α . Fix an isomorphism of M -representations $\varphi_\sigma : \sigma \rightarrow s_\alpha \cdot \sigma$. Moreover, let

$$J'_{\bar{P}|P}(\sigma \otimes \chi) := (\chi(h_\alpha^\vee) - 1)J_{\bar{P}|P}(\sigma \otimes \chi), \quad J'_{P|\bar{P}}(\sigma \otimes \chi) := (\chi^{-1}(h_\alpha^\vee) - 1)J_{P|\bar{P}}(\sigma \otimes \chi).$$

Then both $J'_{\bar{P}|P}(\sigma \otimes \chi)$ and $J'_{P|\bar{P}}(\sigma \otimes \chi)$ do not have a pole at $\chi = 1$. They are invertible for χ in a neighbourhood of $\chi = 1$ in $X_{\text{nr}}(M)$. They are each other's inverse, up to a scalar which depends rationally on χ . Consider, for $\chi \in \text{Hom}(M/Z(G), \mathbb{R}_{>0})$ the composition

$$\begin{aligned} I_P^G(\sigma \otimes \chi) &\xrightarrow{J'_{\bar{P}|P}(\sigma \otimes \chi)} I_{\bar{P}}^G(\sigma \otimes \chi) \xrightarrow{\lambda(s_\alpha)} I_P^G(s_\alpha \cdot \sigma \otimes \chi^{-1}) \xrightarrow{I_{\bar{P}}^G(\varphi_\sigma^{-1} \otimes \text{id})} I_P^G(\sigma \otimes \chi^{-1}) \\ &\xrightarrow{\sim} I_P^G(\overline{\sigma \otimes \chi}^\vee) \xrightarrow{\sim} \overline{I_{\bar{P}}^G(\sigma \otimes \chi)}^\vee. \end{aligned}$$

The last two isomorphisms come from Lemma 4.17. Using the map above, we take

$$(4.13) \quad \begin{aligned} I_P^G(\sigma \otimes \chi) \times I_{\bar{P}}^G(\sigma \otimes \chi) &\xrightarrow{\sim} I_P^G(\sigma \otimes \chi) \times \overline{I_{\bar{P}}^G(\sigma \otimes \chi)}^\vee \rightarrow \mathbb{C} \\ (v, v') &\longmapsto \langle v, v' \rangle_\chi. \end{aligned}$$

This way, we obtain a sesquilinear form, but it is not necessarily Hermitian. Later, we will do some more work to construct a Hermitian form using (4.13).

For now, with (4.13), we have a family of nondegenerate sesquilinear forms on the family of representations $I_P^G(\sigma \otimes \chi)$, $\chi \in \text{Hom}(M/Z(G), \mathbb{R}_{>0})$. It only fails when $J'_{\bar{P}|P}(\sigma \otimes \chi)$ is not invertible. The maps depend rationally on χ , as we will prove below.

Lemma 4.19. *The sesquilinear forms constructed in (4.13) are G -invariant and depend rationally on $\chi \in \text{Hom}(M/Z(G), \mathbb{R}_{>0})$.*

Proof. To show that the sesquilinear forms depend rationally on χ , let us consider each individual map in the composition. It turns out that most of the maps are independent of χ , but let us still describe each of them explicitly.

First of all, we consider the map

$$J'_{\bar{P}|P}(\sigma \otimes \chi) = (\chi(h_\alpha^\vee) - 1)J_{\bar{P}|P} : I_P^G(\sigma \otimes \chi) \rightarrow I_{\bar{P}}^G(\sigma \otimes \chi).$$

