PARAMETERS OF HECKE ALGEBRAS
FOR BERNSTEIN COMPONENTS OF p-ADIC GROUPS

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Abstract. Let $G$ be a reductive group over a non-archimedean local field $F$. Consider an arbitrary Bernstein block $\text{Rep}(G)^s$ in the category of complex smooth $G$-representations. In earlier work the author showed that there exists an affine Hecke algebra $\mathcal{H}(\mathcal{O}, G)$ whose category of right modules is closely related to $\text{Rep}(G)^s$. In many cases this is in fact an equivalence of categories, like for Iwahori-spherical representations.

In this paper we study the $q$-parameters of the affine Hecke algebras $\mathcal{H}(\mathcal{O}, G)$. We compute them in many cases, in particular for principal series representations of quasi-split groups and for classical groups.

Lusztig conjectured that the $q$-parameters are always integral powers of $q_F$ and that they coincide with the $q$-parameters coming from some Bernstein block of unipotent representations. We reduce this conjecture to the case of absolutely simple $p$-adic groups, and we prove it for most of those.

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Date: October 31, 2022.
2010 Mathematics Subject Classification. Primary 22E50, Secondary 20G25, 20C08.
Key words and phrases. representation theory, reductive groups, Hecke algebras, non-archimedean local fields.

The author is supported by a Vidi grant "A Hecke algebra approach to the local Langlands correspondence" (nr. 639.032.528) from Nederlands Wetenschappelijk Onderzoek (NWO).
INTRODUCTION

It is well-known that affine Hecke algebras play an important role in the representation theory of a reductive group $G$ over a non-archimedean local field $F$. In many cases a Bernstein block $\text{Rep}(G)^s$ in the category of smooth complex $G$-representations is equivalent with the module category of an affine Hecke algebra (maybe extended with some finite group). This was first shown for Iwahori-spherical representations [IwMa, Bor] and for depth zero representations [Mor]. With the theory of types [BuKu2] such an equivalence of categories was established for representations of $GL_n(F)$, of inner forms of $GL_n(F)$ [SeSt1, SeSt2] and for inner forms of $SL_n(F)$ [ABPS].

An alternative approach goes via the algebra of $G$-endomorphisms of a progenator $\Pi^s$ of $\text{Rep}(G)^s$. The category of right modules of $\text{End}_G(\Pi^s)$ is naturally equivalent with $\text{Rep}(G)^s$. Heiermann [Hei2, Hei3] showed that for symplectic groups, special orthogonal groups, unitary groups and inner forms of $GL_n(F)$, $\text{End}_G(\Pi^s)$ is always Morita equivalent with an (extended) affine Hecke algebra.

Recently the author generalized this to all Bernstein components of all reductive $p$-adic groups [Sol6]. In the most general setting some subtleties have to be taken into account: the involved affine Hecke algebra must be extended with the group algebra of a finite group, but that group algebra might be twisted by a 2-cocycle. Also, the resulting equivalence with $\text{Rep}(G)^s$ works for finite length representations, but maybe not entirely for representations of infinite length. Nevertheless, the bottom line is that $\text{Rep}(G)^s$ is largely governed by an affine Hecke algebra from $\text{End}_G(\Pi^s)$.

Let $M$ a Levi factor of a parabolic subgroup $P$ of $G$ such that $\text{Rep}(G)^s$ arises by parabolic induction from a supercuspidal representation $\sigma$ of $M$. We denote the variety of unramified twists of $\sigma$ by $\mathcal{O} \subset \text{Irr}(M)$, and the affine Hecke algebra described above by $\mathcal{H}(\mathcal{O}, G)$. If at the same a $s$-type $(J, \rho)$ is known, then the Hecke algebra $\mathcal{H}(G, J, \rho)$ is Morita equivalent with $\text{End}_{\mathcal{O}}(\Pi^s)^{op}$. (In fact [BaSa Appendix A] shows that in most cases $\text{ind}_J^G(\rho)$ is isomorphic with $\Pi^s$.) In this setting $\mathcal{H}(\mathcal{O}, G)$ can also be constructed from $\mathcal{H}(G, J, \rho)$.

The next question is of course: what does $\mathcal{H}(\mathcal{O}, G)$ look like? Like all affine Hecke algebras, it is determined by a root datum and some $q$-parameters. The lattice $X$ (from that root datum) can be identified with the character lattice of $\mathcal{O}$, once the latter has been made into a complex torus by choosing a base point. The root system $\Sigma^\vee_\mathcal{O}$ (also from the root datum) is contained in $X$ and determined by the reducibility points of the family of representations $\{I_P^\mathcal{O}(\sigma') : \sigma' \in \mathcal{O}\}$. Then $\mathcal{H}(\mathcal{O}, G)$ contains a maximal commutative subalgebra $\mathbb{C}[X] \cong \mathbb{C}[\mathcal{O}]$ and a finite dimensional Iwahori–Hecke algebra $\mathcal{H}(W(\Sigma^\vee_\mathcal{O}), q_\lambda^1)$ such that

$$\mathcal{H}(\mathcal{O}, G) = \mathbb{C}[\mathcal{O}] \otimes_{\mathbb{C}} \mathcal{H}(W(\Sigma^\vee_\mathcal{O}), q_\lambda^1)$$

as vector spaces. Here $q_\mathcal{O}$ denotes the cardinality of the residue field of $F$, while $\lambda$ will be defined soon. For every $X_\alpha \in \Sigma^\vee_\mathcal{O}$ there is a $q_\alpha \in \mathbb{R}_{>1}$ such that

$$I_P^\mathcal{O}(\sigma') \text{ is reducible for all } \sigma' \in \mathcal{O} \text{ with } X_\alpha(\sigma') = q_\alpha.$$

Sometimes there is also a number $q_{\alpha*} \in (1, q_\alpha]$ with the property

$$I_P^\mathcal{O}(\sigma') \text{ is reducible for all } \sigma' \in \mathcal{O} \text{ with } X_\alpha(\sigma') = -q_{\alpha*}.$$

When such a real number does not exist, we put $q_{\alpha*} = 1$. These $q$-parameters $q_\alpha$
and \(q_{\alpha^*}\) appear in the Hecke relations of \(\mathcal{H}(W(\Sigma_{\mathcal{O}}^\vee),q_{F}^\lambda)\):

\[
0 = (T_{\alpha} + 1)(T_{\alpha} - q_{F}^{\lambda(\alpha)}) \quad \text{with} \quad q_{F}^{\lambda(\alpha)} = q_{\alpha}q_{\alpha^*} \in \mathbb{R}_{>1}.
\]

Further, we define \(\lambda^*(\alpha) \in \mathbb{R}_{\geq 0}\) by

\[
q_{F}^{\lambda^*(\alpha)} = q_{\alpha}q_{\alpha^*}^{-1}.
\]

Knowing \(q_{\alpha}, q_{\alpha^*}\) is also equivalent to knowing the poles of the Harish-Chandra \(\mu\)-function on \(\mathcal{O}\) associated to \(\alpha\). See Section \[\text{I}\] for more details on the above setup.

The representation theory of \(\mathcal{H}(\mathcal{O}, G)\) depends in a subtle way on the \(q\)-parameters \(q_{\alpha}, q_{\alpha^*}\) for \(X_{\alpha} \in \Sigma_{\mathcal{O}}^\vee\), so knowing them helps to understand \(\text{Rep}(G)^{\mathcal{O}}\). That brings us to the main goal of this paper: determine the \(q\)-parameters of \(\mathcal{H}(\mathcal{O}, G)\) for as many Bernstein blocks \(\text{Rep}(G)^{\mathcal{O}}\) as possible.

The associativity of the algebra \(\mathcal{H}(\mathcal{O}, G)\) puts some constraints on the \(q_{\alpha}\) and \(q_{\alpha^*}\):

- If \(X_{\alpha}, X_\beta \in \Sigma_{\mathcal{O}}^\vee\) are \(W(\Sigma_{\mathcal{O}}^\vee)\)-associate, then \(q_{\alpha} = q_\beta\) and \(q_{\alpha^*} = q_{\beta^*}\).
- \(q_{\alpha^*} > 1\) is only possible if \(X_{\alpha}\) is a short root in a type \(B_n\) root system.

Notice that \(q_{\alpha}\) and \(q_{\alpha^*}\) can be expressed in terms of the "\(q\)-base" \(q_F\) and the labels \(\lambda(\alpha), \lambda^*(\alpha)\). It has turned out \([\text{KalLu}, \text{Sol1}]\) that the representation theory of an affine Hecke algebra hardly changes if one replaces \(q_F\) by another \(q\)-base (in \(\mathbb{R}_{>1}\)) while keeping all labels fixed. If we replace the \(q\)-base \(q_F\) by \(q_{F}'\) and \(\lambda(\alpha), \lambda^*(\alpha)\) by \(\lambda(\alpha)/r, \lambda^*(\alpha)/r\) for some \(r \in \mathbb{R}_{>0}\), then \(q_{\alpha}\) and \(q_{\alpha^*}\) do not change, and in fact \(\mathcal{H}(\mathcal{O}, G)\) is not affected at all. In this way one can always scale one of the labels to 1.

Hence the representation theory of \(\mathcal{H}(\mathcal{O}, G)\) depends mainly on the ratios between the labels \(\lambda(\alpha), \lambda^*(\alpha)\) for \(X_{\alpha} \in \Sigma_{\mathcal{O}}^\vee\).

- For irreducible root systems of type \(A_n, D_n\) and \(E_n\), \(\lambda(\alpha) = \lambda^*(\alpha) = \lambda(\beta)\), for any roots \(X_{\alpha}, X_\beta \in \Sigma_{\mathcal{O}}^\vee\). There is essentially only one label \(\lambda(\alpha)\), and it can be scaled to 1 by fixing \(q_{\alpha}\) but replacing \(q_{F}\) by \(q_{\alpha}\).
- For the irreducible root systems \(C_n, F_4\) and \(G_2\), again \(\lambda(\alpha)\) always equals \(\lambda^*(\alpha)\). There are two independent labels \(\lambda(\alpha)\): one for the short roots and one for the long roots.
- For an irreducible root system of type \(B_n\), \(\lambda^*(\alpha)\) need not equal \(\lambda(\alpha)\) if \(X_{\alpha}\) is short. Here we have three independent labels: \(\lambda(\beta)\) for \(X_\beta\) long, \(\lambda(\alpha)\) for \(X_{\alpha}\) short and \(\lambda^*(\alpha)\) for \(X_{\alpha}\) short.

Lusztig \([\text{Lus5}]\) has conjectured:

**Conjecture A.** Let \(G\) be a reductive group over a non-archimedean local field, with an arbitrary Bernstein block \(\text{Rep}(G)^{\mathcal{O}}\). Let \(\Sigma_{\mathcal{O}, j}^\vee\) be an irreducible component of the root system \(\Sigma_{\mathcal{O}}^\vee\) underlying \(\mathcal{H}(\mathcal{O}, G)\). Then:

(i) the \(q\)-parameters \(q_{\alpha}, q_{\alpha^*}\) are powers of \(q_{F}\), except that for a short root \(\alpha\) in a type \(B_n\) root system the \(q\)-parameters can also be powers of \(q_{F}^{1/2}\) (and then \(q_{\alpha}q_{\alpha^*}^{1/2}\) is still a power of \(q_{F}\)).

(ii) the label functions \(\lambda, \lambda^*\) on \(\Sigma_{\mathcal{O}, j}^\vee\) agree with those obtained in the same way from a Bernstein block of unipotent representations of some adjoint simple \(p\)-adic group, as in \([\text{Lus3}, \text{Lus4}]\).

Conjecture \(\text{A}(i)\) is related to a conjecture of Langlands about Harish-Chandra \(\mu\)-functions \([\text{Sha}, \text{§2}]\). For generic representations of quasi-split reductive groups over \(p\)-adic fields, \([\text{Sha}, \text{§3}]\) reduces Conjecture \(\text{A}(i)\) to a question about poles of adjoint \(\gamma\)-factors. (We do not pursue that special case here.)
Motivation for Conjecture A(ii) comes from the local Langlands correspondence. It is believed [AMS1] that $\text{Irr}(G) \cap \text{Rep}(G)\mathfrak{g}$ corresponds to a Bernstein component $\Phi_e(G)\mathfrak{g}$ of enhanced $L$-parameters for $G$. To $\Phi_e(G)\mathfrak{g}$ one can canonically associate an affine Hecke algebra $\mathcal{H}(\mathfrak{g}_\Phi, q_1^{1/2})$, possibly extended with a twisted group algebra [AMS3, §3.3]. It is expected that the module category of $\mathcal{H}(\mathfrak{g}_\Phi, q_1^{1/2})$ is very closely related to $\text{Rep}(G)\mathfrak{g}$, at least the two subcategories of finite length modules should be equivalent.

The nonextended version $\mathcal{H}^0(\mathfrak{g}_\Phi, \frac{q_1}{2})$ of $\mathcal{H}(\mathfrak{g}_\Phi, q_1^{1/2})$ can be constructed with complex geometry from a connected reductive group $H'$ (the connected centralizer in $G'$ of the image of the inertia group $\mathbf{I}_F$ under the Langlands parameter) and a cuspidal local system $\rho$ on a unipotent orbit for a Levi subgroup $L'$ of $H'$. The exact same data $(H', L', \rho)$ also arise from enhanced Langlands parameters (for some reductive $p$-adic group $G'$) which are trivial on $\mathbf{I}_F$. By the local Langlands correspondence from [Lus3, Lus4, Sol2, Sol4], a Bernstein component of such enhanced $L$-parameters corresponds to a Bernstein component $\text{Rep}(G')\mathfrak{g}'$ of unipotent $G'$-representations.

It follows that $\mathcal{H}^0(\mathfrak{g}_\Phi, \frac{q_1}{2})$ is isomorphic to $\mathcal{H}^0(\mathfrak{g}_\Phi', \frac{q_1}{2})$. By [Sol2, Theorem 4.4], $\mathcal{H}^0(\mathfrak{g}_\Phi', \frac{q_1}{2})$ is isomorphic to $\mathcal{H}(O', G')$, which is an affine Hecke algebra associated to a Bernstein block of unipotent representations of $G'$. If desired one can replace $G'$ by its adjoint group, by [Sol2, Lemma 3.5] that operation changes the affine Hecke algebras a little but preserves the root systems and the $q$-parameters.

Thus, if there exists a local Langlands correspondence with good properties, Conjecture A is a consequence of what happens on the Galois side of the correspondence. Conversely, new cases of Conjecture A might contribute to new instances of a local Langlands correspondence, via a comparison of possible Hecke algebras on both sides as in [Lus3].

We note that the affine root systems in Lusztig’s notation for affine Hecke algebras correspond to affine extensions of our root systems $\Sigma_{\mathcal{O}}$. Now we list all possible label functions from [Lus3 Lus4], for a given irreducible root system:

<table>
<thead>
<tr>
<th>$\Sigma_{\mathcal{O}}$</th>
<th>$\lambda$(long root)</th>
<th>$\lambda$(short root)</th>
<th>$\lambda^*$ (short root)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n, D_n, E_n$</td>
<td>$-$</td>
<td>$\in \mathbb{Z}_{&gt;0}$</td>
<td>$\lambda^* = \lambda$</td>
</tr>
<tr>
<td>$B_n$</td>
<td>1 or 2</td>
<td>$\in \mathbb{Z}_{&gt;0}$</td>
<td>$\in \mathbb{Z}_{&gt;0}$</td>
</tr>
<tr>
<td>$C_n$</td>
<td>$\in \mathbb{Z}_{&gt;0}$</td>
<td>1 or 2</td>
<td>$\lambda^* = \lambda$</td>
</tr>
<tr>
<td>$F_4$</td>
<td>1 or 2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$F_4$</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$F_4$</td>
<td>4</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$G_2$</td>
<td>1 or 3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$G_2$</td>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$G_2$</td>
<td>9</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$B_2$</td>
<td>3</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

An important and accessible class of representations is formed by the principal series representations of quasi-split groups $G$. When $G$ is $F$-split, the Hecke algebras for Bernstein blocks of such representations were already analysed in [Roc1] via
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types, under some mild restrictions on the residual characteristic. To every root of a quasi-split group \( G \) (relative to a maximal \( F \)-split torus) one can associate a splitting field \( F_\alpha \), a finite extension of \( F \).

**Theorem B.** (see Theorem 4.4 and Corollary 4.5)
Conjecture \([A]\) holds for all Bernstein blocks in the principal series of a quasi-split connected reductive group over \( F \). For \( X_\alpha \in \Sigma_\varphi \) (with one exception) \( q_{\alpha*} = 1 \) and \( q_\alpha \) is the cardinality of the residue field of \( F_\alpha \).

As an aside: types and Hecke algebras can be made explicit for quasi-split unitary group, in the spirit of [Roc1]. We present an overview of those Hecke algebras, worked out by Badea [Bade] under supervision of the author.

For parameter computations in Hecke algebras associated to more complicated Bernstein components, we need a reduction strategy. That is the topic of Section 2, which culminates in:

**Theorem C.** (see Corollary 2.5)
Suppose that Conjecture \([A]\) holds for the simply connected cover \( G_{sc} \) of \( G_{der} \). Then it holds for \( G \).

This enables us to reduce the verification of Conjecture \([A]\) to absolutely simple, simply connected groups. For (absolutely) simple groups quite a few results about the parameters of Hecke algebras can be found in the literature, e.g. [BuKu1, Sec, Hei1]. With our current framework we can easily generalize those results, in particular from one group to an isogenous group.

Sécherre and Stevens [Sec, SecSt1, SecSt2] determined the Hecke algebras for all Bernstein blocks for inner forms of \( GL_n(F) \). Together with Theorem C that proves Conjecture \([A]\) for all inner forms of a group of type \( A \).

For classical groups (symplectic, special orthogonal, unitary) we run into the problem that some representation theoretic results have been proven over \( p \)-adic fields but not (yet) over local function fields. We overcome this with the method of close fields [Kaz], which Ganapathy recently generalized to arbitrary connected reductive groups [Gan1, Gan2].

**Theorem D.** (see Corollary 3.7)
Let \( \text{Rep}(G)^\circ \) be a Bernstein block for a reductive group \( G \) over a local function field. Then there exists a Bernstein block \( \text{Rep}(\tilde{G})^\circ \) for a reductive group \( \tilde{G} \) over a \( p \)-adic field, such that:

- \( G \) and \( \tilde{G} \) come from "the same" algebraic group,
- \( \text{Rep}(G)^\circ \cong \text{Rep}(\tilde{G})^\circ \) and \( \mathcal{H}(O,G) \cong \mathcal{H}(\tilde{O},\tilde{G}) \),
- the parameters for both these affine Hecke algebras are the same.

For quasi-split classical groups the parameters of the Hecke algebras were determined in [Hei1], in terms of Mœglin’s classification of discrete series representations [Mo3]. We extend this to pure inner forms of quasi-split classical groups, and to groups isogenous with one of those.

**Theorem E.** (see Paragraph 4.4)
Conjecture \([A]\) holds for all the groups just mentioned. This includes all simple groups of type \( A_n, B_n, C_n, D_n \), except those associated to Hermitian forms on vector spaces over quaternion algebras.
Among classical groups associated to Hermitian forms, Conjecture A only remains open for the forms of quaternionic type. Those groups are precisely the non-pure inner forms of quasi-split classical groups. Unfortunately, the current understanding of their representations does not suffice to carry out our strategies for other groups.

Finally, we consider exceptional groups. For most Bernstein components we can reduce the computation of the Hecke algebra parameters to groups of Lie type $A_n, B_n, C_n$ and $D_n$, but sometimes that does not work. We establish partial results for all simple exceptional groups, most of which can be summarized as follows:

**Theorem F.** (see Paragraphs 4.5, 4.6 and 4.7)
Conjecture A holds for all simple $F$-groups of type $G_2$, $3D_4, E_6^{(3)}, E_6$.

If (for any reductive $p$-adic group $G$) $\Sigma_O^\vee$ has an irreducible component $\Sigma_{O,j}^\vee$ of type $F_4$, then Conjecture A holds for $\Sigma_{O,j}^\vee$.

Our results about $F_4$ are useful in combination with [Sol5, §6]. There we related the irreducible representations of an affine Hecke algebra with arbitrary positive $q$-parameters to the irreducible representations of the analogous algebra that has all $q$-parameters equal to 1. The problem was only that we could not handle certain label functions for type $F_4$ root systems. Theorem F shows that the label functions which could be handled well in [Sol5, §6] exhaust the label functions that can appear for type $F_4$ root systems among affine Hecke algebras coming from reductive $p$-adic groups.

**Acknowledgements.**
We are grateful to Anne-Marie Aubert and Geo Tam for their comments on earlier versions, and in particular for pointing out some problems. We thank the referee for his or her thorough report, which lead to clarification of some issues.

1. **Progenerators and endomorphism algebras for Bernstein blocks**

We fix some notations and recall relevant material from [Sol6]. Let $F$ be a non-archimedean local field with ring of integers $\mathcal{O}_F$ and uniformizer $\varpi_F$. We denote the cardinality of the residue field $k_F = \mathcal{O}_F/\varpi_F$ by $q_F$.

Let $G$ be a connected reductive $F$-group and let $G = G(F)$ be its group of $F$-rational points. We briefly call $G$ a reductive $p$-adic group. We consider the category $\text{Rep}(G)$ of smooth $G$-representations on complex vector spaces. Let $\text{Irr}(G)$ be the set of equivalence classes of irreducible objects in $\text{Rep}(G)$, and $\text{Irr}_{\text{cusp}}(G) \subset \text{Irr}(G)$ the subset of supercuspidal representations.

Let $M$ be a $F$-Levi subgroup of $G$ and write $M = M(F)$. The group of unramified characters of $M$ is denoted $X_{\text{nr}}(M)$. We fix $(\sigma, E) \in \text{Irr}_{\text{cusp}}(M)$. The set of unramified twists of $\sigma$ is

$$\mathcal{O} = \{ \sigma \otimes \chi : \chi \in X_{\text{nr}}(M) \} \subset \text{Irr}(M).$$

It can be identified with the inertial equivalence class $s_M = [M, \sigma]_M$. Let $s = [M, \sigma]_G$ be the associated inertial equivalence class for $G$.

Recall that the supercuspidal support $\text{Sc}(\pi)$ of $\pi \in \text{Irr}(G)$ consists of a Levi subgroup of $G$ and an irreducible supercuspidal representation thereof. Although $\text{Sc}(\pi)$
is only defined up to $G$-conjugacy, we shall only be interested in supercuspidal supports with Levi subgroup $M$, and then the supercuspidal representation is uniquely defined up to the natural action of $N_G(M)$ on $\text{Irr}(M)$.

This setup yields a Bernstein component

$$\text{Irr}(G)^s = \{ \pi \in \text{Irr}(G) : \text{Sc}(\pi) \in (M, O) \}$$

of $\text{Irr}(G)$. It generates a Bernstein block $\text{Rep}(G)^s$ of $\text{Rep}(G)$, see [BeDe]. Let $M^1 \subset M$ be the group generated by all compact subgroups of $M$, so that $X_{nr}(M) = \text{Irr}(M/M^1)$. Then

$$\text{ind}^M_{M^1}(\sigma, E) \cong E \otimes \mathbb{C}[M/M^1] \cong E \otimes \mathbb{C}[X_{nr}(M)],$$

where $\mathbb{C}[M/M^1]$ is the group algebra of the discrete group $M/M^1$ and $\mathbb{C}[X_{nr}(M)]$ is the ring of regular functions on the complex torus $X_{nr}(M)$. Supercuspidality implies that (1.1) is a progenerator of $\text{Rep}(M)^s$. Let $P \subset G$ be a parabolic subgroup with Levi factor $M$, chosen as prescribed by [Sol6, Lemma 9.1]. Let $P_G : \text{Rep}(M) \to \text{Rep}(G)$ be the parabolic induction functor, normalized so that it preserves unitarity. As a consequence of Bernstein’s second adjointness theorem [Ren],

$$\Pi^s := P_G(E \otimes \mathbb{C}[X_{nr}(M)])$$

is a progenerator of $\text{Rep}(G)^s$. That means [Roc2, Theorem 1.8.2.1] that the functor

$$\text{Rep}(G)^s \to \text{End}_G(\Pi^s) - \text{Mod}$$

is an equivalence of categories. This motivates the study of the endomorphism algebra $\text{End}_G(\Pi^s)$, which was carried out in [Roc2, Hei2, Sol6]. To describe its structure, we have to recall several objects which lead to the appropriate root datum. The set

$$X_{nr}(M, \sigma) = \{ \chi \in X_{nr}(M) : \sigma \otimes \chi \cong \chi \}$$

is a finite subgroup of $X_{nr}(M)$. The map

$$X_{nr}(M)/X_{nr}(M, \sigma) \to O : \chi \mapsto \sigma \otimes \chi$$

is a bijection, and in this way we provide $O$ with the structure of a complex variety (a torus, but without a canonical base point). The group

$$M^2_\sigma := \bigcap_{\chi \in X_{nr}(M, \sigma)} \ker \chi$$

has finite index in $M$, and there are natural isomorphisms

$$\text{Irr}(M^2_\sigma/M^1) \cong X_{nr}(M)/X_{nr}(M, \sigma),$$

$$\mathbb{C}[M^2_\sigma/M^1] \cong \mathbb{C}[X_{nr}(M)/X_{nr}(M, \sigma)].$$

Here and later on, the notation $\mathbb{C}[?]$ must be interpreted as in (1.1). The group

$$W(G, M) := N_G(M)/M$$

is a Weyl group in most cases (and if it is not, then it is still very close to a Weyl group). The natural action of $N_G(M)$ on $\text{Rep}(M)$ induces an action of $W(G, M)$ on $\text{Irr}(M)$. Let $N_G(M, O)$ be the stabilizer of $O$ in $N_G(M)$ and write

$$W(M, O) = N_G(M, O)/M.$$
Thus \( W(M, \mathcal{O}) \) acts naturally on the complex algebraic variety \( \mathcal{O} \). This finite group figures prominently in the Bernstein theory, for instance because the centres of \( \text{Rep}(G) \) and of \( \text{End}_C(\mathcal{P}) \) are naturally isomorphic with \( \mathbb{C}[\mathcal{O}]^{W(M, \mathcal{O})} \).

Let \( \mathcal{A}_M \) be the maximal \( F \)-split torus in \( Z(\mathcal{M}) \), put \( \mathcal{A}_M = \mathcal{A}_M(F) \) and let \( X_*(\mathcal{A}_M) = X_*(\mathcal{A}_M) \) be the cocharacter lattice. We write
\[
\mathfrak{a}_M = X_*(\mathcal{A}_M) \otimes \mathbb{Z} \mathbb{R} \quad \text{and} \quad \mathfrak{a}_M^* = X^*(\mathcal{A}_M) \otimes \mathbb{Z} \mathbb{R}.
\]

Let \( \Sigma(G, \mathcal{A}_M) \subset X^*(\mathcal{A}_M) \) be the set of nonzero weights occurring in the adjoint representation of \( \mathcal{A}_M \) on the Lie algebra of \( G \), and let \( \Sigma_{\text{red}}(\mathcal{A}_M) \) be the set of indivisible elements therein.

For every \( \alpha \in \Sigma_{\text{red}}(\mathcal{A}_M) \) there is a Levi subgroup \( \mathcal{L}_\alpha \) of \( G \) which contains \( M \) and whose semisimple rank is one higher than that of \( M \). Let \( \alpha^\vee \in \mathfrak{a}_M \) be the unique element which is orthogonal to \( X^*(\mathcal{A}_M) \) and satisfies \( \langle \alpha^\vee, \alpha \rangle = 2 \).

Recall the Harish-Chandra \( \mu \)-functions from \([\text{Sil2}, \S 1]\) and \([\text{Wall}, \S V.2]\). The restriction of \( \mu^G \) to \( \mathcal{O} \) is a rational, \( W(M, \mathcal{O}) \)-invariant function on \( \mathcal{O} \) \([\text{Wall} \text{ Lemma V.2.1}] \). It determines a reduced root system \([\text{Hei2}, \text{Proposition 1.3}] \)
\[
(1.2) \quad \Sigma_{\mathcal{O}, \mu} = \{ \alpha \in \Sigma_{\text{red}}(\mathcal{A}_M) : \mu^{\mathcal{A}_M}(\sigma \otimes \chi) \text{ has a zero on } \mathcal{O} \}.
\]

For \( \alpha \in \Sigma_{\text{red}}(\mathcal{A}_M) \) the function \( \mu^{\mathcal{A}_M} \) factors through the quotient map \( \mathcal{A}_M \to \mathcal{A}_M/\mathcal{A}_M_\alpha \). The associated system of coroots is
\[
\Sigma_{\mathcal{O}, \mu}^\vee = \{ \alpha^\vee \in \mathfrak{a}_M : \mu^{\mathcal{A}_M}(\sigma \otimes \chi) \text{ has a zero on } \mathcal{O} \}.
\]

By the aforementioned \( W(M, \mathcal{O}) \)-invariance of \( \mu^G \), \( W(M, \mathcal{O}) \) acts naturally on \( \Sigma_{\mathcal{O}, \mu} \) and on \( \Sigma_{\mathcal{O}, \mu}^\vee \). Let \( s_\alpha \) be the unique nontrivial element of \( W(\mathcal{A}_M, \mathcal{A}_\alpha, M) \). By \([\text{Hei2}, \text{Proposition 1.3}] \) the Weyl group \( W(\Sigma_{\mathcal{O}, \mu}) \) can be identified with the subgroup of \( W(G, \mathcal{M}) \) generated by the reflections \( s_\alpha \) with \( \alpha \in \Sigma_{\mathcal{O}, \mu} \), and as such it is a normal subgroup of \( W(M, \mathcal{O}) \).

The parabolic subgroup \( P = \mathcal{M} U \) of \( G \) determines a set of positive roots \( \Sigma_{\mathcal{O}, \mu}^+ \) and a basis \( \Delta_{\mathcal{O}, \mu} \) of \( \Sigma_{\mathcal{O}, \mu} \). Let \( \ell_{\mathcal{O}} \) be the length function on \( W(\Sigma_{\mathcal{O}, \mu}) \) specified by \( \Delta_{\mathcal{O}, \mu} \). Since \( W(M, \mathcal{O}) \) acts on \( \Sigma_{\mathcal{O}, \mu} \), \( \ell_{\mathcal{O}} \) extends naturally to \( W(M, \mathcal{O}) \), by
\[
\ell_{\mathcal{O}}(w) = |w(\Sigma_{\mathcal{O}, \mu}^+) \cap -\Sigma_{\mathcal{O}, \mu}^+|.
\]

The set of positive roots also determines a subgroup of \( W(M, \mathcal{O}) \):
\[
(1.3) \quad R(\mathcal{O}) = \{ w \in W(M, \mathcal{O}) : w(\Sigma_{\mathcal{O}, \mu}^+) = \Sigma_{\mathcal{O}, \mu}^+ \} = \{ w \in W(M, \mathcal{O}) : \ell_{\mathcal{O}}(w) = 0 \}.
\]

As \( W(\Sigma_{\mathcal{O}, \mu}) \subset W(M, \mathcal{O}) \), a well-known result from the theory of root systems says:
\[
W(M, \mathcal{O}) = R(\mathcal{O}) \rtimes W(\Sigma_{\mathcal{O}, \mu}).
\]

Recall that \( X_{\mathcal{M}}(M)/X_{\mathcal{M}}(M, \sigma) \) is isomorphic to the character group of the lattice \( M_{\sigma}^2/M_1 \). Since \( M_{\sigma}^2 \) depends only on \( \mathcal{O} \), it is normalized by \( N_G(M, \mathcal{O}) \). In particular the conjugation action of \( N_G(M, \mathcal{O}) \) on \( M_{\sigma}^2/M_1 \) induces an action of \( W(M, \mathcal{O}) \) on \( M_{\sigma}^2/M_1 \).

Let \( \nu_F : F \to \mathbb{Z} \cup \{ \infty \} \) be the valuation of \( F \). Let \( h_\alpha^\vee \) be the unique generator of \( (M_\alpha^2 \cap M_1^1)/M_1 \cong \mathbb{Z} \) such that \( \nu_F(\alpha(h_\alpha^\vee)) > 0 \). Recall the injective homomorphism \( H_M : M/M_1 \to \mathfrak{a}_M \) defined by
\[
(\langle H_M(m), \gamma \rangle = \nu_F(\gamma(m)) \quad \text{for } m \in M, \gamma \in X^*(M).
\]
In these terms $H_M(h^\vee_\alpha) \in \mathbb{R}_{>0} \alpha^\vee$. Since $M^2_\sigma$ has finite index in $M$, $H_M(M^2_\sigma/M^1)$ is a lattice of full rank in $a_M$. We write

$$(M^2_\sigma/M^1)^\vee = \text{Hom}_Z(M^2_\sigma/M^1, \mathbb{Z}).$$

Composition with $H_M$ and $\mathbb{R}$-linear extension of maps $H_M(M^2_\sigma/M^1) \to \mathbb{Z}$ determines an embedding

$$H_M^\vee : (M^2_\sigma/M^1)^\vee \to a_M^*.$$  

Then $H_M^\vee(M^2_\sigma/M^1)^\vee$ is a lattice of full rank in $a_M^*$.

**Proposition 1.1.** [Sol6 Proposition 3.5]

Let $\alpha \in \Sigma_{\mathcal{O},\mu}$.

(a) For $w \in W(M, \mathcal{O})$: $w(h^\vee_\alpha) = h^\vee_{w(\alpha)}$.
(b) There exists a unique $\alpha^\delta \in (M^2_\sigma/M^1)^\vee$ such that $H_M^\vee(\alpha^\delta) \in \mathbb{R} \alpha$ and $\langle h^\vee_\alpha, \alpha^\delta \rangle = 2$.
(c) Write

$$\Sigma_\mathcal{O} = \{ \alpha^\delta : \alpha \in \Sigma_{\mathcal{O},\mu} \},$$

$$\Sigma_{\mathcal{O}}^\vee = \{ h^\vee_\alpha : \alpha \in \Sigma_{\mathcal{O},\mu} \}.$$

Then $(\Sigma_\mathcal{O}^\vee, M^2_\sigma/M^1, \Sigma_{\mathcal{O}}, (M^2_\sigma/M^1)^\vee)$ is a root datum with Weyl group $W(\Sigma_{\mathcal{O},\mu})$.

(d) The group $W(M, \mathcal{O})$ acts naturally on this root datum, and $R(\mathcal{O})$ is the stabilizer of the basis $\Delta_{\mathcal{O}}^\vee$ determined by $P$.

We note that $\Sigma_\mathcal{O}$ and $\Sigma_{\mathcal{O}}^\vee$ have almost the same type as $\Sigma_{\mathcal{O},\mu}$. Indeed, the roots $H_M^\vee(\alpha^\delta)$ are scalar multiples of the $\alpha \in \Sigma_{\mathcal{O},\mu}$, so the angles between the elements of $\Sigma_\mathcal{O}$ are the same as the angles between the corresponding elements of $\Sigma_{\mathcal{O},\mu}$. It follows that every irreducible component of $\Sigma_{\mathcal{O},\mu}$ has the same type as the corresponding components of $\Sigma_\mathcal{O}$ and $\Sigma_{\mathcal{O}}^\vee$, except that type $B_n/C_n$ might be replaced by type $C_n/B_n$.

For $\alpha \in \Sigma_{\text{red}}(M) \setminus \Sigma_{\mathcal{O},\mu}$, the function $\mu^{M_\alpha}$ is constant on $\mathcal{O}$. In contrast, for $\alpha \in \Sigma_{\mathcal{O},\mu}$ it has both zeros and poles on $\mathcal{O}$. By [Sil2 §5.4.2]

$$(1.5) \quad s_\alpha \cdot \sigma^\prime \cong \sigma^\prime \quad \text{whenever} \quad \mu^{M_\alpha}(\sigma^\prime) = 0.$$  

As $\Delta_{\mathcal{O},\mu}$ is linearly independent in $X^*(A_M)$ and $\mu^{M_\alpha}$ factors through $A_M/A_{M_\alpha}$, there exists a $\tilde{\sigma} \in \mathcal{O}$ such that $\mu^{M_\alpha}(\tilde{\sigma}) = 0$ for all $\alpha \in \Delta_{\mathcal{O},\mu}$. In view of [Sil3 §1] this can even be achieved with a unitary $\tilde{\sigma}$. We replace $\sigma$ by $\tilde{\sigma}$, which means that from now on we adhere to:

**Condition 1.2.** $(\sigma, E) \in \text{Irr}(M)$ is unitary supercuspidal and $\mu^{M_\alpha}(\sigma) = 0$ for all $\alpha \in \Delta_{\mathcal{O},\mu}$.

By (1.5) the entire Weyl group $W(\Sigma_{\mathcal{O},\mu})$ stabilizes the isomorphism class of this $\sigma$. However, in general $R(\mathcal{O})$ need not stabilize $\sigma$. We identify $X_{\text{nr}}(M)/X_{\text{nr}}(M, \sigma)$ with $\mathcal{O}$ via $\chi \mapsto \sigma \otimes \chi$ and we define

$$(1.6) \quad X_\alpha \in \mathbb{C}[X_{\text{nr}}(M)/X_{\text{nr}}(M, \sigma)] \text{ by } X_\alpha(\chi) = \chi(h^\vee_\alpha).$$

For any $w \in W(M, \mathcal{O})$ which stabilizes $\sigma$ in $\text{Irr}(M)$, Proposition 1.1.a implies

$$(1.7) \quad w(X_\alpha) = X_{w(\alpha)} \quad \text{for all } \alpha \in \Sigma_{\mathcal{O},\mu}.$$  

According to [Sil2 §1] there exist $q_\alpha, q_\alpha^\ast \in \mathbb{R}_{\geq 1}, c_\alpha^\prime \in \mathbb{R}_{>0}$ for $\alpha \in \Sigma_{\mathcal{O},\mu}$, such that

$$(1.8) \quad \mu^{M_\alpha}(\sigma \otimes \cdot) = \frac{c_\alpha^\prime (1 - X_\alpha)(1 - X_\alpha^{-1})}{(1 - q_\alpha^{-1}X_\alpha)(1 - q_\alpha^{-1}X_\alpha^{-1})} \frac{(1 + X_\alpha)(1 + X_\alpha^{-1})}{(1 + q_\alpha X_\alpha)(1 + q_\alpha X_\alpha^{-1})}$$

for all $\alpha \in \Sigma_{\mathcal{O},\mu}$.
as rational functions on $X_{nr}(M)/X_{nr}(M,\sigma) \cong \mathcal{O}$. We may modify the choice of $\sigma$ in Condition 1.2 so that, as in [Hei2, Remark 1.7]:

$$(1.9) \quad q_\alpha \geq q_{\alpha*} \text{ for all } \alpha \in \Delta_{O,\mu}.$$ 

Then [Sol6, Lemma 3.4] guarantees that the maps $\Sigma_{O,\mu} \to \mathbb{R}_{\geq 0}$ given by $q_\alpha$ and $q_{\alpha*}$ are $W(M,\mathcal{O})$-invariant.

Comparing (1.8), Condition 1.2 and (1.9), we see that $q_\alpha > 1$ for all $\alpha \in \Sigma_{O,\mu}$. In particular the zeros of $\mu^{M_\alpha}$ occur at

$$\{X_\alpha = 1\} = \{\sigma' \in \mathcal{O} : X_\alpha(\sigma') = 1\}$$

and sometimes at

$$\{X_\alpha = -1\} = \{\sigma' \in \mathcal{O} : X_\alpha(\sigma') = -1\}.$$
The calculations in [Sol6, §6–8] apply also to End_{G}^{p}(\Pi^{\varnothing}) and they imply, as in [Sol6, Corollary 9.4], an equivalence of categories

\[ \text{End}_{G}^{p}(\Pi^{\varnothing}) \cong \text{Mod}_{f} \leftrightarrow \mathcal{H}(O, G) - \text{Mod}_{f}. \]

(1.11)

Here \(-\text{Mod}_{f}\) denotes the category of finite length right modules. To go from \text{End}_{G}^{p}(\Pi^{\varnothing}) - \text{Mod}_{f} to \text{End}_{G}^{p}(\Pi^{\varnothing}) - \text{Mod}_{f} is basically an instance of Clifford theory for a finite group acting on an algebra. In reality it is more complicated [Sol6, §9], but still relatively easy. Consequently the essence of the representation theory of \text{End}_{G}^{p}(\Pi^{\varnothing}) (and thus of \text{Rep}(G)^{p}) is contained in the affine Hecke algebra \mathcal{H}(O, G).

Slightly better results can be obtained if we assume that the restriction of \((\sigma, E)\) to \(M_{1}\) decomposes without multiplicities bigger than one – which by [Roc1, Remark 1.6.1.3] holds for very large classes of reductive \(p\)-adic groups. Assuming it for \((\sigma, E)\), [Sol6, Theorem 10.9] says that there exist:

- a smaller progenerator \((\Pi^{\varnothing}) X_{u}(M, \sigma)\) of \text{Rep}(G)^{p},
- a Morita equivalent subalgebra \text{End}_{G}(\langle (\Pi^{\varnothing}) X_{u}(M, \sigma) \rangle) of \text{End}_{G}(\Pi^{\varnothing}),
- a subalgebra \text{End}_{G}^{p}(\langle (\Pi^{\varnothing}) X_{u}(M, \sigma) \rangle) of \text{End}_{G}(\langle (\Pi^{\varnothing}) X_{u}(M, \sigma) \rangle), which is canonically isomorphic with \mathcal{H}(O, G),
- elements \(J_{r} \in \text{End}_{G}(\langle (\Pi^{\varnothing}) X_{u}(M, \sigma) \rangle)^{X}\) for \(r \in R(O)\), such that

\[
\text{End}_{G}(\langle (\Pi^{\varnothing}) X_{u}(M, \sigma) \rangle) = \bigoplus_{r \in R(O)} \text{End}_{G}^{p}(\langle (\Pi^{\varnothing}) X_{u}(M, \sigma) \rangle) J_{r}.
\]

As announced in the introduction, we want to determine the parameters \(q_{\alpha}, q_{\alpha^{*}}\) for \(\alpha \in \Delta_{O, \mu}\), or equivalently the label functions \(\lambda, \lambda^{*} : \Sigma_{O, \mu} \to \mathbb{R}_{\geq 0}\) of \mathcal{H}(O, G).

When \(\Sigma_{O, \mu}\) is empty, \mathcal{H}(O, G) \cong \mathbb{C}[O] and it does not have parameters or labels. When \(\Sigma_{O, \mu} = \{\alpha, -\alpha\}\), it can already be quite difficult to identify \(q_{\alpha}\) and \(q_{\alpha^{*}}\). For instance, when \(G\) is split of type \(G_{2}\) and \(M\) has semisimple rank one, we did not manage to compute \(q_{\alpha}\) and \(q_{\alpha^{*}}\) for all supercuspidal representations of \(M\). (This was achieved recently in [AuXu].)

Yet, for \mathcal{H}(O, G) this is hardly troublesome. Namely, any affine Hecke algebra \mathcal{H} with \(\Sigma_{O, \mu} = \{\alpha, -\alpha\}\) and \(q_{\alpha}, q_{\alpha^{*}} \in \mathbb{C} \setminus \{0, -1\}\) can be analysed very well. Firstly, one can determine all its irreducible representations directly, as done in [Sol5, §2.2].

Secondly, with [Lus2] the representation theory of \mathcal{H} can be reduced to that of two graded Hecke algebras \(H_{k}\) with root system of rank \(\leq 1\). One of them has label \(k_{\alpha} = \log(q_{\alpha})/\log(q_{F})\) and underlying vector space \(T_{1}(O)\), the other has label \(k_{\alpha^{*}} = \log(q_{\alpha^{*}})/\log(q_{F})\) and underlying vector space \(T_{\chi^{-}}(O)\) (for some \(\chi^{-} \in O\) with \(X_{\alpha}(\chi^{-}) = -1\)).

For graded Hecke algebras with root system \(\{\alpha, -\alpha\}\) and a fixed underlying vector space, there are just two isomorphism classes: one with label \(k \neq 0\) and one with label \(k = 0\). For both there is a nice geometric construction of the irreducible representations of \(H_{k}\), see [Lus1] and [AMS2, Theorem 3.11]. This is an instance of a construction that underlies the representation theory of affine Hecke algebras associated to unipotent representations of \(p\)-adic groups [Lus3, Lus4]. Hence the analysis of \text{Irr}(\mathcal{H}(O, G)) with \(rk(\Sigma_{O, \mu}) \leq 1\) is very close to what is desired in Conjecture [A]. That already looks like a satisfactory answer in such cases.

To proceed, we recall Harish-Chandra’s construction of the function \(\mu^{M_{\sigma}}(\sigma \otimes \chi)\). Let \(\delta_{P} : P \to \mathbb{R}_{>0}\) be the modular function. We realize \(I_{P}^{G}(\sigma \otimes \chi, E)\) on the vector
space
\[ \{ f : G \to E \mid f \text{ is smooth, } \int_{umg} = \sigma(m)(\chi \delta^{1/2})(m)f(g) \forall u \in U, m \in M, g \in G \}, \]
with \( G \) acting by right translations. Let \( P' = MU' \) be another parabolic subgroup of \( G \) with Levi factor \( M \). Following [Wal] §IV.1 we consider the map
\[ (1.12) \quad J_{P'|P}(\sigma \otimes \chi) : I_P^G(\sigma \otimes \chi, E) \to I_{P'}^G(\sigma \otimes \chi, E) \]
\[ f \mapsto [g \mapsto \int_{(U \cap U') \setminus U'} f(u'g)du']. \]
Here \( du' \) denotes a quotient of Haar measures on \( U' \) and \( U \cap U' \). This integral converges for \( \chi \) in an open subset of \( X_{nr}(M) \) (independent of \( f \)). As such it defines a map
\[ X_{nr}(M) \times I_P^G(E) \to I_{P'}^G(E), \]
\[ (\chi, f) \mapsto J_{P'|P}(\sigma \otimes \chi)f, \]
which is rational in \( \chi \) and linear in \( f \) [Wal Théorème IV.1.1]. Moreover it intertwines the \( G \)-representation \( I_P^G(\sigma \otimes \chi) \) with \( I_{P'}^G(\sigma \otimes \chi) \) whenever it converges. Then
\[ J_{P'|P}(\sigma \otimes \chi)J_{P'|P}(\sigma \otimes \chi) \in \text{End}_G(I_P^G(\sigma \otimes \chi, E)) = \mathbb{C} \text{id}, \]
at least for \( \chi \) in a Zariski-open subset of \( X_{nr}(M) \). For any \( \alpha \in \Sigma_{\text{red}}(M) \) there exists by construction [Wal] §IV.3 a nonzero constant such that
\[ (1.13) \quad J_{M_{\alpha} \cap P|s_{\alpha}(M_{\alpha} \cap P)}(\sigma \otimes \chi)J_{s_{\alpha}(M_{\alpha} \cap P)|M_{\alpha} \cap P}(\sigma \otimes \chi) = \frac{\text{constant}}{\mu_{\alpha}(\sigma \otimes \chi)}, \]
as rational functions of \( \chi \in X_{nr}(M) \). We note that
\[ (U \cap s_{\alpha}(U)) \setminus s_{\alpha}(U) = U_{-\alpha} \quad \text{and} \quad (U \cap s_{\alpha}(U)) \setminus U = U_{\alpha}, \]
where \( U_{\pm\alpha} \) denotes a root subgroup with respect to \( A_M \). That allows us to simplify (1.13) to
\[ (1.14) \quad J_{s_{\alpha}(M_{\alpha} \cap P)|M_{\alpha} \cap P}(\sigma \otimes \chi)f = [g \mapsto \int_{U_{-\alpha}} f(u_+ g)du_+], \]
\[ J_{M_{\alpha} \cap P|s_{\alpha}(M_{\alpha} \cap P)}(\sigma \otimes \chi)f = [g \mapsto \int_{U_{\alpha}} f(u_+ g)du_+], \]
where \( du_{\pm} \) is a Haar measure on \( U_{\pm\alpha} \). The numbers \( q_{\alpha}, q_{\alpha}^{-1} \) (and \( q_{\alpha^*}, q_{\alpha^*}^{-1} \) when \( q_{\alpha^*} \neq 1 \)) are precisely the values of \( X_{\alpha}(\chi) = X_{\alpha}(\sigma \otimes \chi) \) at which \( \mu_{\alpha}(\sigma \otimes \chi) \) has a pole, and in view of (1.13) these are also given by the \( \chi \) for which
\[ J_{M_{\alpha} \cap P|s_{\alpha}(M_{\alpha} \cap P)}(\sigma \otimes \chi)J_{s_{\alpha}(M_{\alpha} \cap P)|M_{\alpha} \cap P}(\sigma \otimes \chi) = 0. \]
For other non-unitary \( \sigma \otimes \chi \in \mathcal{O} \) the operators (1.14) are invertible, and by the Langlands classification [Ren, Théorème VII.4.2] \( I_{P'|M_{\alpha}}^G(\sigma \otimes \chi) \) is irreducible.

**Corollary 1.3.** The poles of \( \mu_{\alpha} \) are precisely the non-unitary \( \sigma \otimes \chi \in \mathcal{O} \) for which \( I_{P'|M_{\alpha}}^G(\sigma \otimes \chi) \) is reducible.

### 2. Reduction to simply connected groups

In this section we reduce the analysis of the parameters of \( \mathcal{H}(\mathcal{O}, G) \) to the case where \( G \) is absolutely simple and simply connected. Consider a homomorphism between connected reductive \( F \)-groups \( \eta : \tilde{G} \to \tilde{G} \) such that:
- the kernel of \( d\eta : \text{Lie}(\tilde{G}) \to \text{Lie}(\tilde{G}) \) is central,
- the cokernel of \( \eta \) is a commutative \( F \)-group.
These properties imply \[\text{So13, Lemma 5.1}\] that on the derived groups \(\eta\) restricts to
\[\eta : \tilde{G} = \tilde{G}(F) \to G(F) = G\]
and a pullback functor \(\eta^* : \text{Rep}(G) \to \text{Rep}(\tilde{G})\).

**Lemma 2.1.** Let \(\pi \in \text{Irr}(G)\). Then \(\eta^*(\pi)\) is a finite direct sum of irreducible \(\tilde{G}\)-representations.