We may identify $I_P^G(\sigma \otimes \chi) \cong I_{K \cap P}^K(\sigma)$ and $I_{\bar{P}}^G \cong I_{K \cap \bar{P}}^K(\sigma)$, to consider it as a map

$$J'_{\bar{P}|P}(\sigma \otimes \chi) : I_{K \cap P}^K(\sigma) \rightarrow I_{K \cap \bar{P}}^K(\sigma),$$

and it makes sense to ask if this map is rational in the variable χ , see [Wal03, §IV.1] for details. By [Wal03, Théorème IV.1.1], $J_{\bar{P}|P}$ is rational in χ . Moreover $\chi(h_\alpha^\vee)$ is rational in χ , so it follows that indeed $J'_{\bar{P}|P}(\sigma \otimes \chi) = (\chi(h_\alpha^\vee) - 1)J_{\bar{P}|P}(\sigma \otimes \chi)$ is rational in χ .

The next step is the map

$$\lambda(s_\alpha) : I_{P \cap K}^K(\sigma) \cong I_P^G(\sigma \otimes \chi) \rightarrow I_P^G(s_\alpha \cdot \sigma \otimes \chi^{-1}) \cong I_{P \cap K}^K(s_\alpha \cdot \sigma).$$

It is given by $f \mapsto [g \mapsto f(s_\alpha^{-1}g)]$ which is independent of χ , so rational in χ .

Next up is the map

$$I_P^G(\varphi_\sigma^{-1} \otimes \text{id}) : I_{P \cap K}^K(s_\alpha \cdot \sigma) \cong I_P^G(s_\alpha \cdot \sigma \otimes \chi^{-1}) \rightarrow I_P^G(\sigma \otimes \chi^{-1}) \cong I_{P \cap K}^K(\sigma).$$

It is given by $f \mapsto [g \mapsto (\varphi_\sigma^{-1} \otimes \text{id})f(g)]$ which is again independent of χ .

Then we have the map

$$I_{P \cap K}^K(\sigma) \cong I_P^G(\sigma \otimes \chi^{-1}) \xrightarrow{\sim} I_P^G(\overline{\sigma \otimes \chi}^\vee) \cong I_{P \cap K}^K(\overline{\sigma}^\vee).$$

The isomorphism in the middle comes from unitarity of σ and the assumption that $\chi \in \text{Hom}(M, \mathbb{R}_{>0})$, so $\overline{\chi}^\vee \cong \chi^{-1}$. On the level of vector spaces, this map comes from the pairing $V_\sigma \times V_\sigma \rightarrow \mathbb{C}$ inducing the isomorphism $V_\sigma \cong \overline{V_\sigma}^\vee$, which is independent of χ .

The next map is

$$I_{P \cap K}^K(\overline{\sigma}^\vee) \cong I_P^G(\overline{\sigma \otimes \chi}^\vee) \xrightarrow{\sim} I_P^G(\overline{\sigma \otimes \chi})^\vee \cong I_{P \cap K}^K(\overline{\sigma})^\vee,$$

i.e. the map that describes how taking the contragredient representation commutes with parabolic induction. We have an M -invariant pairing $\overline{\sigma \otimes \chi} \times \overline{\sigma \otimes \chi}^\vee \rightarrow \mathbb{C}$. We use it to build a K -invariant pairing

$$I_{P \cap K}^K(\overline{\sigma \otimes \chi}) \times I_{P \cap K}^K(\overline{\sigma \otimes \chi}^\vee) \rightarrow \mathbb{C}, (f_1, f_2) \mapsto \int_K \langle f_1(k), f_2(k) \rangle dk,$$

which gives an isomorphism $I_{P \cap K}^K(\overline{\sigma}^\vee) \cong I_{P \cap K}^K(\overline{\sigma \otimes \chi}^\vee) \cong I_{P \cap K}^K(\overline{\sigma \otimes \chi})^\vee \cong I_{P \cap K}^K(\overline{\sigma})^\vee$ independent of χ . By the Iwasawa decomposition $G = PK$, $\int_K$ is up to a positive scalar factor the same as $\int_{P \backslash G}$, hence the pairing is also G -invariant, and it induces the isomorphism $I_P^G(\overline{\sigma \otimes \chi}^\vee) \cong I_P^G(\overline{\sigma \otimes \chi})^\vee$, independent of χ .