**Proof.** According to \[\text{Tad, Lemma 2.1}\] this holds for the inclusion of \(G_{\text{der}}\) in \(G\). Taking that into account, \[\text{Sil1}\] says that pullback along \(\eta_{\text{der}} : \tilde{G}_{\text{der}} \to G_{\text{der}}\) has the desired property. This shows that \(\text{Res}_{\tilde{G}}^G \eta^*(\pi)\) is a finite direct sum of irreducible \(\tilde{G}_{\text{der}}\)-representations. As in the proof of \[\text{Tad, Lemma 2.1}\], that implies the same property for \(\eta^*(\pi)\). \(\square\)

By \[\text{(2.1)}\], \(\eta\) induces a bijection
\[
\{\text{Levi subgroups of } G\} \to \{\text{Levi subgroups of } \tilde{G}\}.
\]
One also sees from \[\text{(2.1)}\] that \(\eta\) induces a bijection
\[
\Sigma(G, A_M) \to \Sigma(\tilde{G}, A_{\tilde{M}})
\]
for each \(\alpha \in \Sigma_{\text{red}}(A_M)\) this yields an isomorphism of \(F\)-groups
\[
\eta_{\alpha} : U_\alpha \to U_{\tilde{\alpha}}.
\]
This implies that \(\eta^*\) preserves cuspidality \[\text{Sil1 Lemma 1}\]. Further, pullback along \(\eta\) restricts to an algebraic group homomorphism \(\eta^* : X_{\text{nr}}(M) \to X_{\text{nr}}(\tilde{M})\).

**Proposition 2.2.** Let \((\sigma, E) \in \text{Irr}_{\text{cusp}}(M)\) and let \(\tilde{\sigma} \in \text{Irr}_{\text{cusp}}(\tilde{M})\) be a constituent of \(\eta^*(\sigma)\). For \(\alpha \in \Sigma_{\text{red}}(A_M)\) there exists \(\tilde{\alpha} \in \mathbb{C}^\times\) such that
\[
\mu^{\tilde{\alpha}}(\sigma \otimes \chi) = \tilde{c}_\alpha \mu^\alpha(\tilde{\sigma} \otimes \eta^*(\chi))
\]
as rational functions of \(\chi \in X_{\text{nr}}(M)\).

**Proof.** In view of the explicit shape \[\text{(1.8)}\], it suffices to show that the two rational functions have precisely the same poles. Using the relation \[\text{(1.13)}\], it suffices to show that
\[
J_{s_{\alpha}(M_{\alpha} \cap P)}(\sigma \otimes \chi)J_{s_{\alpha}(M_{\alpha} \cap P)}(\sigma \otimes \chi) = 0 \iff J_{\eta^{-1}(s_{\alpha}(M_{\alpha} \cap P))(\tilde{\sigma} \otimes \eta^*(\chi))}J_{\eta^{-1}(s_{\alpha}(M_{\alpha} \cap P))(\tilde{\sigma} \otimes \eta^*(\chi))} = 0.
\]
Since \(\eta_{\alpha} : U_\alpha \to U_{\tilde{\alpha}}\) is an isomorphism, \[\text{(1.14)}\] shows that the \(J\)-operators on both lines of \[\text{(2.2)}\] do the same thing, namely
\[
f \mapsto [g \mapsto \int_U f(ug) du],
\]
where \(U\) stands for \(U_\alpha\) or \(U_{-\alpha}\). The only real difference between the two lines of \[\text{(2.2)}\] lies in their domain. Since \(\tilde{\sigma} \otimes \eta^*(\chi)\) is a subrepresentation of \(\eta^*(\sigma \otimes \chi)\), it is clear that the implication \(\Rightarrow\) holds.
Conversely, suppose that the second line of (2.2) is 0, for a particular \( \chi \). Let \( \tilde{E} \subset E \) be the subspace on which \( \tilde{\sigma} \) is defined, so that \( I_{\eta-1}^G(\tilde{E}) \cong I_{\tilde{\eta}}^G(\tilde{E}) \) is the vector space underlying \( I_{\eta-1}^G(\tilde{\sigma} \otimes \eta^*(\chi)) \). It is a linear subspace of \( I_{\tilde{\eta}}^G(E) \), on which \( I_{\tilde{\eta}}^G(\sigma \otimes \chi) \) is defined. Then

\[
J_{M_a \cap P(s_{a}(M_a \cap P))}(\sigma \otimes \chi) \cdot J_{s_{a}(M_a \cap P) \cap M_a \cap P}(\sigma \otimes \chi)
\]

coincides on \( I_{\tilde{\eta}}^G(\tilde{E}) \) with

\[
J_{\eta^{-1}(s_{a}(M_a \cap P))}(\tilde{\sigma} \otimes \eta^*(\chi)) \cdot J_{\eta^{-1}(s_{a}(M_a \cap P)) \cap M_a \cap P}(\tilde{\sigma} \otimes \eta^*(\chi)),
\]

so annihilates \( I_{\tilde{\eta}}^G(\tilde{E}) \). But by (1.13), the operator (2.3) is a scalar on \( I_{\tilde{\eta}}^G(E) \), so it annihilates that entire space. \( \square \)

From Proposition 2.2 and (1.2) we deduce:

**Corollary 2.3.** In the setting of Proposition 2.2 write \( \hat{O} = X_{\mathfrak{m}}(\hat{M})\hat{\sigma} \). Then \( \Sigma_{\mathcal{O}, \mu} \) equals

\[
\eta^*(\Sigma_{\mathcal{O}, \mu}) = \{ \tilde{\alpha} = \alpha \circ \eta : \alpha \in \Sigma_{\mathcal{O}, \mu} \}.
\]

We warn that Proposition 2.2 and Corollary 2.3 do not imply that \( q_\alpha = q_{\tilde{\alpha}} \). The problem is that \( X_\alpha \) need not equal \( X_{\tilde{\alpha}} \). To make the relation precise, we have to consider \( h_\alpha^\vee, h_\alpha \) and their images (via \( H_M \) and \( H_{\hat{M}} \)) in \( \mathfrak{a}_M \) and \( \mathfrak{a}_{\hat{M}} \). Note that \( \eta \) induces a linear map \( \mathfrak{a}_\eta : \mathfrak{a}_{\hat{M}} \to \mathfrak{a}_M \). Further, it induces a group homomorphism

\[
\eta : (\tilde{M} \cap M_\lambda^1)/\tilde{M}^1 \to (M \cap M_\lambda^1)/M^1.
\]

Both the source and the target of (2.4) are isomorphic to \( \mathbb{Z} \), so the map is injective.

**Proposition 2.4.** (a) For \( \alpha \in \Sigma_{\mathcal{O}, \mu} \), there exists a \( N_\alpha \in \frac{1}{2}\mathbb{Z}_{>0} \) such that

\[
H_M(h_\alpha) = N_\alpha \mathfrak{a}_\eta(H_{\hat{M}}(h_{\tilde{\alpha}})).
\]

(b) If (2.4) is bijective, then \( N_\alpha \in \mathbb{Z}_{>0} \). This happens for instance when \( \eta \) restricts to a bijection between the almost direct \( F \)-simple factors of \( \hat{G} \) and \( G \) corresponding to \( a_B \) and \( a \).

(c) If \( \eta^*(\sigma) \) is irreducible, then \( N_\alpha \leq 1 \).

(d) Let \( \Sigma_{\mathcal{O}, j}^\vee \) be an irreducible component of \( \Sigma_{\mathcal{O}, j}^\vee \), and regard it as a subset of \( \mathfrak{a}_M \) via \( H_M \). Consider the irreducible component

\[
\Sigma_{\mathcal{O}, j}^\vee = \{ h_\alpha^\vee : h_\alpha \in \Sigma_{\mathcal{O}, j} \}
\]

of \( \Sigma_{\mathcal{O}, j}^\vee \). There are three possibilities:

(i) \( N_\alpha = 1 \) for all \( h_\alpha^\vee \in \Sigma_{\mathcal{O}, j}^\vee \).

(ii) \( \Sigma_{\mathcal{O}, j}^\vee \cong B_n, \Sigma_{\mathcal{O}, j}^\vee \cong C_n, N_\alpha = 1 \) for \( h_\alpha^\vee \in \Sigma_{\mathcal{O}, j}^\vee \) long and \( N_\beta = 1/2 \) for \( h_\beta^\vee \in \Sigma_{\mathcal{O}, j}^\vee \) short. Then

\[
q_\beta \ast = 1, q_\beta = q_{\beta^s} = q_{\beta^s}^{1/2}, \lambda^*(\beta) = 0 \text{ and } \lambda(\beta) = \lambda(\tilde{\beta}) = \lambda^*(\tilde{\beta}).
\]

(iii) \( \Sigma_{\mathcal{O}, j}^\vee \cong C_n, \Sigma_{\mathcal{O}, j}^\vee \cong B_n, N_\alpha = 1 \) for \( h_\alpha^\vee \in \Sigma_{\mathcal{O}, j}^\vee \) short and \( N_\beta = 2 \) for \( h_\beta^\vee \in \Sigma_{\mathcal{O}, j}^\vee \) long. Then

\[
q_\beta \ast = 1, q_\beta^2 = q_{\beta^s}^2 = q_\beta, \lambda^*(\tilde{\beta}) = 0 \text{ and } \lambda(\tilde{\beta}) = \lambda(\beta) = \lambda^*(\beta).
\]

(e) The modifications of the labels in part (d) preserve the class of labels in Table 1.
Proof. (a) It is clear from the constructions that $h^\vee_\alpha$ and $\eta(h^\vee_\alpha)$ lie in $(M \cap M^1_\alpha)/M^1$. The group $H_M(M \cap M^1_\alpha/M^1)$ contains the coroot $\alpha^\vee \in X_*(A_M)$ and is sometimes generated by $\alpha^\vee$ (e.g. when $M_\alpha \cong SL_2$). Let $T \supset A_M$ be a maximal $F$-split torus of $M_\alpha$. Then $\alpha^\vee$ can be extended to a cocharacter of $T$, and it becomes an element of the root datum for $(M_\alpha, T)$. For any $q \in \mathbb{Q}$ such that $q\alpha^\vee \in X_*(T)$, the number $2q = \langle \alpha, q\alpha^\vee \rangle$ is an integer and hence $q \in \frac{1}{2}\mathbb{Z}$. (It is possible that $\alpha^\vee/2$ belongs to $X_*(T)$, e.g. when $M_\alpha \cong PGL_2$.) Therefore either $\alpha^\vee$ or $\alpha^\vee/2$ is a generator of $H_M(M \cap M^1_\alpha/M^1)$. In particular $H_M((h^\vee_\alpha)^2) = 2H_M(h^\vee_\alpha)$ is always an integral multiple of $\alpha^\vee$.

Since $\eta_{\text{der}}$ is a central isogeny, $\alpha^\vee$ lies in the image of

$$X_*(\eta) : X_*(A_{\tilde{M}}) \to X_*(A_M).$$

Now we see that $(h^\vee_\alpha)^2$ lies in the image of (2.4). In particular $\eta^{-1}((h^\vee_\alpha)^2)$ is a well-defined element of $(\tilde{M} \cap \tilde{M}^1_\alpha)/\tilde{M}^1$.

By (2.1) $\eta$ induces an isomorphism between the respective adjoint groups. From $G \to G_{\text{ad}} \to G_\text{ad}$ we get an action of $G$ on $G_\text{ad}$ by “conjugation”. All the $\tilde{M}$-constituents of $\eta^*(\sigma)$ are associated (up to isomorphism) by elements of $M$. For $m \in M, \text{Ad}(m) : \tilde{M} \to \tilde{M}$ does not affect unramified characters of $\tilde{M}$. It follows that any $\chi \in X_{\text{nr}}(\tilde{M})$ which stabilizes $\tilde{\sigma}$, also stabilizes $\eta^*(\sigma)$. That implies

$$\eta^{-1}((M^2_\alpha \cap M^1_\alpha)/M^1) \subset (\tilde{M}^2_\alpha \cap \tilde{M}^1_\alpha)/\tilde{M}^1.$$ 

By definition $h^\vee_\alpha$ generates $(\tilde{M}^2_\alpha \cap \tilde{M}^1_\alpha)/\tilde{M}^1$, so $\eta^{-1}((h^\vee_\alpha)^2)$ is an integral multiple of $h^\vee_\alpha$. Applying $\eta$ and $H_M$, we find that $H_M(h^\vee_\alpha)$ is an integral multiple of $H_M(\eta(h^\vee_\alpha))/2$.

(b) When (2.4) is bijective, the argument for part (a) works without replacing $h^\vee_\alpha$ by $(h^\vee_\alpha)^2$, so in the conclusion we do not have to divide by 2 any more.

(c) If $\chi \in X_{\text{nr}}(M, \sigma)$, then $\eta^*(\sigma) \otimes \eta^*(\chi) = \eta^*(\sigma \otimes \chi)$ is isomorphic with $\eta^*(\sigma)$. Hence

$$\eta^*(X_{\text{nr}}(M, \sigma)) \subset X_{\text{nr}}(\tilde{M}, \eta^*(\sigma)),$$

which implies that $\eta(\tilde{M}^2_\sigma/(\tilde{M}^1_\sigma))/\tilde{M^1}$.

As $h^\vee_\alpha$ generates $(\tilde{M}^2_\alpha \cap \tilde{M}^1_\alpha)/\tilde{M}^1$ and $\eta(h^\vee_\alpha)$ lies in that group, $\eta(h^\vee_\alpha)$ is a multiple of $h^\vee_\alpha$.

(d) From part (a) we know that $X_{\text{nr}} = N_\alpha \eta(X_{\tilde{\alpha}})$. If $N_\alpha \notin \{1/2, 1, 2\}$ this gives a contradiction with the known zeros of $\mu$-functions, as follows.

When $N_\alpha \in \mathbb{Z} > 0$, Proposition 2.2 shows that $\mu^M_{\tilde{\alpha}}$ has a zero at every $\eta^*(\chi)$ for which $X_{\tilde{\alpha}}(\chi) = (N_\alpha X_{\tilde{\alpha}}(\eta^* \chi)\rangle$ equals 1. But by (1.8) these zeros $\tilde{\chi}$ must satisfy $X_{\tilde{\alpha}}(\tilde{\chi}) \in \{1, -1\}$. That forces $N_\alpha \in \{1, 2\}$.

In case $N_\alpha = 2$, $2\eta(X_{\tilde{\alpha}}) = X_{\tilde{\alpha}}$. Then Proposition 2.2 and (1.8) entail $q_\alpha = 1$ and $q_{\tilde{\alpha}} = q_{\tilde{\alpha}} = q_\alpha = q_\alpha = (1/2)$. Notice that this is only possible when $\Sigma^\vee_{\tilde{\alpha},j} \cong C_n$.

If $N_\alpha \in 1/2 + \mathbb{Z}$, then $2N_\alpha \in \mathbb{Z}$ and $2N_\alpha \eta(X_{\tilde{\alpha}}) = 2X_{\tilde{\alpha}}$. The function $\mu^M_{\tilde{\alpha}}$ has zeros at $\{2X_{\tilde{\alpha}}(\chi) = 1\}$, so that $\mu^M_{\tilde{\alpha}}$ has zeros at points where $X_{\tilde{\alpha}}$ is a $2N_\alpha$-th root of unity. That forces $2N_\alpha \in \{1, 2\}$, so $N_\alpha = 1/2$ and $\eta(X_{\tilde{\alpha}}) = 2X_{\tilde{\alpha}}$. For the same reasons as above, $q_\alpha = 1$ and $q_{\tilde{\alpha}} = q_{\tilde{\alpha}} = q_\alpha = q_\alpha = q_\alpha = q_\alpha = (1/2)$. By Sol6 Lemma 3.3 this is only possible if $\Sigma^\vee_{\tilde{\alpha},j}$ has type $B_n$.

(e) Parts (d,ii) and (d,iii) just switch the second line (with $\lambda^* = 0$) and the third line of Table 11. 

\[ \square \]
We remark that examples of case (ii) are easy to find, it already occurs for $SL_2(F) \to PGL_2(F)$ and the unramified principal series (as worked out in Paragraph 4.1). For an instance of case (iii) see Example 4.8.

We can apply Propositions 2.2 and 2.4 in particular with $\tilde{G}$ equal to the simply connected cover $G_{sc}$ of $G_{der}$, that yields:

**Corollary 2.5.** Suppose that Conjecture $A$ holds for $\tilde{G} = G_{sc}$ and $[M, \tilde{\sigma}]_{G_{sc}}$. Then it holds for $G$ and $[M, \sigma]_G$.

Every simply connected $F$-group is a direct product of $F$-simple simply connected groups, say

$$G_{sc} = \prod_i G_{sc}^{(i)}.$$ 

Everything described in Section 1 decomposes accordingly, for instance any $\tilde{\sigma} \in \text{Irr}_{cusp}(G_{sc})$ can be factorized as

$$\tilde{\sigma} = \bigotimes_i \sigma^{(i)} \ \text{with} \ \sigma^{(i)} \in \text{Irr}_{cusp}(G_{sc}^{(i)}).$$

For every $F$-simple simply connected $F$-group $G_{sc}^{(i)}$ there exists a finite separable field extension $F'/F$ and an absolutely simple, simply connected $F'$-group $G_{sc}^{(i)}$, such that $G_{sc}^{(i)}$ is the restriction of scalars from $F'$ to $F$ of $G_{sc}^{(i)}$. Then

$$G_{sc}^{(i)} = G_{sc}^{(i)}(F) = G_{sc}^{(i)}(F') = G_{sc}^{(i)},$$

so $\sigma^{(i)}$ can be regarded as a supercuspidal representation of $G_{sc}^{(i)}$. Of course that last step does not change the parameters $q_{\alpha}$ and $q_{\alpha^*}$ associated to $\sigma^{(i)}$. On the other hand, that step does replace $q_F$ by $q_{F'}$ and changes the labels $\lambda(\alpha)$ and $\lambda^*(\alpha)$ by a factor $\log(q_{F'})/\log(q_F)$. As this is the same scalar factor for all $\alpha \in \mathcal{S}_{\tilde{G}(i), \mu}$, it is innocent. With these steps we reduced the computation of the parameters $q_{\alpha}, q_{\alpha^*}, \lambda(\alpha)$ and $\lambda^*(\alpha)$ to the case where $\tilde{G}$ is absolutely simple and simply connected.

Sometime it is more convenient to study, instead of a simply connected simple group, a reductive group with that as derived group. For instance, the groups $GL_n, U_n, GSpin_n$ are often easier than, respectively, $SL_n, SU_n, Spin_n$. In such situations, the following result comes in handy.

**Proposition 2.6.** [Tad] Propositions 2.2 and 2.7

Suppose that $\tilde{G}$ is a connected reductive $F$-subgroup of $G$ that contains $G_{der}$. For every $\tilde{\pi} \in \text{Irr}(\tilde{G})$ there exists a $\pi \in \text{Irr}(G)$ such that $\text{Res}_{G}^{\tilde{G}}(\pi)$ contains $\tilde{\pi}$. Moreover $\tilde{\pi}$ is supercuspidal if and only if $\pi$ is supercuspidal.

We note that in this setting the inclusion $\iota : \tilde{G} \to G$ satisfies the conditions stated at the start of the paragraph.

**Corollary 2.7.** Let $G, \tilde{G}$ be as in Proposition 2.6. Then Conjecture $A$ holds for $G$ if and only if it holds for $\tilde{G}$.

**Proof.** Let $M$ be a Levi subgroup of $G$ and let $\tilde{\pi} \in \text{Irr}_{cusp}(M \cap \tilde{G})$. An appropriate $\pi$ is obtained from Proposition 2.6 applied to $\iota : M \cap \tilde{G} \to M$. Then $\tilde{\pi}$ is a constituent of $\iota^*(\pi)$. This also works the other way round: if we start with $\pi \in \text{Irr}_{cusp}(M)$ we can choose as $\tilde{\pi}$ any constituent of $\iota^*(\pi)$. Now we can apply Proposition 2.4 which says that the Hecke algebras $\mathcal{H}(X_{nr}(M \cap \tilde{G})\tilde{\pi}, \tilde{G})$ and $\mathcal{H}(X_{nr}(M)\pi, G)$ have root systems and parameters related as in cases (i) or (iii) of Proposition 2.4.d. □
3. Reduction to characteristic zero

For several classes of reductive groups, stronger results are available over $p$-adic fields than over local function fields. With the method of close local fields [Kaz Gan2], will show that all relevant results about affine Hecke algebras associated to Bernstein components can be transferred from characteristic zero to positive characteristic.

We start with an arbitrary local field of characteristic $p$. Choose a $p$-adic field $\tilde{F}$ which is $\ell$-close to $F$, that is

\[ \sigma_F/\varpi_F^\ell \sigma_F \cong \sigma_{\tilde{F}}/\varpi_{\tilde{F}}^\ell \sigma_{\tilde{F}} \quad \text{as rings.} \]

As remarked in [Del], such a field $\tilde{F}$ exists for every given $\ell \in \mathbb{Z}_{>0}$. If (3.1) holds, then it is also valid for every $m < \ell$, and in particular the residue fields $\sigma_F/\varpi_F^m \sigma_F$ and $\sigma_{\tilde{F}}/\varpi_{\tilde{F}}^m \sigma_{\tilde{F}}$ are isomorphic. We note that

\[ F^\times/(1+\varpi_F^m \sigma_F) \cong Z \times \sigma_F^\times/(1+\varpi_F^m \sigma_F) \cong Z \times \sigma_{\tilde{F}}^\times/(1+\varpi_{\tilde{F}}^m \sigma_{\tilde{F}}) = \tilde{F}^\times/(1+\varpi_{\tilde{F}}^m \sigma_{\tilde{F}}). \]

Let $I_{\tilde{F}}^\ell$ be the $\ell$-th ramification subgroup of $\text{Gal}(F_s/F)$. By [Del] (3.5.1) there is a group isomorphism (unique up to conjugation)

\[ \text{Gal}(F_s/F)/I_{\tilde{F}}^\ell \cong \text{Gal}((\tilde{F}_s/\tilde{F})/I_{\tilde{F}}^\ell), \]

and similarly with Weil groups. According to [Del] Proposition 3.6.1, for $m < \ell$ this isomorphism is compatible with the Artin reciprocity map

\[ W_F/I_{\tilde{F}}^\ell \rightarrow F^\times/(1+\varpi_F^m \sigma_F). \]

Let $G$ be a connected reductive $F$-group. We want to exhibit “the same” group over a $p$-adic field. The quasi-split inner form $G^*$ of $G$ is determined by the action of $\text{Gal}(F_s/F)$ on the based absolute root datum of $G$. That action factors through a finite quotient of $\text{Gal}(F_s/F)$, so there exists a $\ell \in \mathbb{Z}_{>0}$ such that $I_{\tilde{F}}^\ell$ acts trivially. The group $G$ is an inner twist of $G^*$, and the inner twists of $G^*$ are parametrized naturally by

\[ (3.4) \quad H^1(F, G^*_\text{ad}) \cong \text{Irr}(Z((G^*_\text{sc}) W_F)). \]

Now we pick a $p$-adic field $\tilde{F}$ which is $\ell$-close to $F$, and we define $\hat{G}$ to be the quasi-split $\tilde{F}$-group with the same based root absolute root datum as $G^*$ and Galois action transferred from that of $G^*$ via (3.3). Then $G^*$ and $\hat{G}$ have the same Langlands dual group (in a form where $I_{\tilde{F}}^\ell$ has been divided out) and hence

\[ (3.5) \quad Z((\hat{G}^*_\text{sc}) W_F) \cong Z((G^*_\text{sc}) W_F). \]

We define $\hat{G}$ to be the inner twist of $G^*$ parametrized by the character of $Z((\hat{G}^*_\text{sc}) W_F)$ that is transformed by (3.5) into the character of $Z((G^*_\text{sc}) W_F)$ that parametrizes $G$.

The following descriptions are based on recent work of Ganapathy [Gan1] Gan2]. It applies when $F$ and $\tilde{F}$ are $\ell$-close with $\ell$ large enough. The relation between $G$ and $\hat{G}$ is the same as in these papers, although over there it is reached in a slightly different way, without (3.4). Let $T \subset G$ be the maximal $F$-torus from which the root datum is built, and let $S \subset T$ be the maximal $F$-split subtorus. In the Bruhat–Tits building $B(G, F)$, $S = S(F)$ determines an apartment $\mathcal{A}_S$.

The same constructions can be performed for $G$. Then $X_s(S) \cong X_s(\hat{S})$ extends to an isomorphism of polysimplicial complexes $\mathcal{A}_S \cong \mathcal{A}_{\hat{S}}$. For every special vertex $x \in \mathcal{A}_S$, we get a special vertex $\hat{x} \in \mathcal{A}_{\hat{S}}$. For every $m \in \mathbb{Z}_{\geq 0}$, there is a refined version of
the Moy–Prasad group $G_{x,m}$, see [Gan1]. It is a compact open normal subgroup of $G_x$, the $G$-stabilizer of $x$. More precisely, there is an $\mathfrak{o}_F$-group scheme $G_x$ (a slightly improved version of the parahoric group schemes constructed in [BrTi]), such that 

$$G_{x,m} = G_x(\varpi^m F) \quad \forall m \in \mathbb{Z}_{\geq 0}.$$ 

By construction [Gan1, §2.D.3], $G_{x,m}$ is totally decomposed in the sense of [Bus, §1]. This means that, for any ordering of the root system $\Sigma(G,S)$, the product map

$$(G_{x,m} \cap Z_G(S)) \times \prod_{\alpha \in \Sigma(G,S)} (G_{x,m} \cap U_\alpha) \rightarrow G_{x,m}$$

is a bijection. Here $U_\alpha$ is the root subgroup of $G$ with respect to $\alpha \in \Sigma(G,S)$ (to be distinguished from the earlier $U_\alpha$ when $M$ is not a minimal Levi subgroup of $G$).

All the above applies to $\tilde{G}$ as well. The following results generalize [Kaz] to non-split groups.

**Theorem 3.1.** [Gan1, Corollary 6.3] Fix $m \in \mathbb{Z}_{>0}$ and let $\ell \in \mathbb{Z}_{>0}$ be large enough. The isomorphisms (3.1) induce an isomorphism of group schemes

$$G_x \times \mathfrak{o}_F \mathfrak{o}_F F / \varpi^m F \cong \tilde{G}_x \times \mathfrak{o}_F \mathfrak{o}_F F / \varpi^m F$$

and group isomorphisms

$$G_{x,0}/G_{x,m} = G_x(\mathfrak{o}_F / \varpi^m F) = \tilde{G}_{x,0}/\tilde{G}_{x,m}.$$ 

We endow $G$ with the Haar measure that gives the parahoric subgroup $G_{x,0}$ volume 1. The vector space $C_c(G_{x,m} \setminus G/G_{x,m})$ with the convolution product is an associative algebra, denoted $\mathcal{H}(G,G_{x,m})$.

**Theorem 3.2.** [Gan2, Theorem 4.1] Fix $m \in \mathbb{Z}_{>0}$ and let $\ell \in \mathbb{Z}_{>0}$ be large enough. The isomorphisms from Theorem 3.1 and the Cartan decomposition give rise to a bijection

$$\zeta_m : G_{x,m} \setminus G/G_{x,m} \rightarrow \tilde{G}_{x,m} \setminus \tilde{G}_{x,m}.$$ 

This map extends to an algebra isomorphism

$$\zeta^{G}_m : \mathcal{H}(G,G_{x,m}) \rightarrow \mathcal{H}(\tilde{G},\tilde{G}_{x,m}).$$

In particular $\zeta_m$ induces a group isomorphism $G/G^1 \rightarrow \tilde{G}/\tilde{G}^1$, and hence a group isomorphism

(3.6) $$\overline{\zeta}_m : X_m(G) = \text{Irr}(G/G^1) \rightarrow \text{Irr}(\tilde{G}/\tilde{G}^1) = \text{Irr}(\tilde{G}).$$ 

Let $\text{Rep}(G,G_{x,m})$ be the category of smooth $G$-representations that are generated by their $G_{x,m}$-fixed vectors. Recall that $G_{x,m}$ is a totally decomposed open normal subgroup of the good maximal compact subgroup $G_x$ of $G$. From [BeDe, §3.7–3.9] we know that there is an equivalence of categories

(3.7) $$\text{Rep}(G,G_{x,m}) \rightarrow \text{Mod}(\mathcal{H}(G,G_{x,m})).$$

From that and Theorem 3.2 we obtain equivalences of categories

(3.8) $$\overline{\zeta}_m^* : \text{Mod}(\mathcal{H}(G,G_{x,m})) \rightarrow \text{Mod}(\mathcal{H}(\tilde{G},\tilde{G}_{x,m})).$$

$$\zeta^{G}_m : \text{Rep}(G,G_{x,m}) \rightarrow \text{Rep}(\tilde{G},\tilde{G}_{x,m}).$$ 

We will need many properties of these equivalences, starting with two easy ones about characters.
Lemma 3.3. (a) Via (3.6), the equivalence of categories $\xi_m$ preserves twists by unramified characters.
(b) $\xi_m$ induces an isomorphism
$$A_G/(A_G \cap G_{x,m}) \to A_{\tilde{G}}/(A_{\tilde{G}} \cap \tilde{G}_{x,m}).$$

The effect of $\xi_m$ on $A_G$-characters of $G$-representations is push-forward along this isomorphism.

Proof. (a) This is clear from (3.6).
(b) Notice that
$$\xi_m: A_G \to A_{\tilde{G}}$$
and similarly for $\tilde{G}$. Since $\xi_m$ comes from the isomorphism $X_*(\tilde{G}) \cong X_*(\tilde{G})$, it induces a linear bijection $X_*(A_G) \to X_*(A_{\tilde{G}})$, and hence an isomorphism from (3.9) to its counterpart for $\tilde{G}$. The $A_G$-characters of representations in $\text{Rep}(G, G_{x,m})$ are precisely the characters of (3.9), and $\xi_m$ pushes them forward along $\xi_m$. □

Let $P$ be a parabolic subgroup of $G$ with a Levi factor $M$, which contains $S$. By [Bus] §1.6 the normalized parabolic functor $I^P_P$ sends $\text{Rep}(M, M_{x,m})$ to $\text{Rep}(G, G_{x,m})$. We will exploit an expression for this functor [Bus] in terms that can be transferred to $\tilde{G}$ with Theorems 3.1 and 3.2.

Let $P$ be the parabolic subgroup of $G$ that is opposite to $P$ with respect to $M$. Let $M_{x,m} = G_{x,m} \cap M$ be the version of $G_{x,m}$ for $M$. Recall that an element $g \in M$ is called $(P, G_{x,m})$-positive if
$$g(G_{x,m} \cap P)g^{-1} \subset G_{x,m} \cap P \quad \text{and} \quad g(G_{x,m} \cap P_{\text{op}})g^{-1} \supset G_{x,m} \cap P_{\text{op}}.$$ Let $\mathcal{H}^+(M, M_{x,m})$ be the subalgebra of $\mathcal{H}(M, M_{x,m})$ consisting of functions that are supported on $(P, G_{x,m})$-positive elements. In [Bus] §3.3, which is based on [BuKn2], a canonical injective algebra homomorphism
$$j_P: \mathcal{H}^+(M, M_{x,m}) \to \mathcal{H}(G, G_{x,m})$$
is given. Let $\tilde{P}$ and $\tilde{M}$ be the subgroups of $\tilde{G}$ corresponding to $P$ and $M$ via the equality of based root data. All the above constructions also work in $\tilde{G}$, and we endow the resulting objects with tildes.

Lemma 3.4. (a) $\xi_m : \mathcal{H}_m$ restricts to an algebra isomorphism from $\mathcal{H}^+(M, M_{x,m})$ to $\mathcal{H}^+(\tilde{M}, \tilde{M}_{x,m})$.
(b) The following diagram commutes:
$$\begin{array}{ccc}
\mathcal{H}(G, G_{x,m}) & \xrightarrow{\xi_m} & \mathcal{H}(\tilde{G}, \tilde{G}_{x,m}) \\
\uparrow j_P & & \uparrow j_P \\
\mathcal{H}^+(M, M_{x,m}) & \xrightarrow{\xi_m} & \mathcal{H}^+(\tilde{M}, \tilde{M}_{x,m})
\end{array}$$

Proof. (a) The property ”$(P, G_{x,m})$-positive” can be expressed in terms of the Cartan decomposition of $M$. Namely, the elements of a double coset $M_{x,0}gM_{x,0}$ with $g \in Z_M(S)$ are $(P, G_{x,m})$-positive if and only if
$$|\alpha(g)|_F \leq 1 \quad \text{for all } \alpha \in \Sigma(G, S) \text{ that appear in } \text{Lie}(P).$$
(Notice that $|\alpha|_F$ extends naturally to a character of $Z_G(S)$ because $S$ is cocompact in $Z_G(S)$.) The map $\xi_m$ from Theorem 3.2 for $M$ preserve the property (3.10),

...
because it comes from the isomorphism $X_\ast(S) \cong X_\ast(\tilde{S})$, which preserves positivity of roots. Thus $\zeta_m$ maps $(P, G_{x,m})$-positive elements to $(\tilde{P}, G_{\tilde{x},m})$-positive elements, and then Theorem 3.2 provides the desired isomorphism.

(b) We endow $M$ (resp. $\tilde{M}$) with the Haar measure that gives $M_{x,0}$ (resp. $\tilde{M}_{\tilde{x},0}$) volume 1. Suppose that $f \in \mathcal{H}^+(M, M_{x,m})$ has support $M_{x,m}gM_{x,m}$ with $g \in M$. The map $j_P$ is characterized by: $j_P f$ has support $G_{x,m}gG_{x,m}$ and

(3.11) \[ j_P f(g) = f(g)\delta_P(g)\mu_M(M_{x,m})\mu_G(G_{x,m})^{-1}. \]

By Theorem 3.1

\[ \mu_G(G_{x,m}) = [G_{x,0} : G_{x,m}]^{-1} = [\tilde{G}_{\tilde{x},0} : \tilde{G}_{\tilde{x},m}]^{-1} = \mu_G(\tilde{G}_{\tilde{x},m}), \]

and similarly for $\mu_M(M_{x,m})$. It is well-known that $\delta_P(g)$ is the product, over all $\alpha \in \Sigma(G, S)$ that appear in $\text{Lie}(P)$, of the factors $|\alpha(g)|^\text{dim}U_\alpha/2\alpha$. The root subgroup $U_\alpha$ contains the root subgroup $U_{2\alpha}$ if $2\alpha$ is also a root, and otherwise $U_{2\alpha} = \{1\}$ by definition. See [Ren, Lemme V.5.4] for a proof (although there a different convention is used, which results in replacing $g$ by $g^{-1}$). By Theorem 3.1 $\text{dim} U_\alpha$ equals $\dim U_{\tilde{\alpha}}$, where $\tilde{\alpha} \in \Sigma(\tilde{G}, \tilde{S})$ corresponds to $\alpha$. Furthermore $\delta_P$ is trivial on compact subgroups, so $\delta_P(g)$ depends only on $M_{x,m}gM_{x,m}$. It follows that

(3.12) \[ \delta_P(M_{x,m}gM_{x,m}) = \delta_P(\zeta_m(M_{x,m}gM_{x,m})). \]

Knowing that, we take another look at (3.11) and we see that $\zeta_m^G \circ j_P = j_P \circ \zeta_m^M$. □

Let $I_{P,m} : \text{Mod}(\mathcal{H}(M, M_{x,m})) \to \text{Mod}(\mathcal{H}(G, G_{x,m}))$ be the composition of $\text{Res}_{H^+(M, M_{x,m})}^{H(M, M_{x,m})}$ and $\text{Mod}(\mathcal{H}^+(M, M_{x,m})) \to \text{Mod}(\mathcal{H}(G, G_{x,m}))$, where $\mathcal{H}(G, G_{x,m})$ is regarded as a left $\mathcal{H}^+(M, M_{x,m})$-module via $j_P$.

**Theorem 3.5.** (a) The equivalences of categories (3.8) are compatible with normalized parabolic induction, in the sense that the following diagram commutes:

\[
\begin{array}{ccc}
\text{Rep}(G, G_{x,m}) & \xrightarrow{\zeta_m^G} & \text{Rep}(\tilde{G}, \tilde{G}_{\tilde{x},m}) \\
\uparrow I_P^G & & \uparrow I_P^{\tilde{G}} \\
\text{Rep}(M, M_{x,m}) & \xrightarrow{\zeta_m^M} & \text{Rep}(\tilde{M}, \tilde{M}_{\tilde{x},m})
\end{array}
\]

(b) The equivalences of categories (3.8) are compatible with normalized Jacquet restriction, in the sense that the following diagram commutes:

\[
\begin{array}{ccc}
\text{Rep}(G, G_{x,m}) & \xrightarrow{\zeta_m^G} & \text{Rep}(\tilde{G}, \tilde{G}_{\tilde{x},m}) \\
\downarrow J_P^G & & \downarrow J_P^{\tilde{G}} \\
\text{Rep}(M, M_{x,m}) & \xrightarrow{\zeta_m^M} & \text{Rep}(\tilde{M}, \tilde{M}_{\tilde{x},m})
\end{array}
\]

(c) $\zeta_m^G$ and its inverse send supercuspidal representations to supercuspidal representation. The same holds for unitary supercuspidal representations.

(d) $\zeta_m^G$ and its inverse preserve temperedness and essential square-integrability.
Proof. (a) Lemma 3.4 ensures that the diagram
\[
\begin{array}{ccc}
\text{Mod}(\mathcal{H}(G, G_{x,m})) & \xrightarrow{(\zeta_G^m)_*} & \text{Mod}(\mathcal{H}(\tilde{G}, \tilde{G}_{x,m})) \\
\uparrow I_{P,m} & & \uparrow I_{\tilde{P},m} \\
\text{Mod}(\mathcal{H}(M, M_{x,m})) & \xrightarrow{(\zeta_M^m)_*} & \text{Mod}(\mathcal{H}(\tilde{M}, \tilde{M}_{x,m}))
\end{array}
\]
commutes. According to [Bus, §4.1], the unnormalized parabolic induction functor \(\text{Ind}_{G}^{P}\) fits in a commutative diagram
\[
\begin{array}{ccc}
\text{Rep}(G, G_{x,m}) & \xrightarrow{\zeta_G^m} & \text{Mod}(\mathcal{H}(G, G_{x,m})) \\
\uparrow \text{Ind}_{G}^{P} & & \uparrow I_{P,m} \\
\text{Rep}(M, M_{x,m}) & \xrightarrow{\zeta_M^m} & \text{Mod}(\mathcal{H}(M, M_{x,m}))
\end{array}
\]
where the horizontal arrows are the equivalences of categories from (3.7). Of course the same holds for \(\tilde{G}\). These two commutative diagrams entail that \(\zeta_G^m \circ \text{Ind}_{G}^{P} = \text{Ind}_{\tilde{G}}^{\tilde{P}} \circ \zeta_M^m\).

In view of (3.12), if we twist this equality on the left hand side by \(\delta_{P}^{1/2}\) and on the right hand side by \(\delta_{\tilde{P}}^{1/2}\), it remains valid. That yields exactly the desired relation with normalized parabolic induction.

(b) By Frobenius reciprocity \(J_{G}^{P}\) is left adjoint to \(I_{G}^{P}\), so by part (a) \(\overline{\zeta}_M^m \circ J_{G}^{P} \circ \overline{\zeta}_G^m^{-1}\) is left adjoint to \(I_{\tilde{P}}^{\tilde{G}}\). Now we use the uniqueness of adjoints.

(c) The first claim follows from part (a), or alternatively from part (b). For the second claim, we note that a supercuspidal \(G\)-representation is unitary if and only if its central character is unitary. As \(A_G\) is cocompact in \(Z(G)\), that is equivalent to: the \(A_G\)-character is unitary. By Lemma 3.3.b, \(\overline{\zeta}_G^m\) preserve the latter property.

(d) For the property ”square integrable modulo centre” one can follow the proof of [Badu, Théorème 2.17.b], reformulated in the setting of [Gan2]. Combining that with Lemma 3.3.a, we find that \(\overline{\zeta}_G^m\) also preserves essential square-integrability.

By [Wall, Proposition III.4.1], every irreducible tempered representation \(\tau \in \text{Rep}(G, G_{x,m})\) is a direct summand of a completely reducible representation of the form \(I_{P}^{G}(\pi)\), where \(\pi \in \text{Rep}(M, M_{x,m})\) is square-integrable modulo centre. By the above and part (a),

\[
(3.13) \quad \overline{\zeta}_m^G(I_{P}^{G}(\pi)) \cong I_{\tilde{P}}^{\tilde{G}}(\overline{\zeta}_m^M(\pi))
\]
is also a direct sum of irreducible tempered representations. As \(\overline{\zeta}_m^G(\tau)\) is a direct summand of (3.13), it is tempered. \(\square\)

Consider an inertial equivalence class \(s = [M, \sigma]_G\), where \(S \subset M\). Choose \(m \in \mathbb{Z}_{>0}\) such that \(\text{Rep}(G)^s \subset \text{Rep}(G, G_{x,m})\), and similarly for all Levi subgroups of \(G\) containing \(M\). This is easy for supercuspidal Bernstein components and possible in general because parabolic induction preserves depths [MoPr, Theorem 5.2]. Fix \(\ell \in \mathbb{Z}_{>m}\) so that Theorems 3.1, 3.2 and 3.5 apply. We may and will assume that \(\sigma\) fulfills Condition 1.2. By Theorem 3.5 d the \(M\)-representation \(\tilde{\sigma} = \overline{\zeta}_m^M(\sigma)\) is unitary and supercuspidal. We write \(\tilde{s}, \tilde{O}\) etc. for objects constructed from \(\tilde{\sigma}\).

**Proposition 3.6.** (a) The bijection \(\Sigma(G, A_M) \to \Sigma(\tilde{G}, A_{\tilde{M}})\), induced by the equality of the root data of \(G\) and \(\tilde{G}\), sends \(\Sigma_{O,\mu}\) onto \(\Sigma_{\tilde{O},\mu}\).
(b) Let $\alpha \in \Sigma_{\mathcal{O},\mu}$ with image $\tilde{\alpha} \in \Sigma_{\widetilde{\mathcal{O}},\mu}$. The pullback of $X_{\tilde{\alpha}}$ along (3.6) is $X_{\alpha}$ and $q_{\alpha} = q_{\tilde{\alpha}}, q_{\alpha^*} = q_{\tilde{\alpha}^*}$.

Proof. Let $\alpha \in \Sigma_{\text{red}}(A_I)$, with image $\tilde{\alpha} \in \Sigma_{\text{red}}(A_{\tilde{I}})$. The groups $\mathcal{M}_\alpha$ and $\tilde{\mathcal{M}}_{\tilde{\alpha}}$ correspond via the equality of root data of $\mathcal{G}$ and $\tilde{\mathcal{G}}$. For $\chi \in X_{\text{nr}}(M_\alpha)$, Theorem 3.5 a implies that
\[ \zeta_{\mathcal{M}}(\sigma \circ \chi) = \tilde{\sigma} \otimes \zeta_{\tilde{\mathcal{M}}}(\chi). \]

By (3.8), $I_{\mathcal{P} \cap \mathcal{M}_\alpha}(\sigma \circ \chi)$ is reducible if and only if $I_{\tilde{\mathcal{P}} \cap \tilde{\mathcal{M}}_{\tilde{\alpha}}}(\tilde{\sigma} \otimes \zeta_{\tilde{\mathcal{M}}}(\chi))$ is reducible. If $\alpha \notin \Sigma_{\mathcal{O},\mu}$, then $I_{\mathcal{P} \cap \mathcal{M}_\alpha}(\sigma \circ \chi)$ is irreducible for all non-unitary $\chi \in X_{\text{nr}}(M_\alpha)$. It follows that $I_{\mathcal{P} \cap \tilde{\mathcal{M}}_{\tilde{\alpha}}}(\tilde{\sigma} \circ \tilde{\chi})$ is irreducible for all non-unitary $\tilde{\chi} \in X_{\text{nr}}(\tilde{M}_{\tilde{\alpha}})$, and by Corollary 1.3 $\tilde{\alpha} \notin \Sigma_{\tilde{\mathcal{O}},\mu}$.

On the other hand, suppose that $\alpha \in \Sigma_{\mathcal{O},\mu}$. Then $I_{\mathcal{P} \cap \mathcal{M}_\alpha}(\sigma \circ \chi)$ is reducible for a $\chi \in X_{\text{nr}}(M_\alpha)$ with $X_\alpha(\chi) = q_\alpha > 1$. It is clear from the construction of $X_{\tilde{\alpha}}$ in (3.6) that $X_{\tilde{\alpha}} \circ \zeta_{\mathcal{M}}$ is a multiple of $X_\alpha$. Consequently $I_{\tilde{\mathcal{P}} \cap \tilde{\mathcal{M}}_{\tilde{\alpha}}}(\tilde{\sigma} \otimes \zeta_{\tilde{\mathcal{M}}}(\chi))$ is reducible and $X_{\tilde{\alpha}}(\zeta_{\tilde{\mathcal{M}}}(\chi)) \in \mathbb{R}_{>0} \setminus \{1\}$. With Corollary 1.3 we conclude that $\tilde{\alpha} \in \Sigma_{\tilde{\mathcal{O}},\mu}$.

(b) By (3.8) and Theorem 3.2 the bijection $\zeta_m$ induces a bijection
\[ (M_2^2 \cap M_1^4)/M_1 \longrightarrow (\tilde{M}_2^2 \cap \tilde{M}_1^4)/\tilde{M}_1 \cong \mathbb{Z}. \]

The element $h_{\alpha}^\vee$ generates $(M_2^2 \cap M_1^4)/M_1$, while $h_{\tilde{\alpha}^\vee}$ generates $(\tilde{M}_2^2 \cap \tilde{M}_1^4)/\tilde{M}_1$. These generators are determined by conditions $\nu_F(\alpha(h_{\alpha}^\vee)) > 0$ and $\nu_F(\tilde{\alpha}(h_{\tilde{\alpha}}^\vee)) > 0$, respectively. As $\nu_F \circ \tilde{\alpha} \circ \zeta_m = \nu_F \circ \alpha$, we can conclude that
\[ \zeta_m(h_{\alpha}^\vee) = h_{\tilde{\alpha}^\vee} \quad \text{and} \quad X_\alpha = X_{\tilde{\alpha}} \circ \zeta_m^{M_\alpha}. \]

Then $X_{\tilde{\alpha}}(\zeta_{\mathcal{M}}(\chi)) = q_\alpha$ and $I_{\mathcal{P} \cap \mathcal{M}_\alpha}(\tilde{\sigma} \otimes \zeta_{\mathcal{M}}(\chi))$ is reducible, so $q_\alpha = q_{\tilde{\alpha}}$

If $q_{\alpha^*} > 1$, then $I_{\mathcal{P} \cap \mathcal{M}_\alpha}(\sigma \circ \chi')$ is reducible for a $\chi' \in X_{\text{nr}}(M_\alpha)$ with $X_\alpha(\chi') = -q_{\alpha^*}$. In that case $I_{\mathcal{P} \cap \tilde{\mathcal{M}}_{\tilde{\alpha}}}(\tilde{\sigma} \otimes \zeta_{\mathcal{M}}(\chi'))$ is also reducible and $X_{\tilde{\alpha}}(\zeta_{\mathcal{M}}(\chi')) = -q_{\alpha^*}$, so by Corollary 1.3 $q_{\tilde{\alpha}^*} = q_{\alpha^*}$. When $q_{\alpha^*} = 1$, $I_{\mathcal{P} \cap \mathcal{M}_\alpha}(\sigma \circ \chi')$ is irreducible for all $\chi' \in X_{\text{nr}}(M_\alpha)$ with $X_{\alpha'}(\chi') \in \mathbb{R}_{<1}$. That translates to $\tilde{M}_{\tilde{\alpha}}$, and then Corollary 1.3 implies that $q_{\tilde{\alpha}^*} = 1$.

We summarise the conclusions of this sections:

**Corollary 3.7.** Let $\text{Rep}(G)^{\mathfrak{g}}$ be an arbitrary Bernstein block for a connected reductive group $\mathcal{G}$ over a local function field $F$. There exist:

- a $p$-adic field $\tilde{F}$, sufficiently close to $F$,
- a connected reductive $\tilde{F}$-group $\tilde{\mathcal{G}}$ with the same based root datum as $\mathcal{G}$,
- a Bernstein block $\text{Rep}(\tilde{\mathcal{G}})^{\mathfrak{g}}$ for $\tilde{\mathcal{G}}$,

such that:

- $\text{Rep}(G)^{\mathfrak{g}}$ is equivalent with $\text{Rep}(\tilde{G})^{\mathfrak{g}}$,
- $\mathcal{H}(\mathcal{O}, G)$ is isomorphic with $\mathcal{H}(\tilde{\mathcal{O}}, \tilde{G})$,
- whenever $\alpha \in \Sigma_{\mathcal{O},\mu}$ and $\tilde{\alpha} \in \Sigma_{\tilde{\mathcal{O}},\mu}$ correspond (via Proposition 3.6), $\lambda(\alpha) = \lambda(\tilde{\alpha})$ and $\lambda^*(\alpha) = \lambda^*(\tilde{\alpha})$.

**Proof.** It only remains to establish the isomorphism of affine Hecke algebras. From (3.8) and Theorem 3.2 we get the bijection $M_2^2/M_1 \rightarrow \tilde{M}_2^2/M_1$. From (3.14) we
obtain the bijection $\Sigma(\mathcal{O}) \to \Sigma(\mathcal{O})$. Dualizing these two bijections, we obtain an isomorphism from the root datum underlying $\mathcal{H}(\mathcal{O}, G)$ to the root datum underlying $\mathcal{H}(\tilde{\mathcal{O}}, \tilde{G})$. It respects the bases because $\mathcal{G}$ and $\tilde{\mathcal{G}}$ have the same based root datum. By Proposition 3.6.b the parameters $q_\alpha, q_{\alpha^*}$ are the same on both sides. As $q_F = [\omega_F : \varpi_F \omega_F] = [\omega_F : \varpi_F \omega_F] = q_F$, also the label functions $\lambda, \lambda^*$ on both sides correspond via $\alpha \mapsto \tilde{\alpha}$. 

\[ \square \]

4. HECKE ALGEBRA PARAMETERS FOR SIMPLE GROUPS

4.1. Principal series of split groups.

The affine Hecke algebras for Bernstein blocks in the principal series of split groups were worked out in [Roc1], under some mild assumptions on the residual characteristic of $F$. In particular, for roots $\alpha \in \Sigma_{\mathcal{O}, \mu}$ one finds $q_\alpha = q_F$ and $q_{\alpha^*} = 1$. We will derive the same conclusion in a different way, which avoids any restrictions on the residual characteristic.