Next, we need to consider the map

$$I_{P \cap K}^K(\overline{\sigma})^\vee \cong I_P^G(\overline{\sigma \otimes \chi})^\vee \xrightarrow{\sim} \overline{I_P^G(\sigma \otimes \chi)}^\vee \cong \overline{I_{P \cap K}^K(\sigma)}^\vee,$$

which shows that taking the complex conjugate representation commutes with parabolic induction. Again, this map is independent of χ .

Thus, the forms $\langle \cdot, \cdot \rangle_\chi$ depend rationally on χ . The sesquilinear forms are G -invariant because in the construction in (4.13), the first arrow is a composition of homomorphisms of G -representations (i.e. the homomorphisms described above), so it is G -equivariant, and the second arrow is G -invariant by definition. $\square$

We use the sesquilinear forms in (4.13) to build a family of Hermitian forms on the representations $I_P^G(\sigma \otimes \chi)$, which is positive definite on $I_P^G(\sigma)$, and nondegenerate if $\mu(\sigma \otimes \chi) \neq \infty$. We do this by multiplying with a suitable function $f(\chi)$, which depends continuously on χ .

Lemma 4.20. *Suppose $\chi \in \text{Hom}(M/Z(G), \mathbb{R}_{>0})$ and $\mu(\sigma \otimes \chi) \neq \infty$. Then $I_P^G(\sigma \otimes \chi)$ is irreducible, and on this representation we have the sesquilinear form as constructed in (4.13). We can multiply it by $f(\chi) \in S^1$, depending continuously on χ , such that*

$$(4.14) \quad \langle v, v' \rangle_{I_P^G(\sigma \otimes \chi)} := f(\chi) \langle v, v' \rangle_\chi$$

is a G -invariant Hermitian form on $I_P^G(\sigma \otimes \chi)$ which is nondegenerate, and positive definite for $\chi = 1$.

Proof. First we consider the case $\chi = 1$. Since σ is unitary, $I_P^G(\sigma)$ is also unitary, see [Ren10, §IV.2.3]. The zero $\mu(\sigma) = 0$ corresponds to a pole of order 1 of $J_{\bar{P}|P}(\sigma \otimes \chi)$ at $\chi = 1$. The construction of $J'_{\bar{P}|P}(\sigma \otimes \chi)$ gets rid of the pole at $\chi = 1$, therefore $J'_{\bar{P}|P}(\sigma)$ is invertible. Thus, the construction in (4.13) gives us a G -invariant sesquilinear form $\langle \cdot, \cdot \rangle_{\chi=1}$ on $I_P^G(\sigma)$.

By [Tan17, Proposition 6.3], $I_P^G(\sigma)$ is irreducible. By irreducibility of $I_P^G(\sigma)$, there is up to scalars only one Hermitian form on $I_P^G(\sigma)$, see [Ren10, p. 112]. By unitarity of $I_P^G(\sigma)$, there exists a positive definite Hermitian form $\langle \cdot, \cdot \rangle'_{\chi=1}$ on $I_P^G(\sigma)$. Then $\langle \cdot, \cdot \rangle_{\chi=1}$ must be a complex scalar multiple of $\langle \cdot, \cdot \rangle'_{\chi=1}$, since it is G -invariant, sesquilinear and $I_P^G(\sigma)$ is irreducible. We may pick $f(1) \in S^1$ such that $f(1)\langle \cdot, \cdot \rangle_{\chi=1}$ equals $\langle \cdot, \cdot \rangle'_{\chi=1}$ multiplied by a positive real scalar. In this way, we obtain a Hermitian form on $I_P^G(\sigma)$ which is positive definite.