Let $\mathcal{G}$ be a split connected reductive $F$-group. We may assume that $\mathcal{G}$ is a Chevalley group, so defined over $\mathbb{Z}$. Let $T$ be a maximal $F$-split torus of $\mathcal{G}$ and write $T = T(F)$. We consider an inertial equivalence class $s = [T, \sigma]_G$, where $\sigma$ is a character of $T$ that fulfills Condition 1.2.

For $\alpha \in \Sigma(\mathcal{G}, T)$ the group $M_\alpha$ is generated by $T$ and the root subgroups $U_\alpha, U_{-\alpha}$. It has root system $\Sigma(M_\alpha, T) = \{\alpha, -\alpha\}$ and parabolic subgroups $P_\alpha = \langle T, U_\alpha \rangle$, $P_{-\alpha} = \langle T, U_{-\alpha} \rangle$. Let $u_\alpha : F \to U_\alpha$ and $u_{-\alpha} : F \to U_{-\alpha}$ be the coordinates coming from the Chevalley model.

We assume that $s_\alpha \cdot \sigma = \sigma$, a condition which by [1.5] is necessary for $\sigma \in \Sigma_{\mathcal{O}, \mu}$. Then $\sigma \circ \alpha^\vee = (\sigma \circ \alpha^\vee)^{-1}$, so $\sigma \circ \alpha^\vee$ has order $\leq 2$ in $\text{Irr}(F^\times)$. When the residual characteristic of $F$ is not 2, this implies that $\sigma \circ \alpha^\vee$ has depth zero.

We start the search for $q_\alpha$ with elements of $I_{P_\alpha}^{M_\alpha}(\sigma \otimes \chi)$ that are as close as possible to fixed by the Iwahori subgroup

$$I = u_\alpha(\varpi_F)T(\varpi_F)u_{-\alpha}(\varpi_F).$$

For $x \in F^\times$ we write

$$s_\alpha(x) = u_\alpha(-x^{-1})u_{-\alpha}(x)u_{\alpha}(-x^{-1}) \in N_{M_\alpha}(T).$$

It follows quickly from the Iwasawa decomposition of $M_\alpha$ that $M_\alpha = P_\alpha I \sqcup P_\alpha s_\alpha I$, where $s_\alpha = s_\alpha(1)$.

Consider the elements $f_1, f_s \in I_{P_\alpha}^{M_\alpha}(\sigma \otimes \chi)$ defined by

$$\text{supp}(f_1) = P_\alpha I \quad f_1(u_\alpha(x)tu_{-\alpha}(y)) = (\sigma \chi^{\frac{1}{2}})_{P_\alpha}(t) \quad x \in F, t \in T, y \in \varpi_F \omega_F,$$

$$\text{supp}(f_s) = P_\alpha s_\alpha I \quad f_s(u_\alpha(x)tu_{-\alpha}(y)s_\alpha) = (\sigma \chi^{\frac{1}{2}})_{P_\alpha}(t) \quad x \in F, t \in T, y \in \varpi_F.$$

We endow $F$ with the Haar measure that gives $\varpi_F$ volume 1. We compute

\begin{equation}
J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi)f_1(1_G) = \int_F f_1(u_{-\alpha}(x))dx = \text{vol}(\varpi_F \omega_F) = q_F^{-1},
\end{equation}
Although our computations for intertwining operators were certainly known already to a

As \(\chi^{1/2}_{\alpha} \) is unramified and \(\sigma \circ \alpha^\vee \) is quadratic,

If \((4.3)\) is nontrivial, then

In that case \(J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi)f_1(s_\alpha) = 0\). On the other hand, when \((4.3)\) is trivial:

Similar calculations show that

Case I: \(\sigma \circ \alpha^\vee \) is unramified

Here \(\text{Rep}(M_\alpha)^I\) is the Iwahori-spherical Bernstein block and \(J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi)\) restricts to a \(\mathcal{H}(M_\alpha, I)\)-homomorphism

Although our computations for intertwining operators were certainly known already in this case, we did not find them in the literature. The space \(I_{P_{-\alpha}|P_\alpha}^M(\sigma \otimes \chi)^I\) has a basis \(f'_1, f'_s\) where \(\text{supp}(f'_1) = P_{-\alpha}I\) and \(\text{supp}(f'_s) = P_{-\alpha}s_\alpha I\). Abbreviating \(z_\alpha = \)}
Similarly one checks that $J_{P_α|P_{-α}}(σ ⊗ χ)$ restricts to

$$
\begin{pmatrix}
1 & 1 - q_F^{-1} \\
1 - q_F^{-1} & 1 - z_α^{-1}
\end{pmatrix} : I_{P_α}^M(σ ⊗ χ)^I → I_{P_α}^M(σ ⊗ χ)^I.
$$

We find that $J_{P_α|P_{-α}}(σ ⊗ χ)J_{P_{-α}|P_α}(σ ⊗ χ)$ restricts to

$$
(4.6) \quad \left(q_F^{-1} + \frac{(1 - q_F^{-1})^2}{(1 - z_α)(1 - z_α^{-1})}\right) : I_{P_α}^M(σ ⊗ χ)^I → I_{P_α}^M(σ ⊗ χ)^I.
$$

We already know that $J_{P_α|P_{-α}}(σ ⊗ χ)J_{P_{-α}|P_α}(σ ⊗ χ)$ is a scalar multiple of the identity on $I_{P_α}^M(σ ⊗ χ)$, so (4.6) gives that scalar. We note that (4.6) has a pole at $z_α = 1$ and that (4.6) is zero if and only if $z_α = q_F$ or $z_α = q_F^{-1}$. As $σ$ is unitary and $χ \in \text{Hom}(M_α, ℜ_{>0})$, this is equivalent to

$$
(4.7) \quad σ \circ α^\vee = 1 \quad \text{and} \quad χ \circ α^\vee(ω_F) ∈ \{q_F, q_F^{-1}\}.
$$

Since $M_α^2 = T$, $h_α^\vee$ generates $T/T^1$. If $α^\vee(ω_F) = h_α^\vee$, (4.7) says that $q_α = q_F$ and $q_{α^*} = 1$. If $α^\vee(ω_F) = 2h_α^\vee$, then (4.7) means $q_α = q_F^2 = q_{α^*}$. But in that case we can also define $X_α(χ) = χ(α^\vee(ω_F))$ instead of $X_α(χ) = χ(h_α^\vee)$. These new $X_α$ also form a root system, which embeds naturally in $R(\mathcal{G}, \mathcal{T})^\vee$. From the presentation on page 10 one sees that this redefinition does not change the affine Hecke algebra. Hence we can achieve $q_α = q_F, q_{α^*} = 1$ in all these cases.

Case II: $σ \circ α^\vee$ is ramified
For $r ∈ ℤ_{>0}$, $M_α$ has compact open subgroups

$$
J_r = x_α(ω_F^r ϕ_F)\mathcal{T}(ω_F^r ϕ_F)x_{-α}(ω_F^r ϕ_F),
$$

$$
H_r = x_α(ω_F^{2r-1} ϕ_F)\mathcal{T}(ω_F^r ϕ_F)x_{-α}(ω_F ϕ_F).
$$

Here $\mathcal{T}(ω_F^r ϕ_F)$ is a shorthand for the kernel of $\mathcal{T}(ϕ_F) → \mathcal{T}(ϕ_F/ω_F^r ϕ_F)$.

Lemma 4.1. There exists $r ∈ ℤ_{>0}$ such that $\mathcal{T}(ω_F^r ϕ_F) ⊂ \ker(σ)$ and $\text{Rep}(M_α)^{s_r}$ is a direct factor of

$$
\text{Rep}(M_α, H_r) ∼= \text{Mod}(\mathcal{H}(M_α, H_r)).
$$

Proof. Choose an odd $r ∈ ℤ_{>0}$ such that $\mathcal{T}(ω_F^r ϕ_F) ⊂ \ker(σ)$ and $I_{P_α}^M(σ)_{J_r} ≠ 0$. Then $I_{P_α}^M(σ ⊗ χ)_{J_r} ≠ 0$ for any $χ ∈ \text{Xur}(T)$ because $J_r$ is compact. Hence

$$
\text{Rep}(M_α)^{s_r} ⊂ \text{Rep}(M_α, J_r).
$$

We note that $J_r$ is a normal subgroup of the hyperspecial parahoric subgroup $M_α(ϕ_F)$ of $M_α$. It is known from [BeDe] that $\text{Rep}(M_α, J_r)$ is a direct product of finitely many Bernstein blocks of $\text{Rep}(M_α)$, and that

$$
(4.8) \quad \text{Rep}(M_α, J_r) → \text{Mod}(\mathcal{H}(M_α, J_r)) : V ↦ V_{J_r}.
$$
is an equivalence of categories. Consider conjugation by $\alpha^\vee (\varpi_F^{(r-1)/2})$. This sends $J_r$ to $H_r$ and induces equivalences of categories

$$\text{Rep}(M_\alpha, J_r) \cong \text{Rep}(M_\alpha, H_r), \quad \text{Mod}(\mathcal{H}(M_\alpha, J_r)) \cong \text{Mod}(\mathcal{H}(M_\alpha, H_r)).$$

Lemma 4.1 tells us that most aspects of $I_{P_\alpha}^M(\sigma \otimes \chi)$ can already be detected on $I_{P_\alpha}^M(\sigma \otimes \chi)^{H_r}$.

**Lemma 4.2.** The double cosets in $P_\alpha \backslash M_\alpha / H_r$ can be represented by

$$\{1_G\} \cup \{(u_\alpha(z)s_\alpha : z \in \mathfrak{O}_F / \varpi_F^{2r-1} \mathfrak{O}_F\}.$$ 

Similarly $P_{-\alpha} \backslash M_\alpha / H_r$ can be represented by $\{s_\alpha\} \cup \{u_\alpha(\mathfrak{O}_F)/u_\alpha(\varpi_F^{2r-1} \mathfrak{O}_F)\}$.

**Proof.** From the Iwasawa decomposition $M_\alpha = P_\alpha M_\alpha(\mathfrak{O}_F)$ we get

$$P_\alpha \backslash M_\alpha / H_r \cong (P_\alpha \cap M_\alpha(\mathfrak{O}_F) \backslash M_\alpha(\mathfrak{O}_F))/H_r.$$

Recall that by the Bruhat decomposition of $M_\alpha(k_F)$:

$$M_\alpha(\mathfrak{O}_F) = I \sqcup Is_\alpha I = I \sqcup u_\alpha(\mathfrak{O}_F)T(\mathfrak{O}_F)u_\alpha(\varpi_F \mathfrak{O}_F)s_\alpha.$$ 

Furthermore, we note that $(P_\alpha \cap M_\alpha(\mathfrak{O}_F))H_r = I$ and

$$(P_\alpha \cap M_\alpha(\mathfrak{O}_F))u_\alpha(z)s_\alpha H_r = (P_\alpha \cap M_\alpha(\mathfrak{O}_F))u_\alpha(z + \varpi_F^{2r-1} \mathfrak{O}_F)s_\alpha, \quad z \in \mathfrak{O}_F.$$ 

In combination with (4.9) and (4.10) that yields the desired representatives for (4.9).

The representatives for the second double coset space are found in analogous fashion, now using

$$M_\alpha(\mathfrak{O}_F) = s_\alpha I \sqcup s_\alpha Is_\alpha I = u_\alpha(\mathfrak{O}_F)T(\mathfrak{O}_F)u_\alpha(\varpi_F \mathfrak{O}_F)s_\alpha \sqcup u_\alpha(\mathfrak{O}_F)T(\mathfrak{O}_F)u_\alpha(\mathfrak{O}_F)$$

instead of (4.10). □

It follows from Lemma 4.2 that $I_{P_\alpha}^M(\sigma \otimes \chi)^{H_r}$ has a basis $\{f_1\} \cup \{f_{zs} : z \in \mathfrak{O}_F / \varpi_F^{2r-1} \mathfrak{O}_F\}$. Here $\text{supp}(f_1) = P_\alpha H_r = P_\alpha I$ as before and

$$\text{supp}(f_{zs}) = P_\alpha u_\alpha(z)H_r = P_\alpha x_\alpha(z + \varpi_F^{2r-1} \mathfrak{O}_F)s_\alpha,$$

$$f_{zs}(u_\alpha(x)tu_\alpha(y)s_\alpha) = (\sigma \chi^\Delta_{P_\alpha}/2)(t), \quad x \in F, y \in z + \varpi_F^{2r-1} \mathfrak{O}_F, t \in T.$$ 

**Proposition 4.3.** Recall that $\sigma \circ \alpha^\vee$ is ramified and $s_\alpha \cdot \sigma = \sigma$.

(a) The functions $J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi)f_1$ and $J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi)f_{zs}$ (with $z \in \mathfrak{O}_F / \varpi_F^{2r-1} \mathfrak{O}_F$) of $\chi \in X_{\text{nr}}(M_\alpha)$ do not have any poles.

(b) $\alpha \notin \Sigma_{\text{nr}}$.

**Proof.** (a) Note that $J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi)$ preserves the $H_r$-invariance of an element $f$ of the given basis. By Lemma 4.2 it suffices to check the values of $J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi)f$ at $\{s_\alpha\} \cup u_\alpha(\mathfrak{O}_F)$. From the earlier computations (4.1) and (4.2) we know that $J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi)f_1$ does not have poles at $1_G$ or at $s_\alpha$. For $y \in \mathfrak{O}_F \setminus \varpi_F^{2r-1} \mathfrak{O}_F$ the multiplication rules in $SL_2(F)$ (which surjects on $M_{\alpha,\text{der}}$) enable us to compute

$$J_{P_{-\alpha}|P_\alpha}(\sigma \otimes \chi)f_1(u_\alpha(y)) = \int_F f_1(u_\alpha(x)u_\alpha(y))dx$$

$$= \int_F f_1(u_\alpha(\frac{y}{1 + xy})\alpha^\vee(\frac{1}{1 + xy})u_\alpha(\frac{x}{1 + xy}))dx$$

$$= \int_F (\sigma \chi^\Delta_{P_\alpha}/2)(\alpha^\vee(\frac{1}{1 + xy}))f_1(u_\alpha(\frac{x}{1 + xy}))dx.$$

(4.11)
In terms of the new variable \(x' := 1 + xy\) this becomes
\[
|y|^{-1} \int_F (\sigma^1(\delta P_\alpha)(\alpha^\vee(x')^{-1}) f_1(u_{-\alpha}(-x' \frac{1}{y})) \, dx'.
\]
The integrand is nonzero if and only if \(\frac{x' - 1}{y} \in \mathbb{Z}_F \mathfrak{o}_F\), which is equivalent to
\[
(x' - 1)/x' \in y \mathbb{Z}_F \mathfrak{o}_F \subset \mathbb{Z}_F \mathfrak{o}_F.
\]
That is only possible when \(|x'| = 1\), so (4.11) becomes an integral of a continuous function over the compact set \(\mathfrak{o}_F^\times\). In particular it converges and \(J_{P_\alpha | P_\alpha}(\sigma \otimes \chi) f_1\) does not have any poles.

With calculations as in (4.2) we check the other basis elements \(f_{zs}\):
\[
J_{P_\alpha | P_\alpha}(\sigma \otimes \chi) f_{zs}(s_\alpha) = \int_F f_{zs}(u_{-\alpha}(x)s_\alpha) \, dx = \text{vol}(z + \mathbb{Z}_F^{2r-1} \mathfrak{o}_F) = q_F^{1-2r},
\]
\[
J_{P_\alpha | P_\alpha}(\sigma \otimes \chi) f_{zs}(u_\alpha(y)) = \int_F f_{zs}(u_{-\alpha}(x)u_\alpha(y)) \, dx
\]
\[
= \int_F f_{zs}(u_\alpha(-x^{-1})u_{-\alpha}(x)u_{-\alpha}(y + x^{-1})) \, dx
\]
(4.12)
\[
= \int_F f_{zs}(s_\alpha(x)u_\alpha(y + x^{-1})) \, dx
\]
\[
= \int_F (\sigma^1(\delta P_\alpha)(\alpha^\vee(x^{-1})) f_{zs}(s_\alpha u_\alpha(y + x^{-1})) \, dx
\]
\[
= \int_F (\sigma^1(\delta P_\alpha)(\alpha^\vee(x^{-1})) f_{zs}(s_\alpha(x)u_\alpha(y + x^{-1})) \, dx
\]
\[
= \int_F (\sigma^1(\delta P_\alpha)(\alpha^\vee(x^{-1})) f_{zs}(u_{-\alpha}(-y - x^{-1})s_\alpha) \, dx.
\]

When \(-y \notin z + \mathbb{Z}_F^{2r-1} \mathfrak{o}_F\), this integral is supported on a compact subset of \(F\), and it converges. When \(-y \in z + \mathbb{Z}_F^{2r-1} \mathfrak{o}_F\), the support condition on \(x\) becomes \(|x| \geq q_F^{2r-1}\), and the integral reduces to
\[
\sum_{n=2r-1}^{\infty} \int_{\mathbb{Z}_F^{n} \mathfrak{o}_F} (\sigma^1(\delta P_\alpha)(\alpha^\vee(x^{-1})) \, dx.
\]
Since \(\sigma \circ \alpha^\vee\) is ramified and quadratic, it is nontrivial on \(\mathfrak{o}_F^\times\). Then (4.3) and (4.4) show that every term of the above sum is zero. We conclude that \(J_{P_\alpha | P_\alpha}(\sigma \otimes \chi) f_{zs}\) also does not have any poles.

(b) Part (a) and Lemma 4.2 show that \(J_{P_\alpha | P_\alpha}(\sigma \otimes \chi)\) does not have any poles on \(I_{P_\alpha}^M(\sigma \otimes \chi)^H_r\). Similar computations (which we omit) show that \(J_{P_\alpha | P_\alpha}(\sigma \otimes \chi)\) does not have any poles on \(I_{P_\alpha}^M(\sigma \otimes \chi)^H_r\). By Lemma 4.1 they neither have poles on, respectively, \(I_{P_\alpha}^M(\sigma \otimes \chi)\) and \(I_{P_\alpha}^M(\sigma \otimes \chi)\). Then (1.13) says that \(\mu^M(\sigma \otimes \chi)\) is nonzero for \(\chi \in X_{nr}(T)\), which by definition means \(\alpha \notin \Sigma_{O,\mu}\).

Let us combine the conclusions for all possible \(\sigma \circ \alpha^\vee\):

**Theorem 4.4.** Suppose that \(\alpha \in \Sigma_{O,\mu}\), for a principal series Bernstein component of a \(F\)-split group \(G\). Define \(X_\alpha(\chi) = \chi(\alpha^\vee(\mathbb{Z}_F))\). Then \(\sigma \circ \alpha^\vee = 1\) and \(q_\alpha = q_F, q_{\alpha^*} = 1\).
4.2. Principal series of quasi-split groups.

We consider a quasi-split non-split connected reductive $F$-group $\mathcal{G}$. By Section 2 we may suppose that $\mathcal{G}$ is absolutely simple. Then it is an outer form of Lie type $A_n, D_n$ or $E_6$.

Let $T$ be the centralizer of a maximal $F$-split torus $S$ in $\mathcal{G}$, and let $\sigma$ be a character of $T$ satisfying Condition 1.2. Let $\text{Gal}(F_s/\bar{F})$ be the normal subgroup of $\text{Gal}(F_s/F)$ that acts trivially on $X^*(T)$, so that $\bar{F}/F$ is a minimal Galois extension splitting $T$.

Consider a root $\alpha \in \Sigma_{\mathcal{O}, \mu}$. By a suitable choice of a basis of $\Sigma(G, S) \subset \Sigma(G, A_T)$, we may assume that $\alpha$ is simple. It corresponds to a unique Galois orbit $W_F\alpha_T$ in $\Sigma(\mathcal{G}, T)$. Then

$$U_\alpha(F) = \left( \prod_{\beta \in W_F\alpha_T} U_{\beta_T}(F_s) \right)^{W_F} \cong U_{\alpha_T}(F_s)^{W_{F, \alpha_T}} \cong F_s^{W_{F, \alpha_T}} =: F_\alpha.$$  

The field $F_\alpha$ does not depend on the choice of $\alpha_T$ (up to isomorphism) and is known as a splitting field for $\alpha$.

By construction the numbers $q_\alpha, q_{\alpha*}$ depend only on the group $M_\alpha$. Parts (b–c) of Proposition 2.4 apply, so we may even replace $M_\alpha$ by its derived subgroup $M_{\alpha, \text{der}}$.

Suppose for the moment that the elements of $W_F\alpha_T \subset \Sigma(\mathcal{G}, T)$ are mutually orthogonal. Then $M_{\alpha, \text{der}}$ is isomorphic to the restriction of scalars, from $F_\alpha$ to $F$, of $SL_2$ or $PGL_2$. Now $q_\alpha$ and $q_{\alpha*}$ can be computed in $SL_2(F_\alpha)$ or $PGL_2(F_\alpha)$, as in Paragraph 4.1. (Recall that even for $PGL_2$ we insisted that $X_\alpha$ is based on $\alpha^\vee$ rather than on $k_\alpha^\vee$.) By Theorem 4.4, $\sigma \circ \alpha^\vee = 1$, $q_{\alpha*} = 1$ and $q_\alpha = q_{F_\alpha}$ is the cardinality of the residue field of $F_{\alpha_T}$.

From Galois theory for local fields [Ser] it is known that

$$|W_F\alpha_T| = [W_F : W_{F, \alpha_T}] = e_{F_\alpha/F}f_{F_\alpha/F} = [I_F : I_F \cap W_{F, \alpha_T}] \cdot [W_F/I_F : W_{F, \alpha_T}] = |I_{F, \alpha_T}| \cdot f_{F_\alpha/F}.$$  

Since $I_F$ is normal in $W_F$, the number

$$q_{F_\alpha} = q_{f_{F_\alpha/F}} = q_{F}^{[W_F, W_{F, \alpha_T}]}$$

depends only on $\alpha$, and not on the choice of $\alpha_T$. This leads to the possibilities for the Dynkin diagrams (with Galois action indicated by arrows), the relative root systems and the $q_\alpha$ in Table 2. We stress that the parameters $q_\alpha$ only come into play when $\alpha \in \Sigma_{\mathcal{O}, \mu}$, for $\alpha \in \Sigma(A_T)/\Sigma_{\mathcal{O}, \mu}$ they are not defined. In Table 2 $q = q_F$ and $q' \in \{q_F, q_F^2\}$, according to (4.15). For a $F$-group of type $3D_4$, $[\bar{F} : F]$ can be of degree 3 or 6. In both cases $[F_\alpha : F] = 3$ for the roots $\alpha$ not fixed by $W_F$, so $q'' \in \{q_F, q_F^3\}$. Thus Conjecture A holds in all these cases.

It remains to consider the case where the elements of $W_F\alpha_T$ are not orthogonal. From the above diagrams we see that this happens only once for absolutely simple groups, namely for one pair of simple roots in type $2A_2$. With Proposition 2.4 we can transfer the determinant of $q_\alpha$ and $q_{\alpha*}$ (which no longer needs to be 1) to the simply connected cover of $M_{\alpha, \text{der}}$, which is isomorphic to $SU_3$. This does not change the $q$-parameters, by Proposition 2.4 (b–c). Because we cannot reduce the issue to $SL_2$ or $PGL_2$, the necessary computations are more involved. With Section 2 we can transfer the issue to $U_3(F)$, which is easier. Indeed, for that group all the $q$-parameters can be found in Paragraph 4.4, as well as a proof of Conjecture A.

Let us state the above conclusions concisely:
Corollary 4.5. Conjecture [A] holds for all Bernstein blocks in the principal series of a quasi-split connected reductive group $G$ over $F$. When we base $X_\alpha$ on $\alpha^\vee$, $q_{\alpha^*} = 1$ and $q_\alpha = q_{F_\alpha}$ (except for one root in type $^2A_{2n}$).

The Hecke algebras of types for principal series Bernstein blocks of split groups were analysed in depth by Roche [Roc1]. This analysis was generalized to quasi-split unitary groups by the author’s PhD student Badea [Bade]. We take the opportunity to report on that work, which provides much more explicit results than those obtained for the same Bernstein blocks in Paragraph 4.4.

For technical reasons we assume in the remainder of this paragraph that the residue field of $F$ does not have characteristic 2. Let $\tilde{F}/F$ be a quadratic extension of non-archimedean local fields and denote the nontrivial automorphism of $\tilde{F}/F$ by $x \mapsto \bar{x}$. Let $\tilde{F}^1 = \{x \in \tilde{F} : x\bar{x} = 1\}$ be the group of norm 1 elements in $\tilde{F}$. Consider the quasi-split unitary group $G = U_N(F)$ and its diagonal torus

$$T \cong (\tilde{F}^\times)^{|N/2|} \times \tilde{F}^1$$

(where the factor $\tilde{F}^1$ only appears if $N$ is odd). For every character $\tau$ of $T$ and $s = [T, \tau]_G$, the Bernstein block $\text{Rep}(G)_s$ admits a type $(J, \rho)$ [MiSt]. The Hecke algebra $\mathcal{H}(G, J, \rho)$ was analysed in [Bade], by means of direct, rather involved computations with $G, J$ and $\rho$.

The group $X_{nr}(T, \tau)$ is trivial (because $T$ is a torus), and then [Solo] Theorem 10.9 implies that $\mathcal{H}(G, J, \rho)$ is in fact isomorphic with $\text{End}_G(\Pi^\rho)^\text{op}$ (but this was not yet known when [Bade] was written). We can write $\tau : T \to \mathbb{C}^\times$ in the form

$$\tau_1 \otimes \cdots \otimes \tau_{|N/2|} \otimes \tau_0,$$
with $\tau_0$ a character of $\tilde{F}^1$ (if $N$ odd).

The character $\tau(\tau_0^{-1}\circ \det)$ of $T$ gives rise to the same Hecke algebras as $\tau$, because $\tau_0\circ \det$ extends to a character of $G$. Indeed, pointwise multiplication with $\tau_0\circ \det$ provides an algebra isomorphism

$$\mathcal{H}(G,J,\rho) \longrightarrow \mathcal{H}(G,J,\rho \otimes \tau_0^{-1}\circ \det).$$

Replacing $\tau$ by $\tau(\tau_0^{-1}\circ \det)$, may assume that $\tau_0 = 1$. Next we bring $\tau$ in a standard form (with adjustments that do not change the Hecke algebras).

- Tensoring by an unramified character of $T$, we assume that components $\tau_1, \tau_j$ which differ by an unramified character of $\tilde{F}_0^\times$ are actually equal.
- With the coordinate change $a_i \mapsto \pi_i^{-1}$ on $T$, we may exchange $\tau_i$ for $\tau_i^{-1}$.
- By permuting the coordinates $1, \ldots, [N/2]$ of $T$, we can bring equal coordinates of $\tau$ in adjacent position.
- Divide the coordinates $\tau_i$ in three classes: not skew, skew and nontrivial on $\sigma_k^\times$ (can only occur if $\tilde{F}/F$ is ramified), skew and trivial on $\sigma_k^\times$. When $N$ is odd, we treat the trivial character of $\tilde{F}_0^\times$ as a fourth class.

Now we rewrite

$$\tau = \tau_1^{\otimes N_1} \otimes \cdots \otimes \tau_d^{\otimes N_d} \otimes 1^{\otimes 1+N_0}$$

(4.16)

where $\tau_i$ is not skew for $1 \leq i \leq k$, skew and nontrivial on $\sigma_k^\times$ for $k < i \leq k'$, skew and trivial on $\sigma_k^\times$ for $k' < i \leq d$, and the factor $1^{\otimes N_0}$ is only present if $N$ is odd.

(For even $N$, we consider $1$ just as a skew character of $\tilde{F}_0^\times$.) For consistency:

$$N_1 + \cdots + N_d + N_0 = [N/2]$$

Root system: $R(G,T)$ has type $C_{N/2}$ if $N$ even and has type $BC_{[N/2]}$ if $N$ is odd. The stabilizer of $\tau$ in $W(G,T) = W(R(G,T))$ is

$$W_\tau = \prod_{i=1}^k S_{N_i} \times \prod_{i=k+1}^d W(C_{N_i}) \times W(B_{N_0}),$$

where $B_{N_0}$ is only present if $N$ odd. Due to the special form of $\tau$, the Bernstein group $W(M,\mathcal{O}) \subset N_G(T)/T$ equals $W_\tau$. The lattice $M^1_2/M^1$ can be identified with the cocharacter lattice $X_\ast(T) \cong \mathbb{Z}^{[N/2]}$. From (4.16) we get a factorization

$$X^\ast(T) = \prod_{i=1}^k \mathbb{Z}^{N_i} \times \prod_{i=k+1}^d \mathbb{Z}^{N_i} \times \mathbb{Z}^{N_0}.$$

(4.17)

The further structure of the Hecke algebra $\mathcal{H}(G,J,\rho) \cong \text{End}_G(\Pi^\ast)^{op}$ depends on the ramification of $\tilde{F}/F$.

$\tilde{F}/F$ unramified [Bade, Chapter 4]

Here every skew character of $\tilde{F}_0^\times$ is trivial on $\sigma_k^\times$, so $k' = k$. Root system:

$$\Sigma^\times_C = \prod_{i=1}^k A_{N_i-1} \times \prod_{i=k+1}^d B_{N_i} \times B_{N_0}$$

($B_{N_0}$ only if $N$ odd). Labels for Hecke algebras:

<table>
<thead>
<tr>
<th>$A_{N_i-1}$</th>
<th>$B_{N_i}$</th>
<th>$B_{N_0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda(\alpha) = \lambda^\ast(\alpha) = 2$</td>
<td>$\lambda(\alpha) = \lambda^\ast(\alpha) = 2$</td>
<td>$\lambda(\alpha) = \lambda^\ast(\alpha) = 2$</td>
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<tr>
<td>$\lambda(\beta) = \lambda^\ast(\beta) = 1$</td>
<td>$\lambda(\beta) = \lambda^\ast(\beta) = 1$</td>
<td>$\lambda(\beta) = 3, \lambda^\ast(\beta) = 1$</td>
</tr>
</tbody>
</table>
Here $\alpha = \pm e_j \pm e_{j'}$ ($j \neq j'$) and $\beta = \pm e_j$. In the affine Hecke algebras the quadratic relations for the simple reflections become

$$(T_{s_\alpha} + 1)(T_{s_\alpha} - q_F^2) = 0, \quad (T_{s_\beta} + 1)(T_{s_\beta} - q_F^{\lambda(\beta)}) = 0, \quad (T_{s_\beta^*} + 1)(T_{s_\beta^*} - q_F) = 0,$$

where $\beta^*$ denotes the additional simple affine root $\beta^*$ in $B_{N_0}$.

Affine Hecke algebra $\mathcal{H}(G, J, \rho)$: tensor product of the factors

$$\mathcal{H}(A_{N_{-1}}, \mathbb{Z}^{N_i}, A_{N_{-1}}, \mathbb{Z}^{N_i}, q_F^2), \quad \mathcal{H}(B_{N_i}, \mathbb{Z}^{N_i}, C_{N_{-1}}, \mathbb{Z}^{N_i}, q_F^2, q_F)$$

and $\mathcal{H}(B_{N_0}, \mathbb{Z}^{N_0}, C_{N_0}, \mathbb{Z}^{N_0}, q_F^2, q_F^3, q_F)$ (still $B_{N_0}$ only if $N$ odd).

$\tilde{F}/F$ ramified \cite{Bad}: Chapter 5]

Root system (with $A_0$ and $D_1$ regarded as empty):

$$\Sigma_\text{O} = \prod_{i=1}^{k} A_{N_{i-1}} \times \prod_{i=k+1}^{k'} C_{N_{i}} \times \prod_{i=k'+1}^{d} D_{N_{i}} \times B_{N_0}.$$  

As before, $B_{N_0}$ occurs only if $N$ is odd. We have roots $\alpha = \pm e_j \pm e_{j'}$ ($j \neq j'$) and $\beta = \pm 2e_j$ (type $C$) or $\beta = \pm e_j$ (type $B$).

Labels for Hecke algebras: $\lambda = \lambda^* = 1$ in all cases.

For any simple reflection $\gamma$ (also the affine one in $B_{N_0}$), the quadratic relation in the affine Hecke algebra becomes

$$(T_{s_\gamma} + 1)(T_{s_\gamma} - q_F) = 0.$$

For every type $D$ factor the Hecke algebra contains a generator $T_{s_{2\beta}}$ where $s_{2\beta} = s_{\beta}$ fixes $\tau$ but $2\beta = 2e_{N_{i}}$ (a simple root for $C_{N_{i}}$) does not lie in the root system. Then $T_{s_{2\beta}}^2 = 1$ and it generates a copy of the group algebra of $(s_{2\beta})$. This group acts on $\mathcal{H}(D_{N_i}, \mathbb{Z}^{N_i}, D_{N_i}, \mathbb{Z}^{N_i}, q_F)$ and we can form the crossed product. It is easy to see that there is a support-preserving isomorphism of algebras

$$\mathcal{H}(D_{N_i}, \mathbb{Z}^{N_i}, D_{N_i}, \mathbb{Z}^{N_i}, q_F) \rtimes (s_{2\beta}) \cong \mathcal{H}(C_{N_{i}}, \mathbb{Z}^{N_{i}}, B_{N_{i}}, \mathbb{Z}^{N_{i}}, q_F, 1),$$

where the 1 indicates that the long roots of $C_{N_{i}}$ get parameter $q_F^0 = 1.$

Affine Hecke algebra $\mathcal{H}(G, J, \rho)$: tensor product of the factors

$$\mathcal{H}(A_{N_{i-1}}, \mathbb{Z}^{N_{i}}, A_{N_{i-1}}, \mathbb{Z}^{N_{i}}, q_F), \quad \mathcal{H}(C_{N_{i}}, \mathbb{Z}^{N_{i}}, B_{N_{i}}, \mathbb{Z}^{N_{i}}, q_F),$$

$$\mathcal{H}(D_{N_{i}}, \mathbb{Z}^{N_{i}}, D_{N_{i}}, \mathbb{Z}^{N_{i}}, q_F) \rtimes (s_{2\beta}), \quad \mathcal{H}(B_{N_0}, \mathbb{Z}^{N_0}, C_{N_0}, \mathbb{Z}^{N_0}, q_F).$$

4.3. Inner forms of Lie type $A_n$.

We consider simple $F$-groups $\mathcal{G}$ that are inner forms of a split group of type $A_{n-1}$. The simply connected cover of $\mathcal{G}$ is an inner form of $SL_n$, so isomorphic to the derived subgroup of an inner form of $GL_n$. In view of Section 2$^\mathbb{2}$ it suffices to consider the latter case, so with $G$ isomorphic to $GL_m(D)$ for a division algebra $D$ with centre $F$ and $\dim_F(D) = (n/m)^2$.

For every Bernstein block $\text{Rep}(G)^\mathbb{2}$ there exists a type $(J, \rho)$ \cite{Ste2}. We can write $s = [M, \sigma]_G$ in the form

$$M = \prod_i GL_{m_i}(D)^{e_i}, \quad \sigma = \otimes_i \sigma_i^{\otimes e_i},$$

where the various $\sigma_i$ differ by more than an unramified character. The associated Hecke algebra $\mathcal{H}(G, J, \rho)$ is a tensor product of affine Hecke algebras of type $GL_{e_i}$ \cite{Ste1}, so the underlying root system has irreducible components of type $A_{e_i-1},$
for suitable $c_i \leq n$. The same result was obtained around the same time in [Hei2], using $\Pi^\circ$. The parameters of such a type $GL_{c_i}$ affine Hecke algebra were determined explicitly in [Sec Théorème 4.6], they are of the form $q_0 = q_{f}^t, q_{\alpha} = 1$ for a specific positive integer $f$. Thus $\lambda$ and $\Lambda^*$ are constant and equal to $f$ on the underlying root system $A_{c_i-1}$. From [Hei2 1.13–1.15] or [SéSt2] we also see that

\begin{equation}
W(M, O) = W(\Sigma_{O,\mu}) \cong \prod_{c_i} S_{c_i},
\end{equation}

and $R(O) = \{1\}$. From that, (1.11) and

\[
\text{Mod} - \mathcal{H}(G, J, \rho) \cong \text{Rep}(G)^{\circ} \cong \text{End}_{G}(\Pi^\circ) - \text{Mod}
\]

we deduce that $\mathcal{H}(G, J, \rho)$ is Morita equivalent with $\mathcal{H}(G, O)^{\circ}$. These are both affine Hecke algebras, and then Morita equivalence implies that $\mathcal{H}(G, J, \rho)$ and $\mathcal{H}(G, O)^{\circ}$ are isomorphic. We summarise:

**Theorem 4.6.** [Heiermann, Sécherre–Stevens]

Let $G$ be an inner form of a simple $F$-split group of type $A_{n-1}$, and let $s$ be an inertial equivalence class for $G$. Then the root system underlying $\mathcal{H}(O, G)$ has irreducible components of type $A_{c-1}$ with $c \leq n$. The label functions $\lambda, \Lambda^*$ are constant on $A_{c-1}$, and equal to an integer $f$.

We note that such parameters already occur for Iwahori-spherical representations. Namely, consider $GL_e(D)$ where $\text{dim}_F(D) = f^2$. Its Iwahori–Hecke algebra is isomorphic with an affine Hecke algebra of type $GL_e$ with parameters $q^f_{f'}$. More explicit information about $f$ comes from [SéSt2 Introduction]. Every type $GL_e$ affine Hecke algebra as above comes from a supercuspidal representation $\pi^{\circ}_{\text{Sc}}$ of $GL_{m'/e}(D)$ for some $m' \leq m$. Then $f$ equals the torsion number $t_\pi = |X_m^e(GL_{m'/e}(D), \pi)|$ times the reducibility number $s_\pi$. With the Jacquet–Langlands correspondence [Badu, DKV] one can relate the torsion and reducibility numbers of $\pi$ to the same numbers for a specific discrete series representation $JL(\pi)$ of $GL_{nm'/c}(F)$. More information about those numbers is already known from [BuKu1]. From that or from a comparison with Langlands parameters as in [AMS3 p. 57], one sees that $s_\pi$ divides $\frac{nm'}{me}$ and that $t_\pi$ divides $\frac{nm'}{me}$. Therefore

\begin{equation}
f = s_\pi t_\pi \quad \text{divides} \quad \frac{nm'}{me} \leq \frac{n}{e}.
\end{equation}

We note that in all these cases $\alpha^\vee$ generates $H_M(M_\alpha^1/M^1)$, because the derived groups are simply connected. The torsion number $t_\pi$ says precisely that

\[ H_M(M_\alpha^2 \cap M_\alpha^1/M^1) = \mathbb{Z} t_\pi \alpha^\vee. \]

Consider a $F$-split connected reductive group $M_\alpha$ with root system of type $A_{n+m-1}$. Let $M$ be the standard $F$-Levi subgroup of $M_\alpha$ obtained by omitting a simple root $\alpha$, with root system of type $A_{n-1} \times A_{m-1}$. Then the simply connected cover of $M_{\text{der}}$ is isomorphic to $SL_n(F) \times SL_m(F)$.

Put $s = [M, \sigma]_{M_\alpha}$ for some $\sigma \in \text{Irr}_{\text{cusp}}(M)$. The inflation of $\sigma|_{M_{\text{der}}}$ to the simply connected cover $M_{sc}$ of $M_{\text{der}}$ can be written as a finite direct sum

\[ \bigoplus_i \sigma_i \boxtimes \sigma'_i \quad \text{with} \quad \sigma_i \in \text{Irr}_{\text{cusp}}(SL_n(F)), \sigma'_i \in \text{Irr}_{\text{cusp}}(SL_m(F)). \]
Corollary 4.7. Let $M_\alpha$, $M$ and $\sigma$ be as above.

(a) If $n \neq m$, then $s_\alpha$ does not give rise to an element of $N_{M_\alpha}(M)/M$.

(b) Suppose that $n = m$ and that, for any $i$, $\sigma_i$ and $\sigma_i'$ are not isomorphic. Then $s_\alpha$ does not give rise to an element of $W(M, \mathcal{O})$.

(c) Suppose that $n = m$ and that, for at least one $i$, $\sigma_i$ and $\sigma_i'$ are isomorphic. Then $\Sigma_{O, \mu} = \{\alpha, -\alpha\}$ and $s_\alpha$ gives rise to an element of $W(M, \mathcal{O})$ that exchanges the two almost direct simple factors of $M_{\text{der}}$.

When $M_\alpha = GL_{2n}(F)$, the $q$-parameters for $\mathcal{H}(O, M_\alpha)$ are $q_{\alpha^*} = 1$ and $q_\alpha = q_f$. Here $f$ is the torsion number $t_{\sigma_i} \in \mathbb{Z}_{>0}$, which divides $n$.

Proof. (a) This is clear, because such an element would have to exchange the two almost direct simple factors of $M_{\text{der}}$.

(b) Now $s_\alpha$ does give an element of $N_{M_\alpha}(M)/M$, which exchanges the two almost direct simple factors of $M_{\text{der}}$. By Proposition 2.2 we may lift to the simply connected cover $M_{\text{ac}}$, picking one irreducible constituent $\sigma_i \otimes \sigma_i'$ of the inflation of $\sigma|_{M_{\text{der}}}$. As $M_{\text{ac}}$ does not have nontrivial unramified characters, stabilizing $\alpha$ has become stabilizing $\sigma_i \otimes \sigma_i'$. Clearly $s_\alpha$ does that if and only if $\sigma_i$ and $\sigma_i'$ are isomorphic.

(c) This follows from Theorem 4.6 \hfill $\square$

In part (c) for $M_\alpha \neq GL_{2n}(F)$, it may still be necessary to apply Proposition 2.3d to obtain the precise parameters.

Example 4.8. Consider the inclusion $\eta : SL_4(F) \rightarrow GL_4(F)$ and the Levi subgroups $M = GL_2(F)^2$ and $\mathcal{M} = SL/GL_2(F)^2)$. Let $\sigma \in \text{Irr}_{\text{cusp}}(GL_2(F))$ with

$$X_{\text{nr}}(GL_2(F), \sigma) = \{1, \chi_\eta\}.$$  

We may assume that $\sigma|_{SL_2(F)}$ decomposes as a direct sum of two irreducible representations, both stable under $\text{diag}(a, b) \in GL_2(F)$ for all $a, b \in \sigma_F^\times$. Then $\sigma \otimes \sigma \in \text{Irr}(M)$ and $\eta^*(\sigma \otimes \sigma)$ is a direct sum of two irreducible $\tilde{M}$-representations $\tilde{\sigma}_1, \tilde{\sigma}_2$, permuted by $\text{diag}(\varpi^\times_F, 1) \in M$. Here

$$\eta^*(X_{\text{nr}}(M, \sigma \otimes \sigma)) = \{1, \chi_\eta \otimes 1\}$$

but tensoring by $\chi_\eta \otimes 1$ exchanges $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$. It follows that

$$X_{\text{nr}}(\tilde{M}, \tilde{\sigma}_1) = X_{\text{nr}}(\tilde{M}, \tilde{\sigma}_2) = \{1\}.$$  

The root systems of the Hecke algebras are $\{\alpha, -\alpha\}$ and $\{\tilde{\alpha}, -\tilde{\alpha}\}$, while $h_\alpha^\prime = \eta(h_\alpha^\prime)^2 \in M/M^1$. So this is an instance of Proposition 2.4d.(iii).

4.4. Classical groups.

We look at classical groups associated to Hermitian forms on $F$-vector spaces. Let $G^*$ be a symplectic group or a special orthogonal group (not necessarily split). It was shown in [Hei2] that $\text{End}_G(\Pi^\alpha)$ is Morita equivalent with the crossed product of $\mathcal{H}(O, G)$ and $R(O)$, where $\mathcal{H}(O, G)$ is a tensor product of affine Hecke algebras with lattice $\mathbb{Z}_c^e$ and root system $A_{c-1}, B_e, C_e$ or $D_e$. When $G^*$ is $F$-split, the parameters are computed in [Hei1], relying on [Mo2]. Later the (quasi-)split assumption in [Mo2] was lifted in [MoRe], which means that [Hei1] also applies to non-split groups.

We also allow $G^*$ to be a special unitary group. With Section 2 we reduce that to $U_n$, a unitary group $U_n$ which splits over a separable quadratic extension $\tilde{F}/F$.  

According to [Hei3] Theorem 1.8 and §C.5, the above description of \( \mathcal{H}(O, G) \) is also valid for \( U_n \). Unfortunately there is no real proof of these claims in [Hei3], but it is similar to [Hei2] and in fact an instance of the more general results of [Sol6].

Also according to [Hei3] §C, the parameters of these affine Hecke algebras can be computed as in [Hei1]. This uses the results of [Mœ1, Mœ2, Mœ3].

Recall that every \( F \)-Levi subgroup of \( G^* \) is of the form

\[
\mathcal{M}^*(F) \cong \prod_i GL_{n_i}(F') \times \mathcal{H}^*(F),
\]

where \( \mathcal{H}^* \) is of the same type as \( G^* \), but of smaller rank. Here \( F' = \tilde{F} \) for (special) unitary groups and \( F' = F \) otherwise. Let \( f \) be residue degree of \( F'/F \), so 2 for unramified (special) unitary groups and 1 otherwise.

Let \( G \) be a group isogenous to \( G^* \) and let \( \mathcal{M} \) be a \( F \)-Levi subgroup of \( G \).

**Theorem 4.9.** Let \( G \) be isogenous to \( G^* \) as above, and consider an inertial equivalence class \( s = [M, \sigma]_G \). Fix an irreducible component \( \Sigma_{O,j}^\vee \) of \( \Sigma_O^\vee \) as in Section 2. Associate a supercuspidal representation \( \rho \) of \( GL_{d_\rho}(F') \) to \( \Sigma_{O,j}^\vee \) as in [Hei1] §2.1 and §3, and let \( t = t_\rho \in \mathbb{Z}_{>0} \) be its torsion number, a divisor of \( d_\rho \).

(a) When \( \Sigma_{O,j}^\vee \cong C_e \) and \( h_\alpha^\vee \) is a long root, there exists an integer \( a_+ \in \mathbb{Z}_{>0} \) such that \( q_a = q_{F'}^{t_\alpha}, q_{a+} = 1 \) and \( \lambda(\alpha) = \lambda^*(\alpha) = ft_{a+} \).

(b) When \( \Sigma_{O,j}^\vee \cong B_e \) and \( h_\alpha^\vee \) is a short root, there exist integers \( a \geq a_- \geq -1 \) such that \( q_a = q_{F'}^{t_\alpha/2}, q_{a+} = 1 \) and \( \lambda(\alpha) = \lambda^*(\alpha) = ft_\alpha \).

(c) For all other \( h_\alpha^\vee \in \Sigma_{O,j}^\vee \), \( q_a = q_{F'}^{t_\alpha}, q_{a+} = 1 \) and \( \lambda(\alpha) = \lambda^*(\alpha) = ft_\alpha \).

Suppose that \( M \) is isogenous to (4.20), and that the complex dual group of \( \mathcal{H}^* \) consists of matrices of size \( N^\vee \). Then

\[
\left[ \left( \frac{a+1}{2} \right)^2 + \left( \frac{a_-+1}{2} \right)^2 \right] \leq N^\vee d_\rho^{-1}
\]

in case (b) and \( a_+^2 \leq 2N^\vee d_\rho^{-1} + 1 \) in case (a).

We note that for a maximal Levi subgroup \( M^* = GL_{d_\rho}(F') \times H^* \) of \( G^* \), the root system \( \Sigma_O^\vee \) has type \( B_1 \) or is empty.

**Proof.** By Corollary 3.7 we may assume that \( F \) has characteristic zero.

For \( G^* \) the claims (b) and (c) follow from [Hei1] Proposition 3.4 and [Hei3] §C.5. The role of the torsion number \( t \) is to replace the sublattice \( \mathbb{Z}^e \) of \( M/M^1 \) corresponding to \( \Sigma_{O,j}^\vee \) by the \( (t\mathbb{Z})^e \), which is a direct summand of \( M^2_2/M^1 \). In this process, all the labels \( \lambda(\alpha) \) and \( \lambda^*(\alpha) \) are multiplied by \( t \).

The numbers \( a, a_- \) come from [Mœ1 Proposition 4], [Mœ2 §1.3–1.4] and [Mœ3 Théorème 3.1], where they are computed in terms of reducibility of the parabolic induction of a supercuspidal representation \( \rho \otimes \pi \) of \( GL_{d_\rho}(F') \times H^* \). This shows that in general we have to use \( F' \) instead of \( F \). For (special) unitary groups, the factors \( GL_{d_\rho}(F) \) in (4.20) cause another factor \( f \) in all the parameters, as explained in [Hei3] §C.

Recall that the Jordan block of \( \pi \in \text{Irr}(H^*) \) is built from the pairs \((\rho, a)\) that we consider (but those with \( a \leq 0 \) omitted), by adding new pairs according to the rule

\[
\text{if } (\rho, a) \in \text{Jord}(\pi) \text{ and } a > 2 \text{ then } (\rho, a - 2) \in \text{Jord}(\pi).
\]
It was shown in \cite{Mo2} §1.4 and \cite{Mo1} Proposition 4) that

\begin{equation}
\sum_{(\rho, a) \in \text{Jord}(\pi)} ad_{\rho} = N^{\vee}.
\end{equation}

We fix a \(\rho\) and let \(\rho_-\) be the unramified twist of \(\rho\) from which \(a_-\) is determined. Isolating the terms with \(\rho\) and \(\rho_-\) in (4.21), we obtain

\begin{equation}
N^{\vee} \geq \sum_{a':(\rho, a') \in \text{Jord}(\pi)} a' d_{\rho} + \sum_{a':(\rho_- , a') \in \text{Jord}(\pi)} a' d_{\rho_-}.
\end{equation}

Case (a) is not mentioned explicitly in \cite{Heil}, it is an instance of case (b) when we focus on the Weyl group (not on the root system). As the lattice containing \(\Sigma_{\text{root}}\) is isomorphic to \(\mathbb{Z}^e\), the construction of \(h_{\alpha}^{\vee}\) entails that \(h_{\alpha}^{\vee} \notin 2\mathbb{Z}^e\), so that \(\Sigma_{\text{root}}^{\vee}\) does not have type \(C_e\). Still, this root system occurs if \(\Sigma_{\text{root}}^{\vee} \cong C_e\). We find \(\Sigma_{\text{root}}^{\vee} \cong C_e\). Then we can replace \(h_{\alpha}^{\vee}\) by \(h_{\alpha/2}^{\vee}\), \(X_{\alpha}\) by \(X_{\alpha}^2\), \(B_{e}\) by \(C_{e}\), \(q_{\alpha}\) by \(q_{\alpha}^2 = q_{\alpha}q_{\alpha^*}\) and \(q_{\alpha^*}\) by \(1\), without changing the Hecke algebra \(H(\mathcal{O}, G)\). We find

\[\lambda(\alpha/2) = \lambda^*(\alpha/2) = \lambda(\alpha) + \lambda^*(\alpha) = tf(a + 1),\]

so the \(a_+\) for \(2h_{\alpha}^{\vee}\) is 1 plus the \(a\) from \(B_{e}\). When \(a_+\) is odd, the previously established bound on \(a = a_-\) directly yields the new bound on \(a_+\). When \(a_+\) is even, (4.22) says

\[2\left(\frac{a + 1}{2}\right)^2 - \frac{1}{4} \leq N^{\vee}d_{\rho}^{-1}, \text{ so } 2a_+^2 = (a + 1)^2 \leq 2N^{\vee}d_{\rho}^{-1} + 1.\]

When \(\mathcal{G}\) is a quotient of \(G^*\), Section 2 enables us to reduce to \(G^*\). According to Proposition 2.3 in the process the labels for \(\alpha\) must be multiplied by some \(N_{\alpha} \in \{1/2, 1, 2\}\). But for type \(A\) roots nothing really changes in \(G^* \to G\) (the computations can be placed entirely in a general linear group) so \(N_{\alpha} = 1\). For other roots \(\beta\) either \(N_{\beta} = 1\) or (if \(\Sigma_{\text{root}}^{\vee} \cong C_e\)) \(N_{\beta} = 2\) or (if \(\Sigma_{\text{root}}^{\vee} \cong B_{e}\)) \(N_{\beta} = 1/2\). In the last two cases types \(C_e\) and \(B_{e}\) are exchanged, and the relations between the parameters are the same as between cases (a) and (b) of the theorem.