Now we consider $\chi \neq 1$. For $\chi \in \text{Hom}(M/Z(G), \mathbb{R}_{>0})$, we have that $s_\alpha(\chi) = \chi^{-1}$. For $\chi \neq 1$, we have $\mu(\sigma \otimes \chi) \neq 0$ by Proposition 4.15 and $s_\alpha(\sigma \otimes \chi) \cong \sigma \otimes \chi^{-1} \not\cong \sigma \otimes \chi$. Hence, by [Tan17, Proposition 6.2], $I_P^G(\sigma \otimes \chi)$ is irreducible. Since $\mu(\sigma \otimes \chi) \neq 0$ and $\mu(\sigma \otimes \chi) \neq \infty$, we have that $J'_{\bar{P}|P}(\sigma \otimes \chi)$ is invertible. Therefore, we have the G -invariant sesquilinear form $\langle \cdot, \cdot \rangle_\chi$ on $I_P^G(\sigma \otimes \chi)$, as constructed in (4.13).

By Lemma 4.17, there exists a nondegenerate G -invariant Hermitian form $\langle \cdot, \cdot \rangle'_\chi$ on $I_P^G(\sigma \otimes \chi)$, which is unique up to real scalars. For every χ , fix one such $\langle \cdot, \cdot \rangle'_\chi$ and scale it, so that we can write

$$\langle \cdot, \cdot \rangle_\chi = e^{i\phi(\chi)} \langle \cdot, \cdot \rangle'_\chi.$$

Then the choice of $\langle \cdot, \cdot \rangle'_\chi$ is unique up to multiplication by ± 1 . Hence, $e^{i\phi(\chi)}$ is also unique up to multiplication by ± 1 . Moreover, since $\langle \cdot, \cdot \rangle_\chi$ depends continuously on χ ,

$$|\langle \cdot, \cdot \rangle_\chi| = |e^{i\phi(\chi)}| |\langle \cdot, \cdot \rangle'_\chi| = |\langle \cdot, \cdot \rangle'_\chi|$$

depends continuously on χ . Recall that we already fixed $f(1) = e^{-i\phi(1)}$ such that $f(1)\langle \cdot, \cdot \rangle_{\chi=1} = \langle \cdot, \cdot \rangle'_{\chi=1}$ is positive definite. Consider $0 \neq v \in I_{P \cap K}^K(V_\sigma)$, then $\langle v, v \rangle'_{\chi=1} > 0$. Consider a neighbourhood of 1 such that for all χ in this neighbourhood, $\langle v, v \rangle'_\chi \neq 0$. This is possible since $|\langle \cdot, \cdot \rangle'_\chi|$ depends continuously on χ . The $\langle \cdot, \cdot \rangle'_\chi$ were chosen uniquely up to a sign, so it is possible to choose them in such a way that $\langle v, v \rangle'_\chi > 0$ for all χ in this neighbourhood of 1. In this way, $\langle \cdot, \cdot \rangle'_\chi$ depends continuously on χ in this neighbourhood. In fact, for all χ we can pick the sign of $\langle \cdot, \cdot \rangle'_\chi$ such that the Hermitian forms will depend continuously on χ . Then also $f(\chi) = e^{-i\phi(\chi)}$ depends continuously on χ , and we are done. $\square$

Lemma 4.21. *Let (π, V) be a smooth Hermitian representation of G . Let C be a compact open subgroup of G and let V^C be the space of C -fixed vectors in*

V . Then the nondegenerate G -invariant Hermitian form on V restricts to a nondegenerate C -invariant Hermitian form on V^C .

Proof. Since C is compact, we can write

$$V = V^C \oplus V_C$$

where V_C is spanned by the vectors $c \cdot v - v$ for $c \in C$, $v \in V$. Since (π, V) is Hermitian, we have a G -invariant nondegenerate Hermitian form on V ,

$$\langle \cdot, \cdot \rangle : V \times V = (V^C \oplus V_C) \times (V^C \oplus V_C) \rightarrow \mathbb{C}.$$

Consider the projection operator

$$p_C := \frac{1}{|C|} \int_C \pi(c) dc = \begin{cases} 1 & \text{on } V^C \\ 0 & \text{on } V_C. \end{cases}$$

Since the Hermitian form on V is G -invariant, in particular C -invariant, we have that

$$\langle c \cdot v, v' \rangle = \langle v, c^{-1} \cdot v' \rangle$$

for all $v, v' \in V$, $c \in C$. It follows that for all $v \in V^C$, $v' \in V_C$,

$$\langle v, v' \rangle = \langle p_C v, v' \rangle = \langle v, p_C v' \rangle = \langle v, 0 \rangle = 0.$$

So the pairing between V^C and V_C is zero, but the form on the whole space V is nondegenerate, therefore the form restricted to V^C must be nondegenerate. $\square$

For a Hermitian form on a finite dimensional vector space V , we can consider its signature $(\dim(V^+), \dim(V^0), \dim(V^-))$, where $V = V^+ \oplus V^0 \oplus V^-$ and the form is positive definite on V^+ , zero on V^0 , negative definite on V^- . For Hermitian forms on admissible representations, we can consider the signature by restricting to open compact subgroups, which will be done to prove the next result.

Lemma 4.22. *Suppose $\mu(\sigma) = 0$, $\chi \in \text{Hom}(M/Z(G), \mathbb{R}_{>0})$. If $\mu(\sigma \otimes \chi^t)$ does not have a pole for $t \in [0, \tau]$, $\tau \in \mathbb{R}_{>0}$, then $I_P^G(\sigma \otimes \chi^\tau)$ is unitary.*

Proof. By Lemma 4.20, for each $t \in [0, \tau]$, the representation $I_P^G(\sigma \otimes \chi^t)$ is Hermitian, and we have the Hermitian form as constructed in (4.14). The representations are realized on the same vector space, say $I_{P \cap K}^K(V_\sigma)$. By Lemma 4.20, the Hermitian form on $I_P^G(\sigma)$ is positive definite, i.e. $\langle f, f \rangle_{I_P^G(\sigma)} > 0$. The function $t \mapsto \langle f, f \rangle_{I_P^G(\sigma \otimes \chi^t)}$ is continuous for $f \in I_{P \cap K}^K(V_\sigma)$, see Lemma 4.19 and Lemma 4.20. Let $C \subset K$ be a compact open subgroup fixing f . Recall that $I_P^G(\sigma \otimes \chi^t)$ is irreducible, hence admissible by [FP22, Theorem 6.3], therefore $\dim I_{P \cap K}^K(V_\sigma)^C$ is finite. By Lemma 4.21, the G -invariant nondegenerate Hermitian form on $I_P^G(\sigma \otimes \chi^t)$ constructed in (4.14) restricts to a C -invariant nondegenerate Hermitian form on $I_{P \cap K}^K(V_\sigma)^C$. We have a continuous family of such forms parametrized by $t \in [0, \tau]$. For $t = 0$, recall that we have a positive definite form, so the form has signature $(\dim I_{P \cap K}^K(V_\sigma)^C, 0, 0)$. For any $t \in [0, \tau]$, the form is nondegenerate, so with signature $(p_t, 0, q_t)$ with $p_t + q_t = \dim I_{P \cap K}^K(V_\sigma)^C$. If for some t we have $q_t > 0$, then since the forms

vary continuously, we must have that for some t' the form is degenerate, a contradiction. So all the Hermitian forms on $I_{P \cap K}^K(V_\sigma)^C$ for $t \in [0, \tau]$ are positive definite. We can do this for all f and C fixing f , hence all the representations $I_P^G(\sigma \otimes \chi^t)$ for $t \in [0, \tau]$ are unitary. $\square$

We have just shown that if $\mu(\sigma \otimes \chi^t)$ does not have a pole for any $t \in \mathbb{R}_{\geq 0}$, then all representations $I_P^G(\sigma \otimes \chi^t)$ are unitary. We want to show that this is not possible, so that we obtain a pole for some t .