In the remaining cases \(\mathcal{G}\) is a spin group (or a half-spin group, but by passing to the simply connected cover we reduce that case to a spin group). With Section 2 and the above arguments we can handle all Bernstein blocks of \(G\) associated to supercuspidal representations that are pulled back from a Levi subgroup of \(G^*\). However, \(G\)-representations on which \(\ker(G \to G^*)\) acts nontrivially cannot be studied in this way.

Fortunately, \(\mathcal{G}\) can be embedded in a general spin group \(G_{\text{Spin}}^{\alpha}\) — not necessarily split, but at least a pure inner form of a quasi-split group. The Levi subgroups of \(G_{\text{Spin}}^{\alpha}\) follow the same pattern as for \(SO_n\), and their discrete series representations can be classified as for special orthogonal groups \(\text{Mo2, Mo3}\). Consequently the results of \cite{Heil, Hei2} also hold for \(G_{\text{Spin}}^{\alpha}\), with the only difference that the lattices in the Hecke algebras have rank one higher than for \(SO_n\). Thus Theorem 4.9 holds for \(G_{\text{Spin}}^{\alpha}\), possibly with correction factors \(N_{\beta} \in \{1/2, 1, 2\}\) as above for quotients of \(G^*\). By Corollary 2.7 the theorem also holds for the derived group \(\mathcal{G}\) of \(G_{\text{Spin}}^{\alpha}\). \(\square\)

In the generality of Theorem 4.9 it is hard to make the integers \(a_+, a\) and \(a_-\) more explicit, since they depend in a very subtle way on the involved supercuspidal representations. If one restricts to specific classes of Bernstein components, more can be said about the Hecke algebra parameters. In particular, for the principal
series representations of quasi-split classical groups the method in Paragraphs 4.1 and 4.2 yields the concrete $q$-parameters.

To check the integrality of the label functions $\lambda, \lambda^*$, we analyse the parity of $a$ and $a_-$. As explained in [Hei1 §1], this boils down to comparing the Langlands parameter $\rho$ of a supercuspidal representation of $GL_k(F_0)$ with the Langlands parameter of a supercuspidal representation of $H$ (a classical group of the same type as $G$).

For $G$ of Lie type $B_n, C_n, D_n$ or $2D_n$, $\rho$ is self-dual. Then $a$ is odd if and only if $\rho$ and the complex dual group $H^\vee$ of $H$ have the same type (orthogonal or symplectic).

For $G$ of Lie type $2A_{n-1}$, $\rho$ is conjugate-dual, that is, the contragredient $\rho^\vee$ is isomorphic to $s \cdot \rho$ for $s \in W_F \setminus W_{\bar{F}}$. From [GGP Theorem 8.1] we see that the standard representation of $H^\vee$ is conjugate-orthogonal or conjugate-symplectic, depending on the size of $H^\vee$. Just as above, $a$ is odd if and only if $\rho$ and this standard representation have the same type.

The parity of $a_-$ is determined by analogous considerations, starting from a different self-dual or conjugate-dual representation $\rho \otimes \chi$ with $\chi$ an unramified character. More specifically, let $t_\rho$ be the torsion number of $\rho \in \text{Irr}(W_F)$, that is, the number of unramified characters $\chi$ such that $\rho \cong \rho \otimes \chi$. Then there exist precisely $2t_\rho$ unramified characters $\chi$ of $W_F$ such that $\rho \otimes \chi$ is self-dual, namely those with $\chi^{2t_\rho} = 1$. When moreover $\chi^{t_\rho} \neq 1$, $\rho \otimes \chi \not\cong \rho$. That provides a unique (up to isomorphism) $\rho_- = \rho \otimes \chi$ which is self-dual and not isomorphic to $\rho$. The number $a_-$ is computed from this $\rho_-$. The same applies to conjugate-dual representations of $W_{\bar{F}}$.

The following result extends [Hei1 Proposition 1.3].

**Proposition 4.10.** (a) Let $\rho \in \text{Irr}(W_F)$ be a self-dual and let $\rho_-$ be as above.

- If $t_\rho$ is odd, then $\rho$ and $\rho_-$ have the same type (orthogonal or symplectic).
- If $t_\rho$ is even, then $\rho$ and $\rho_-$ can have the same or opposite type.

(b) Let $\tilde{\rho} \in \text{Irr}(W_{\bar{F}})$ be a conjugate-dual and let $\tilde{\rho}_-$ be the unique (up to isomorphism) conjugate-dual twist of $\tilde{\rho}$ by an unramified character.

- If $\bar{F}/F$ is ramified and $t_{\tilde{\rho}}$ is odd, then $\tilde{\rho}$ and $\tilde{\rho}_-$ have the same type (conjugate-orthogonal or conjugate-symplectic).
- If $\bar{F}/F$ is ramified and $t_{\tilde{\rho}}$ is even, then $\tilde{\rho}$ and $\tilde{\rho}_-$ can have the same or opposite type.
- If $\bar{F}/F$ is unramified, then $\tilde{\rho}$ and $\tilde{\rho}_-$ have opposite type.

**Proof.** (a) Since the inertia group $I_F$ is normal and $W_F/I_F \cong \mathbb{Z}$, $\rho$ can be analysed well by restriction to $I_F$. Clifford theory tells us that there exist mutually inequivalent irreducible $I_F$-representations $\rho_1, \ldots, \rho_t$ such that

\begin{equation}
\text{Res}_{I_F}^{W_F} \rho \cong \rho_1 \oplus \cdots \oplus \rho_t
\end{equation}

and a Frobenius element $\text{Frob}$ of $W_F$ permutes the $\rho_i$ cyclically. The unramified characters $\chi$ that stabilize $\rho$ are precisely those for which $\chi(\text{Frob}^i)$ acts trivially on $\rho_1$, so $t$ equals the torsion number $t_\rho$.

If $\rho_1$ is self-dual, then so are all the $\rho_i$, and the $W_F$-invariant bilinear form on $\rho$ is a direct sum of $I_F$-invariant bilinear forms on the $\rho_i$. Then the type of $\rho$ is the same as the type of $\rho_1$, which depends only on $I_F$ and is not affected by twisting with unramified characters. This can happen for even $t$ and for odd $t$.

If $\rho_1$ is not-self-dual, then none of the $\rho_i$ is self-dual. In that case $t$ is even and the dual of $(\rho_i, V_i)$ is isomorphic to $\rho_{iv}$ for a unique integer $i$. Further the $W_F$-invariant bilinear form on $\rho$ restricts on $\rho_i \times \rho_{iv}$ to $z$ times the canonical pairing,
for some $z \in \mathbb{C}^\times$. Similarly it restricts on $\rho_{i\nu} \times \rho_i$ to $z^\nu$ times the canonical pairing. It is easy to check that the representation $\rho_{i\nu} \oplus \rho_i$ of $I_F \rtimes (\text{Frob}^{i/2})$ is self-dual and $z^\nu = \pm z$ where $\pm$ indicates the type of the representation. Then the type of $\rho$ is the same as the type of $\rho_{i\nu} \oplus \rho_i$.

By self-duality of $\rho$ and $\rho_- = \rho \otimes \chi$ and $\rho \not\cong \rho_-$, we must have $\chi(\text{Frob}^i) = -1$ and $\chi(\text{Frob}^{i/2}) = \pm i$. In particular the representation $(\rho_{i\nu} \oplus \rho_i) \otimes \chi$ of $I_F \rtimes (\text{Frob}^{i/2})$ is not self-dual with respect to the same bilinear form as $\rho_{i\nu} \oplus \rho_i$. To make $(\rho_{i\nu} \oplus \rho_i) \otimes \chi$ self-dual, we can take the bilinear form where in the above description $z^\nu$ is replaced by $-z^\nu$. This changes the sign of the bilinear form, so $\rho$ and $\rho_-$ have opposite type.

(b) When $\widetilde{F}/F$ is ramified, we can pick a representative for $W_F/W_{\widetilde{F}}$ in $I_F$. Then the notions conjugate-dual, conjugate-orthogonal and conjugate-symplectic can be defined in the same way for $I_{\widetilde{F}}$-representations. The proof of part (a) applies to $\tilde{\rho} \in \text{Irr}(W_{\widetilde{F}})$, when we replace self-dual by conjugate-dual. The conclusion is that $\tilde{\rho}$ and $\tilde{\rho}_-$ have the same type.

When $\widetilde{F}/F$ is unramified, we pick a representative $s$ for $W_F/W_{\widetilde{F}}$ so that $s^2$ is a Frobenius element of $W_{\widetilde{F}}$. Conjugate-duality is still defined for $I_{\widetilde{F}}$-representations (because $I_{\widetilde{F}}$ is normal in $W_F$), but the type of such a representation is not (because $s^2 \not\in I_{\widetilde{F}}$). Nevertheless, we can still decompose $\tilde{\rho} \in \text{Irr}(W_{\widetilde{F}})$ as an $I_{\widetilde{F}}$-representation like in (4.23). We see that the $W_{\widetilde{F}}$-invariant bilinear pairing between $\tilde{\rho}$ and $s \cdot \tilde{\rho}$ restricts to a pairing between $\tilde{\rho}_i, V_i$ and $(\tilde{\rho}_{i\nu}, V_{i\nu})$ for a unique $i\nu$. By definition [GGP, §3], the type of $\tilde{\rho}$ is given by the sign $\pm$ in

$$\langle v, v' \rangle = \pm \langle v', \tilde{\rho}(s^2)v \rangle \quad \forall v, v' \in V_{\tilde{\rho}}.$$  

We consider $v$ in $V_1$ and $v' \in V_{1\nu}$ such that the pairing (4.24) is nonzero. Then $\tilde{\rho}(s^2)v$ must also belong to $V_1$, but at the same time $\tilde{\rho}(s^2)$ permutes the $\tilde{\rho}_i$ cyclically. That renders (4.24) impossible, unless $t_{\tilde{\rho}} = 1$. But then $\tilde{\rho}|_{I_{\widetilde{F}}}$ is irreducible and isomorphic to $s \cdot \tilde{\rho}'|_{I_{\widetilde{F}}}$. In this situation the bilinear pairing between $\tilde{\rho}$ and $s \cdot \tilde{\rho}$ is already determined by their structure as $I_{\widetilde{F}}$-representations.

The same applies to $\tilde{\rho}_- = \tilde{\rho} \otimes \tilde{\chi}$. Then the conjugate-duality of $\tilde{\rho}$ and $\tilde{\rho}_-$ implies that $\tilde{\chi}$ is quadratic. It cannot be trivial because $\tilde{\rho}$ and $\tilde{\rho}_-$ are not isomorphic, so $\tilde{\chi}$ is the unique unramified character of $W_{\widetilde{F}}$ of order two. By [GGP, Lemma 3.4] $\tilde{\chi}$ is conjugate-symplectic, and by [GGP, Lemma 3.5.ii] $\tilde{\rho}$ and $\tilde{\rho}_-$ have opposite type. □

Proposition 4.10 is the key to the following result.

Lemma 4.11. We assume the setting of Theorem 4.9b.

(a) When $G^*$ is a special unitary group which splits over an unramified extension $F'/F$, a and $a_-$ have different parity.

(b) For all other eligible $G^*$: if $t_\rho$ is odd, then $a$ and $a_-$ have the same parity.

(c) All the labels $\lambda(\alpha), \lambda^*(\alpha)$ in Theorem 4.9 are integers.

Proof. (b) Assume first that $G$ does not have Lie type $2A_{n-1}$. In the proof of Theorem 4.9 we saw how the issue can be reduced from $G$ to $G^*$ or $G\text{Spin}_n$. To $G^*$ and $G\text{Spin}_n$ we apply Proposition 4.10a and the remarks above it.

(a) When $G$ does have Lie type $2A_{n-1}$, Section 2 allows to reduce to $G^* = SU_n$, and then to $U_n$. Now we apply Proposition 4.10b and the remarks above it.

(c) It is clear that the labels in parts (a) and (c) of Theorem 4.9 are integers. We recall that the labels in Theorem 4.9b are

$$\lambda(\alpha) = t_\rho f(a + a_- + 2)/2 \quad \text{and} \quad \lambda^*(\alpha) = t_\rho f(a - a_-)/2.$$
These are integers, except possibly when \( a \) and \( a_+ \) have different parity. In the cases where \( \mathcal{G}^* \) is an unramified special unitary group, \( f = 2 \) and again the labels are integers. In the other cases with \( a \) and \( a_+ \) of different parity, part (b) of the current lemma tells us that \( t_\rho \) is even, which makes the labels integral.

Having checked Conjecture \( A \)(i), we turn to Conjecture \( A \)(ii).

**Lemma 4.12.** Consider a root system of type \( A_{e-1}, B_e, C_e \) or \( D_e \), with label functions \( \lambda, \lambda^* \) as in Theorem 4.9. There exist:

- a simple group \( \mathcal{G} \) over a nonarchimedean local field \( \mathcal{F} \),
- a Bernstein block \( \text{Rep}(\mathcal{G})^s \), which consists of unipotent representations of \( \mathcal{G} = \mathcal{G}(\mathcal{F}) \),
- a \( s \)-type \((J, \rho)\),

such that \( \mathcal{H}(\mathcal{G}, J, \rho) \) is an affine Hecke algebra with the given root system and the given label functions.

**Proof.** When the root system has type \( A_{e-1} \) (resp. \( D_e \)) we take \( G = GL_e \) (resp. \( SO_{2e} \)). Choose a non-archimedean local field \( \mathcal{F} \) with residue field of order \( q_{\mathcal{F}} = q_{\mathcal{F}}^t \).

We take the Iwahori-spherical Bernstein block and let \( J \) be an Iwahori subgroup of \( \mathcal{G} = G(\mathcal{F}) \). Then \((J, \text{triv})\) is a type and \( \mathcal{H}(\mathcal{G}, J, \text{triv}) \) is an affine Hecke algebra with parameters \( q_{\mathcal{F}}^t \). We obtain labels \( \lambda(\alpha) = \lambda^*(\alpha) = ft \).

Suppose that the root system is \( C_e \) (or \( B_e \) with \( a_+ = -1 \), that boils down to the same thing). We choose the \( q \)-base \( q_{\mathcal{F}}^{ft} \), which can be achieved by considering \( \mathcal{F} \)-groups. Thus reduce to the situation where the short root \( \alpha \) has label 1 and the long root \( \beta \) has label \( a_+ \in \mathbb{Z}_{>0} \). Now see [Lus3, 7.40–7.42] when \( a_+ \) is even and [Lus3, 7.56] when \( a_+ \) is odd. In each case, a type for the associated Bernstein component is produced in [Lus3, §1].

Suppose that the root system is \( B_e \) and that \( a_+ \geq 0 \). Let \( \beta \) be a short root and \( \alpha \) a long root. When \( a_+ + a_- \) is even we take the \( q \)-base \( q_{\mathcal{F}}^{ft} \) and we reduce to the labels \( \lambda(\alpha) = 2, \lambda(\beta) = a_+ + a_- + 2, \lambda^*(\beta) = a_+ - a_- \).

Depending on the parities of \( \lambda(\beta) \) and \( \lambda^*(\beta) \), see [Lus3, 7.38–7.39] (both even) or [Lus3, 7.48–7.49] (both odd) or [Lus3, 7.54–7.55] (one even, one odd).

When \( a_+ + a_- \) is odd we take the \( q \)-base \( q_{\mathcal{F}}^{ft/2} \) (a power of \( q_{\mathcal{F}} \) by Lemma 4.11) and we reduce to the labels

\[
\lambda(\alpha) = 1, \quad \lambda(\beta) = (a_+ + a_- + 2)/2, \quad \lambda^*(\beta) = (a_+ - a_-)/2.
\]

See [Lus4, 11.2–11.3] for an appropriate Bernstein component consisting of unipotent representations.

We covered all simple groups of type \( A_n, 2A_n \) or \( B_n \), but some simple groups of Lie type \( C_n, D_n \) or \( 2D_n \) remain, namely:

- the non-split (non-pure) inner twist of a symplectic group,
- the two non-pure inner twists of a split even special orthogonal group,
- the non-pure inner twist of a quasi-split even special orthogonal group,
- groups isogenous to one of the above.

We note that (apart from the last entry) this list consists of classical groups associated to Hermitian forms on vector spaces over quaternionic division algebras. As far as we are aware, much less is known about the representation theory of these
groups. They are ruled out in [Mœ1, Mœ2, Mœ3], so it is not clear which Hecke algebra labels can arise.

For unipotent representations, this is known completely [Lus3, Lus4, Sol2, Sol4], and that indicates that Theorem 4.9 might hold for these groups. The relevant label functions $\lambda, \lambda^*$ in the tables [Lus3, 7.44–7.46 and 7.51–7.53], occur also in Theorem 4.9 (with $a - a_-$ odd, like for unitary groups).

4.5. Groups of Lie type $G_2$.

Up to isogeny, there are three absolutely simple $F$-groups whose relative root system has type $G_2$:

- the split group $G_2$,
- the quasi-split group $3D_4$, which splits over a Galois extension $\tilde{F}/F$ of degree 3 or 6,
- the non-split inner forms $E_6^{(3)}$, which split over the cubic unramified extension $F^{(3)}/F$.

Let $G = \mathcal{G}(F)$ denote the rational points of one of these groups. Let $M$ be a Levi subgroup of $G$ and write $s = [M, \sigma]_G, \mathcal{O} = X_m(M)\sigma$.

When the semisimple rank of $M$ is $\geq 1$, $\Sigma_{\mathcal{O}, \mu}$ has rank $\leq 1$. For those cases we refer to page 11.

Otherwise $\mathcal{M}$ is a minimal $F$-Levi subgroup of $\mathcal{G}$. For $G = G_2(F)$, $\text{Rep}(G)^{s}$ consists of principal series representations. In Theorem 4.4 we proved that $q_\alpha = q_F$ and $q_{\alpha^*} = 1$ for $\alpha \in \Sigma_{\mathcal{O}, \mu}$. For $G = 3D_4(F)$ $\text{Rep}(G)^{s}$ also belongs to the principal series. We showed in (4.15) that $q_{\alpha^*} = 1, q_\alpha = q_F$ for long roots $\alpha \in \Sigma_{\mathcal{O}, \mu}$ and $q_{\beta^*} = 1, q_{\beta} \in \{q_F, q_F^3\}$ for short roots $\beta \in \Sigma_{\mathcal{O}, \mu}$. Notice that in $\Sigma_{\mathcal{O}}$ the lengths of the roots are reversed.

The group $G = E_6^{(3)}$ involves a central simple $F$-algebra $D$ of dimension $3^2 = 9$. Assume for the moment that $\mathcal{G}$ is simply connected, so that we can apply the reduction steps from Section 2 with isogenies that are isomorphisms on the derived groups. For a short root $\alpha \in \Sigma_{\mathcal{O}, \mu}$, the inclusion $M \to M_\alpha$ is isogenous to $S(GL_1(D)^2) \times GL_1(F) \to S(GL_1(D)^2) \times SL_2(F)$.

In particular the coroot $\alpha^\vee$ is orthogonal to $M_{\text{der}}$ and the restriction of $\sigma$ to the image of $\alpha^\vee$ is a direct sum of finitely many characters. Hence the same computations as in Paragraph 4.1 apply here, with $M$ instead of $T$. Thus $q_\alpha = q_F$ and $q_{\alpha^*} = 1$. On the other hand, for a long root $\beta \in \Sigma_{\mathcal{O}, \mu}$ the inclusion $M \to M_\beta$ is isogenous to $S(GL_1(D)^2) \times GL_1(F) \to SL_2(D) \times GL_1(F)$.

Again with Section 2 the computation of the parameters can be transferred to $GL_1(D)^2 \to GL_2(D)$, which is discussed in Paragraph 4.3. Then Theorem 4.6 and (4.19) show that $q_{\beta^*} = 1$ and $q_\beta = q_F^f$ where $f$ divides 3. All this based on an $X_\alpha$ defined in terms of $\alpha^\vee$. We still have to take the effect of the isogenies

(4.25) $M_\beta \leftarrow SL_2(D) \times GL_1(F) \to GL_2(D)$

into account. As worked out in Proposition 2.4, this goes via changing $h_\alpha^\vee$. Since the derived groups are simply connected, no $\alpha^\vee/2$ can be involved, and this effect comes only from changes in the torsion number $|X_m(M, \sigma)|$. That boils down to the torsion number of a representation of $GL_1(D)$, so it can only be 1 or 3. In terms of cocharacter lattices both maps in (4.25) are index 2 inclusions, and 2 is coprime to 3,
so actually the torsion numbers do not change along these inclusions. We conclude that the labels are \( \lambda(\alpha) = 1 \) and \( \lambda(\beta) \in \{1, 3\} \) (and the same for \( \lambda^* \)).

When \( G \) is not simply connected, we can apply Proposition 2.2 to compare with its simply connected cover. If \( \Sigma_{O,\mu} \) has rank \( > 1 \), then it is isomorphic to \( A_1 \times A_1, A_2 \) or \( G_2 \). In the latter two cases Proposition 2.4 tells us that the parameters do not change when we pass from \( G \) to its simply connected cover. In the first case there could be a change as in Proposition 2.4.d.(ii), but that does not bother us because we already understand affine Hecke algebras of type \( A_1 \) completely.

4.6. Groups of Lie type \( F_4 \).

Just as for \( G_2 \) we will analyse all possibilities for the parameters, by reduction to earlier cases. Up to isogeny there are three absolutely simple \( F \)-groups with relative root system of type \( F_4 \):

- the split group \( F_4 \),
- the quasi-split group \( ^2E_6 \), split over a separable quadratic extension \( F'/F \),
- the non-split inner form \( E_7^{(2)} \), split over the unramified quadratic extension \( F^{(2)}/F \).

Supported by Section 2, we only consider the simply connected version of these groups. We number the bases of \( F_4 \) and \( E_7 \) as follows:

\[
F_4: 1 2 3 4 5 6 7 \quad \quad E_7: 1 2 3 4 5 6 7
\]

Let \( D \) be a central simple \( F \)-algebra of dimension \( 2^2 = 4 \). The anisotropic kernel of \( E_7^{(2)}(F) \) corresponds to the labels 1, 5, 7 and is isomorphic to \( SL_1(D)^3 \).

Let \( G = G(F) \) denote the rational points of one of the above groups. Fix a maximal \( F \)-split torus \( S = S(F) \) and let \( \Delta \) be a basis of \( \Sigma(G,S) \). Let \( M_J = M_J(F) \) be the standard Levi subgroup associated to \( J \subset \Delta \). Write \( g = [M_J,\sigma]_G \) and \( O = X_{nr}(M_J)\sigma \).

\[ |J| = 3 \text{ or } |J| = 4 \]

In these cases \( \Sigma_{O,\mu} \) has rank \( \leq 1 \), and we refer to page 11.

For the other subsets \( J \) of \( \Delta \) we make use of the data collected in [How, p. 74]. There we find the normalizer of \( W_J \) in \( W(F_4) \) and the root system underlying

\[ N_{W(F_4)}(W_J)/W_J \cong \text{Stab}_{W(F_4)}(J). \]

\( J \) equals \( \{\alpha_1, \alpha_4\} \) or \( \{\alpha_1, \alpha_3\} \) or \( \{\alpha_2, \alpha_4\}, \Sigma_{\text{red}}(A_{M_J}) \cong A_1 \times A_1 \)

These three subsets of \( \Delta \) are associate under the Weyl group \( W(F_4) \), so it suffices to consider \( J = \{\alpha_1, \alpha_4\} \). Up to a sign there are just two possibilities for \( \alpha \in \Sigma_{O,\mu} \cong A_1 \times A_1 \). These can be represented by \( \alpha_2 \) and \( \alpha_4 \). The inclusion \( M_J \rightarrow M_{J \cup \{\alpha_4\}} \) is isogeneous the following inclusion (depending on the type of \( G \))

\[
\begin{align*}
F_4 \quad & SL_2(F) \times GL_2(F) \times GL_1(F) \rightarrow SL_2(F) \times GL_3(F) \\
^2E_6 \quad & SL_2(F) \times GL_2(F') \times GL_1(F') \rightarrow SL_2(F) \times GL_3(F') \\
E_7^{(2)} \quad & SL_2(F) \times GL_2(D) \times GL_1(D) \rightarrow SL_2(F) \times GL_3(D)
\end{align*}
\]
Here the direct factors $SL_2(F)$ do not influence the other direct factors, so the parameters $q_{\alpha_4}$ and $q_{\alpha_4^*}$ can be computed ignoring these factors $SL_2(F)$. Then \[\text{(4.18)}\] reveals that $\alpha_4$ does not contribute to $\Sigma_{O,\mu}$, so $q_{\alpha_4} = q_{\alpha_4^*} = 1$.