For the next result, we consider the maximal C^* -algebra $C^*(G)$ of G , which is a completion of $\mathcal{H}(G)$ such that the irreducible smooth unitary representations of G are in bijection with the irreducible representations of $C^*(G)$. This bijection is given by

$$\text{Irr}(G)^{\text{unit}} \rightarrow \text{Irr}(C^*(G)), (\pi, V) \mapsto (\pi, \mathcal{V}),$$

where $\mathcal{V}$ denotes the Hilbert space completion of V , and the action of $C^*(G)$ on $\mathcal{V}$ is (for functions $f \in \mathcal{H}(G)$) given by

$$(4.15) \quad \pi(f)(v) = \int_G f(g) \pi(g) v dg.$$

Every unitary representation (π, V) of G gives a homomorphism of C^* -algebras $\pi : C^*(G) \rightarrow \mathcal{B}(\mathcal{V})$, where $\mathcal{B}(\mathcal{V})$ is the C^* -algebra of bounded linear operators on $\mathcal{V}$. Since it is a homomorphism of C^* -algebras, it must be norm-decreasing, see [Mur90, Theorem 2.1.7].

Lemma 4.23. *Suppose $\mu(\sigma) = 0$ and let $\chi \in \text{Hom}(M/Z(G), \mathbb{R}_{>0})$ such that $\chi(h_\alpha^\vee) > 1$. Then $\mu(\sigma \otimes \chi^t)$ has a pole for some $t \in \mathbb{R}_{>0}$.*

Proof. Suppose not, then by Lemma 4.22, $\pi^t := I_P^G(\sigma \otimes \chi^t)$ is a unitary G -representation for all $t \in \mathbb{R}_{\geq 0}$. Let $v \in I_{P \cap K}^K(V_\sigma) \cong I_P^G(V_\sigma)$ such that $\text{supp}(v) = (P \cap K) \cdot U_{-\alpha, r}$ for some r , and such that v is right $U_{-\alpha, r}$ -invariant. Then $v(\bar{u}) = v(1)$ for all $\bar{u} \in U_{-\alpha, r}$. Let C be a compact open subgroup of G fixing v (this is possible because $I_P^G(V_\sigma)$ is smooth) and put $f = 1_{h_\alpha^\vee C}$. Then $f \in \mathcal{H}(G) \subset C^*(G)$, where $C^*(G)$ is the maximal C^* -algebra of G , so $\|f\|_{C^*(G)} \in \mathbb{R}_{\geq 0}$. We can view π^t as a representation of $C^*(G)$, as written in (4.15). We have

$$\begin{aligned} \pi^t(f)v &= \int_G f(g) \pi^t(g) v dg = \int_{h_\alpha^\vee C} \pi^t(g) v dg = \int_{h_\alpha^\vee C} (\sigma \otimes \chi^t)(g) v dg \\ &= \int_{h_\alpha^\vee C} (\sigma \otimes \chi^t)(h_\alpha^\vee) v dg = \text{vol}(h_\alpha^\vee C) \cdot (\sigma \otimes \chi^t)(h_\alpha^\vee) v, \end{aligned}$$

where the second to last equality holds because C is a compact open subgroup fixing v , so the only nontrivial contribution of the action of $h_\alpha^\vee C$ on v comes from $h_\alpha^\vee$. Note that $\pi^t(f)v$ is again right $U_{-\alpha, r}$ -invariant, and supported in $(P \cap K) \cdot U_{-\alpha, r}$.

Since $\pi^t(f)v$ lies in $I_{P \cap K}^K(V_\sigma)$, it also lies in the Hilbert space completion $\mathcal{V}$ of $I_{P \cap K}^K(V_\sigma)$. Let $\mathcal{B}(\mathcal{V})$ be the C^* -algebra of bounded linear operators on

$\mathcal{V}$, with the operator norm denoted by $\|\cdot\|_{\text{op}}$. The operator norm of $\pi^t(f)$ is given by

$$\|\pi^t(f)\|_{\text{op}} = \sup_{v' \neq 0} \frac{\|\pi^t(f)v'\|_{\mathcal{V}}}{\|v'\|_{\mathcal{V}}}, \text{ where } \|v'\|_{\mathcal{V}}^2 = \int_K \|v'(k)\|^2 dk.$$

Let v with $\text{supp}(v) = (P \cap K) \cdot U_{-\alpha, r}$ as above. Then

$$\begin{aligned} \|v\|_{\mathcal{V}}^2 &= \int_K \|v(k)\|^2 dk = \int_{(P \cap K) \cdot U_{-\alpha, r}} \|v(k)\|^2 dk = \text{vol}(U_{-\alpha, r}) \int_{P \cap K} \|v(k)\|^2 dk \\ &= \text{vol}(U_{-\alpha, r}) \int_{P \cap K} \|\sigma(k)v(1)\|^2 dk = \text{vol}(U_{-\alpha, r}) \text{vol}(P \cap K) \|v(1)\|^2, \end{aligned}$$

since $v \in I_{P \cap K}^K(V_{\sigma})$ is right $U_{-\alpha, r}$ -invariant and σ is unitary. Similarly, we have

$$\begin{aligned} \|\pi^t(f)v\|_{\mathcal{V}}^2 &= \int_K \|\pi^t(f)v(k)\|^2 dk = \int_{(P \cap K) \cdot U_{-\alpha, r}} \|\pi^t(f)v(k)\|^2 dk \\ &= \text{vol}(U_{-\alpha, r}) \text{vol}(h_{\alpha}^{\vee} C)^2 \int_{P \cap K} \|(\sigma \otimes \chi^t)(h_{\alpha}^{\vee})v(k)\|^2 dk \\ &= \text{vol}(U_{-\alpha, r}) \text{vol}(h_{\alpha}^{\vee} C)^2 (\chi^t(h_{\alpha}^{\vee}))^2 \int_{P \cap K} \|\sigma(h_{\alpha}^{\vee})\sigma(k)v(1)\|^2 dk \\ &= \text{vol}(U_{-\alpha, r}) \text{vol}(h_{\alpha}^{\vee} C)^2 (\chi^t(h_{\alpha}^{\vee}))^2 \text{vol}(P \cap K) \|v(1)\|^2 \\ &= \text{vol}(h_{\alpha}^{\vee} C)^2 (\chi^t(h_{\alpha}^{\vee}))^2 \|v\|_{\mathcal{V}}^2. \end{aligned}$$

Recall that since π^t is unitary, it gives a homomorphism of C^* -algebras $\pi^t : C^*(G) \rightarrow \mathcal{B}(\mathcal{V})$, which is norm-decreasing. Since $\chi(h_{\alpha}^{\vee}) > 1$, we have for $t \rightarrow \infty$ that

$$\|\pi^t(f)v\|_{\mathcal{V}}/\|v\|_{\mathcal{V}} = \text{vol}(h_{\alpha}^{\vee} C) \chi^t(h_{\alpha}^{\vee}) \rightarrow \infty, \text{ hence } \lim_{t \rightarrow \infty} \|\pi^t(f)\|_{\text{op}} = \infty$$

which contradicts $\|\pi^t(f)\|_{\text{op}} \leq \|f\|_{C^*(G)}$. So $\mu(\sigma \otimes \chi^t) = \infty$ for some $t > 0$. $\square$

Now we are ready to prove the main result. We drop the assumption $\sigma = \sigma_0$ that was used to prove the results in this subsection so far.