We also list inclusions isogenous to $M_J \to M_{J \cup \{\alpha_2\}}$:

<table>
<thead>
<tr>
<th>Group</th>
<th>Inclusion</th>
<th>$q_{\alpha_4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_4$</td>
<td>$GL_1(F) \times Sp_4(F) \times GL_1(F) \to GL_1(F) \times Sp_6(F)$</td>
<td>$q_F^{\alpha_4}$</td>
</tr>
<tr>
<td>$^{2}E_6$</td>
<td>$GL_1(F) \times S_0^4(F) \times GL_1(F) \to SO_8^*(F) \times GL_1(F)$</td>
<td>$q_F^{\alpha_4^*}$</td>
</tr>
<tr>
<td>$E_7^{(2)}$</td>
<td>$GL_2(F) \times SO_6^*(F) \times GL_1(D) \to SO_1^4(F) \times GL_1(D)$</td>
<td>$q_F^{\alpha_4}$</td>
</tr>
</tbody>
</table>

Here $SO_{2n}^*$ denotes a quasi-split special orthogonal group, while $SO_{2n}'$ stands for a non-split inner form of $SO_{2n}$. For the parameter computations, the direct factors $GL_1(F)$ and $GL_1(D)$ can be ignored. In all three cases Theorem 4.9.b shows that $q_{\alpha_2} = q_F^{t(\alpha+1)/2}$ and $q_{\alpha_2^*} = q_F^{t(\alpha-1)/2}$, where $t \in \{1,2\}$. A small correction might still come from the involved isogenies via Proposition 2.4.

$J = \{\alpha_2, \alpha_3\}, \Sigma_{\text{red}}(A_{M_J}) \cong B_2$

Here $\alpha_1$ gives rise to a long root and $\alpha_4$ to a short root of $\Sigma_{\text{red}}(A_{M_J})$. In the root system $\Sigma^Y_\alpha$ these lengths are reversed, so there exists an $x \in M_\alpha^2/M_1$ with $X_{\alpha_4}(x) = -1$. Then the proof of [Sol6] Lemma 3.3 shows that $\mu^{M_{\alpha_4}}$ does not have a pole at $\{X_{\alpha_4} = -1\}$, so that $q_{\alpha_{4^*}} = 1$. Considering the inclusion $M_J \to M_{J \cup \{\alpha_4\}}$ up to isogenies and applying Theorem 4.9.a we find:

Here $a_+, a_+^* \leq 3$. For $E_7^{(2)}$ this involves quaternionic special orthogonal groups, which we could not handle in Theorem 4.9. We expect that the effect of replacing $F$ by $D$ on the parameters is just squaring, so that $q_{\alpha_4}$ should be $q_F^{2a_4}$ for some integer $a \in [0,3]$.

Up to isogenies, the inclusions $M_J \to M_{J \cup \{\alpha_1\}}$ are:

<table>
<thead>
<tr>
<th>Group</th>
<th>Inclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_4$</td>
<td>$GL_1(F) \times SO_5(F) \times GL_1(F) \to SO_7(F) \times GL_1(F)$</td>
</tr>
<tr>
<td>$^{2}E_6$</td>
<td>$GL_1(F) \times S_0^4(F) \times GL_1(F') \to SO_8^*(F) \times GL_1(F')$</td>
</tr>
<tr>
<td>$E_7^{(2)}$</td>
<td>$GL_1(F) \times SO_6^*(F) \times GL_1(D) \to SO_1^4(F) \times GL_1(D)$</td>
</tr>
</tbody>
</table>

In all three cases Theorem 4.9.b shows that $q_{\alpha_1} = q_F^{(\alpha+1)/2}$ and $q_{\alpha_1^*} = q_F^{(\alpha-1)/2}$. We expect that these integers $a, a_-$ are related to the $a_+$ for $\alpha_4$, but working that out would lead us too far afield. For unipotent representations of $F_4(F)$, [Lus3] \[\text{§7.31}\] shows that $q_{\alpha_1} = q_F^3, q_{\alpha_1} = q_F^2$ and $q_{\alpha_1^*} = q_F$, a pattern that does not fit with Theorem 4.9.

$J = \{\alpha_1, \alpha_2\}, \Sigma_{\text{red}}(A_{M_J}) \cong G_2$

Now $\alpha_3$ gives rise to a short root of $\Sigma_{\text{red}}(A_{M_J})$, and to a long root of $\Sigma^Y_\alpha$. Analysing
the inclusion $M_J \to M_{J,\cup(\alpha_1)}$ up to isogeny, we obtain:

<table>
<thead>
<tr>
<th>group</th>
<th>inclusion</th>
<th>$q_{\alpha_4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_4$</td>
<td>$SL_3(F) \times GL_1(F)^2$ $\to$ $SL_3(F) \times GL_2(F)$</td>
<td>$q_F$</td>
</tr>
<tr>
<td>$2E_6$</td>
<td>$SL_3(F) \times GL_1(F)^2$ $\to$ $SL_3(F) \times GL_2(F)$</td>
<td>$q_F^3$</td>
</tr>
<tr>
<td>$E_7^{(2)}$</td>
<td>$SL_3(F) \times SL_1(D) \times GL_1(D)^2$ $\to$ $SL_3(F) \times SL_1(D) \times GL_2(D)$</td>
<td>$q_F^t$</td>
</tr>
</tbody>
</table>

where $f \in \{1, 2\}$. In all three cases $q_{\alpha_4} = 1$ by Theorem 4.6. Since $\alpha_4$ is orthogonal to $M_{J,\text{der}}$, the computations behind these $q$-parameters work equally well in $M_{J,\cup(\alpha_1)}$, no corrections from isogenies are needed.

For $\alpha_3$ we find

<table>
<thead>
<tr>
<th>group</th>
<th>inclusion</th>
<th>$q_{\alpha_3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_4$</td>
<td>$GL_3(F) \times GL_1(F)$ $\to$ $SO_6(F) \times GL_1(F)$</td>
<td>$q_F^{(a-1)/2}$</td>
</tr>
<tr>
<td>$2E_6$</td>
<td>$GL_3(F) \times SO_4(F) \times GL_1(F)$ $\to$ $SO_6^*(F) \times GL_1(F)$</td>
<td>$q_F^{(a-1)/2}$</td>
</tr>
<tr>
<td>$E_7^{(2)}$</td>
<td>$GL_3(F) \times SO_4(F) \times GL_1(D)$ $\to$ $SO_4^0(F) \times GL_1(D)$</td>
<td>$q_F^{(a-1)/2}$</td>
</tr>
</tbody>
</table>

where $t \in \{1, 3\}$. The parameter $q_{\alpha_3}$ equals $q_F^{(a-1)/2}$. Recall the bound on $a$ and $a_-$ from Theorem 4.9.

When $\Sigma_{\O,\mu}$ has type $G_2$, [Sol6, Lemma 3.3] says that $q_{\alpha_3} = 1$. Then $a_- = -1$ and Lemma 4.11 tells us that $a$ is odd. For $F_4$ and $E_6^{(2)}$ that means $a = -1$ and $q_{\alpha_3} = 1$, so that actually $\Sigma_{\O,\mu}$ does not have type $G_2$. For $E_7^{(2)}$ it would still be possible that $a = 1$, so that $q_{\alpha_3} = q_F^4$. But then the Langlands parameter of a representation of $SO_4^0(F)$ would be the sum of a three-dimensional and a one-dimensional representation of $W_F$ which is not compatible with the isogeny to $SL_1(D)$. Hence this case does not arise, and we conclude that for $J = \{\alpha_1, \alpha_2\}$ the root system $\Sigma_{\O,\mu}$ has rank $\leq 1$.

$J = \{\alpha_3, \alpha_4\}$, $\Sigma_{\text{red}}(A_{M_J}) \cong G_2$

Now a long root of $\Sigma_{\text{red}}(A_{M_J})$ comes from $\alpha_1$, and in $\Sigma_{\O}^{\vee}$ a short root comes from $\alpha_1$. The inclusion $M_J \to M_{J,\cup(\alpha_1)}$ is isogenous to:

- $F_4$: $GL_1(F)^2 \times SL_3(F) \to GL_2(F) \times SL_3(F)$
- $2E_6$: $GL_1(F)^2 \times SL_3(F') \to GL_2(F) \times SL_3(F')$
- $E_7^{(2)}$: $GL_1(F)^2 \times SL_3(D) \to GL_2(F) \times SL_3(D)$

In each case the parameters can be analysed already with $GL_1(F)^2 \to GL_2(F)$, and Theorem 4.6 tells us that $q_{\alpha_1} = q_F^4$, $q_{\alpha_1} = 1$.

Let us also consider the inclusion $M_J \to M_{J,\cup(\alpha_2)}$ up to isogenies:

<table>
<thead>
<tr>
<th>group</th>
<th>inclusion</th>
<th>$q_{\alpha_2}$</th>
<th>$q_{\alpha_2*}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_4$</td>
<td>$GL_1(F) \times GL_3(F)$ $\to$ $GL_1(F) \times SO_7(F)$</td>
<td>$q_F^{t(a+1)/2}$</td>
<td>$q_F^{t(a-1)/2}$</td>
</tr>
<tr>
<td>$2E_6$</td>
<td>$GL_1(F) \times GL_3(F)$ $\to$ $GL_1(F) \times SO_7(F)$</td>
<td>$q_F^{t(a+1)/2}$</td>
<td>$q_F^{t(a-1)/2}$</td>
</tr>
<tr>
<td>$E_7^{(2)}$</td>
<td>$GL_1(F) \times GL_3(D)$ $\to$ $GL_1(F) \times SO_6(D)$</td>
<td>$q_F^{t(a+1)/2}$</td>
<td>$q_F^{t(a-1)/2}$</td>
</tr>
</tbody>
</table>

Here $t \in \{1, 3\}$ and by Theorem 4.9 $0 \geq a \geq a_- \geq -1$. When $\Sigma_{\O,\mu} \cong G_2$, we know from [Sol6, Lemma 3.3] that $q_{\alpha_2*} = 1$. With Lemma 4.11 that implies $q_{\alpha_2} = 1$ for $F_4$ and for $2E_6$ if $F'/F$ is ramified. For $2E_6$ with $F'/F$ unramified, it is still possible that $a = 0$, so that $q_{\alpha_2} = q_F^2 = q_F^4$. For the same reasons as after (4.25), no corrections from isogenies are needed. For $E_7^{(2)}$ the analysis involves quaternionic
special orthogonal groups, a case which remains open.

\[ J = \{\alpha_1\} \text{ or } J = \{\alpha_2\}, \Sigma_{\text{red}}(A_{M_J}) \cong C_3 \]

These two are \( W(F_I) \)-conjugate, so it suffices to consider \( J = \{\alpha_1\} \). Then a long root of \( \Sigma_{\text{red}}(A_{M_J}) \) is associated to \( \alpha_2 \). The parameters for \( \alpha_3 \) and \( \alpha_4 \) are the same as those for \( \alpha_4 \) in the case \( J = \{\alpha_1, \alpha_2\} \). The inclusion \( M_J \rightarrow M_{J \cup \{\alpha_2\}} \) is isogenous to:

\[
\begin{align*}
F_4 & \quad GL_2(F) \times GL_1(F)^2 \rightarrow SL_3(F) \times GL_1(F)^2 \\
2E_6 & \quad GL_2(F) \times GL_1(F')^2 \rightarrow SL_3(F) \times GL_1(F)^2 \\
E_7^{(2)} & \quad GL_2(F) \times SL_3(D) \rightarrow SL_3(F) \times SL_3(D)
\end{align*}
\]

This reduces to \( GL_2(F) = S(GL_2(F) \times GL_1(F)) \rightarrow SL_3(F) \) and from there to \( GL_2(F) \times GL_1(F) \rightarrow GL_3(F) \). From Corollary 4.7 we see that \( \alpha_2 \) does not contribute to \( \Sigma_{O,\mu} \), so \( \Sigma_{O} \) has type \( A_2 \) or \( A_1 \) (or is empty).

\[ J = \{\alpha_3\} \text{ or } J = \{\alpha_4\}, \Sigma_{\text{red}}(A_{M_J}) \cong B_4 \]

These two are \( W(F_I) \)-conjugate, so it suffices to consider \( J = \{\alpha_4\} \). Here a short root of \( \Sigma(A_{M_J}) \) comes from \( \alpha_3 \). The parameters for \( \alpha_1 \) and \( \alpha_2 \) are the same as the parameters for \( \alpha_1 \) in the case \( J = \{\alpha_3, \alpha_4\} \), so \( q_{\alpha_1} = q_{\alpha_2} = q_F \) and \( q_{\alpha_1^*} = q_{\alpha_2^*} = 1 \).

The inclusion \( M_J \rightarrow M_{J \cup \{\alpha_3\}} \) is isogenous to:

\[
\begin{align*}
F_4 & \quad GL_1(F)^2 \times GL_2(F) \rightarrow GL_1(F) \times SL_3(F) \\
2E_6 & \quad GL_1(F)^2 \times GL_2(F') \rightarrow GL_1(F)^2 \times SL_3(F') \\
E_7^{(2)} & \quad GL_1(F) \times GL_1(D) \times GL_2(D) \rightarrow GL_1(F) \times GL_3(D)
\end{align*}
\]

Now Corollary 4.7 shows that \( \alpha_3 \) does not contribute to \( \Sigma_{O,\mu} \). Hence \( \Sigma_{O} \) has type \( A_2 \) or \( A_1 \) (or is empty).

\[ J \text{ is empty, } \Sigma_{\text{red}}(A_{M_J}) \cong F_4 \]

For \( G_2 \) and \( ^2E_6 \), \( \text{Rep}(G)^a \) consists of principal series representations. For \( G = G_2(F) \) we proved in Theorem 4.4 that \( q_{\alpha} = q_F \) and \( q_{\alpha^*} = 1 \) for all \( \alpha \in \Sigma_{O,\mu} \).

For \( G = ^2E_6(F) \), we showed in (4.15) that \( q_{\alpha^*} = 1, q_{\alpha} = q_F \) for long roots \( \alpha \in \Sigma_{O,\mu} \) and \( q_{\beta^*} = 1, q_{\beta} \in \{q_F, q_F^2\} \) for short roots \( \beta \in \Sigma_{O,\mu} \). Notice that in \( \Sigma_{O}^\vee \), the lengths of the roots are reversed.

For \( G = E_7^{(2)}(F) \) and \( \alpha \in \{\alpha_1, \alpha_2\} \), the reasoning as for \( J = \{\alpha_4\} \) and for \( J = \{\alpha_3, \alpha_4\} \) shows that \( q_{\alpha} = q_F \) and \( q_{\alpha^*} = 1 \). For \( \alpha \in \{\alpha_3, \alpha_4\} \), the arguments from the cases \( J = \{\alpha_1\} \) and \( J = \{\alpha_1, \alpha_2\} \) entail that \( q_{\alpha^*} = 1 \) and \( q_{\alpha} \in \{q_F, q_F^2\} \).

Summarising: we checked our main conjecture for absolutely simple groups with relative root system of type \( F_4 \), except for the subset \( J = \{\alpha_2, \alpha_3\} \) of \( \Delta \). For the group \( E_7^{(2)} \) we are also not sure when \( J = \{\alpha_3, \alpha_4\} \).

4.7. Groups of Lie type \( E_6, E_7, E_8 \)

We consider simply connected \( F \)-split groups of type \( E_n \). We number \( E_6 \) and \( E_8 \) (or rather their bases \( \Delta \)) as

\[
\begin{align*}
E_6 & \quad \text{Diagram} & \quad E_8 & \quad \text{Diagram}
\end{align*}
\]

and \( E_7 \) similarly (as on page 40). The number of inequivalent Levi subgroups is
quite large, which renders a case-by-case analysis as for $G_2$ and $F_4$ elaborate. An advantage is of course that all these Levi subgroups are simply connected and $F$-split, so the analysis of Hecke algebra parameters for $E_n$ consists of the principal series (dealt with in Paragraph 4.1) and contributions from split groups of lower rank. For Levi subgroups of semisimple rank $n - 1$ the root system $\Sigma_{\mathcal{O},\mu}$ has rank $\leq 1$, and on page 11 we discussed all such cases.

For $E_6$ and Levi subgroups of semisimple rank at most 4, the $q$-parameters can be computed via inclusions $M \rightarrow M_{\alpha}$ where $M_{\alpha}$ has semisimple rank at most 5. These $M_{\alpha}$ are not exceptional, so the $q$-parameters can be found in Paragraphs 4.3 and 4.4. We only consider the subsets $J \subset \Delta$ such that $\Sigma_{\text{red}}(A_{M_J})$ has a component of type $B_n, C_n, F_4$ or $G_2$ – for irreducible root systems Conjecture $A$ is trivial anyway. The possible $J$ can be found by inspecting the tables on [How, p.75–77].

\[
J = \{\alpha_1, \alpha_3\}, \Sigma_{\text{red}}(A_{M_J}) \cong B_3
\]
The long simple roots $\alpha \in \Sigma_{\text{red}}(A_{M_J})$ correspond to $\alpha_5, \alpha_6 \in \Delta$, which are orthogonal to $M_{\beta,\text{der}}$. Hence the computations reduce to those in Paragraph 4.1 and yield $q_{\alpha \ast} = 1, q_{\mu} = q_F$ (or $\alpha \not\in \Sigma_{\mathcal{O},\mu}$).

The short simple root $\beta$ of $\Sigma_{\text{red}}(A_{M_J})$ comes from $\alpha_4 \in \Delta$. Here $M_{\beta,\text{der}} \cong SL_4(F)$ and $M \cap M_{\beta,\text{der}} \cong S(GL_1(F)^2)$. If $\beta \in \Sigma_{\mathcal{O},\mu}$, then Corollary 4.7 associates to $GL_1(F)^2 \rightarrow GL_2(F)$ the parameters $q_{\beta \ast} = 1$ and $q_{\beta} = q_F^f$ with $f \in \{1, 2\}$. Under the isogenies that transfer back to $M \rightarrow M_{\beta}$, $h_{\beta}^\mu$ remains equal to $\beta^\nu$, so the $q$-parameters do not change.

\[
J = \{\alpha_1, \alpha_3, \alpha_4\}, \Sigma_{\text{red}}(A_{M_J}) \cong B_2
\]
The long simple root $\alpha_1$ of $\Sigma_{\text{red}}(A_{M_J})$ comes from $\alpha_6 \in \Delta$, which is orthogonal to $M_{\beta,\text{der}}$. Hence $q_{\alpha} = q_F$ and $q_{\alpha \ast} = 1$.

The short simple root $\beta$ of $\Sigma_{\text{red}}(A_{M_J})$ comes from $\alpha_5 \in \Delta$. Here $M_{\beta,\text{der}} \cong SL_6(F) \times GL_1(F)$. The $q$-parameters for this setting are known from Theorem 4.9:

\[
q_{\beta} = q_F^{(a+1)/2}, q_{\beta \ast} = q_F^{(a-1)/2}
\]

so $a \leq 4$. When we apply Proposition 2.4 to $M_{\beta,\text{der}} \rightarrow M_{\beta}$, the parameters stay the same or (only when $a = a_-$) Proposition 2.4.d.(iii) applies.

\[
J = \{\alpha_2, \alpha_3, \alpha_5, \alpha_6\}, \Sigma_{\text{red}}(A_{M_J}) \cong G_2
\]
The long simple root $\alpha_1 \in \Sigma_{\text{red}}(A_{M_J})$ comes from $\alpha_1 \in \Delta$. That one is orthogonal to $M_{\beta,\text{der}}$, so by Paragraph 4.1 $q_{\alpha} = q_F$ and $q_{\alpha \ast} = 1$.

The short simple root $\beta \in \Sigma_{\text{red}}(A_{M_J})$ comes from $\alpha_4 \in \Delta$. Now $M_{\beta,\text{der}} \cong SL_6(F)$ and $M \cap M_{\beta,\text{der}} \cong S(GL_3(F)^2)$. The same arguments as above for $J = \{\alpha_1, \alpha_3\}$ shows that here (if $\beta \in \Sigma_{\mathcal{O},\mu}$) $q_{\beta \ast} = 1$ and $q_{\alpha} = q_F^f$ with $f \in \{1, 3\}$.

Having checked Conjecture $A$ for $E_6$, we turn to the simply connected split $F$-groups of type $E_7$ and $E_8$. For most $J \subset \Delta$, the $q$-parameters of $H(\mathcal{O}, G)$ can be analysed as before. However, some $J$ behave like $\{\alpha_2, \alpha_3\}$ for $F_4$, where we found it hard to relate the parameters of the two simple roots to each other. For other $J$ (only in $E_8$) the computation of the $q$-parameters can only be reduced to inclusions
of Lie type $A_2 \times A_1 \times A_2 \rightarrow E_6$ or $D_6 \rightarrow E_7$ or $E_6 \rightarrow E_7$, and we do not know an effective method in these cases.

Therefore we settle for a modest goal: we want to prove our main conjecture whenever the root system $\Sigma_{O,\mu}$ has a component of type $F_4$. From [How p.75–77] one sees that this happens in only very few cases. For any root $\alpha$ in a type $F_4$ root system, [Sol6 Lemma 3.3] shows that $q_{\alpha*} = 1$, and then Proposition 2.4 entails that no involved isogeny can change the parameters.

For $G = E_7(F)$ there is only one $J$ with $\Sigma_{red}(A_{M_j}) \cong F_4$, namely $J = \{\alpha_1, \alpha_5, \alpha_7\}$. The $q$-parameters can be obtained in the same way as for $E_7^{(2)}(F)$ and $J = \emptyset$, as treated in Paragraph 4.6. The only difference is that an inclusion $S(GL_1(D)^2) \rightarrow SL_2(D)$ must be replaced be an inclusion $S(GL_2(F)^2) \rightarrow SL_4(F)$, but from Paragraph 4.3 we know that exactly the same $q$-parameters can occur for both these inclusions. Thus $q_\alpha = q_F$, $q_{\alpha*} = 1$ for any long root $\alpha \in \Sigma_{O,\mu} \cong F_4$ and $q_{\beta*} = 1, \beta \in \{q_F, q_F^2\}$ for any short root $\beta \in \Sigma_{O,\mu}$.

For $E_8(F)$ and $J = \{\alpha_1, \alpha_5, \alpha_7\}$ we also have $\Sigma_{red}(A_{M_j}) \cong F_4$. This case can be handled just as for $E_7$, and leads to the same $q$-parameters.

For $E_8(F)$ and $J = \{\alpha_1, \alpha_3, \alpha_5\}$ we have $\Sigma_{red}(A_{M_j}) \cong F_4 \times A_1$. The long simple roots of $F_4$ come from $\alpha_7, \alpha_8 \in \Delta$. These are orthogonal to $M_{\text{der}}$, so $q_\alpha = q_F$ and $q_{\alpha*} = 1$. According to [How p. 75] the short simple roots $\beta$ of $F_4$ are associated to an inclusion $S(GL_2(F)^2) \rightarrow SL_4(F)$. We can use the same $X_\beta$ as for $GL_2(F)^2 \rightarrow GL_4(F)$, for which Corollary 4.7 shows that $q_{\beta*} = 1$ and $q_\beta \in \{q_F, q_F^2\}$.

The only remaining case with $\Sigma_{red}(A_{M_j}) \cong F$ is $J = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5\}$. Like in the previous case $q_\alpha = q_F$, $q_{\alpha*} = 1$ for any long simple root $\alpha \in \Sigma_{O,\mu}$. Both short simple roots $\beta$ of $F_4$ come from a non-simple root in $E_8$, for which $M \cap M_{\beta,\text{der}} \rightarrow M_{\beta,\text{der}}$ is isomorphic to the inclusion of a double cover of $SO_8(F) \times GL_1(F)$ in Spin$_{10}(F)$. According to Theorem 4.9 the resulting $q$-parameters are

$$q_\beta = q_F^{(a+1)/2}, q_{\beta*} = q_F^{(a-1)/2},$$

where $[\frac{a+1}{2}] + [\frac{a+1}{2}] \leq 8$.

Since $\Sigma_{O,\mu}$ has type $F_4$, $q_{\beta*} = 1$ and $a_- = -1$. From Lemma 4.11 we know that $a$ and $a_-$ have the same parity, so $a$ is odd. The estimate shows that $a < 5$, so $a \in \{1, 3\}$ and $q_\beta \in \{q_F, q_F^2\}$ as desired.

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[Bor] A. Borel, “Admissible representations of a semi-simple group over a local field with vectors fixed under an Iwahori subgroup”, Inv. Math. 35 (1976), 233–259


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