Theorem 4.24. *There exists a unitary $\sigma_0 \in \text{Irr}(M)_{[M, \sigma]}$ such that*

$$\mu(\sigma_0 \otimes \chi) = \mu(\sigma_0, z) = c_{\mu} \cdot \frac{(1-z)(1-z^{-1})}{(1-qz)(1-qz^{-1})} \frac{(1+z)(1+z^{-1})}{(1+q'z)(1+q'z^{-1})},$$

where $c_{\mu} \in \mathbb{R}_{>0}$ and $q \geq 1, q' \geq 1$.

Proof. Suppose that $\mu(\sigma, z)$ is constant, then we can take $q = q' = 1$. Suppose that $\mu(\sigma, z)$ is not constant, then there exists a unitary $\sigma_0 \in \text{Irr}(M)_{[M, \sigma]}$ as in Proposition 4.15. By Lemma 4.4, $\mu(\sigma_0, z)$ has the form

$$\mu(\sigma_0, z) = c_{\mu} \prod_i (z - \lambda_i)^{n_i} (z^{-1} - \overline{\lambda_i})^{n_i},$$

where $\lambda_i \in \mathbb{C}$, $n_i \in \mathbb{Z}$, $c_{\mu} \in \mathbb{R}_{>0}$. By Proposition 4.12, $\mu(\sigma_0, z)$ is not zero at $z = 0$ and $z = \infty$. By Proposition 4.15, the zero at $z = 1$ is of order 2. So $\mu(\sigma_0, z)$ has a factor $(1-z)(1-z^{-1})$ in the numerator. By Lemma 4.23, $\mu(\sigma_0) =$

0 implies that $\mu(\sigma_0 \otimes \chi^t)$ has a pole for some $t \in \mathbb{R}_{>0}$, $\chi \in \text{Hom}(M/Z(G), \mathbb{R}_{>0})$ with $\chi(h_\alpha^\vee) > 1$. So $\mu(\sigma_0, z)$ has a pole at $q = \chi^t(h_\alpha^\vee) \in \mathbb{R}_{>1}$ and μ has a factor $(1 - qz)(1 - qz^{-1})$ in the denominator. The only other possible zero of $\mu(\sigma_0, z)$ is at $z = -1$, and again of multiplicity 2, so μ has a factor $(1 + z)(1 + z^{-1})$ in the numerator (if μ is not zero at $z = -1$, we can cancel out this factor by taking $q' = 1$ in the formula). Suppose that $\mu(\sigma_0, -1) = 0$ and let χ' be a unitary character such that $z = \chi'(h_\alpha^\vee) = -1$. Then $\mu(\sigma_0 \otimes \chi') = 0$, and by Lemma 4.23, $\mu((\sigma_0 \otimes \chi') \otimes \chi^t) = \mu(\sigma_0 \otimes \chi' \chi^t)$ has a pole for some $\chi \in \text{Hom}(M/Z(G), \mathbb{R}_{>0})$, $t \in \mathbb{R}_{>0}$. This implies that $\mu(\sigma_0, z)$ has a pole at $z = \chi'(h_\alpha^\vee) \chi^t(h_\alpha^\vee) = -1 \cdot q'$ with $q' \in \mathbb{R}_{>1}$. So μ has a factor $(1 + q'z)(1 + q'z^{-1})$ in the denominator.

It remains to show that $\mu(\sigma_0, z)$ does not have any additional factors. More factors in the numerator would mean that $\mu(\sigma_0, z)$ has extra zeros in $\mathbb{C}^\times$, which is not possible. More factors in the denominator would mean that $\mu(\sigma_0, \infty) = \mu(\sigma_0, 0) = 0$, which is not possible by Proposition 4.12. $\square$

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IMAPP, Radboud Universiteit, Heyendaalseweg 135, 6525AJ Nijmegen, the Netherlands
Email address: janet.flikkema@ru.nl

IMAPP, Radboud Universiteit, Heyendaalseweg 135, 6525AJ Nijmegen, the Netherlands
Email address: m.solleveld@science.ru.nl